Hilbert space with reproducing kernel and uniform distribution preserving maps, II

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Introduction

- $\Phi(\mathbf{x})$ – uniform distribution preserving map
- the mean square worst-case error

$$\int_{[0,1]^s} \sup_{\substack{f\in H \\ ||f||\le 1}} \left| \frac{1}{N}\sum_{n=0}^{N-1} f(\Phi(\mathbf{x}_n \oplus \boldsymbol{\sigma})) - \int_{[0,1]^s} f(\mathbf{x})\mathrm{d}\mathbf{x} \right|^2 \mathrm{d}\boldsymbol{\sigma} \qquad (1)$$

H – Hilbert space with reproducing kernel $K(\mathbf{x}, \mathbf{y})$ with respect to a sequence $\mathbf{x}_0, \dots, \mathbf{x}_{N-1}$ shifted to $\Phi(\mathbf{x}_0 \oplus \boldsymbol{\sigma}), \dots, \Phi(\mathbf{x}_{N-1} \oplus \boldsymbol{\sigma})$. Applying the method of Fourier-Walsh expansion (Hickernell 2002), we found

$$\sum_{\substack{\mathbf{k}\in\mathbb{N}_0^s \\ \mathbf{k}\neq\mathbf{0}}} \widehat{K}_1(\mathbf{k},\mathbf{k}) \left| \frac{1}{N}\sum_{n=0}^{N-1} \mathrm{wal}_{\mathbf{k}}(\mathbf{x}_n) \right|^2, \qquad (2)$$

where $\widehat{K}_1(\mathbf{k},\mathbf{k})$ are Fourier-Walsh coefficients of $K(\Phi(\mathbf{x}), \Phi(\mathbf{y}))$.

- J.F.Hickernell (2002)
- L.L. Cristea, J. Dick, G. Leobacher and F. Pillichshammer (2007)

Basic notions

- $x = \frac{x_0}{b} + \frac{x_1}{b^2} + \dots$ is a b-adic representation of $x \in [0,1)$
- $\sigma = \frac{\sigma_0}{b} + \frac{\sigma_1}{b^2} + \dots$
- $x \oplus \sigma = \frac{x_0+\sigma_0 \pmod b}{b} + \frac{x_1+\sigma_1 \pmod b}{b^2} + \dots$.
- $\mathbf{x} = (x_1, x_2, \dots, x_s)$, $\mathbf{y} = (y_1, y_2, \dots, y_s)$, and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \dots, \sigma_s)$ are vectors in $[0,1)^s$ then $\mathbf{x} \oplus \boldsymbol{\sigma} = (x_1 \oplus \sigma_1, x_2 \oplus \sigma_2, \dots, x_s \oplus \sigma_s)$
- $\Psi(\mathbf{x}, \boldsymbol{\sigma})$ is an u.d.p. map of the form $\Phi(\mathbf{x} \oplus \boldsymbol{\sigma})$ or $\Phi(\mathbf{x} + \boldsymbol{\sigma} \bmod 1)$, where $\Phi(\mathbf{x})$ is an arbitrary u.d.p. map;
- $\boldsymbol{\sigma}_i$, $i = 0, 1, \dots,$ is a u.d. sequence in $[0,1)^s$;
- $g_{m,n}(\mathbf{x}, \mathbf{y})$ is a distribution function (d.f.) of the sequence $(\mathbf{x}_m \oplus \boldsymbol{\sigma}_i, \mathbf{x}_n \oplus \boldsymbol{\sigma}_i)$, $i = 0, 1, 2, \dots,$ if $\Psi(\mathbf{x}, \boldsymbol{\sigma}) = \Phi(\mathbf{x} \oplus \boldsymbol{\sigma})$ or the sequence $(\mathbf{x}_m + \boldsymbol{\sigma}_i \bmod 1, \mathbf{x}_n + \boldsymbol{\sigma}_i \bmod 1)$, $i = 0, 1, 2, \dots$ if $\Psi(\mathbf{x}, \boldsymbol{\sigma}) = \Phi(\mathbf{x} + \boldsymbol{\sigma} \bmod 1)$.
- $H = \{f : [0,1] \to \mathbb{R}; f$ absolutely continuous , $f(1) = 0$, $\int_0^1 (f'(x))^2 \mathrm{d}x < \infty\}$, with scalar product $f(x) \cdot g(x) = \int_0^1 f'(x) g'(x) \mathrm{d}x$ and reproducing kernel $K(x, y) = 1 - \max(x, y)$.

