Spherical Nets and their Relatives — Optimal order Quasi-Monte Carlo methods on the sphere

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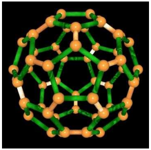
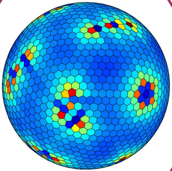
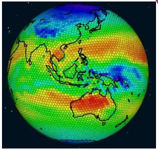
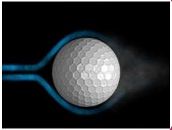
based on joint work with Ed Saff (Vanderbilt) and
Josef Dick, Ian Sloan, Rob Womersley (UNSW) and
Christoph Aistleitner (TU-Graz)

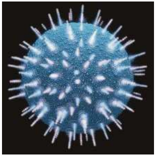
Mathematics & Statistics in all Spheres of Life











Rob Womersley (http://web.maths.unsw.edu.au/~rsw/)

Diploma thesis ’Throwable Camera Array for Capturing Spherical Panoramas’: Jonas Pfeil. 2010. Advisors: Marc Alexa, Carsten Gremzow.

*How do you **“uniformly”** arrange*
*N **nodes***
*on the **unit sphere** …*

…that will be ***"good"*** ***integration nodes***
for ***qMC rules***
on the ***unit sphere***

$$\frac{1}{N}\sum_{\mathbf{x}\in X_N} f(\mathbf{x}) \approx \int_{\mathbb{S}^d} f(\mathbf{x})\,\mathrm{d}\sigma_d(\mathbf{x})$$

Outline

Uniform Distribution on the unit sphere

Worst-case error (WCE)

- Sums of Distances

Approximate Spherical Designs

- Low-discrepancy configurations

Explicit Point Constructions

- **Spherical** Digital Nets and Fibonacci points

Uniform Distribution
on the unit sphere

Uniform Distribution on the Unit Sphere $\mathbb{S}^d$ in $\mathbb{R}^{d+1}$

Definition

A sequence (X_N) is **asymptotically uniformly distributed over** $\mathbb{S}^d$ if

$$\lim_{N\to\infty} \frac{|X_N \cap B|}{N} = \sigma_d(B)$$

for every Riemann measurable set B in $\mathbb{S}^d$.

Informally: A reasonable set gets a fair share of points as N becomes large.

$$\mathrm{I}[f] := \int_{\mathbb{S}^d} f \,\mathrm{d}\,\sigma_d, \qquad \mathrm{Q}_N[f] := \frac{1}{N}\sum_{k=1}^{N} f(\mathbf{x}_k)$$

Equiv.: (X_N) is **asymptotically uniformly distributed over** $\mathbb{S}^d$ if [a]

$$\lim_{N\to\infty} \mathrm{Q}_N[f] = \mathrm{I}[f] \qquad \text{for every } f \in C(\mathbb{S}^d).$$

[a] I.e., $\nu(X_N^{(s)}) \to \sigma_d$ as $N \to \infty$ (in the weak-$*$ limit).

Quantification
of
Uniform Distribution

Worst-case error (WCE)

$$\mathrm{wce}(\mathrm{Q}_N; B) := \sup_{\substack{f \in B, \\ \|f\|_B \le 1}} \left| \mathrm{Q}_N[f] - \mathrm{I}[f] \right|$$

Sobolev Space over $\mathbb{S}^d$

- sequence of positive real numbers $(a_\ell^{(s)})_{\ell \geq 0}$ satisfying
$$a_\ell^{(s)} \asymp [\ell + (d-1)/2]^{-2s},$$
- **inner product** for functions f and g in $C^\infty(\mathbb{S}^d)$ by means of
$$(f, g)_{\mathbb{H}^s} := \sum_{\ell=0}^{\infty} \sum_{k=1}^{Z(d,\ell)} \frac{1}{a_\ell^{(s)}} \widehat{f}_{\ell,k} \, \widehat{g}_{\ell,k}$$
with Laplace-Fourier coefficients
$$\widehat{f}_{\ell,k} = (f, Y_{\ell,k})_{\mathbb{L}_2(\mathbb{S}^d)} = \int_{\mathbb{S}^d} f(\mathbf{x}) Y_{\ell,k}(\mathbf{x}) \, \mathrm{d}\sigma_d(\mathbf{x}).$$

The **Sobolov space** $\mathbb{H}^s(\mathbb{S}^d)$ then is the completion of the space
$$\left\{ f \in C^\infty(\mathbb{S}^d) : \sum_{\ell=0}^{\infty} \sum_{k=1}^{Z(d,\ell)} \frac{1}{a_\ell^{(s)}} \left[\widehat{f}_{\ell,k}\right]^2 < \infty \right\}$$
with respect to the Sobolev norm $\|\cdot\|_{\mathbb{H}^s} := \sqrt{(\cdot,\cdot)_{\mathbb{H}^s}}$.

Cui and Freeden's **Generalized Discrepancy**

[SIAM J. SCI. COMPUT. 18:2 (1997)]

…based on pseudodifferential operators yielding

A Koksma-Hlawka like inequality

$$\left| \mathrm{Q}_N[f] - \mathrm{I}[f] \right| \leq \sqrt{6}\, \mathrm{D}_{\mathrm{CF}}(X_N) \, \|f\|_{\mathbb{H}^{3/2}(\mathbb{S}^2)},$$

where f is from the Sobolev space $\mathbb{H}^{3/2}(\mathbb{S}^2)$.

$\mathrm{D}_{\mathrm{CF}}(X_N)$ expressible in terms of elementary function:

$$4\pi \left[\mathrm{D}_{\mathrm{CF}}(X_N)\right]^2 = 1 - \frac{2}{N^2} \sum_{j=1}^{N} \sum_{k=1}^{N} \log\left(1 + \left|\mathbf{x}_j - \mathbf{x}_k\right|/2\right).$$

Definition (Equidistribution in $\mathbb{H}^{3/2}(\mathbb{S}^2)$)

A sequence (X_N) is **equidistributed in** $\mathbb{H}^{3/2}(\mathbb{S}^2)$ if $\lim_{N\to\infty} \mathrm{D}_{\mathrm{CF}}(X_N) = 0$.

…an asymptotically uniformly distributed (X_N) is 'equidistributed in $C(\mathbb{S}^2)$'.

Interpretation as Worst Case Error (WCE)

Sloan and Womersley [Adv. Comput. Math. 21 (2004)]

" . . . show that $\mathrm{D}_{\mathrm{CF}}(X_N)$ has a natural interpretation, as the worst-case error (apart from a constant factor) for the equal-weight cubature rule

$$E_n(f) = \frac{4\pi}{d_n} \sum_{k=1}^{d_n} f(\mathbf{x}_k)$$

applied to a function $f \in B(H)$, where $B(H)$ is the unit ball in a certain Hilbert space H. "

Worst Case Error (WCE) in a Sobolev Space $\mathbb{H}^s(\mathbb{S}^d)$

$$K^{(s)}(\mathbf{x}\cdot\mathbf{y}) := \sum_{\ell=0}^{\infty} a_\ell^{(s)} \sum_{k=1}^{Z(d,\ell)} Y_{\ell,k}(\mathbf{x}) Y_{\ell,k}(\mathbf{y}) = \sum_{\ell=0}^{\infty} a_\ell^{(s)} Z(d,\ell) P_\ell^{(d)}(\mathbf{x}\cdot\mathbf{y}).$$

Note: $K^{(s)}(z)$ depends on $(a_\ell^{(s)})_{\ell\geq 0}$ satisfying $a_\ell^{(s)} \asymp [\ell + (d-1)/2]^{-2s}$.

The worst-case error of $Q_N[f]$

$$\mathrm{wce}(Q_N; \mathbb{H}^s(\mathbb{S}^d)) = \sup_{\substack{f\in\mathbb{H}^s(\mathbb{S}^d),\\ \|f\|_{\mathbb{H}^s}\leq 1}} \left| Q_N[f] - I[f] \right|, \qquad s > d/2.$$

By Cauchy-Schwarz inequality

$$\left[\mathrm{wce}(Q_N; \mathbb{H}^s(\mathbb{S}^d))\right]^2 = \frac{1}{N^2}\sum_{j=1}^{N}\sum_{k=1}^{N} K^{(s)}(\mathbf{x}_j\cdot\mathbf{x}_k) - \int_{\mathbb{S}^d}\int_{\mathbb{S}^d} K^{(s)}(\mathbf{x}\cdot\mathbf{y})\,\mathrm{d}\sigma_d(\mathbf{x})\,\mathrm{d}\sigma_d(\mathbf{y}).$$

Spherical Cap centered at **z** with 'opening angle' θ

$$C(\mathbf{z};\theta) := \left\{\mathbf{x} \in \mathbb{S}^d : \mathbf{x} \cdot \mathbf{z} \geq \cos\theta\right\}, \qquad \mathbf{z} \in \mathbb{S}^d, -1 \leq t \leq 1.$$

Spherical cap $\mathbb{L}_2$-discrepancy:

$$D_{\mathbb{L}_2}^C(X_N) := \left\{ \int_0^\pi \int_{\mathbb{S}^d} \left| \frac{|X_N \cap C(\mathbf{x};\theta)|}{N} - \sigma_d(C(\mathbf{x};\theta)) \right|^2 \mathrm{d}\,\sigma_d(\mathbf{x}) \sin\theta \,\mathrm{d}\,\theta \right\}^{1/2}.$$

Theorem (Stolarsky [Proc. of the AMS 41:2 (1973)])

$$\frac{1}{N^2} \sum_{j,k=1}^{N} |\mathbf{x}_j - \mathbf{x}_k| + \frac{1}{C_d} \left[D_{\mathbb{L}_2}^C(X_m) \right]^2 = \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} |\mathbf{x} - \mathbf{y}| \,\mathrm{d}\,\sigma_d(\mathbf{x}) \,\mathrm{d}\,\sigma_d(\mathbf{y}).$$

Stolarsky's proof makes use of Haar integrals over $\mathrm{SO}(d+1)$.

