Directional Discrepancy

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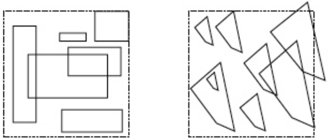
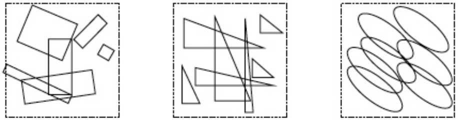
Discrepancy bounds in dimension $d = 2$

	LOWER BOUND	UPPER BOUND
Axis-parallel rectangles		
L^∞	$\log N$	$\log N$
L^2	$\log^{\frac{1}{2}} N$	$\log^{\frac{1}{2}} N$
Rotated rectangles		
	$N^{1/4}$	$N^{1/4}\sqrt{\log N}$
Circles		
	$N^{1/4}$	$N^{1/4}\sqrt{\log N}$
Convex Sets		
	$N^{1/3}$	$N^{1/3}\log^4 N$

- No rotations: discrepancy $\approx \log N$
  (K. Roth, W. Schmidt, van der Corput)



- All rotations: discrepancy $\approx N^{1/4}$
  (J. Beck, H. Montgomerry)



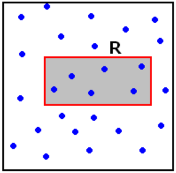
Pictures taken from Matoušek's "Geometric Discrepancy: An Illustrated Guide"

Higher dimensions: $d \geq 3$

LOWER BOUND		UPPER BOUND
Axis-parallel boxes		
L^∞	$(\log N)^{\frac{d-1}{2}+\eta}$	$(\log N)^{d-1}$
L^2	$(\log N)^{\frac{d-1}{2}}$	$(\log N)^{\frac{d-1}{2}}$
Rotated boxes		
	$N^{\frac{1}{2}-\frac{1}{2d}}$	$N^{\frac{1}{2}-\frac{1}{2d}}\sqrt{\log N}$
Balls		
	$N^{\frac{1}{2}-\frac{1}{2d}}$	$N^{\frac{1}{2}-\frac{1}{2d}}\sqrt{\log N}$
Convex Sets		
	$N^{1-\frac{2}{d+1}}$	$N^{1-\frac{2}{d+1}}\log^c N$

Directional Discrepancy: Setting

$\mathcal{P}_N$ – a set of N points in $[0,1]^d$
$R \subset [0,1]^d$ – a measurable set.



$$D(\mathcal{P}_N, R) = \sharp\{\mathcal{P}_N \cap R\} - N \cdot vol\,(R)$$

- Let $\Omega \subset [0, \pi/2)$
- $\mathcal{A}_\Omega = \{\text{rectangles pointing in directions of } \Omega\}$

$$D_\Omega(N) = \inf_{\mathcal{P}_N} \sup_{R \in \mathcal{A}_\Omega} |D(\mathcal{P}_N, R)|$$

No rotations: axis-parallel rectangles

- $\Omega = \{0\}$

Theorem (Lerch; Schmidt)

$$D_\Omega(N) \approx \log N$$

Theorem (L^2: Roth; Davenport)

$$D_\Omega^{(2)}(N) \approx \sqrt{\log N}$$

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- Ω finite $\longrightarrow$ same (Beck-Chen, Chen-Travaglini)

All rotations

- $\Omega = [0, \pi/2)$

Theorem (J. Beck)

$$N^{1/4} \lesssim D_\Omega(N) \lesssim N^{1/4}\sqrt{\log N}$$

Irrational Lattice

Example

Let $\theta \in [0, \pi]$. Denote by $\mathcal{P}_N^\theta$ the lattice $\frac{1}{\sqrt{N}}\mathbb{Z}^2$ rotated by θ and intersected with $[0, 1]^2$.

Theorem

If $\tan\theta$ *is badly approximable, i.e.*

$$\left|\tan(\theta) - \frac{p}{q}\right| \gtrsim \frac{1}{q^2}$$

for all $p \in \mathbb{Z}$, $q \in \mathbb{N}$, *then the discrepancy of* $\mathcal{P}_N^\theta$ *with respect to axis-parallel rectangles satisfies*

$$D(\mathcal{P}_N^\theta) \approx \log N.$$

Finitely many directions: Davenport-Cassels lemma

Lemma (Davenport; Cassels)

For all $n > 0$ there exists $c = c(n) > 0$ so that for any $\phi_1, \phi_2, ..., \phi_n$ there exists θ such that for all $j = 1, ..., n$

$$\left|\tan(\theta - \phi_j) - \frac{p}{q}\right| > \frac{c}{q^2}$$

for all $p \in \mathbb{Z}$, $q \in \mathbb{N}$.

