Function systems in the theory of u.d. mod 1

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Underlying ideas

This talk has three underlying ideas:

1. from algebra:
   addition (of real numbers, of digit vectors, . . .)
2. from harmonic analyis:
   the dual group and associated function systems on $[0, 1[^s$
3. from applications:
   hybrid sequences

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Two questions

D a nonempty set,
f a function on D, integrable in some suitable sense,
$\omega = (x_n)_{n\geq 0}$ a sequence in D.

Question 1: a limit relation

Under which conditions on f and ω do we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} f(x_n) = \int_D f \text{ ?} \tag{1}$$

Question 2: an error estimate

$\omega = (x_n)_{n=0}^{N-1}$ a finite sequence

What can be said about the error

$$\left| \frac{1}{N}\sum_{n=0}^{N-1} f(x_n) - \int_D f \right| ? \tag{2}$$

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Several answers

These two questions are studied in several fields

Aspects

- Theory of dynamical systems: Q1
  $T : D \to D$ structure preserving,
  $\omega = (T^n x)_{n \geq 0}$ the orbit of $x \in D$

  keywords: ergodic theorem
- Stochastics: Q1, Q2
  f a random variable on the probability space D, ω a given sample;

  keywords: LLN, applied statistics
- Numerical mathematics: Q2

  keywords: quadrature formulae
- Metric Number Theory: Q1, Q2
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Monte Carlo and Quasi-Monte Carlo Methods

The underlying space

$\mathbb{R}^s/\mathbb{Z}^s$ ... s-dimensional torus, identified with $[0,1[^s$, $s \geq 1$
λ_s ... Haar measure on $[0,1[^s$ (s-dimensional Lebesgue measure)

Applications

Suppose $\int_{[0,1[^s} f$ cannot be computed.

Approach: generate a suitable finite sequence $\omega = (x_n)_{n=0}^{N-1}$ such that, for large N,

$$\frac{1}{N}\sum_{n=0}^{N-1} f(x_n) \quad \text{is close to} \quad \int_{[0,1[^s} f \, d\lambda_s.$$

Two techniques to construct such sequences ω in $[0,1[^s$:

1. random points x_n: Monte Carlo method (MC)
2. deterministic points x_n:
   - pseudorandom numbers (PRN): mimic random points
   - low-discrepancy points (LDP): quasi-Monte Carlo method (QMC)

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Notation

Sample means

$\omega = (x_n)_{n\geq 0}$ ... sequence in $[0,1[^s$
$f:\ [0,1[^s \to \mathbb{R}(\mathbb{C})$ ... Riemann-integrable function

$$S_N(f,\omega) \;=\; \frac{1}{N}\sum_{n=0}^{N-1} f(\mathbf{x}_n) \qquad \text{...sample mean}$$

Intervals in $[0,1[^s$

$$\mathcal{J} \;=\; \{\prod_{i=1}^{s}[u_i,v_i[:\ 0\leq u_i<v_i\leq 1\},$$

$$\mathcal{J}^* \;=\; \{\prod_{i=1}^{s}[0,v_i[:\ 0<v_i\leq 1\}.$$

For $J\in\mathcal{J}$, define

$\mathbf{1}_J$ ... the indicator function of J, $\qquad \lambda_s(J)$ ... the volume of J.

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Uniform distribution of sequences

Remark

$$S_N(\mathbf{1}_J - \lambda_s(J), \omega) \;=\; S_N(\mathbf{1}_J, \omega) - \lambda_s(J).$$

Definition

$\omega = (x_n)_{n\geq 0}$ uniformly distributed in $[0,1[^s$ (u.d.), if

$$\forall J \in \mathcal{J}: \qquad \lim_{N\to\infty} S_N(\mathbf{1}_J - \lambda_s(J), \omega) \;=\; 0.$$

Definition

Discrepancy of ω:

$$D_N(\omega) \;=\; \sup_{J\in\mathcal{J}} \left|S_N(\mathbf{1}_J - \lambda_s(J), \omega)\right|,$$

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Results

Theorem

$\omega = (x_n)_{n\geq 0}$ uniformly distributed in $[0,1[^s$ if and only if one of the following is true:

1. *for all subintervals J of $[0,1[^s$:*

$$\lim_{N\to\infty} S_N(\mathbf{1}_J,\omega) = \int_{[0,1[^s} \mathbf{1}_J \, d\lambda_s,$$

2. *for all functions f Riemann integrable on $[0,1[^s$:*

$$\lim_{N\to\infty} S_N(f,\omega) = \int_{[0,1[^s} f \, d\lambda_s,$$

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Tasks

Questions

1. Prove the u.d. mod 1 of a given sequence ω
2. Measure the u.d. mod 1 of a given sequence ω
3. Construct sequences ω with good uniform distribution

Comments

1. On the 1st question: sometimes we have to resort to counting . . .
2. On the 2nd question: cannot compute discrepancy exactly (in many cases) . . .
3. On the 3rd question: this is an art . . .

Goals

We look for general techniques to study these questions.

Examples of sequences

Example (Kronecker sequences)

$\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_s) \in \mathbb{R}^s$
$\omega = (\{n\boldsymbol{\alpha}\})_{n \geq 0}$ ...sequence of fractional parts of the vector $n\boldsymbol{\alpha}$
If $\alpha = \mathbf{g}/N$, $\mathbf{g} \in \mathbb{Z}^s$: good lattice points and lattice rules.