Distribution functions method

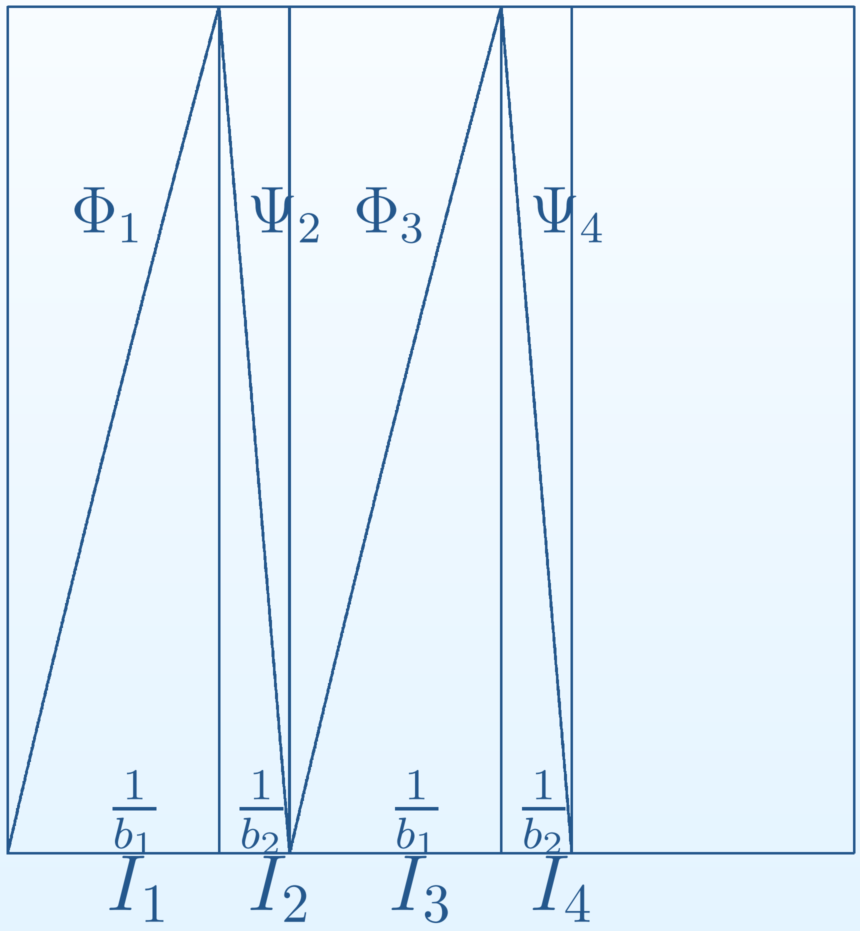
We have the following basic equation.

$$\int_{[0,1]^s} \sup_{\substack{f\in H\\ ||f||\le 1}} \left| \frac{1}{N}\sum_{n=0}^{N-1} f(\Psi(\mathbf{x}_n,\boldsymbol{\sigma})) - \int_{[0,1]^s} f(\mathbf{x})\mathrm{d}\mathbf{x}\right|^2 \mathrm{d}\boldsymbol{\sigma}$$

$$= \frac{1}{N^2}\sum_{n,m=0}^{N-1}\int_{[0,1]^s} K(\Phi(\mathbf{x}),\Phi(\mathbf{y}))\mathrm{d}_{\mathbf{x}}\mathrm{d}_{\mathbf{y}} g_{m,n}(\mathbf{x},\mathbf{y}) - \int_{[0,1]^{2s}} K(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}. \tag{3}$$