Reproved using reproducing kernel Hilbert space techniques (B-Dick [Proc. of the AMS, in press])

WCE in $\mathbb{H}^s(\mathbb{S}^d)$ for $s = (d+1)/2$, $d \geq 2$

Distance integral:

$$V_{-1}(\mathbb{S}^d) = \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} |\mathbf{x} - \mathbf{y}| \, \mathrm{d}\,\sigma_d(\mathbf{x}) \, \mathrm{d}\,\sigma_d(\mathbf{y}).$$

Theorem (B.–Womersley)

$\mathbb{H}^s(\mathbb{S}^d)$ *with reproducing kernel*

$$K^{(s)}(\mathbf{x} \cdot \mathbf{y}) = 2V_{-1}(\mathbb{S}^d) - |\mathbf{x} - \mathbf{y}|, \qquad \mathbf{x}, \mathbf{y} \in \mathbb{S}^d.$$

Then

$$\left[\mathrm{wce}(Q_N; \mathbb{H}^s(\mathbb{S}^d))\right]^2 = V_{-1}(\mathbb{S}^d) - \frac{1}{N^2} \sum_{i,j=1}^{N} |\mathbf{x}_i - \mathbf{x}_j| = \frac{1}{C_d} \left[\mathrm{D}_{\mathbb{L}_2}^{\mathrm{C}}(X_N)\right]^2.$$

Proposition (by results of Stolarsky and Beck)

$$D_{\mathbb{L}_2}^{C}(X_N^*) \asymp N^{-s/d}.$$

WCE in $\mathbb{H}^s(\mathbb{S}^d)$ for $d/2 < s < d/2+1$, $d \geq 2$

Generalized distance integral:

$$V_{d-2s}(\mathbb{S}^d) = \int_{\mathbb{S}^d}\int_{\mathbb{S}^d} |\mathbf{x}-\mathbf{y}|^{2s-d}\,\mathrm{d}\sigma_d(\mathbf{x})\,\mathrm{d}\sigma_d(\mathbf{y}).$$

Theorem (B.–Womersley)

$\mathbb{H}^s(\mathbb{S}^d)$ *with reproducing kernel*

$$K_{\mathrm{gd}}^{(s)}(\mathbf{x}\cdot\mathbf{y}) = 2V_{d-2s}(\mathbb{S}^d) - |\mathbf{x}-\mathbf{y}|^{2s-d}, \qquad \mathbf{x},\mathbf{y}\in\mathbb{S}^d.$$

Then

$$\left[\mathrm{wce}(Q_N;\mathbb{H}^s(\mathbb{S}^d))\right]^2 = V_{d-2s}(\mathbb{S}^d) - \frac{1}{N^2}\sum_{i,j=1}^{N}|\mathbf{x}_i-\mathbf{x}_j|^{2s-d}.$$

Proposition (by results of G. Wagner)

$$V_{d-2s}(\mathbb{S}^d) - \frac{1}{N^2}\sum_{j=1}^{N}\sum_{k=1}^{N}\left|\mathbf{x}_{j,N}^*-\mathbf{x}_{k,N}^*\right|^{2s-d} \asymp N^{-2s/d}, \quad 0 < 2s-d < 2.$$

WCE in $\mathbb{H}^s(\mathbb{S}^d)$ for $d/2 + L < s < d/2 + L + 1$

$(-1)^{L+1}|\mathbf{x} - \mathbf{y}|^{2s-d}$ is conditionally positive definite of order $L + 1$:

$$(-1)^{L+1}|\mathbf{x} - \mathbf{y}|^{2s-d} = \sum_{\ell=0}^{L} a_\ell \, Z(d,\ell) \, P_\ell^{(d)}(z) + \sum_{\ell=L+1}^{\infty} \underbrace{a_\ell}_{>0} \, Z(d,\ell) \, P_\ell^{(d)}(z).$$

WCE in $\mathbb{H}^s(\mathbb{S}^d)$ for $d/2 + L < s < d/2 + L + 1$

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Observation

A spherical L-design **annihilates** $\sum_{\ell=1}^{L} a_\ell \, Z(d,\ell) \, P_\ell^{(d)}(z)$.

WCE in $\mathbb{H}^s(\mathbb{S}^d)$ for $d/2 + L < s < d/2 + L + 1$

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Observation

A spherical L-design **annihilates** $\sum_{\ell=1}^{L} a_\ell \, Z(d,\ell) \, P_\ell^{(d)}(z)$.

WCE of a spherical L-design

$$\left[\mathrm{wce}(\mathrm{Q}_{N,L}; \mathbb{H}^s(\mathbb{S}^d))\right]^2 = (-1)^{L+1} \left[\frac{1}{N^2} \sum_{j=1}^{N} \sum_{k=1}^{N} |\mathbf{x}_j - \mathbf{x}_k|^{2s-d} - V_{d-2s}(\mathbb{S}^d) \right].$$

Geometric interpretation of this WCE

Let $2s - d$ be and odd integer $2M - 1$. Then

$$\mathrm{wce}(Q_{N,L}; \mathbb{H}^s(\mathbb{S}^d)) = \frac{1}{\sqrt{C_d(M)}} \left\{ \int_{-1}^{1} \int_{\mathbb{S}^d} \left| \Delta_{X_N,2M}(\mathbf{z}, t) \right|^2 \mathrm{d}\,\sigma_d(\mathbf{z}) \,\mathrm{d}\,t \right\}^{1/2}$$

for an explicit constant $C_d(M)$ with the **local discrepancy function**

$$\Delta_{X_N,\beta}(\mathbf{z}, t) = \frac{1}{N} \sum_{j=1}^{N} (\mathbf{x}_j \cdot \mathbf{z} - t)_+^{\beta-1} - \int_{\mathbb{S}^d} (\mathbf{y} \cdot \mathbf{z} - t)_+^{\beta-1} \,\mathrm{d}\sigma_d(\mathbf{y}).$$

This follows from the kernel representations
(B-Dick [arXiv:1203.5157v1 [math.NA]])

$$\begin{aligned} \mathcal{K}_\beta(\mathbf{x}, \mathbf{y}) &:= \int_{-1}^{1} \int_{\mathbb{S}^d} (\mathbf{x} \cdot \mathbf{z} - t)_+^{\beta-1} (\mathbf{y} \cdot \mathbf{z} - t)_+^{\beta-1} \,\mathrm{d}\,\sigma_d(\mathbf{z}) \,\mathrm{d}\,t \\ &= \mathcal{Q}_{M-1}(\mathbf{x}, \mathbf{y}) + (-1)^M c_d(M) \left| \mathbf{x} - \mathbf{y} \right|^{2M-1}. \end{aligned}$$

Approximate Spherical Designs

- Low-discrepancy configurations

Spherical Designs

Definition (Delsarte, Goethals and Seidel 1977)

Spherical t-designs $\mathbf{x}_1, \ldots, \mathbf{x}_{N(t)}$ with $N(t)$ points have the property

$$\int_{\mathbb{S}^d} P(\mathbf{x}) \,\mathrm{d}\,\sigma_d(\mathbf{x}) = \frac{1}{N(t)} \sum_{j=1}^{N(t)} P(\mathbf{x}_j)$$

for all polynomials P with $\deg P \leq t$.

Sequences of spherical designs have a known fast-convergence property in Sobolev spaces.

Bounding the WCE

Theorem

Let $s > d/2$. For any N-point integration rule Q_N we have

$$\frac{c(s,d)}{N^{s/d}} \leq \mathrm{wce}(\mathbb{H}^s, Q_N) \qquad \text{for all } N \geq 1.$$

Further

$$\mathrm{wce}(\mathbb{H}^s, \widehat{Q}_N) \leq \frac{C(s,d)}{t^s} \qquad \text{for all } t \geq 1,$$

where $\widehat{Q}_N$ uses an $N(t)$-point spherical t-design as integration nodes.

Remark

For $N(t) \asymp t^d$ (optimal(!) order of points) both bounds match.

$\mathbb{H}^{3/2}(\mathbb{S}^2)$: K. Hesse & I. H. Sloan [Bull. Austral. Math. Soc. **71** (2005)]
$\mathbb{H}^s(\mathbb{S}^d)$, $s > d/2$: K. Hesse [Numer. Math. **103** (2006)]
$\mathbb{H}^s(\mathbb{S}^2)$, $s > 1$: K. Hesse & I. H. Sloan [J. Approx. Theory **141** (2006)]
$\mathbb{H}^s(\mathbb{S}^d)$, $s > d/2$: B. & K. Hesse [Constr. Approx. **25** (2007)]

Approximate Spherical Designs

Definition

Let $s > d/2 \geq 1$.
A sequence (X_N) of N-point configurations on $\mathbb{S}^d$ with $N \to \infty$ is said to be a **sequence of approximate spherical designs for** $\mathbb{H}^s(\mathbb{S}^d)$ if there exists $c(s,d) > 0$, independent of N, such that

$$\sup_{\substack{f \in \mathbb{H}^s(\mathbb{S}^d), \\ \|f\|_{\mathbb{H}^s} \leq 1}} \left| \frac{1}{N} \sum_{\mathbf{x} \in X} f(\mathbf{x}) - \int_{\mathbb{S}^d} f(\mathbf{x}) \, \mathrm{d}\sigma_d(\mathbf{x}) \right| \leq \frac{c(s,d)}{N^{s/d}}.$$

X_N need not exist for all natural numbers N.

