Bases with small representation function

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Definitions

In general, a set $A \subseteq \mathbb{N} \cup \{0\}$ is called a *basis* of $B \subseteq \mathbb{N} \cup \{0\}$ if every element of B belongs to the sumset

$$A + A = \{a + a' \ : \ a, a' \in A\}.$$

Usually, B is $\mathbb{N} \cup \{0\}$, but we also consider the case $B = \{0, 1, 2, \ldots, n\}$.

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Let A be a finite or infinite subset of the set $\mathbb{N} \cup \{0\}$. The square of the power series

$$f_A(x) = \sum_{a \in A} x^a$$

associated with A is given by the formulae

$$f_A(x)^2 = \sum_{n=0}^{\infty} r_A(n) x^n,$$

where $r_A(n)$ stands for the number of representations of the integer $n \geqslant 0$ by the sum $a + b$ with $a, b \in A$, namely,

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Partial results on the Erdős-Turán conjecture

It is known that for any $A \subseteq \mathbb{N} \cup \{0\}$ the values of $r_A(n)$, where $n \geqslant 0$, cannot all lie in the interval $[1, 5]$

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For instance, all $r_A(n)$ cannot all lie in the interval $[7, 13]$, because

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Related results

In the opposite direction, Erdős answered a question of Sidon and showed that there exists a basis A of $\mathbb{N} \cup \{0\}$ such that

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for some positive constant c_1 and each $n \geqslant 2$.

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In 1990 Ruzsa proved that there is a basis A of $\mathbb{N} \cup \{0\}$ whose representation function $r_A(n)$ is bounded on average, namely,

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Theorem 1

For each ε satisfying $0 < \varepsilon < 1/2$ there is a positive constant $c(\varepsilon)$ and a basis A of $\mathbb{N} \cup \{0\}$ such that

$$0.1\varepsilon^2 \log n \leqslant r_A(n) \leqslant (2e + \varepsilon) \log n + c(\varepsilon) \tag{3}$$

for every $n \geqslant 2$.

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The version of the ET problem with nonnegative coefficients for series has a trivial answer.
Consider the series

$$g_2(z) := (1-z)^{-1/2} = \sum_{n=0}^{\infty} (-z)^n \binom{-1/2}{n},$$

where

$$(-1)^n \binom{-1/2}{n} = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2^n n!}$$

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Under additional assumption of f being a *reciprocal* polynomial, namely, $f(x) = x^n f(1/x)$, it was proved

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that if the coefficients of $f(x)^2$ are all at least 1 then the largest coefficient of $f(x)^2$ must be at least $\kappa_{\rm rec}(n)$, where

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For a general Newman polynomial we prove that

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In terms of sumsets Theorem 2 asserts that for each $\varepsilon > 0$ and each sufficiently large n there is subset A of the set $\{0, 1, \ldots, n\}$ such that $A + A = \{0, 1, \ldots, 2n\}$ and the number of representations of each given $k \in \{0, 1, \ldots, 2n\}$ by the sum $a + a'$, with $a, a' \in A$, is at most $(1+\varepsilon)(4/\pi)(\log n)^2$, i.e.,

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$$\{0, 1, \ldots, \lfloor (2-\varepsilon)n \rfloor\} \subseteq A + A$$

Theorem 1 gives a stronger bound with $(\log n)^2$ replaced by $\log n$:

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For each $\varepsilon > 0$ there is a positive constant $C = C(\varepsilon)$ such that for every integer $n \geqslant 2$ there is a set $A \subseteq \{0, 1, \ldots, n\}$ for which the sumset $A + A$ contains the set $\{0, 1, \ldots, \lfloor (2-\varepsilon)n \rfloor\}$ and

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Reciprocal Newman polynomial

For a reciprocal Newman polynomial the correct growth is of the order $\sqrt{n}$:

Theorem 4

For each reciprocal Newman polynomial $f(x)$ of degree n whose square has all of its $2n+1$ coefficients at least 1, the middle coefficient for x^n in $f(x)^2$ must be at least $2\sqrt{n}-3$. On the other hand, for each $n \in \mathbb{N}$ there is a reciprocal Newman polynomial of degree n such that the coefficients of its square are all in the interval $[1, 2\sqrt{2n}+4]$.