Example (Digital sequences)

$b \geq 2$ integer
$n \in \mathbb{N}_0$: $\vec{n}$ denotes digit vector of n in base b
$\omega = ((\varphi_b(C_1 \vec{n}^T), \ldots, \varphi_b(C_s \vec{n}^T)))_{n \geq 0}$, with suitably chosen matrices $C_1, \ldots, C_s$ and some output function φ_b

a finite digital sequence in base b: a finite sequence ω on the grid

$$b^{-m}\mathbb{Z}^s \pmod 1.$$

Example (Van der Corput sequence)

$\omega = (\sum_{j \geq 0} n_j 2^{-j-1})_{n \geq 0}$, where $n = \sum_{j \geq 0} n_j 2^j$ dyadic expansion

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Classical approach

Trigonometric functions

$\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{Z}^s$, $\mathbf{x} = (x_1, \ldots, x_s) \in [0,1[^s$: $\quad e_{\mathbf{k}}(\mathbf{x}) := \prod_{j=1}^{s} e^{2\pi i k_j x_j}$

$$\mathcal{T}^{(s)} = \{e_{\mathbf{k}} : \mathbf{k} \in \mathbb{Z}^s\} \quad \ldots \text{trigonometric function system on } [0,1[^s.$$

Theorem (Weyl criterion)

$\omega = (\mathbf{x}_n)_{n \geq 0}$ *uniformly distributed in* $[0,1[^s$ *if and only if*

$$\forall \mathbf{k} \neq \mathbf{0}: \lim_{N \to \infty} S_N(e_{\mathbf{k}}, \omega) = 0.$$

Theorem (Inequality of Erdös-Turán-Koksma)

$$D_N(\omega) \leq C_s \left(\frac{1}{M} + \sum_{\mathbf{k} \in \Delta^*(M)} \frac{1}{r(\mathbf{k})} \cdot |S_N(e_{\mathbf{k}}, \omega)| \right)$$

$r(\mathbf{k}) = \prod_{i=1}^{s} \max\{1, |k_i|\}$.
$\Delta^*(M) = \{\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{Z}^s \setminus \{\mathbf{0}\} : ||\mathbf{k}||_\infty = \max_{1 \leq i \leq s} |k_i| < M\}$,
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Observations

Problems

- $\mathcal{T}^s$ is well suited to study Kronecker sequences:

$$S_N(e_{\mathbf{k}}, \omega) = \frac{1}{N} \frac{e_{\mathbf{k}}(\alpha)^N - 1}{e_{\mathbf{k}}(\alpha) - 1}.$$

- $\mathcal{T}^s$ is not suited to study digital sequences or van der Corput sequence!

Diagnosis

The type of addition used in the construction principle has to match the function system used in the analysis of the u.d. behavior of the sequence.

Consequence

Let us study addition!

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$$S_N(e_{\mathbf{k}}, \omega) = \frac{1}{N} \frac{e_{\mathbf{k}}(\alpha)^N - 1}{e_{\mathbf{k}}(\alpha) - 1}.$$

- $\mathcal{T}^s$ is not suited to study digital sequences or van der Corput sequence!

Diagnosis

The type of addition used in the construction principle has to match the function system used in the analysis of the u.d. behavior of the sequence.

Consequence

Let us study addition!

Construction principles for sequences

The general setting

Given the state space $\mathcal{Z}$,

1. generate a sequence $(z_n)_{n\geq 0}$ in $\mathcal{Z}$,
2. map this sequence to $\omega = (x_n)_{n\geq 0}$ in $[0,1[$ by an output function

$$\varphi : \mathcal{Z} \to [0,1[,$$

   with $x_n = \varphi(z_n)$, $n = 0, 1 \ldots$.

Need some structure on $\mathcal{Z}$ in order to derive properties of $(z_n)_{n\geq 0}$ and, in consequence, of the final sequence ω.

Algebraic properties

Type of addition used in the construction principles:

- Kronecker sequences: addition modulo 1
- digital sequences: addition without carry
- van der Corput: ?

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Adding binary vectors, Approach 1

Given

Two binary vectors of length m:

$$x = (x_0, \ldots, x_{m-1}),$$
$$y = (y_0, \ldots, y_{m-1})$$

Question

How to define addition of these two vectors?

Approach I: XOR

$$x \oplus y = (x_0 + y_0, \ldots, x_{m-1} + y_{m-1})$$

Consequences

Underlying group: $(\mathbb{F}_2^m, \oplus)$
Appropriate function system: Walsh functions in base 2

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b-adic representations

b-adic representation

For $k \in \mathbb{N}_0$, let

$$k = \sum_{j \geq 0} k_j b^j, \quad k_j \in \{0, 1, \dots, b-1\}.$$

For $x \in [0, 1[$, let

$$x = \sum_{j \geq 0} x_j b^{-j-1}, \quad x_j \in \{0, 1, \dots, b-1\}.$$

Representation is unique under the condition $x_j \neq b-1$ for infinitely many j.

Let

$$\mathrm{e}(y) = \mathrm{e}^{2\pi i y}, \quad y \in \mathbb{R}.$$

Definition

$k \in \mathbb{N}_0$, $k = \sum_{j \geq 0} k_j b^j$, $\quad x \in [0, 1[$, $x = \sum_{j \geq 0} x_j b^{-j-1}$

k-th Walsh function in base b:

$$w_k(x) = e\left(\left(\sum_{j \geq 0} k_j x_j\right)/b \right).$$

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Walsh functions on $[0,1[^s$

Definition

$\mathbf{b} = (b_1, \dots, b_s)$ a vector of not necessarily distinct integers $b_i \geq 2$
$\mathbf{k} = (k_1, \dots, k_s) \in \mathbb{N}_0^s, \quad \mathbf{x} = (x_1, \dots, x_s) \in [0,1[^s$

k-th Walsh function in base **b**:

$$w_{\mathbf{k}}(\mathbf{x}) = \prod_{i=1}^{s} w_{k_i}(x_i), \quad w_{k_i} \text{ in base } b_i.$$

$\mathcal{W}_{\mathbf{b}}^{(s)} = \{w_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}_0^s\}$ ... the Walsh system in base **b**

ONB property

$\mathcal{W}_{\mathbf{b}}^{(s)}$ is an ONB of $L^2([0,1[^s)$.