Approximate Spherical Designs . . .

. . . EXIST: MINIMIZERS OF THE WCE ARE NATURAL CANDIDATES

. . . ARE BETTER THAN AVERAGE

. . . A CHARACTERIZATION

Low-discrepancy sequences on the unit sphere

J. Beck

There are $c_1, c_2 > 0$ s.t. to any N-point set Z_N on $\mathbb{S}^d$ there exists a spherical cap $C \subset \mathbb{S}^d$ s.t.

$$c_1 \, N^{-1/2-1/(2d)} < \left| \frac{|Z_N \cap C|}{N} - \sigma_d(C) \right|$$

and (by a probabilistic argument) there exist N-point sets Z_N^* on $\mathbb{S}^d$ s.t.

$$\left| \frac{|Z_N^* \cap C|}{N} - \sigma_d(C) \right| < c_2 \, N^{-1/2-1/(2d)} \sqrt{\log N}$$

for every spherical cap C.

Definition

A sequence (Z_N) of point sets on $\mathbb{S}^d$ is said to be a **low-discrepancy sequence on** $\mathbb{S}^d$ if there exists $c_1 > 0$ independent of N s.t. for all Z_N

$$D_{\mathbb{L}_\infty}^{C}(Z_N) \leq c_1 \, \frac{\sqrt{\log N}}{N^{1/2+1/(2d)}}.$$

Theorem

Let $d \geq 2$. Any low-discrepancy sequence (Z_N) on $\mathbb{S}^d$ is a sequence of approximate spherical designs for $\mathbb{H}^s(\mathbb{S}^d)$ with

$$d/2 < s < (d+1)/2.$$

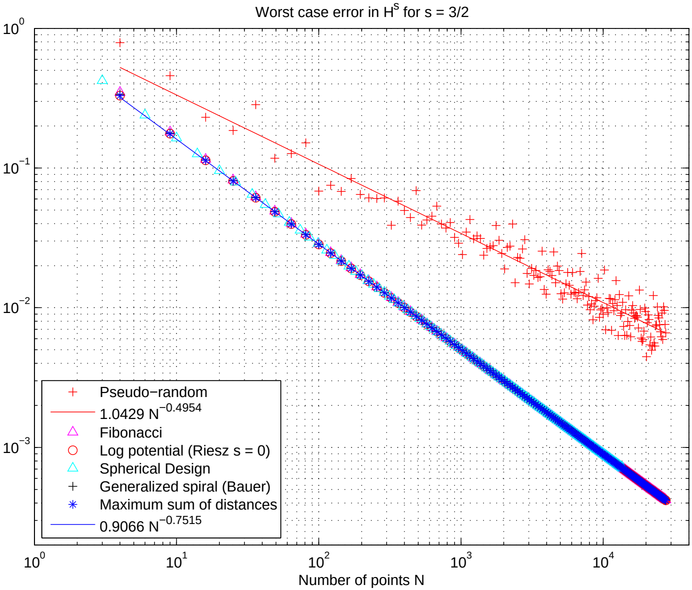
Remark

A sequence of maximizers of

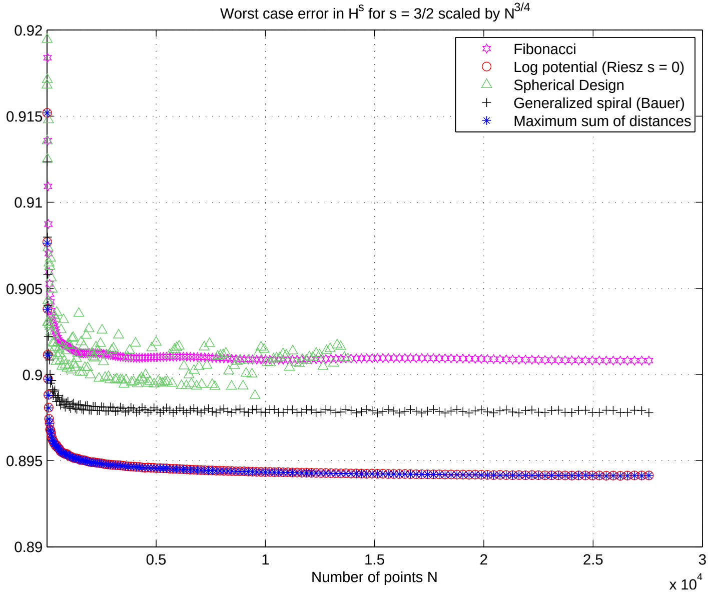
$$\sum_{j=1}^{N} \sum_{k=1}^{N} |\mathbf{x}_j - \mathbf{x}_k|^{2s-d}, \qquad (d+1)/2 < s < d/2+1 \text{ fixed,}$$

beats any given low-discrepancy sequence (Z_N) on $\mathbb{S}^d$.

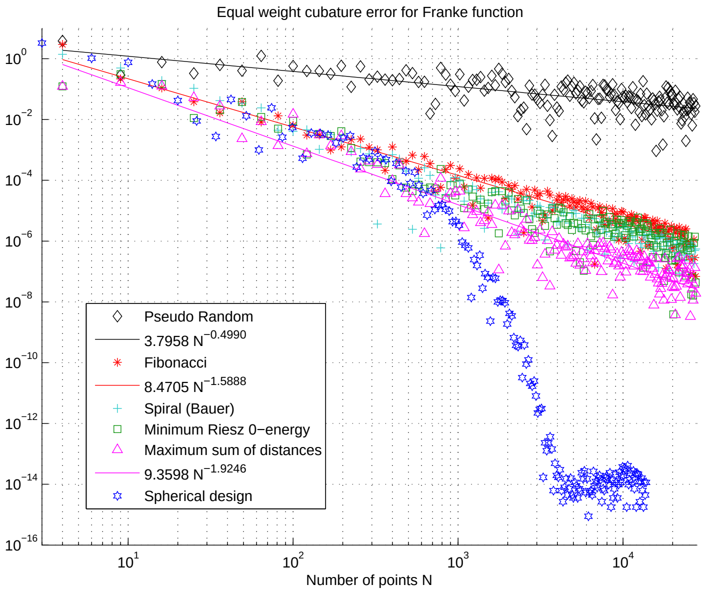
Numerics, I



Worst case error in H^s for s = 3/2
10^0
10^−1
10^−2
10^−3
10^0
10^1
10^2
10^3
10^4
Number of points N
Pseudo−random
1.0429 N^−0.4954
Fibonacci
Log potential (Riesz s = 0)
Spherical Design
Generalized spiral (Bauer)
Maximum sum of distances
0.9066 N^−0.7515


Worst case error in H^s for s = 3/2 scaled by N^3/4
0.92
0.915
0.91
0.905
0.9
0.895
0.89
0.5
1
1.5
2
2.5
3
Number of points N
x 10^4
Fibonacci
Log potential (Riesz s = 0)
Spherical Design
Generalized spiral (Bauer)
Maximum sum of distances

Numerics, II


Equal weight cubature error for Franke function
10^0
10^−2
10^−4
10^−6
10^−8
10^−10
10^−12
10^−14
10^−16
10^1
10^2
10^3
10^4
Number of points N
Pseudo Random
3.7958 N^−0.4990
Fibonacci
8.4705 N^−1.5888
Spiral (Bauer)
Minimum Riesz 0−energy
Maximum sum of distances
9.3598 N^−1.9246
Spherical design

EXPLICIT POINT CONSTRUCTIONS

- **Spherical** Digital Nets and Fibonacci points

Area preserving map to $\mathbb{S}^2$ & Consequences

Lambert cylindrical equal-area projection

Map to $\mathbb{S}^d$

$$\mathbf{x} := \mathbf{\Phi}_s(\mathbf{y}) := \widetilde{\mathbf{\Phi}}_s(\phi_1(y_1), \phi_2(y_2)) \in \mathbb{S}^2,$$

where $\phi_1(y_1) := 2\pi y_1$, $\phi_2(y_2) := 1 - 2y_2$, and

$$\widetilde{\mathbf{\Phi}}_s(\phi, t) := \left(\sqrt{1-t^2}\cos\phi, \sqrt{1-t^2}\sin\phi, t\right).$$

Lemma (B-Dick, Numerische Mathematik (2011))

$$D_N^{(*)}(\mathbb{S}^2, \Omega^{(*)}; Z_N) = D_N^{(*)}([0,1)^2, \mathcal{R}^{(*)}; \mathcal{Z}_N).$$

Theorem (B-Dick, Numerische Mathematik (2011))

Sequence $(Z_N)_{N \geq 2}$ *s.t.* $Z_N = \mathbf{\Phi}_2(\mathcal{Z}_N)$, $\mathcal{Z}_N \subseteq [0,1)^2$.

$$\left[\mathrm{wce}(Q_N; \mathbb{H}^{3/2}(\mathbb{S}^2))\right]^2 \leq \left(\frac{24}{\sqrt{3}} + 2\sqrt{2}\right) D_N^*([0,1)^2, \mathcal{R}^{(*)}; \mathcal{Z}_N).$$