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To prove the second part of Theorem 4 we use the following explicit example

$$\sum_{i=0}^{t-1}(x^i + x^{n-i}) + \sum_{j=1}^{s}(x^{jt} + x^{n-jt}) + \delta(x), \qquad (5)$$

where

$$t := \lfloor\sqrt{n/2}\rfloor, \quad s := \lceil n/2t\rceil - 1,$$

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The constants -3 and 4 in Theorem 4 can be easily improved. However, we do not know for which constant in the interval $[2, 2\sqrt{2}]$ both parts of Theorem 4 hold, so we ask for the best possible constant κ for $\sqrt{n}$ in the sense that for each $\varepsilon > 0$ and each sufficiently large $n \in \mathbb{N}$ the first statement of Theorem 4 holds with $(\kappa - \varepsilon)\sqrt{n}$ instead of $2\sqrt{n} - 3$ while the second holds with $(\kappa + \varepsilon)\sqrt{n}$ instead of $2\sqrt{2n} + 4$.

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Probabilistic construction

For the proof of Theorems 1 and 2 we define mutually independent random variables Y_k and Y_k^* taking only values 0 and 1, by

$$\mathbb{P}(Y_0 = 1) = \mathbb{P}(Y_0^* = 1) = \mathbb{P}(Y_1 = 1)$$

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$$\mathbb{P}(Y_k = 1) = \mathbb{P}(Y_k^* = 1) = p_k := \lambda\sqrt{\frac{2\log k}{\pi k}} \tag{6}$$

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so that $0 < p_k \leqslant p_3 < 2\sqrt{\frac{2\log 3}{3\pi}} < 0.97 < 1$ for $k \geqslant 3$, by (6) and (7). For convenience, we shall also use the notation $p_0 = p_1 = p_2 = 1$, so, by (6),

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Suppose $X_1, \ldots, X_s$ are s independent Bernoulli trials, where

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for $i = 1, \ldots, s$. Set $X := X_1 + \cdots + X_s$ and

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Then Chernoff's inequality asserts that

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For any $\delta > 0$ *we have*

$$\mathbb{P}(X > (1+\delta)\mathbb{E}(X)) \leqslant e^{-((1+\delta)\log(1+\delta)-\delta)\mathbb{E}(X)} \tag{9}$$

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Preparation for the proof of Theorem 2

In the proof of Theorem 2 we will consider the random Newman polynomial

$$f(x) := \sum_{k=0}^{n} U_k x^k = \sum_{0 \leqslant k < n/2} (Y_k x^k + Y_k^* x^{n-k}) + Y_{n/2} x^{n/2}, \quad (11)$$

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Note that $U_k = Y_k$ for $k \leqslant n/2$ and $U_k = Y^*_{n-k}$ for $n/2 < k \leqslant n$. The square of f is given by

$$f(x)^2 = \sum_{m=0}^{2n} Z_m x^m, \tag{12}$$

where

$$Z_m := 2 \sum_{0 \leqslant k < m/2} U_k U_{m-k} + U_{m/2} \tag{13}$$

for $m \leqslant n$.

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Note that $U_k = Y_k$ for $k \leqslant n/2$ and $U_k = Y^*_{n-k}$ for $n/2 < k \leqslant n$. The square of f is given by

$$f(x)^2 = \sum_{m=0}^{2n} Z_m x^m, \tag{12}$$

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By symmetry (see (6), (11) and (12)), for each interval $I \subseteq \mathbb{R}$ we must have

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$U_k U_{m-k}$, where $0 \leqslant k < m/2$, are mutually independent random variables taking only values 0 and 1, so we will be able to apply Lemma 5 to V_m. For $k \leqslant n/2$ we have $U_{m-k} = Y_{m-k}$ if $m-k \leqslant n/2$ and $U_{m-k} = Y^*_{n-m+k}$ if $m-k > n/2$. Hence, by (6),

$$\mathbb{P}(U_k U_{m-k} = 1) = \mathbb{P}(Y_k U_{m-k} = 1)$$

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Thus the expectation of V_m for $m \leqslant n$ is

$$\mathbb{E}(V_m) = T_m := \sum_{0 \leqslant k < m/2} p_k p_{\min\{m-k, n-m+k\}}. \tag{16}$$

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