Generalized tent function

Extend u.d.p. $\Phi(x) = 1 - |2x-1|$ to $\Phi_{b_1,b_2}(x)$



One-dimensional case

$$\int_0^1 \sup_{\substack{f\in H\\ ||f||\le 1}} \left| \frac{1}{N}\sum_{n=0}^{N-1} f(\Phi_{b_1,b_2}(x_n\oplus\sigma)) - \int_0^1 f(x)\mathrm{d}x \right|^2 \mathrm{d}\sigma$$

$$= -\frac{1}{3} + \frac{1}{N^2}\sum_{m,n=0}^{N-1}\Bigg[\sum_{\substack{i,j=1\\ i,j-\text{even}}}^{k} b_2 \int_{\frac{i}{k}-\frac{1}{b_2}}^{\frac{i}{k}} g_{m,n}\left(x, x+\frac{j-i}{k}\right)\mathrm{d}x$$

$$+ \sum_{\substack{i,j=1\\ i,j-\text{odd}}}^{k} b_1 \int_{\frac{i-1}{k}}^{\frac{i-1}{k}+\frac{1}{b_1}} g_{m,n}\left(x, x+\frac{j-i}{k}\right)\mathrm{d}x$$

$$- \sum_{\substack{i,j=1\\ i-\text{even},j-\text{odd}}}^{k} b_2 \int_{\frac{i}{k}-\frac{1}{b_2}}^{\frac{i}{k}} g_{m,n}\left(x, \frac{b_2}{b_1}\left(\frac{i}{k}-x\right)+\frac{j-1}{k}\right)\mathrm{d}x$$

$$- \sum_{\substack{i,j=1\\ i-\text{odd},j-\text{even}}}^{k} b_1 \int_{\frac{i-1}{k}}^{\frac{i-1}{k}+\frac{1}{b_1}} g_{m,n}\left(x, \frac{b_1}{b_2}\left(\frac{i-1}{k}-x\right)+\frac{j}{k}\right)\mathrm{d}x\Bigg], \qquad (4)$$

Cont.

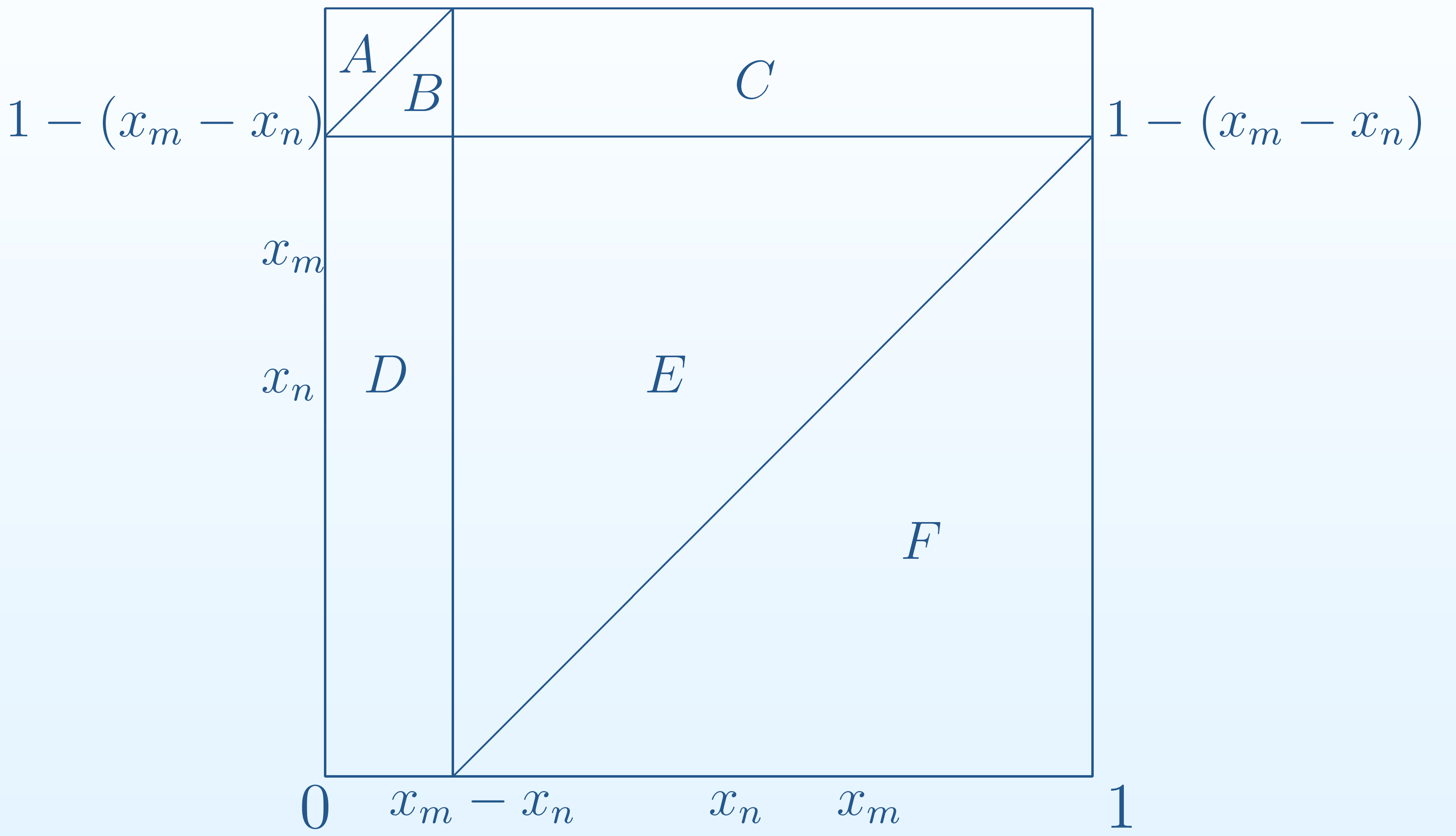
where $k/2 = 1/b_1 + 1/b_2$, $2|k$. The bases b_1, b_2 for u.d.p. map $\Phi_{b_1,b_2}(x)$ are independent of the base of the shift $x_n \oplus \sigma$. The equation (4) also holds for $\Phi_{b_1,b_2}(x_n + \sigma \mod 1)$.