Definition (Isotropic Discrepancy)

$$J_N(\mathcal{Z}_N) := D_N([0,1]^2, \mathcal{A}; \mathcal{Z}_N),$$

where $\mathcal{A}$ is the family of convex subsets of $[0,1]^2$

Theorem (Aistleitner-B-Dick, submitted)

$$D^C(\mathbf{\Phi}(\mathcal{Z}_N)) \leq 11 J_N(\mathcal{Z}_N), \qquad \mathcal{Z}_N \subseteq [0,1)^2.$$

DIGITAL NETS AND
SPHERICAL DIGITAL NETS

Digital Nets

Definition (Star Discrepancy and local Discrepancy)

$$D_N^*(X_N) = \sup_{\mathbf{x}\in[0,1]^s} |\delta_{X_N}(\mathbf{x})|, \qquad \delta_{X_N}(\mathbf{x}) = \frac{|X_N \cap [\mathbf{0},\mathbf{x})|}{N} - \mathrm{VOL}([\mathbf{0},\mathbf{x}))$$

Digital Net (Characterization)

A (t, m, s)-net in base b with $N = b^m$ points has the property that each **elementary interval** contains exactly b^t points.

$$\prod_{i=1}^{s}\left[\frac{a_i}{b^{d_i}}, \frac{a_i+1}{b^{d_i}}\right) \qquad \begin{array}{c} 0 \le a_i < b^{d_i} \\ d_1 + \cdots + d_s = m - t \\ 0 \le d_1, \ldots, d_s \in \mathbb{Z} \end{array} \qquad \text{(elementary interval)}$$

Example   Construction

Theorem

X_N *(t, m, s)-net in base b:* $\mathrm{D}_N^*(X_N) \le C\,(m-t)^{s-1}/b^{m-t}$.

$$D_N^*(X_N) = \sup_{\mathbf{x}\in[0,1]^s} |\delta_{X_N}(\mathbf{x})| , \qquad \delta_{X_N}(\mathbf{x}) = \frac{|X_N \cap [\mathbf{0},\mathbf{x})|}{N} - \mathrm{VOL}([\mathbf{0},\mathbf{x}))$$

THM.: X_N $(0, m, s)$-net in base b: $\mathrm{D}_N^*(X_N) \leq C\, m^{s-1}/b^{m-1}$.

HAMMERSLEY [Ann. New York Acad. Sci. 86 1960] and HALTON [Numer. Math. 2 1960]

(i) for any $N \geq 2$ there exists $\mathbf{x}_1, \ldots, \mathbf{x}_N \in [0,1)^s$ s.t.

$$D_N^*((\mathbf{x}_1, \ldots, \mathbf{x}_N)) = \mathcal{O}(\frac{(\log N)^{s-1}}{N})$$

(ii) there exists a sequence $(\mathbf{x}_n)_{n\geq 1}$ in $[0,1)^s$ s.t.

$$D_N^*((\mathbf{x}_n)_{n\geq 1}) = \mathcal{O}(\frac{(\log N)^{s}}{N})$$

Spherical Rectangle discrepancy

Families of 'half-open' axis-parallel rectangles 'lifted' to $\mathbb{S}^2$

$$\Omega^* = \{\mathbf{\Phi}_s(R_{\mathbf{0},\mathbf{y}}) : \mathbf{0} \preccurlyeq \mathbf{y} \preccurlyeq \mathbf{1}\}.$$

Definition (**Spherical Rectangle (Star) Discrepancy**)

$$\mathrm{D}_N^*(\Omega^*; X_N) := \sup_{\mathcal{R} \in \Omega^*} |\delta_N(\mathcal{R}; X_N)|.$$

Theorem (B-Dick, Numerische Mathematik)

For $\mathcal{Z}_N$ (a $(0, m, 2)$-net in base b lifted to $\mathbb{S}^2$) Recall

$$D_{b^m}^*(\Omega^*, \mathcal{Z}_N) \leq \frac{b^2/4}{b+1}\frac{m}{b^m} + \frac{1}{b^m}\left(\frac{9}{4} + \frac{1}{b}\right) + \frac{1}{b^{2m}}\left(\frac{b}{2} - \frac{1}{4} - \frac{4b+3}{4(b+1)^2}\right).$$

By Roth [Mathematika (1954)]

$$D_N^*(\Omega^*, X_N) \geq (\lfloor \log_2 N \rfloor + 3)/(2^8 N).$$

Isotropic & Spherical Cap Discrepancy

Theorem (Aistleitner-B-Dick, submitted)

- $\mathcal{Z}_N \ldots (0, m, 2)$*-net in base b* ($N = b^m$)

$$J_N(\mathcal{Z}_N) \leq \frac{4\sqrt{2b}}{\sqrt{N}}.$$

- $\mathcal{Z}_N \ldots$ *first N points of* $(0, 2)$*-sequence*

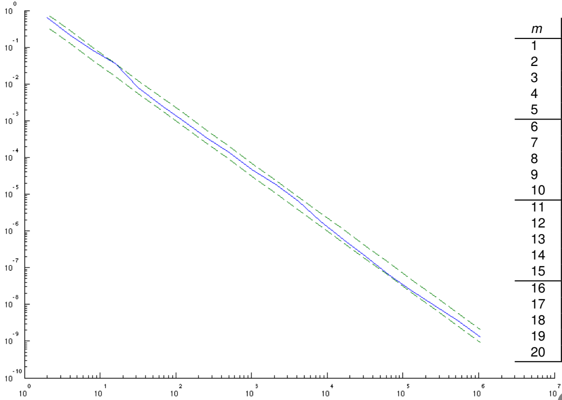
$$J_N(\mathcal{Z}_N) \leq \frac{4\sqrt{2}(b^2 + b^{3/2})}{\sqrt{N}}.$$

Remark

Optimal rate of convergence for J_N (cf. Beck and Chen, 1987 & 2008):

$$N^{-2/3}(\log N)^c \qquad \text{for some } 0 \leq c \leq 4!$$

WCE of Numerical Integration Rules based on Digital Nets (Sobol' points) on Spheres



m	sWCE	$N^{3/2}$ sWCE
1	6.2622e-01	1.7712
2	2.1149e-01	1.6920
3	8.1448e-02	1.8430
4	3.5091e-02	2.2459
5	8.0526e-03	1.4577
6	2.6309e-03	1.3470
7	9.4336e-04	1.3661
8	3.4501e-04	1.4132
9	1.3374e-04	1.5495
10	4.6029e-05	1.5083
11	1.8846e-05	1.7468
12	6.4670e-06	1.6953
13	1.7873e-06	1.3252
14	5.6815e-07	1.1915
15	1.9912e-07	1.1811
16	6.3194e-08	1.0602
17	2.4122e-08	1.1447
18	9.1906e-09	1.2335
19	3.7001e-09	1.4047
20	1.3068e-09	1.4032

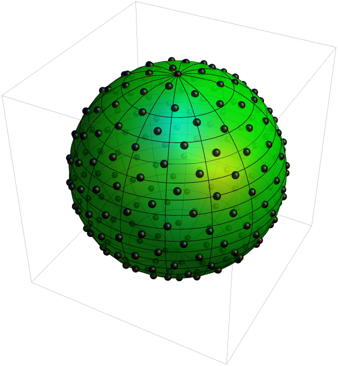
▸ $d = 2$ ▸ $d = 200$

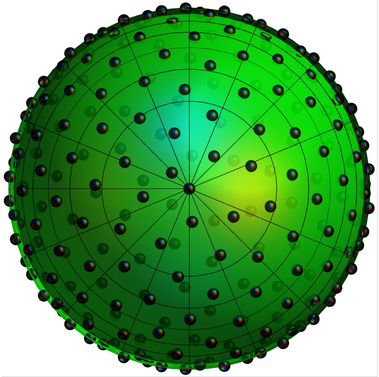
Conjecture (B-Dick, Numerische Mathematik)

A sequence of Sobol' points in $[0,1)^s$ ($s \geq 2$) lifted to the s-sphere achieves optimal convergence rate of the sum of distances.

Fibonacci Lattice and

Spherical Fibonacci Lattices





Fibonacci lattice points in $[0,1]^2$

Fibonacci sequence:

$$F_1 = 1, \qquad F_2 = 1, \qquad \text{and} \qquad F_{n+1} = F_n + F_{n-1}, \quad n \geq 1.$$

Fibonacci lattice in $[0,1]^2$

$$\boldsymbol{f}_k := \left(\frac{k}{F_n}, \left\{ \frac{kF_{n-1}}{F_n} \right\} \right), \quad 0 \leq k < F_n,$$

where $\{x\} = x - \lfloor x \rfloor$ for nonnegative real numbers x. The set

$$\mathcal{F}_n := \{\boldsymbol{f}_0, \ldots, \boldsymbol{f}_{F_n - 1}\}$$

is called a **Fibonacci lattice point set**.

Fibonacci lattice points on the unit sphere $\mathbb{S}^2$

Lambert transformation $\boldsymbol{\Phi} : [0,1]^2 \to \mathbb{S}^2$

$$\boldsymbol{\Phi}(x,y) = \left(2\cos(2\pi y)\sqrt{x-x^2}, 2\sin(2\pi y)\sqrt{x-x^2}, 1-2x\right).$$

Note: Area preserving map.