Remark

For $\mathcal{W}_{\mathbf{b}}^{(s)}$, we have a Weyl criterion as well as an Erdös-Turán-Koksma inequality.

Adding binary vectors, Approach 2

Given

Two binary vectors of length m:

$$x = (x_0, \ldots, x_{m-1}),$$
$$y = (y_0, \ldots, y_{m-1})$$

Approach II: integer addition

$$x \mapsto int(x) = x_0 + x_1 2 + \cdots + x_{m-1} 2^{m-1}$$
$$y \mapsto int(y) = y_0 + y_1 2 + \cdots + y_{m-1} 2^{m-1}$$
$$x + y = \text{digit vector of } \{int(x) + int(y) \pmod{2^m}\}$$

Consequences

Underlying group: $(\mathbb{Z}_{2^m}, +)$ ...generalize to $(\mathbb{Z}_2, +)$
Appropriate function system: ?

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Appropriate function system: ?

Conclusion

Remark

Adding two binary vectors of length m:

Essentially, there are only those two types of addition!

The b-adic integers

Definition

The compact abelian group of b-adic integers $\mathbb{Z}_b$:

$$\mathbb{Z}_b \;=\; \left\{ z = \sum_{j \geq 0} z_j\, b^j, \quad \text{with digits } z_j \in \{0, 1, \ldots, b-1\} \right\},$$

Remark

$\mathbb{Z}$ is embedded in $\mathbb{Z}_b$. In particular, $-1 = \sum_{j\geq 0}(b-1)\, b^j$.

Definition

The dual group $\hat{\mathbb{Z}}_b$:

$$\begin{aligned} \hat{\mathbb{Z}}_b \;&= \{\chi_0\} \\ &\quad \cup \Big\{ z \mapsto \underbrace{e\left(\frac{a}{b^g}(z_0 + z_1 b + z_2 b^2 + \cdots)\right)}_{\chi(a,g;z)} : \; 1 \leq a < b^g,\, a, g \in \mathbb{N} \Big\}, \end{aligned}$$

χ_0 ...the trivial character, $\forall z \in \mathbb{Z}_b : \; \chi_0(z) = 1$.

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Monna's map

Definition (Monna's map)

$$\varphi_b : \quad \mathbb{Z}_b \to [0,1[,$$
$$\varphi_b\Big(\sum_{j\geq 0} z_j\, b^j\Big) \;=\; \sum_{j\geq 0} z_j\, b^{-j-1} \pmod 1.$$

Properties

φ_b is continuous and surjective, but not injective.

$\hat{\mathbb{Z}}_b$ enumerated

$$\hat{\mathbb{Z}}_b \;=\; \{z \mapsto \underbrace{e\left(\varphi_b(k)(z_0 + z_1 b + z_2 b^2 + \cdots)\right)}_{\chi_k(z)} \;:\; k \in \mathbb{N}_0\},$$

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Pseudo-inverse of Monna map

Definition (Pseudo-inverse of Monna's map)

$$\varphi_b^+ : \; [0,1[\to \mathbb{Z}_b,$$
$$\varphi_b^+(x) = \sum_{j \geq 0} x_j b^j,$$
$$\text{if } x = \sum_{j \geq 0} x_j b^{-j-1} \text{ with } x_j \neq b-1 \text{ infinitely often}$$

Definition (The b-adic function system in dimension $s = 1$)

$k \in \mathbb{N}_0$

$$\gamma_k : \; [0,1[\to \{c \in \mathbb{C} : \; |c| = 1\},$$
$$\gamma_k(x) = \chi_k(\varphi_b^+(x)).$$

Put $\Gamma_b^{(1)} = \{\gamma_k : \; k \in \mathbb{N}_0\}$ … the b-adic function system on $[0,1[$

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Put $\Gamma_b^{(1)} = \{\gamma_k : \; k \in \mathbb{N}_0\}$ ... the b-adic function system on $[0, 1[$

The b-adic function system

Definition (The b-adic function system in dimension $s \geq 1$)

$\mathbf{b} = (b_1, \ldots, b_s)$ a vector of s not necessarily distinct integers $b_i \geq 2$

$\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$

$$\gamma_{\mathbf{k}} : \ [0,1[^s \to \{c \in \mathbb{C} : \ |c| = 1\},$$

$$\gamma_{\mathbf{k}}(\mathbf{x}) \ = \ \prod_{i=1}^{s} \gamma_{k_i}(x_i), \quad \gamma_{k_i} \in \Gamma_{b_i}.$$

Put $\Gamma_{\mathbf{b}}^{(s)} = \{\gamma_{\mathbf{k}} : \ \mathbf{k} \in \mathbb{N}_0^s\}$ ...the **b**-adic function system on $[0,1[^s$

ONB property

$\Gamma_{\mathbf{b}}^{(s)}$ is an ONB of $L^2([0,1[^s)$.