$g_{m,n}(x, y)$

In the case $x_m = x_n$ we have $g_{m,m}(x, y) = \min(x, y)$ and for $x_m \neq x_n$ we have

$$g_{m,n}(x,y) = \begin{cases} x & \text{if } (x,y) \in A, \\ y - (1 - |x_m - x_n|) & \text{if } (x,y) \in B, \\ x + y - 1 & \text{if } (x,y) \in C, \\ 0 & \text{if } (x,y) \in D, \\ x - |x_m - x_n| & \text{if } (x,y) \in E, \\ y & \text{if } (x,y) \in F, \end{cases} \tag{5}$$

where



$\Phi(x) = bx \mod 1$

If $b_2 \to \infty$, k =constant, then $b_1 \to b = \frac{k}{2}$ and $\Phi_{b_1,b_2}(x) \to \Phi(x)$. The following limits hold:
Let $K(x,y) = 1 - \max(x,y)$ be the kernel, let $\Phi(x) = bx \bmod 1$ be the u.d.p. map and $k = 2b$. Then we have

$$\int_0^1 \sup_{\substack{f\in H\\ ||f||\leq 1}} \left| \frac{1}{N}\sum_{n=0}^{N-1} f(\Phi(x_n \oplus \sigma)) - \int_0^1 f(x)\mathrm{d}x \right|^2 \mathrm{d}\sigma$$

$$= -\frac{1}{3} + \frac{1}{N^2}\sum_{m,n=0}^{N-1}\Bigg[\sum_{\substack{i,j=1\\ i,j-\text{even}}}^{k} g_{m,n}\left(\frac{i}{k},\frac{j}{k}\right) + \sum_{\substack{i,j=1\\ i,j-\text{odd}}}^{k} \frac{k}{2}\int_{\frac{i-1}{k}}^{\frac{i}{k}+\frac{2}{k}} g_{m,n}\left(x, x+\frac{j-i}{k}\right)\mathrm{d}x$$

$$- \sum_{\substack{i,j=1\\ i-\text{even},j-\text{odd}}}^{k} \frac{k}{2}\int_{\frac{j-1}{k}}^{\frac{j-1}{k}+\frac{2}{k}} g_{m,n}\left(\frac{i}{k}, x\right)\mathrm{d}x - \sum_{\substack{i,j=1\\ i-\text{odd},j-\text{even}}}^{k} \frac{k}{2}\int_{\frac{i-1}{k}}^{\frac{i-1}{k}+\frac{2}{k}} g_{m,n}\left(x, \frac{j}{k}\right)\mathrm{d}x\Bigg]. \tag{6}$$

Cont.

which holds also for $\Phi(x_n + \sigma \bmod 1)$. The base b for u.d.p. map $\Phi(x)$ is independent of the base of the shift $x_n \oplus \sigma$.