Fibonacci lattice on $\mathbb{S}^2$

The spherical Fibonacci lattice points are then given by

$$\boldsymbol{z}_k = \boldsymbol{\Phi}(\boldsymbol{f}_k), \quad 0 \le k < F_n,$$

and the point set

$$\mathcal{Z}_n = \{\boldsymbol{z}_0, \ldots, \boldsymbol{z}_{F_n - 1}\}$$

is the **spherical Fibonacci lattice point set**.

Separation results (B–Dick, work in progress)

Minimum distance:

$$d_{\min}(\mathcal{P}) = \min_{0\le k<\ell<N} |\boldsymbol{y}_k - \boldsymbol{y}_\ell|, \qquad \mathcal{P} = \{\boldsymbol{y}_0, \dots, \boldsymbol{y}_{N-1}\} \subset \mathbb{R}^{d+1}.$$

Theorem (Fibonacci points in $[0,1]^2$)

$$d_{\min}(\mathcal{F}_{2m+1}) = \frac{1}{\sqrt{F_{2m+1}}}, \quad m \ge 1, \qquad d_{\min}(\mathcal{F}_{2m}) = \frac{\sqrt{2}F_m}{F_{2m}}, \quad m \ge 2.$$

Theorem (Fibonacci points in $\mathbb{S}^2$)

$$d_{\min}(\mathcal{Z}_{2m+1}) \ge \frac{1}{F_{2m+1}}, \qquad m \ge 1.$$

Conjecture (Fibonacci points in $\mathbb{S}^2$)

$$d_{\min}(\mathcal{Z}_n) = \frac{2}{\sqrt{F_n}}, \qquad n \ge 2.$$

n	F_n	$\sqrt{Z_n} d_{\min}(\mathcal{Z}_n)$
3	2	2 ± 10^{-100}
4	3	2 ± 10^{-100}
5	5	2 ± 10^{-100}
6	8	2 ± 10^{-100}
7	13	2 ± 10^{-100}
8	21	2 ± 10^{-100}
9	34	2 ± 10^{-100}
10	55	2 ± 10^{-100}
11	89	2 ± 10^{-100}
12	144	2 ± 10^{-100}

Isotropic & Spherical Cap Discrepancy

Corollary (Aistleitner-B-Dick, submitted)

$$D^C(Z_{F_m}) \leq \begin{cases} 44\sqrt{2/F_m} & \textit{if } m \textit{ is odd,} \\ 44\sqrt{8/F_m} & \textit{if } m \textit{ is even.} \end{cases}$$

Table : $\tilde{D}(Z_{F_m})$ . . . maximum of the absolute values of the local discrepancies of Z_{F_m} at the spherical caps centered at the spherical Fibonacci points Z_{F_m}.

m / F_m	17 / 1597	18 / 2584	19 / 4181	20 / 6765	21 / 10946	22 / 17711	23 / 28657	24 / 46368
$\frac{\tilde{D}(Z_{F_m})*F_m^{3/4}}{\sqrt{\log F_m}}$	0.6729	0.6373	0.6228	0.6661	0.6953	0.6890	0.7427	0.6900
$\frac{\tilde{D}(Z_{F_m})*F_m^{3/4}}{\log F_m}$	0.2477	0.2273	0.2156	0.2243	0.2279	0.2203	0.2318	0.2105
m / F_m	25 / 75025	26 / 121393	27 / 196418	28 / 317811	29 / 514229	30 / 832040	31 / 1346269	32 / 2178309
$\frac{\tilde{D}(Z_{F_m})*F_m^{3/4}}{\sqrt{\log F_m}}$	0.6957	0.7249	0.7531	0.7205	0.8562	0.7455	0.7862	0.8082
$\frac{\tilde{D}(Z_{F_m})*F_m^{3/4}}{\log F_m}$	0.2076	0.2118	0.2157	0.2024	0.2361	0.2019	0.2092	0.2115

Spherical Cap $\mathbb{L}_2$-discrepancy (B–Dick, work in progress)

$$4\left[D_{\mathbb{L}_2}^{C}(\mathcal{Z}_n)\right]^2 = \frac{4}{3} - \frac{1}{F_n^2}\sum_{j,k=0}^{F_n-1}\left|\boldsymbol{z}_j - \boldsymbol{z}_k\right|$$

n	F_n	$4\left[D_{\mathbb{L}_2}^{C}(\mathcal{Z}_n)\right]^2$	$F_n^{-3/2}$	$4F_n^{3/2}\left[D_{\mathbb{L}_2}^{C}(\mathcal{Z}_n)\right]^2$
3	2	6.2622e-01	3.5355e-01	1.7712
4	3	3.2188e-01	1.9245e-01	1.6725
5	5	1.2865e-01	8.9442e-02	1.4384
6	8	5.7129e-02	4.4194e-02	1.2926
7	13	2.4622e-02	2.1334e-02	1.1540
8	21	1.1107e-02	1.0391e-02	1.0688
9	34	5.0965e-03	5.0440e-03	1.0103
10	55	2.3683e-03	2.4516e-03	0.9660
11	89	1.1064e-03	1.1910e-03	0.9289
12	144	5.2192e-04	5.7870e-04	0.9018
13	233	2.4792e-04	2.8116e-04	0.8817
14	377	1.1837e-04	1.3661e-04	0.8665
15	610	5.6680e-05	6.6375e-05	0.8539
16	987	2.7240e-05	3.2249e-05	0.8446
17	1597	1.3119e-05	1.5669e-05	0.8372
18	2584	6.3331e-06	7.6130e-06	0.8318
19	4181	3.0598e-06	3.6989e-06	0.8272
20	6765	1.4808e-06	1.7972e-06	0.8239
21	10946	7.1699e-07	8.7320e-07	0.8211
22	17711	3.4756e-07	4.2426e-07	0.8192
23	28657	1.6848e-07	2.0613e-07	0.8173
24	46368	8.1756e-08	1.0015e-07	0.8162
25	75025	3.9663e-08	4.8662e-08	0.8150
26	121393	1.9257e-08	2.3643e-08	0.8145
27	196418	9.3470e-09	1.1487e-08	0.8136
28	317811	4.5399e-09	5.5814e-09	0.8133
29	514229	2.2041e-09	2.7118e-09	0.8128
30	832040	1.0708e-09	1.3176e-09	0.8127
31	1346269	5.1999e-10	6.4018e-10	0.8122
				0.7985

cf. B [Uniform Distribution Theory **6**:2 (2011)]

Sum of distances for Spherical Fibonacci points

$$\frac{1}{F_n^2}\sum_{j=0}^{F_n-1}\sum_{k=0}^{F_n-1}|\mathbf{z}_j-\mathbf{z}_k|^{2s-2}=V_{2s-2}+V_{2s-2}\sum_{\ell=1}^{\infty}\frac{(1-s)_\ell}{(1+s)_\ell}(2\ell+1)\left|\frac{1}{F_n}\sum_{k=0}^{F_n-1}P_\ell(1-\frac{2k}{F_n})\right|^2$$

$$+2V_{2s-2}\sum_{\ell=1}^{\infty}\frac{(1-s)_\ell}{(1+s)_\ell}(2\ell+1)\sum_{m=1}^{\ell}\frac{(\ell-m)!}{(\ell+m)!}\left|\frac{1}{F_n}\sum_{k=0}^{F_n-1}P_\ell^m(1-\frac{2k}{F_n})\,e^{2\pi i\,m\,kF_{n-1}/F_n}\right|^2.$$

On the right-hand side one has (the error of) the numerical integration rule

$$0=\int_{-1}^{1}P_\ell(x)\,\mathrm{d}\,x\approx\frac{1}{F_n}\sum_{k=0}^{F_n-1}P_\ell(1-\frac{2k}{F_n}),\qquad \ell\geq 1,$$

with equally spaced nodes in $[-1,1]$ for the Legendre polynomials $P_\ell(x)$ and the *Fibonacci lattice rule*

$$0=\int_0^1\int_0^1 P_\ell^m(1-2x)\,e^{2\pi i\,m\,y}\,\mathrm{d}\,x\,\mathrm{d}\,y\approx\frac{1}{F_n}\sum_{k=0}^{F_n-1}P_\ell^m(1-\frac{2k}{F_n})\,e^{2\pi i\,m\,kF_{n-1}/F_n}$$

based on the Fibonacci lattice points in the unit square $[0,1]^2$ for functions

$$f_\ell^m(x,y):=P_\ell^m(1-2x)\,e^{2\pi i\,m\,y},\qquad \ell\geq 1, 1\leq|m|\leq\ell.$$

The End

Thank You!