- $c(n) \approx 1/n^2$
- Davenport (1964)
- Cassels (1956): without *tangent*.
- Hall (1947): any real number is representable as a sum of two continued fractions with partial quotients at most 4.

Lacunary directions

- Lacunary sequence of directions $\Omega = \{\phi_n\}_{n=1}^{\infty}$
- $\frac{|\phi_n - \phi|}{|\phi_{n+1} - \phi|} \geq \lambda > 1$ for some ϕ.
- E.g., $\Omega = \{2^{-n}\}_{n=1}^{\infty}$

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Lemma (DB, X. Ma, J. Pipher, C. Spencer)

There exists θ such that for all $\phi \in \Omega$

$$\left| \tan(\theta - \phi) - \frac{p}{q} \right| \gtrsim \frac{1}{q^2 \log^2 q}$$

for all $p \in \mathbb{Z}$, $q \in \mathbb{N}$.

Lacunary of finite order

- Lacunary set of order M: union of a lacunary set $\mathcal{L}$ of order $M-1$ and lacunary sequences converging to points of $\mathcal{L}$.
- Harmonic analysis: Maximal function with respect to rectangles in $\mathcal{A}_\Omega$ is bounded in $L^p(\mathbb{R}^2)$ iff Ω is a finite union of lacunary sets of finite order (M. Bateman, 2007)
- Example: $\Omega = \{2^{-n_1} + 2^{-n_2} + \ldots + 2^{-n_M}\}_{n_k \in \mathbb{N}}$

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for all $p \in \mathbb{Z}$, $q \in \mathbb{N}$.

Minkowski dimension

- Upper Minkowski dimension: infimum of exponents d such that for every $0 < \delta \ll 1$ the set Ω can be covered by $\mathcal{O}(\delta^{-d})$ intervals of length δ.

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Let Ω have upper Minkowski dimension at most d, $0 \leq d < 1$. Then there exists θ such that for all $\phi \in \Omega$

$$\left| \tan(\theta - \phi) - \frac{p}{q} \right| \gtrsim \frac{1}{q^{\frac{2}{(1-d)^2} + \varepsilon}}$$

for all $p \in \mathbb{Z}$, $q \in \mathbb{N}$ and any $\varepsilon > 0$.

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Examples:

- Cantor-type sets (standard Cantor set has dimension $d = \log_3 2$)
- Sequence $\{n^{-\alpha}\}$ has Minkowski dimension $d = \frac{1}{\alpha+1}$

Proof of the lemmas

- We construct a sequence $..I_{n-1} \supset I_n \supset ...$ with $|I_n| \to 0$ so that

$$\left|\tan(\theta - \phi) - \frac{p}{q}\right| \geq \frac{c(n)}{q^2} \qquad (*)$$

for all $\theta \in I_n$, $\phi \in \Omega$, and $R(n) \leq q \leq R(n+1)$

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- We construct a sequence $..I_{n-1} \supset I_n \supset ...$ with $|I_n| \to 0$ so that

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- Upper Minkowski dimension d: $N \leq C_\varepsilon \left(\frac{1}{\delta}\right)^{d+\varepsilon}$

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- Assume $(\theta_1, \phi_1, p_1, q_1)$ and $(\theta_2, \phi_2, p_2, q_2)$ with $\phi_i \in B_k$, $\theta_i \in I_{n-1}$, $R(n) \leq q \leq R(n+1)$ violate $(*)$

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- $$\left|\frac{p_1}{q_1} - \frac{p_2}{q_2}\right| \leq \frac{2c(n)}{R(n)^2} + C|I_{n-1}| + C\delta(n)$$

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- Then there is at most one p/q with $R(n) \leq q_i \leq R(n+1)$, which violates $(*)$ for each B_k.