Open problem

An open problem is to find sequences $x_0, x_1, \dots, x_{N-1})$ to minimize (1) with u.d.p. $\Phi(x) = bx \bmod 1$ or a generalized tent function.

For $\Phi(x) = x$ we have following result:
Let $K(x,y) = 1 - \max(x,y)$, $\Phi(x + \sigma \bmod 1) = \{x + \sigma\}$, $0 \le x_0 < x_1 < \cdots < x_{N-2} < x_{N-1} \le 1$ and put $t_i = x_{i+1} - x_i$, $i = 0, 1, \dots, N-2$. Then the minimum of mean square worst-case error (3) is attained at $t_i = \frac{1}{N}$, $i = 0, 1, \dots, N-2$ and

$$\int_0^1 \sup_{\substack{f \in H \\ ||f|| \le 1}} \left| \frac{1}{N} \sum_{n=0}^{N-1} f(\{x_n + \sigma\}) - \int_0^1 f(x) \mathrm{d}x \right|^2 \mathrm{d}\sigma = \frac{1}{6N^2}. \qquad (7)$$

Sketch of the proof, step 1^0

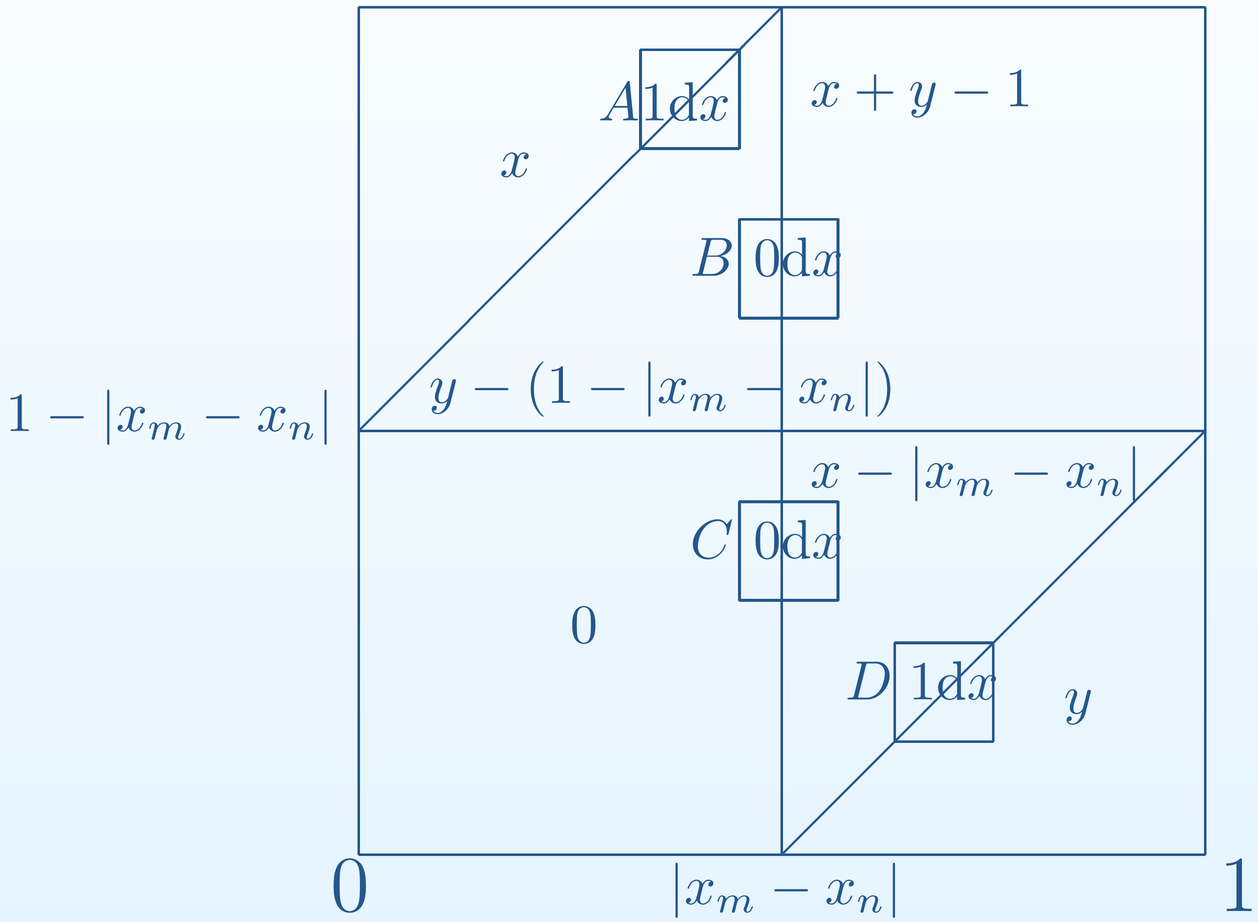
In the 1^0 step we shall express the integral in (3) for one-dimensional case