Numerical Integration in the Unit Cube

Koksma-Hlawka Inequality

$$\left|\frac{1}{N}\sum_{n=0}^{N-1} f(\mathbf{x}_n) - \int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}\,\mathbf{x}\right| \le D_N^*(\mathbf{x}_0, \mathbf{x}_1, \dots)\, V_{\mathrm{HK}}(f).$$

Star-Discrepancy

$$D_N^*(\mathbf{x}_0, \mathbf{x}_1, \dots) = \sup_{\mathbf{x}\in[0,1]^s}\left|\frac{|X_N \cap [\mathbf{0},\mathbf{x})|}{N} - \mathrm{VOL}([\mathbf{0},\mathbf{x}))\right|.$$

Return

Reproducing Kernel Properties

$$K^{(s)}(\cdot, \mathbf{y}) \in \mathbb{H}^s(\mathbb{S}^d),$$

$$\left(f, K^{(s)}(\cdot, \mathbf{y})\right)_{\mathbb{H}^s} = f(\mathbf{y}), \qquad f \in \mathbb{H}^s(\mathbb{S}^d).$$

Return

Indeed (B-Dick [Proc. of the AMS, in press]) . . .

$$K_{\mathrm{C}}^{(s)}(\mathbf{x},\mathbf{y}) := \int_0^\pi \int_{\mathbb{S}^d} \chi_{C(\mathbf{z};\theta)}(\mathbf{x}) \chi_{C(\mathbf{z};\theta)}(\mathbf{y}) \,\mathrm{d}\sigma_d(\mathbf{z}) \,\sin\theta \,\mathrm{d}\theta, \qquad \mathbf{x},\mathbf{y} \in \mathbb{S}^d$$

. . . symmetric, positive definite $\Rightarrow$ reproducing kernel for some $\mathbb{H}(\mathbb{S}^d)$.

$$K_{\mathrm{C}}^{(s)}(\mathbf{x},\mathbf{y}) = 1 - C_d \left|\mathbf{x}-\mathbf{y}\right|, \qquad \mathbf{x},\mathbf{y} \in \mathbb{S}^d$$

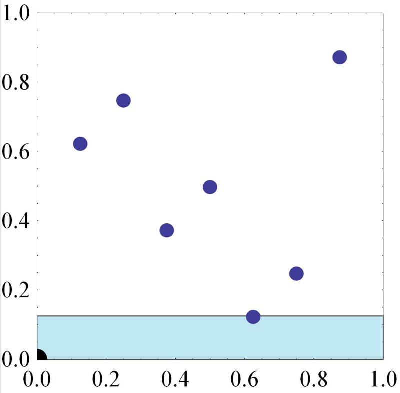
. . . reproducing kernel for $\mathbb{H}^s(\mathbb{S}^d)$, $s = (d+1)/2$.

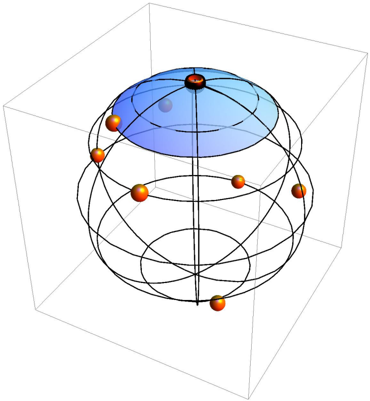
$$\left[\mathrm{wce}(\mathrm{Q}_N; \mathbb{H}^s(\mathbb{S}^d))\right]^2 = V_{-1}(\mathbb{S}^d) - \frac{1}{N^2} \sum_{j=1}^N \sum_{k=1}^N \left|\mathbf{x}_j - \mathbf{x}_k\right|,$$

$$\left[\mathrm{wce}(\mathrm{Q}_N; \mathbb{H}^s(\mathbb{S}^d))\right]^2 = \frac{1}{C_d} \left[D_{\mathbb{L}_2}^C(X_m)\right]^2 .$$

Return

Example (Digital $(0, 3, 2)$-net in base 2 with 8 points)





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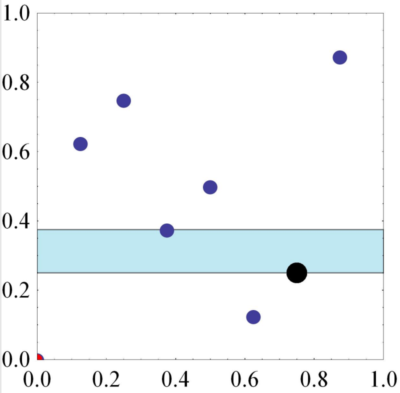

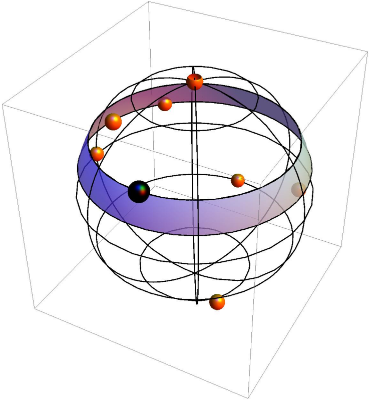

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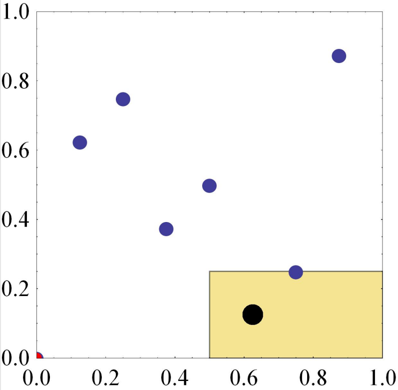

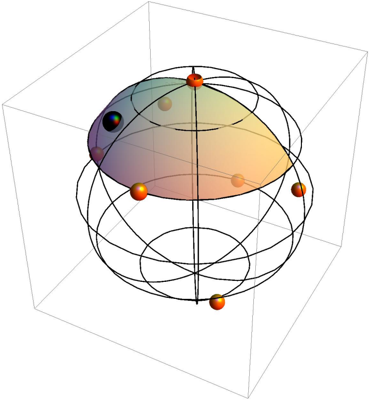

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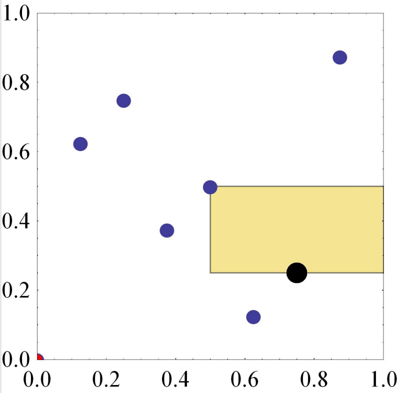
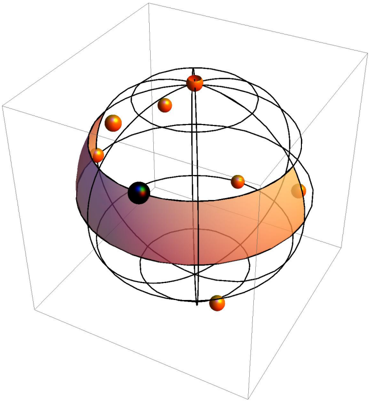
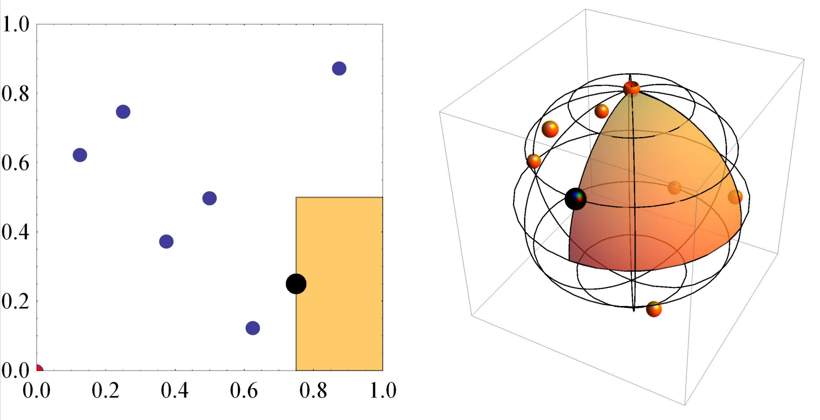
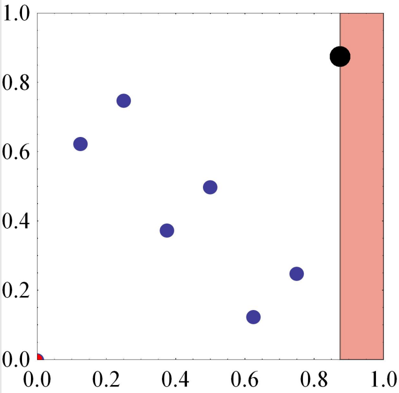


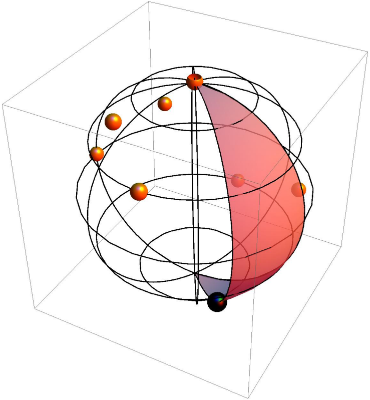


Example (Digital $(0, 3, 2)$-net in base 2 with 8 points)



Example (Digital $(0, 3, 2)$-net in base 2 with 8 points)





Return

Construction of (0, m, s)-nets by means of Sobol' pts

Let

$$\ell = b_1 + b_2 2 + \cdots + b_n 2^n \leq 2^n.$$

The j-th component of the ℓ-th point in the **Sobol' sequence** is

$$x_{\ell,j} = 2^{-n}\left[b_1 v_{1,j} \oplus \cdots \oplus b_n v_{n,j}\right], \qquad v_{k,j} = m_{k,j} 2^{n-k}.$$

The **direction numbers** $\{m_{1,j}, m_{2,j}, \cdots\}$ are recursively defined by

$$m_{k,j} = \bigoplus_{\nu=1}^{s_j-1} 2^\nu a_{\nu,j} m_{k-\nu,j} \oplus 2^{s_j} m_{k-s_j,j} \oplus m_{k-s_j,j}, \qquad k \geq s_j + 1,$$

in terms of the coefficients of the primitive polynomial

$$x^{s_j} + a_{1,j} x^{s_j-1} + \cdots + a_{s_j-1,j} x + 1$$

over $\mathbb{Z}_2$ * and the initial direction numbers $m_{1,j}, \ldots, m_{s_j,j}$.
(For the first component one may chose $m_{k,1} = 1$ for all k.)