Remark

For $\Gamma_{\mathbf{b}}^{(s)}$, we have a Weyl criterion.

b-adic Weyl criterion

Theorem (**b**-adic Weyl Criterion)

$\mathbf{b} = (b_1, \ldots, b_s)$ *a vector of* s *not necessarily distinct integers* $b_i \geq 2$

Then ω *u.d. in* $[0,1[^s$ *if and only if*

$$\forall \mathbf{k} \neq \mathbf{0} : \quad \lim_{N\to\infty} S_N(\gamma_{\mathbf{k}}, \omega) \; = \; 0.$$

Remark

Van der Corput sequence in base b has the form

$$\omega = (\varphi_b(n))_{n\geq 0}.$$

Corollary

For $\omega = (\varphi_b(n))_{n\geq 0}$ the van der Corput sequence in base b: Then

$$S_N(\gamma_k, \omega) = \frac{1}{N}\frac{\mathrm{e}(\varphi_b(k))^N - 1}{\mathrm{e}(\varphi_b(k)) - 1}, \quad k \in \mathbb{N}.$$

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$\mathbb{Z}_b$ and the vdC-sequence

The setting

For the van der Corput sequence ω in base b:

1. State space: $\mathcal{Z} = \mathbb{Z}_b$,
2. Generating principle: $T : \mathbb{Z}_b \to \mathbb{Z}_b$, $Tz = z + 1$
3. Put $z_n = T^n 0 = n$, $n = 0, 1, \ldots$
4. Output function: $\varphi_b : \mathbb{Z}_b \to [0, 1[$,
5. vdC: $\omega = (x_n)_{n \geq 0}$ in $[0, 1[$ with $x_n = \varphi_b(n)$, $n = 0, 1, \ldots$

Remark

The exponential sums $S_N(\gamma_k, \omega)$ for $\Gamma_b^{(1)}$ are of the form

$$1/N \sum_{n=0}^{N-1} e(\varphi_b(k))^n, \quad k = 0, 1, \ldots$$

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$\mathbf{b}$-adic Inequality of Erdös-Turán-Koksma

Theorem

ω a sequence in $[0,1[^s$,
$\mathbf{b} = (b_1, \ldots, b_s) \in \mathbb{N}^s$, $b_i \geq 2$, not necessarily distinct,
$\mathbf{g} = (g_1, \ldots, g_s) \in \mathbb{N}^s$, arbitrary:

$$D_N(\omega) \leq 2s \max_{1 \leq i \leq s} b_i^{-g_i} + \sum_{\mathbf{k} \in \Delta^*_{\mathbf{b}}(\mathbf{g})} \rho_{\mathbf{b}}(\mathbf{k}) \left| S_N(\gamma_{\mathbf{k}}, \omega) \right| \,,$$

where, for $b \geq 2$ integer:

$$\rho_b(k) = \begin{cases} 1 & \text{if} \quad k = 0 \\ \dfrac{2}{b^t \sin(\pi k_{t-1}/b)} & \text{if} \quad b^{t-1} \leq k < b^t, \ t \in \mathbb{N}, \end{cases}$$

$$\rho_{\mathbf{b}}(\mathbf{k}) = \prod_{i=1}^{s} \rho_{b_i}(k_i), \quad \mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s,$$

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b-adic Inequality of Erdös-Turán-Koksma

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Main lemma

Question

Where do the weights $\rho_{\mathbf{b}}(\mathbf{k})$ come from?

Lemma

Choose $\mathbf{g} = (g_1, \ldots, g_s) \in \mathbb{N}_0^s$ *and consider* $f(\mathbf{x}) = \mathbf{1}_I(\mathbf{x}) - \lambda_s(I)$,
where $I = \prod_{i=1}^s \left[a_i b_i^{-g_i}, d_i b_i^{-g_i}\right[, \quad 0 \leq a_i < d_i \leq b_i^{g_i}, \; g_i \geq 1.$

Then

1. $\forall\, \mathbf{k} \in \mathbb{N}_0^s \setminus \Delta_{\mathbf{b}}^*(\mathbf{g})$,
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2. $\forall\, \mathbf{k} \in \Delta_{\mathbf{b}}^*(\mathbf{g})$,
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3. *for all* $\mathbf{x} \in [0,1[^s$:
$$f(\mathbf{x}) = \sum_{\mathbf{k} \in \Delta_{\mathbf{b}}^*(\mathbf{g})} \hat{\mathbf{1}}_I(\mathbf{k})\, \gamma_{\mathbf{k}}(\mathbf{x}).$$

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Another notion

Definition

Uniform distribution determining subclass $\mathcal{F} = \{\xi_{\mathbf{k}} : \mathbf{k} \in \Xi\}$ of the class of all Riemann integrable functions on $[0,1[^s$:

$$\left(\forall \xi_{\mathbf{k}} \in \mathcal{F} : \lim_{N\to\infty} S_N(\xi_{\mathbf{k}}, \omega) = \int_{[0,1[^s} \xi_{\mathbf{k}}\, d\lambda_s\right) \iff \omega \text{ is unif. distr. in } [0,1[^s.$$

Example

1. Trigonometric function system on $[0,1[^s$, $\Xi = \mathbb{Z}^s$.
2. Walsh function system $\mathcal{W}_{\mathbf{b}}^{(s)}$ on $[0,1[^s$, $\Xi = \mathbb{N}_0^s$.
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Question

What other function systems are u.d. determining?

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Weyl arrays

ω a sequence in $[0,1[^s$,
$\mathcal{F} = \{\xi_{\mathbf{k}} : \mathbf{k} \in \Xi\}$ u.d. determining class on $[0,1[^s$, $\Xi \in \{\mathbb{Z}^s, \mathbb{N}_0^s, \ldots\}$.