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- For each $k = 1, \ldots, N(n)$ remove from I_{n-1} the set

$$\tan^{-1}\left(\frac{p}{q} - \frac{c(n)}{R(n)^2}, \frac{p}{q} - \frac{c(n)}{R(n)^2}\right) + B_k$$

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- $$|I_{n-1}| - N(n) \cdot \left(\frac{2c(n)}{R(n)^2} + \delta(n)\right)$$

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- $$|I_{n-1}| - N(n) \cdot \left(\frac{2c(n)}{R(n)^2} + \delta(n)\right) \geq (N(n)+1)|I_n|$$

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- $$|I_{n-1}| - N(n) \cdot \left(\frac{2c(n)}{R(n)^2} + \delta(n)\right) \geq (N(n)+1)|I_n|$$

- Then the remainder contains an interval of length $|I_n|$

Proof of the lemmas

- We construct a sequence $\ldots I_{n-1} \supset I_n \supset \ldots$ with $|I_n| \to 0$ so that
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Proof of the lemmas

- We construct a sequence $\dots I_{n-1} \supset I_n \supset \dots$ with $|I_n| \longrightarrow 0$ so that
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for all $\theta \in I_n$, $\phi \in \Omega$, and $R(n) \leq q \leq R(n+1)$

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- $\frac{2c(n)}{R(n)^2} + C|I_{n-1}| + C\delta(n) \leq \frac{1}{R(n+1)^2}$
- Finite: $R(n) = a^n$, $c(n) = c$, $\delta(n) = 0$, $N(n) = N$

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- $\frac{2c(n)}{R(n)^2} + C|I_{n-1}| + C\delta(n) \leq \frac{1}{R(n+1)^2}$
- Lacunary: $R(n) = n^{n/2} \log^{n/2} n$, $c(n) = \frac{c}{n^2 \log^2 n}$

$$c(n) \approx \log^{-2} R(n)$$

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for all $\theta \in I_n$, $\phi \in \Omega$, and $R(n) \leq q \leq R(n+1)$

- $|I_{n-1}| - N(n)\cdot\left(\frac{2c(n)}{R(n)^2} + \delta(n)\right) \geq (N(n)+1)|I_n|$
- $\frac{2c(n)}{R(n)^2} + C|I_{n-1}| + C\delta(n) \leq \frac{1}{R(n+1)^2}$
- Lacunary of order M: $R(n) = n^{nM/2}$, $c(n) = \frac{c}{n^{2M}\log^{2M} n}$
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- $\frac{2c(n)}{R(n)^2} + C|I_{n-1}| + C\delta(n) \leq \frac{1}{R(n+1)^2}$
- Minkowski dimension d: $R(n) = 2^{a^n}$,

$c(n) = c2^{-2a^n(a^2-1)} \approx R(n)^{-(2a^2-1)}$, where $a = \frac{1}{1-d-\varepsilon}$

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for all $p \in \mathbb{Z}$, $q \in \mathbb{N}$.

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Theorem (DB, X. Ma, J. Pipher, C. Spencer)

There exists θ such that for all $\phi \in \Omega$

$$\left| \tan(\theta - \phi) - \frac{p}{q} \right| \gtrsim F^{-1}\left(F^{-1}\left(\frac{1}{q^2} \right) \right)$$

for all $p \in \mathbb{Z}$, $q \in \mathbb{N}$.

- Ω finite: $N = \#\Omega$, $F(x) \approx Nx$,

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- Ω lacunary: $N(x) \approx \log \frac{1}{x}$, $F(x) \approx x \cdot \log \frac{1}{x}$, $F^{-1}(x) \approx \frac{x}{\log(1/x)}$,

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Erdős-Turan inequality

- For a sequence $\omega = (\omega_n) \subset [0,1]$:

$$D_N(\omega) = \sup_{x \subset [0,1]} \left| \#\{n \leq N : \omega_n \leq x\} - Nx \right|$$

Theorem (Erdős-Turan)

For any sequence $\omega \subset [0,1]$ we have

$$D_N(\omega) \lesssim \frac{N}{m} + \sum_{h=1}^{m} \frac{1}{h} \left| \sum_{n=1}^{N} e^{2\pi i h \omega_n} \right|$$

for all natural numbers m.