Return

*decoded as the integer $a_{1,j} 2^{s_j-1} + \cdots + a_{s_j-1,j} 2$

Example

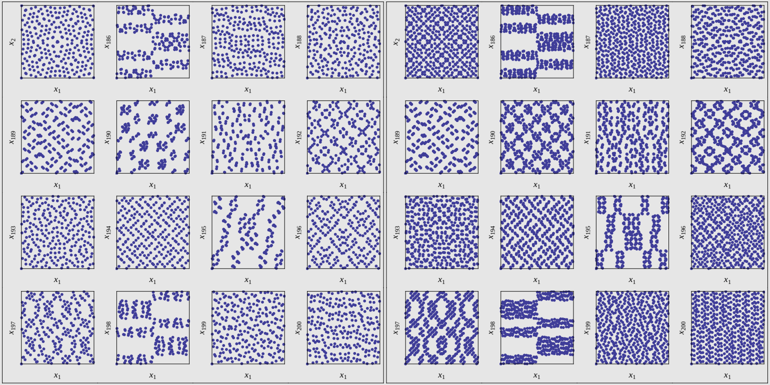


Figure : 2-dim. projections of Sobol' points in $[0,1)^{200}$ with $N = 256, 512$.

Return

Proof.

Substitution of

$$\mathrm{d}\,\sigma_d(\mathbf{x}_d) = \frac{\omega_{d-1}}{\omega_d}\left(1-\tau_d^2\right)^{d/2-1}\mathrm{d}\,\tau_d\;\mathrm{d}\,\sigma_{d-1}(\mathbf{x}_{d-1}) \quad d = 2, 3, \ldots,$$

yields $\sigma_1(\mathbf{\Phi}_1(R_{\mathbf{y}})) = \int_0^{2\pi y_1}\mathrm{d}\,\phi/(2\pi) = y_1$ and

$$\sigma_d(\mathbf{\Phi}_d(R_{\mathbf{y}})) = \frac{\omega_{d-1}}{\omega_d}\int_{t_d}^{1}\left(1-\tau_d^2\right)^{d/2-1}\mathrm{d}\,\tau_d\int_{\mathbf{\Phi}_{d-1}(R_{\mathbf{y}})}\mathrm{d}\,\sigma_{d-1}(\mathbf{x}_{d-1})$$
$$= T_d(t_d)\,\sigma_{d-1}(\mathbf{\Phi}_{d-1}(R_{\mathbf{y}})),$$

where (since $2^{d-1}\omega_{d-1}/\omega_d = 1/B(d/2, d/2)$)

$$T_d(t) = \frac{\omega_{d-1}}{\omega_d}\int_t^1\left(1-\tau^2\right)^{d/2-1}\mathrm{d}\,\tau = 2^{d-1}\frac{\omega_{d-1}}{\omega_d}\int_0^{(1-t)/2}u^{d/2-1}\,\mathrm{d}\,u$$
$$= \mathrm{I}_{(1-t)/2}(d/2, d/2).$$

Note that $T_2(t) = (1-t)/2$ and $t_d = 1 - 2y_d$ $(2 \leq d \leq s)$. □

Return

$A(J; X_N)$ ... **number of points of X_N in J**

Definition (Local Discrepancy)

$$\delta_N(J; X_N) := \frac{A(J; X_N)}{N} - \mathrm{VOL}(J).$$

Return

Beck (1984)

To any N-point cfg. $\{\mathbf{x}_1, \dots, \mathbf{x}_N\}$ on $\mathbb{S}^d$ there exists a spherical cap C s.t.

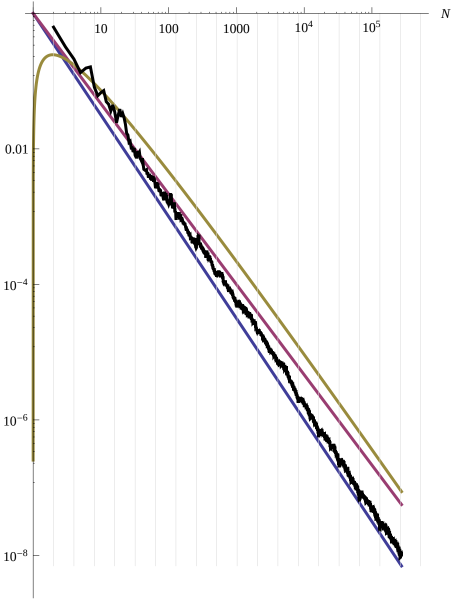
$$c_1 \, N^{-1/2-1/(2d)} < \left| \frac{\#\{k : \mathbf{x}_k \in C\}}{N} - \sigma_d(C) \right|.$$

There exists an N-point cfg. $\{\mathbf{x}_1, \dots, \mathbf{x}_N\}$ on $\mathbb{S}^d$ s.t. for any spherical cap C

$$\left| \frac{\#\{k : \mathbf{x}_k \in C\}}{N} - \sigma_d(C) \right| < c_2 \, N^{-1/2-1/(2d)} \sqrt{\log N}.$$

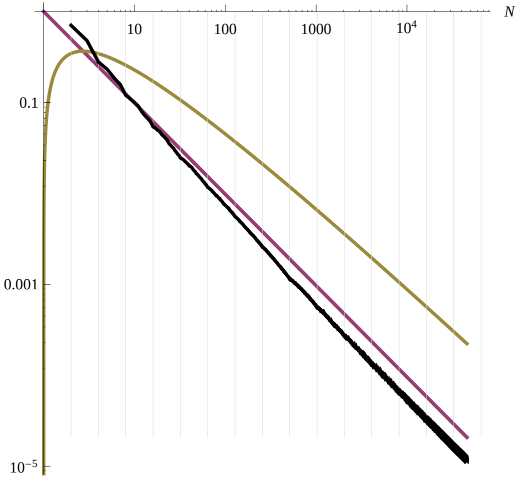
Return

$d = 2; N = 2, \ldots, 273200$



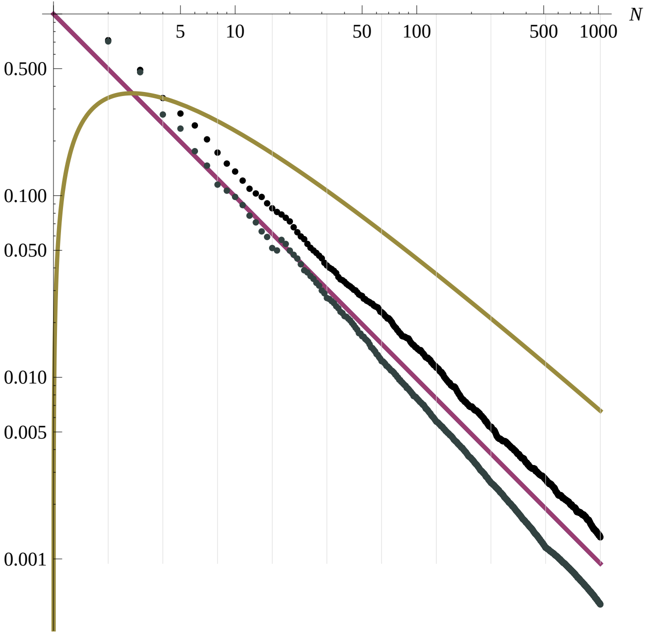
Return

$d = 200$; $N = 2, \ldots, 45701$



Comparison, pseudo-random points

Return


N
5
10
50
100
500
1000
0.500
0.100
0.050
0.010
0.005
0.001

Spherical Designs

Much progress has been made since their introduction by Delsarte, Goethals and Seidel [Geometriae Dedicata 6 (1977)]:

- existence of spherical designs in general and
- their construction by algebraic and by variational means

culminating in a recently proposed proof of the

- t^2-conjecture (or t^d-conjecture) of Korevaar and Meyers [Integral Transform. Spec. Funct. 1 (1993)] that

$N(t) = \mathcal{O}(t^d)$ points are not only necessary but also **sufficient** to form a spherical t-design

by Andriy Bondarenko, Danylo Radchenko and Maryna Viazovska (arXiv:1009.4407v3 [math.MG]).

Return

Approximate Spherical Designs **exist:** **Minimizers of the WCE are natural candidates . . .**

Corollary

If a spherical t-design Y_t on $\mathbb{S}^d$ with $|Y_t| = N \asymp t^d$ exists, then

$$\mathrm{wce}(\mathrm{Q}_N^*; \mathbb{H}^s(\mathbb{S}^d)) \leq \mathrm{wce}(\mathrm{Q}_N; \mathbb{H}^s(\mathbb{S}^d)) \leq \frac{C(s,d)}{N^{s/d}},$$

*where Q_N^* has nodes that **minimize** the functional*

$$\frac{1}{N^2} \sum_{j=1}^{N} \sum_{k=1}^{N} \widetilde{K}^{(s)}(\mathbf{x}_j \cdot \mathbf{x}_k), \qquad \mathbf{x}_1, \dots, \mathbf{x}_N \in \mathbb{S}^d,$$

and Q_N is the exact rule defined by Y_t.