Definition

Define the array of functions $A_{\mathcal{F}} = (\xi_{\mathbf{k}})_{\mathbf{k}\in\Xi}$.
The Weyl array of the first N points of ω with respect to $\mathcal{F}$ is defined as

$$S_N(A_{\mathcal{F}}, \omega) = (S_N(\xi_{\mathbf{k}}, \omega))_{\mathbf{k}\in\Xi}.$$

Let $||\cdot||$ be an arbitrary norm on $\mathbb{R}^s$, and let $\rho : \Xi \to \mathbb{R}$ be a weight function such that

1. $\forall \mathbf{k} \in \Xi : \quad \rho(\mathbf{k}) > 0$,
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Define the array of weights $A_\rho = (\rho(\mathbf{k}))_{\mathbf{k}\in\Xi}$.

Definition

Define the weighted Weyl array as

$$S_N(A_\rho \otimes A_{\mathcal{F}}, \omega) = (\rho(\mathbf{k}) S_N(\xi_{\mathbf{k}}, \omega))_{\mathbf{k}\in\Xi}.$$

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Spectral test and diaphony

Definition

ω, $\mathcal{F}$, and ρ as before

Let $\Xi^* = \Xi \setminus \{\mathbf{0}\}$ and $A^*_{\mathcal{F}} = (\xi_{\mathbf{k}})_{\mathbf{k}\in\Xi^*}$, $A^*_\rho = (\rho(\mathbf{k}))_{\mathbf{k}\in\Xi^*}$.

The spectral test of the first N points of ω w.r.t. $\mathcal{F}$ and ρ is defined as

$$\sigma_N(\omega) = \frac{1}{||A^*_\rho||_\infty}\, ||S_N(A^*_\rho \otimes A^*_{\mathcal{F}}, \omega)||_\infty,$$

and the diaphony of the first N points of ω w.r.t. $\mathcal{F}$ and ρ as

$$F_N(\omega) = \frac{1}{||A^*_\rho||_2}\, ||S_N(A^*_\rho \otimes A^*_{\mathcal{F}}, \omega)||_2.$$

Theorem

σ_N and F_N are normalized and are u.d. measures:

- $0 \le \sigma_N(\omega), F_N(\omega) \le 1$
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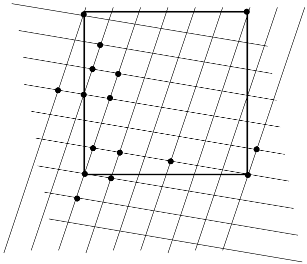
Lattices

Definition (Integration Lattice)

A d-dimensional integration lattice is a lattice in $\mathbb{R}^d$ that contains $\mathbb{Z}^d$ as a sublattice.

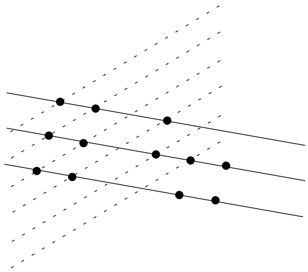


Some points of an integration lattice in $\mathbb{R}^2$.

Spectral Test: Aspects

Remark

- Geometric Interpretation of the spectral test (for $\mathcal{T}$):
  Maximum distance between adjacent parallel hyperplanes that cover L
- Efficient computation:
  Algorithms due to Dieter (1975), Knuth (1981), L'Ecuyer and Couture (1997)



b-adic Diaphony

Example (**b**-adic Diaphony)

$\mathbf{b} = (b_1, \ldots, b_s)$ a vector of s not necessarily distinct integers $b_i \geq 2$
ω a sequence in $[0, 1[^s$

$$F_N(\omega) = \frac{1}{(\sigma - 1)^{1/2}} \left(\sum_{\mathbf{k} \neq \mathbf{0}} \rho_{\mathbf{b}}(\mathbf{k})^2 \left| S_N(\gamma_{\mathbf{k}}, \omega) \right|^2 \right)^{1/2}$$

where, for an integer $b \geq 2$:

$$\rho_b(k) = \begin{cases} 1 & \text{if} \quad k = 0 \\ \frac{1}{b^{t-1}} & \text{if} \quad b^{t-1} \leq k < b^t, \ t \in \mathbb{N}, \end{cases}$$

$$\rho_{\mathbf{b}}(\mathbf{k}) = \prod_{i=1}^{s} \rho_{b_i}(k_i), \quad \mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s,$$

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Results I

Theorem (H., 2010)

$\mathbf{p} = (p, \ldots, p)$, *p prime*
$g \in \mathbb{N}$,
$\omega = (1/p^g)\mathbb{Z}^s \pmod 1$ *the regular grid of* p^{gs} *points in* $[0,1[^s$.
Put $N = p^{gs}$.
Then

$$C_1 \frac{1}{N^{1/s}} \leq F_N(\omega) \leq C_2 \frac{\log N}{N^{1/s}}, \quad C_1, C_2 \text{ explicit constants}$$

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Theorem (Inequality of Erdös-Turán-Koksma)

ω *a sequence in* $[0,1[^s$,
$\mathbf{g} \in \mathbb{N}^s$ *arbitrary*

$$F_N^2(\omega) \leq \frac{\sigma}{\sigma - 1} s\delta + \frac{1}{\sigma - 1} \sum_{\mathbf{k} \in \Delta_{\mathbf{b}}^*(\mathbf{g})} \rho_{\mathbf{b}}(\mathbf{k})^2 \left| S_N(\gamma_{\mathbf{k}}, \omega) \right|^2 ,$$

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Corollary

b-*adic Weyl criterion follows immediately.*

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Hybrid sequences

Definition

Hybrid sequence $\omega = (\mathbf{x}_n)_{n \geq 0}$ in $[0,1[^s$:
mix at least two lower-dimensional sequences of different types s.t. certain coordinates of $\mathbf{x}_n$ stem from the first sequence, certain of the remaining coordinates stem from the second sequence, and so on.