- If $\omega_n = \{n\alpha\}$

$$\left| \sum_{n=1}^{N} e^{2\pi i h n \alpha} \right| \leq \frac{2}{|e^{2\pi i h \alpha} - 1|} = \frac{1}{|\sin(\pi h \alpha)|} = \frac{1}{\sin(\pi \|h\alpha\|)} \lesssim \frac{1}{\|h\alpha\|}.$$

From Diophantine properties to discrepancy

Let ψ be a non-decreasing function on $\mathbb{R}_+$. We say that $\alpha \in \mathbb{R}$ is of type $< \psi$ if for all $q \in \mathbb{N}$ we have $q\|q\alpha\| > 1/\psi(q)$, i.e.

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Finitely many directions: $\psi(q) = \text{const}$

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Minkowski dimension d: $\psi(q) \approx q^{\frac{2}{(1-d)^2} - 2 + \varepsilon}$

Partial quotients

- Let $\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\dots}}} = [a_0;\, a_1,\, a_2,\, a_3, \dots]$ and consider its convergents $\frac{p_n}{q_n} = [a_0;\, a_1,\, a_2, \dots, a_n]$.
- For the Kronecker sequence ω: $\omega_n = \{n\alpha\}$

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- One can deduce that

$$D_N(\omega) \lesssim \log N \cdot (\log\log N)^2,$$

while Erdős-Turan would only yield $\log^2 N$.

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- *Lacunary directions*

$$D_\Omega(N) \lesssim \log^3 N$$

- *Lacunary of order M*

$$D_\Omega(N) \lesssim \log^{2M+1} N$$

- *Ω has upper Minkowski dimension $0 \leq d < 1$*

$$D_\Omega(N) \lesssim N^{\frac{\tau}{2(\tau+1)}+\varepsilon},$$

where $\tau = \frac{2}{(1-d)^2} - 2$.

Critical dimension

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- So $(*)$ is interesting only for $\frac{\tau}{2(\tau+1)} < \frac{1}{4}$, i.e. $\tau < 1$,

$$d < 1 - \left(\frac{2}{3}\right)^{\frac{1}{2}} \approx 0.1835\ldots$$

Directional Discrepancy in L^2

Theorem (DB, Ma, Pipher, Spencer)

Let μ be any probability measure on $\mathcal{A}_\Omega$. Then

1) If Ω is a lacunary sequence, there exists $\mathcal{P} \subset [0,1]^2$, $\#\mathcal{P} = N$

$$\left(\int_{\mathcal{A}_\Omega} |D_\Omega(\mathcal{P}, R)|^2 d\mu(R) \right)^{\frac{1}{2}} \lesssim \log^{\frac{5}{2}} N.$$

2) If Ω is lacunary of order M, there exists $\mathcal{P} \subset [0,1]^2$, $\#\mathcal{P} = N$

$$\left(\int_{\mathcal{A}_\Omega} |D_\Omega(\mathcal{P}, R)|^2 d\mu(R) \right)^{\frac{1}{2}} \lesssim \log^{2M+\frac{1}{2}} N.$$

3) If Ω has upper Minkowski dimension $0 \le d < 1$, there exists $\mathcal{P}$

$$\left(\int_{\mathcal{A}_\Omega} |D_\Omega(\mathcal{P}, R)|^2 d\mu(R) \right)^{\frac{1}{2}} \lesssim N^{\frac{1}{2} - \frac{1}{2(\tau+1)} + \varepsilon},$$

for any $\varepsilon > 0$, if $\tau = \frac{2}{(1-d)^2} - 2 < 1$

Lemma

Let I be a finite interval of consecutive integers.

1) Assume that α satisfies $\nu\|\nu\alpha\| > \frac{c}{\log^{2M}\nu}$, for all $\nu \in \mathbb{N}$. Then

$$\sum_{\nu=1}^{\infty} \frac{1}{\nu^2} \left| \sum_{n \in I} e^{-2\pi i \nu n \alpha} \right|^2 \lesssim \log^{4M+1} |I|.$$

2) Assume that α satisfies $\nu\|\nu\alpha\| > c\nu^{-\tau+\varepsilon}$, for all $\varepsilon > 0$, where $0 \leq \tau < 1$. Then

$$\sum_{\nu=1}^{\infty} \frac{1}{\nu^2} \left| \sum_{n \in I} e^{-2\pi i \nu n \alpha} \right|^2 \lesssim |I|^{\frac{2\tau}{\tau+1}+\varepsilon'}, \text{ where } \varepsilon' = \mathcal{O}(\varepsilon).$$