Theorem

Given $d/2 < s < d/2 + 1$,
a sequence of N-point configurations maximizing

$$\sum_{j=1}^{N} \sum_{k=1}^{N} |\mathbf{x}_j - \mathbf{x}_k|^{2s-d}$$

form a sequence of approximate spherical designs for $\mathbb{H}^s(\mathbb{S}^d)$.

Recall:

Proposition (by results of G. Wagner)

$$V_{d-2s}(\mathbb{S}^d) - \frac{1}{N^2} \sum_{j=1}^{N} \sum_{k=1}^{N} |\mathbf{x}_{j,N}^* - \mathbf{x}_{k,N}^*|^{2s-d} \asymp N^{-2s/d}, \quad 0 < 2s - d < 2.$$

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Approximate Spherical Designs **are better than average**

Theorem

Given $d \geq 2$ and $s > d/2$,

$$\sqrt{\mathbb{E}\left[\sup_{\substack{f \in \mathbb{H}^s(\mathbb{S}^d),\\ \|f\|_{\mathbb{H}^s} \leq 1}} \left| \frac{1}{N} \sum_{j=1}^{N} f(\mathbf{x}_j) - \int_{\mathbb{S}^d} f(\mathbf{x}) \, \mathrm{d}\,\sigma_d(\mathbf{x}) \right|^2 \right]} = \frac{a(s,d)}{N^{1/2}}$$

for some constant $a(s,d) > 0$.

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Approximate Spherical Designs, **a characterization**

Variational Characterization of spherical t-designs

$\mathbf{x}_1, \ldots, \mathbf{x}_N$ form a spherical t-design iff

$$\frac{1}{N^2} \sum_{j=1}^{N} \sum_{k=1}^{N} \widetilde{K}_t^{(s)}(\mathbf{x}_j \cdot \mathbf{x}_k) = 0.$$

Theorem

*Let $s > d/2 \geq 1$. Let $\mathbb{H}^s(\mathbb{S}^d)$ be provided with $K^{(s)}$. Then (X_N) satisfying **Property R** is a sequence of approximate spherical designs for $\mathbb{H}^s(\mathbb{S}^d)$ iff*

$$\frac{1}{N^2} \sum_{j=1}^{N} \sum_{k=1}^{N} \widetilde{K}_t^{(s)}(\mathbf{x}_j \cdot \mathbf{x}_k) = \mathcal{O}(N^{-s/d}) \qquad \text{as } N \to \infty.$$

Property R

Definition (Property R)

A sequence (Z_N) of point configurations on $\mathbb{S}^d$ is said to have the Property R if there exist positive numbers c_0 and c_1 independent of N, such that for all Z_N and all $\mathbf{z} \in \mathbb{S}^d$ the nodes satisfy

$$|Z_N \cap C(\mathbf{z}; c_1 |Z_N|^{-1/d})| \leq c_0,$$

where $C(\mathbf{z}; \theta) := \{\mathbf{y} \in \mathbb{S}^d : \mathbf{y} \cdot \mathbf{z} \geq \cos\theta\}$.

The regularity Property R expresses a natural requirement, that a spherical cap whose radius is of order $|Z_N|^{-1/d}$, and hence whose area is of order $|Z_N|^{-1}$, should contain no more than an appropriate number of points for such an area, namely a number of points of order $|Z_N| \times |Z_N|^{-1} = 1$.

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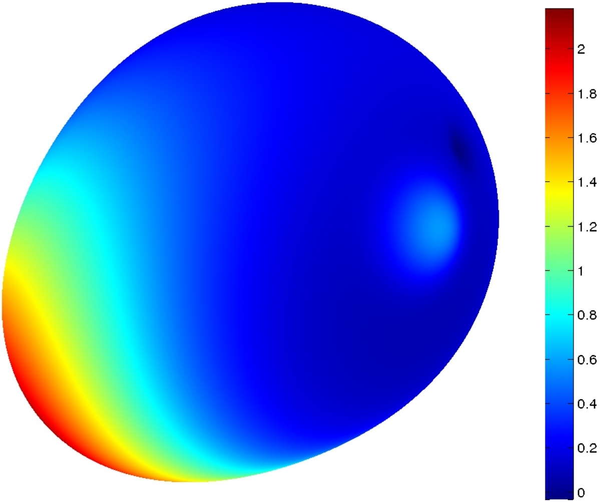
Franke test function



$$
\begin{aligned}
tmp1 &= (9*x-2)^2 + (9*y-2)^2 + (9*z-2)^2;\\
tmp2 &= (9*x+1)^2/49 + (9*y+1)/10 + (9*z+1)/10;\\
tmp3 &= (9*x-7)^2 + (9*y-3)^2 + (9*z-5)^2;\\
tmp4 &= (9*x-4)^2 + (9*y-7)^2 + (9*z-5)^2;\\
f &= 0.75*exp(-tmp1/4) + 0.75*exp(-tmp2)\\
&\quad + 0.5*exp(-tmp3/4) - 0.2*exp(-tmp4);\\
lex &= 6.69618222007361795 23;
\end{aligned}
$$

Return

Lifting Digital Nets to the Unit Sphere in $\mathbb{R}^{d+1}$, $d \geq 2$

Cylindrical Coordinates on the *d*-sphere

$$\mathbf{x} = \mathbf{x}_s = \left(\sqrt{1-t_s^2}\,\mathbf{x}_{s-1}, t_s\right),$$
$$\cdots$$
$$\mathbf{x}_2 = \left(\sqrt{1-t_2^2}\,\mathbf{x}_1, t_2\right),$$
$$\mathbf{x}_1 = \left(\cos\phi, \sin\phi\right)$$

$$-1 \leq t_d \leq 1, \mathbf{x}_d \in \mathbb{S}^d \quad (d = 1, \ldots, s) \quad 0 \leq \phi \leq 2\pi.$$

Arbitrary $\mathbf{x} \in \mathbb{S}^s$ represented through **angle** ϕ and **heights** $t_2, \ldots, t_s$

$$\mathbf{x} = \mathbf{x}(\phi, t_2, \ldots, t_s), \qquad 0 \leq \phi \leq 2\pi, \ -1 \leq t_2, \ldots, t_s \leq 1.$$

A point $\mathbf{y} = (y_1, \ldots, y_s) \in [0,1)^s$ mapped to $\mathbf{x} \in \mathbb{S}^s$ using

$$\varphi_1(y_1) = 2\pi y_1, \qquad \varphi_d(y_d) = 1 - 2y_d \quad (d = 2, 3, \ldots, s)$$

and cylinder coordinates

$$\mathbf{x} = \mathbf{\Phi}_s(\mathbf{y}) = \mathbf{x}\left(\varphi_1(y_1), \varphi_2(y_2), \ldots, \varphi_s(y_s)\right).$$

Area preserving map

Normalized surface area measure σ_d on $\mathbb{S}^d$

$$\mathrm{d}\,\sigma_1(\mathbf{x}_1) = \mathrm{d}\,\phi/(2\pi),$$

$$\mathrm{d}\,\sigma_d(\mathbf{x}_d) = \frac{\omega_{d-1}}{\omega_d}\left(1 - t_d^2\right)^{d/2-1}\mathrm{d}\,t_d\,\mathrm{d}\,\sigma_{d-1}(\mathbf{x}_{d-1}), \quad d = 2, 3, \ldots.$$

Let $R_{\mathbf{y}} = [0, y_1) \times \cdots [0, y_s) \subseteq [0, 1)^s$.

Proposition

$$\sigma_s(\mathbf{\Phi}_s(R_{\mathbf{y}})) = y_1 \prod_{d=2}^{s} \mathrm{I}_{y_d}(d/2, d/2).$$

Proof

Regularized Beta function

$$\mathrm{I}_z(a, b) = \mathrm{B}_z(a, b)/\,\mathrm{B}(a, b), \qquad \mathrm{B}_z(a, b) = \int_0^z u^{a-1}\,(1-u)^{b-1}\,\mathrm{d}\,u.$$

Let $\mathrm{VOL}(R_{\mathbf{z}}) = z_1 z_2 \cdots z_s$ for $\mathbf{z} \in [0,1)^s$.

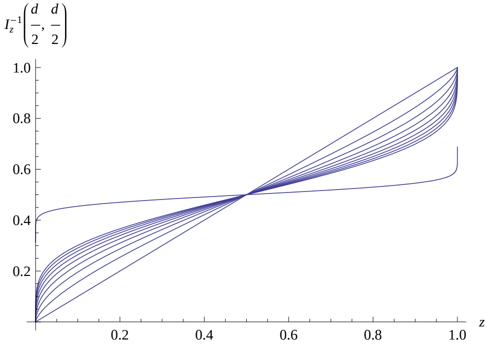
- Chose $y_1 = z_1$, $y_2 = z_2$ and y_d such that $\mathrm{I}_{y_d}(d/2, d/2) = z_d$.

Then again $\sigma_s(\mathbf{\Phi}_s(R_{\mathbf{y}})) = z_1 z_2 \cdots z_s$.

Area preserving map

$$\Psi_s : (z_1, \ldots, z_s) \mapsto \mathbf{\Phi}_s((h_d^{-1}(z_1), h_d^{-1}(z_2), \ldots, h_d^{-1}(z_s))).$$

$h_d^{-1}(y)$ is the inverse of $h_1(y) = y$ and $h_d(y) = \mathrm{I}_y(d/2, d/2)$, $d \geq 2$.



$d = 2, \ldots, 10$ and $d = 200$.

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