References

Spanier (1997); Niederreiter (2009 –)

Problem

Need new tools to analyze hybrid sequences.

Situation

We have different function systems on $[0,1[^s$ at our disposition:

$$\mathcal{T}^{(s)}, \mathcal{W}_{\mathbf{b}}^{(s)}, \Gamma_{\mathbf{b}}^{(s)}$$

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Hybrid function systems

Notation

$s \geq 1$: $s = s_1 + s_2 + s_3, \quad s_i \in \mathbb{N}_0$
$\mathbf{y} = (y_1, \ldots, y_s) \in \mathbb{R}^s$:

$$\mathbf{y} = (\mathbf{y}^{(1)}, \mathbf{y}^{(2)}, \mathbf{y}^{(3)}),$$

with $\mathbf{y}^{(1)} = (y_1, \ldots, y_{s_1})$, $\mathbf{y}^{(2)} = (y_{s_1+1}, \ldots, y_{s_1+s_2})$, and so on.

Definition (Hybrid function system)

$s \geq 1,\ s = s_1 + s_2 + s_3,\ s_i \in \mathbb{N}_0,$
$\mathbf{b} = (b_1, \ldots, b_{s_2})$ a vector of s_2 not necessarily distinct integers $b_i \geq 2$,
$\mathbf{b}' = (b'_1, \ldots, b'_{s_3})$ a vector of s_3 not necessarily distinct integers $b'_i \geq 2$.
$\mathcal{T}^{(s_1)}$ the trigonometric function system in dimension s_1,
$\mathcal{W}_{\mathbf{b}}^{(s_2)}$ the Walsh function system in base $\mathbf{b}$ in dimension s_2,
$\Gamma_{\mathbf{b}'}^{(s_3)}$ the $\mathbf{b}'$-adic function system in dimension s_3.

Hybrid function system $\mathcal{F} = \mathcal{T}^{(s_1)} \otimes \mathcal{W}_{\mathbf{b}}^{(s_2)} \otimes \Gamma_{\mathbf{b}'}^{(s_3)} = \{\xi_{\mathbf{k}} : \mathbf{k} \in \mathbb{Z}^{s_1} \times \mathbb{N}_0^{s_2} \times \mathbb{N}_0^{s_3}\}$,

$$\xi_{\mathbf{k}}(\mathbf{x}) = e_{\mathbf{k}^{(1)}}(\mathbf{x}^{(1)}) \cdot w_{\mathbf{k}^{(2)}}(\mathbf{x}^{(2)}) \cdot \gamma_{\mathbf{k}^{(3)}}(\mathbf{x}^{(3)})$$

(here, $\Xi = \mathbb{Z}^{s_1} \times \mathbb{N}_0^{s_2} \times \mathbb{N}_0^{s_3}$)

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with $\mathbf{y}^{(1)} = (y_1, \ldots, y_{s_1})$, $\mathbf{y}^{(2)} = (y_{s_1+1}, \ldots, y_{s_1+s_2})$, and so on.

Definition (Hybrid function system)

$s \geq 1,\ s = s_1 + s_2 + s_3,\ s_i \in \mathbb{N}_0,$
$\mathbf{b} = (b_1, \ldots, b_{s_2})$ a vector of s_2 not necessarily distinct integers $b_i \geq 2$,
$\mathbf{b}' = (b'_1, \ldots, b'_{s_3})$ a vector of s_3 not necessarily distinct integers $b'_i \geq 2$.
$\mathcal{T}^{(s_1)}$ the trigonometric function system in dimension s_1,
$\mathcal{W}_{\mathbf{b}}^{(s_2)}$ the Walsh function system in base $\mathbf{b}$ in dimension s_2,
$\Gamma_{\mathbf{b}'}^{(s_3)}$ the $\mathbf{b}'$-adic function system in dimension s_3.

Hybrid function system $\mathcal{F} = \mathcal{T}^{(s_1)} \otimes \mathcal{W}_{\mathbf{b}}^{(s_2)} \otimes \Gamma_{\mathbf{b}'}^{(s_3)} = \{\xi_{\mathbf{k}} : \mathbf{k} \in \mathbb{Z}^{s_1} \times \mathbb{N}_0^{s_2} \times \mathbb{N}_0^{s_3}\}$,

$$\xi_{\mathbf{k}}(\mathbf{x}) = e_{\mathbf{k}^{(1)}}(\mathbf{x}^{(1)}) \cdot w_{\mathbf{k}^{(2)}}(\mathbf{x}^{(2)}) \cdot \gamma_{\mathbf{k}^{(3)}}(\mathbf{x}^{(3)})$$

(here, $\Xi = \mathbb{Z}^{s_1} \times \mathbb{N}_0^{s_2} \times \mathbb{N}_0^{s_3}$)

Properties

Remark

Clearly, $\mathcal{F}$ is an ONB of $L^2([0,1[^s)$.

Remark

The concept of Weyl arrays applies, which leads to a hybrid spectral test and to hybrid diaphony.

Theorem (Hybrid Weyl Criterion)

$s \geq 1,\ s = s_1 + s_2 + s_3,\ s_i \in \mathbb{N}_0,\ \mathbf{b}$ and $\mathbf{b}'$ as before

The the associated hybrid function system $\mathcal{F}$ is u.d. determining.