$$\int_0^1 \int_0^1 K(\Phi(x), \Phi(y)) \mathrm{d}_x \mathrm{d}_y g_{m,n}(x, y)$$
$$= \int_0^{|x_m - x_n|} K(\Phi(x), \Phi(x + 1 - |x_m - x_n|) \mathrm{d}x$$
$$+ \int_{|x_m - x_n|}^1 K(\Phi(x), \Phi(x - |x_m - x_n|) \mathrm{d}x. \qquad (8)$$

The differential $\mathrm{d}_x \mathrm{d}_y g_{m,n}(x, y)$ is defined by

$$\mathrm{d}_x \mathrm{d}_y g_{m,n}(x, y) = g_{m,n}(x, y) + g_{m,n}(x + \mathrm{d}x, y + \mathrm{d}y) - g_{m,n}(x, y + \mathrm{d}y) - g_{m,n}(x + \mathrm{d}x, y).$$

Computation of $\mathrm{d}_x \mathrm{d}_y g_{m,n}(x, y)$



Step 2^0

In the 2^0 step, for $\Phi(x) = x$ and putting $T_{m,n} = |x_m - x_n| = \sum_{k=m}^{n-1} t_k$ for $m < n$ we will prove

$$\frac{\partial}{\partial t_i}\left(\int_0^{T_{m,n}} K(x, x+1-T_{m,n})\mathrm{d}x + \int_{T_{m,n}}^{1} K(x, x-T_{m,n})\mathrm{d}x\right) = 2T_{m,n} - 1 \tag{9}$$

assuming that $T_{m,n}$ contains term t_i.

Step 3^0

In the 3^0 step we shall see that the zero partial derivative

$$\frac{\partial}{\partial t_i}\left(\sum_{\substack{m<n\\ m,n=0}}^{N-1}\int_0^1\int_0^1 K(x,y)\mathrm{d}_x\mathrm{d}_y g_{m,n}(x,y)\right)=\sum_{\substack{m<n,m,n=0\\ T_{m,n}\text{ contains } t_i}}^{N-1}\left(2T_{m,n}-1\right)=0 \tag{10}$$

is equivalent to the following linear equation

$$\begin{aligned} &t_0(N-2-i+1)+t_1 2(N-2-i+1)+\cdots+t_k(k+1)(N-2-i+1)\\ &+\cdots+t_i(i+1)(N-2-i+1)+t_{i+1}(i+1)(N-2-(i+1)+1)+\dots\\ &+t_s(i+1)(N-2-s+1)+\cdots+t_{N-2}(i+1)=\frac{1}{2}(i+1)(N-2-i+1). \end{aligned} \tag{11}$$

Step 4^0 and 5^0

In part 4^0 we prove that the system (11), $i = 0, 1, \ldots, N-2$, is regular and has the unique solution $t_i = \frac{1}{N}$ for $i = 0, 1, \ldots, N-2$.

In the final part 5^0 we compute the minimum of mean square worst-case error for $t_i = \frac{1}{N}$.

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