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Theorem (Hybrid Weyl Criterion)

$s \geq 1, \; s = s_1 + s_2 + s_3, \, s_i \in \mathbb{N}_0$, $\mathbf{b}$ *and* $\mathbf{b}'$ *as before*

The the associated hybrid function system $\mathcal{F}$ *is u.d. determining.*

Some details of the proof

Definition

A **b**-adic elementary interval, or **b-adic elint** for short:
a subinterval $I_{\mathbf{c},\mathbf{g}}$ of $[0,1[^s$ of the form

$$I_{\mathbf{c},\mathbf{g}} = \prod_{i=1}^{s} \left[\varphi_{b_i}(c_i), \varphi_{b_i}(c_i) + p_i^{-g_i}\right[,$$

$\mathbf{g} = (g_1, \ldots, g_s) \in \mathbb{N}_0^s$, $\mathbf{c} = (c_1, \ldots, c_s) \in \mathbb{N}_0^s$: $0 \leq c_i < b_i^{g_i}$, $1 \leq i \leq s$.

Remark

Written in the 'classical' form:

$$I_{\mathbf{c},\mathbf{g}} = \prod_{i=1}^{s} \left[a_i b_i^{-g_i}, (a_i+1) b_i^{-g_i}\right[,$$

where $\varphi_{b_i}(c_i) = a_i b_i^{-g_i}$, with $a_i \in \mathbb{N}_0$, $0 \leq a_i < b_i^{g_i}$, $1 \leq i \leq s$.

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Two lemmata

For $\mathbf{b} = (b_1, \ldots, b_s) \in \mathbb{N}^s$, $b_i \geq 2$ for all i, and $\mathbf{g} \in \mathbb{N}_0^s$:

$$\Delta_{\mathbf{b}}(\mathbf{g}) = \left\{\mathbf{k} = (k_1, \ldots, k_s) :\ 0 \leq k_i < b_i^{g_i}, 1 \leq i \leq s\right\},$$
$$\Delta_{\mathbf{b}}^*(\mathbf{g}) = \Delta_{\mathbf{b}}(\mathbf{g}) \setminus \{\mathbf{0}\}.$$

Lemma (H., 1994)

$I_{\mathbf{c},\mathbf{g}}$ an arbitrary $\mathbf{b}$-adic elint, $f = \mathbf{1}_{I_{\mathbf{c},\mathbf{g}}} - \lambda_s(I_{\mathbf{c},\mathbf{g}})$:

$$f(\mathbf{x}) = \sum_{\mathbf{k} \in \Delta_{\mathbf{b}}^*(\mathbf{g})} \hat{\mathbf{1}}_{I_{\mathbf{c},\mathbf{g}}}(\mathbf{k}) w_{\mathbf{k}}(\mathbf{x}), \quad \forall \mathbf{x} \in [0,1[^s.$$

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A lemma of Niederreiter

Lemma (Niederreiter, 2009)

J an arbitrary subinterval of the form $[u, v[$, $0 < u < v \leq 1$, of $[0, 1[$.
For every $H \in \mathbb{N}$, there is a trigonometric polynomial

$$P_J(x) = \sum_{k=-H}^{H} c_J(k) e_k(x), \quad x \in [0, 1[,$$

with coefficients $c_J(k)$, $c_J(0) = \lambda(J)$, and $\mid c_J(k) \mid < 1/ \mid k \mid$ for all $k \neq 0$, such that

$$|\mathbf{1}_J(x) - P_J(x)| \leq \frac{1}{H+1} \sum_{k=-H}^{H} u_J(k) e_k(x), \quad \forall x \in [0, 1[,$$

with some complex numbers $u_J(k)$, where $\mid u_J(k) \mid \leq 1$ for all k and $u_J(0) = 0$.

Idea of proof

Definition (**b**-adic subinterval)

For $\mathbf{b} = (b_1, \ldots, b_s) \in \mathbb{N}^s$, $b_i \geq 2$ for all i, and $\mathbf{g} \in \mathbb{N}_0^s$:

$$\prod_{i=1}^{s} \left[\frac{a_i}{b_i^{g_i}}, \frac{c_i}{b_i^{g_i}} \right[, \quad 0 \leq a_i < c_i \leq b_i^{g_i},\ g_i \geq 0,\ a_i, c_i, g_i \in \mathbb{N}_0$$

Idea of the proof

Assume that ω be uniformly distributed in $[0,1[^s$. Each function $\xi_{\mathbf{k}}$ is Riemann-integrable, hence $S_N(\xi_{\mathbf{k}}, \omega) = 0$, for all $\mathbf{k} \neq 0$.

Assume that $\lim_{N\to\infty} S_N(\xi_{\mathbf{k}}, \omega) = 0$, for all $\mathbf{k} \neq 0$.
Let J by a subinterval of $[0,1[^s$ of the form $J = J^{(1)} \times J^{(2)} \times J^{(3)}$, where
$J^{(1)}$ is an arbitrary subinterval of $[0,1[^{s_1}$,
$J^{(2)}$ is an arbitrary **b**-adic subinterval of $[0,1[^{s_2}$, and
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We show that $\lim_{N\to\infty} S_N(\mathbf{1}_J - \lambda_s(J), \omega) = 0$, which will prove the uniform distribution of the sequence ω.

Idea of proof

Definition ($\mathbf{b}$-adic subinterval)

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We show that $\lim_{N\to\infty} S_N(\mathbf{1}_J - \lambda_s(J), \omega) = 0$, which will prove the uniform distribution of the sequence ω.

Details of the proof: I

$$
\begin{aligned}
S_N(\mathbf{1}_J - \lambda_s(J)) =& \frac{1}{N}\sum_{n=0}^{N-1} \mathbf{1}_J(\mathbf{x}_n) - \lambda_s(J) \\
=& \frac{1}{N}\sum_{n=0}^{N-1} \mathbf{1}_{J^{(1)}}(\mathbf{x}_n^{(1)})\mathbf{1}_{J^{(2)}}(\mathbf{x}_n^{(2)})\mathbf{1}_{J^{(3)}}(\mathbf{x}_n^{(3)}) \\
& - \lambda_{s_1}(J^{(1)})\lambda_{s_2}(J^{(2)})\lambda_{s_3}(J^{(3)}) \\
=& \frac{1}{N}\sum_{n=0}^{N-1} \left(\mathbf{1}_{J^{(1)}}(\mathbf{x}_n^{(1)}) - P_{J^{(1)}}(\mathbf{x}_n^{(1)})\right) \mathbf{1}_{J^{(2)}}(\mathbf{x}_n^{(2)})\mathbf{1}_{J^{(3)}}(\mathbf{x}_n^{(3)}) \\
& + \frac{1}{N}\sum_{n=0}^{N-1} P_{J^{(1)}}(\mathbf{x}_n^{(1)})\mathbf{1}_{J^{(2)}}(\mathbf{x}_n^{(2)})\mathbf{1}_{J^{(3)}}(\mathbf{x}_n^{(3)}) \\
& - \lambda_{s_1}(J^{(1)})\lambda_{s_2}(J^{(2)})\lambda_{s_3}(J^{(3)})
\end{aligned}
$$

Details of the proof: II

Define

$$\Sigma_1 = \frac{1}{N}\sum_{n=0}^{N-1}\left(\mathbf{1}_{J^{(1)}}(\mathbf{x}_n^{(1)}) - P_{J^{(1)}}(\mathbf{x}_n^{(1)})\right)\mathbf{1}_{J^{(2)}}(\mathbf{x}_n^{(2)})\mathbf{1}_{J^{(3)}}(\mathbf{x}_n^{(3)}),$$

$$\Sigma_2 = \frac{1}{N}\sum_{n=0}^{N-1} P_{J^{(1)}}(\mathbf{x}_n^{(1)})\mathbf{1}_{J^{(2)}}(\mathbf{x}_n^{(2)})\mathbf{1}_{J^{(3)}}(\mathbf{x}_n^{(3)}) - \lambda_{s_1}(J^{(1)})\lambda_{s_2}(J^{(2)})\lambda_{s_3}(J^{(3)}).$$

Then, by application of Niederreiter's lemma above,

$$|\,\Sigma_1\,| \leq \mathcal{O}_s\left(\frac{1}{H} + \sum_{\mathbf{k}^{(1)}\neq\mathbf{0}^{(1)}:M(\mathbf{k}^{(1)})\leq H}\frac{1}{r(\mathbf{k})}\,|\,S_N(e_{\mathbf{k}},\omega)\,|\right),$$

where $\mathcal{O}_s$ denotes a constant that depends only on s
and $M(\mathbf{k}^{(1)}) = \max_{i:1\leq i\leq s_1}|k_i|$.

This implies

$$\lim_{N\to\infty}\Sigma_1 = 0.$$

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Details of the proof: III

For Σ_2, we note that

$$P_{J^{(1)}}(\mathbf{x}^{(1)})\mathbf{1}_{J^{(2)}}(\mathbf{x}^{(2)})\mathbf{1}_{J^{(3)}}(\mathbf{x}^{(3)}) = \sum_{\mathbf{k}^{(1)}:0\le M(\mathbf{k}^{(1)})\le H} \sum_{\mathbf{k}^{(2)}\in\Delta_{\mathbf{b}}(\mathbf{g})} \sum_{\mathbf{k}^{(3)}\in\Delta_{\mathbf{p}}(\mathbf{h})} c_{J^{(1)}}(\mathbf{k}^{(1)})\hat{\mathbf{1}}_{J^{(2)}}(\mathbf{k}^{(2)})\hat{\mathbf{1}}_{J^{(3)}}(\mathbf{k}^{(3)})\xi_{\mathbf{k}}(\mathbf{x}),$$

for all $\mathbf{x} \in [0,1[^s$.

Hence,

$$S_N(P_{J^{(1)}}\mathbf{1}_{J^{(2)}}\mathbf{1}_{J^{(3)}},\omega) = \sum_{\substack{\mathbf{k}: \\ 0\le M(\mathbf{k}^{(1)})\le H, \\ \mathbf{k}^{(2)}\in\Delta_{\mathbf{b}}(\mathbf{g}), \\ \mathbf{k}^{(3)}\in\Delta_{\mathbf{p}}(\mathbf{h})}} c_{J^{(1)}}(\mathbf{k}^{(1)})\hat{\mathbf{1}}_{J^{(2)}}(\mathbf{k}^{(2)})\hat{\mathbf{1}}_{J^{(3)}}(\mathbf{k}^{(3)})S_N(\xi_{\mathbf{k}},\omega).$$

We observe that

$$c_{J^{(1)}}(\mathbf{0}^{(1)})\hat{\mathbf{1}}_{J^{(2)}}(\mathbf{0}^{(2)})\hat{\mathbf{1}}_{J^{(3)}}(\mathbf{0}^{(3)}) = \lambda_{s_1}(J^{(1)})\lambda_{s_2}(J^{(2)})\lambda_{s_3}(J^{(3)}).$$

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Details of the proof: IV

As a consequence,

$$\Sigma_2 = \sum_{\substack{\mathbf{k}\neq\mathbf{0}: \\ 0\leq M(\mathbf{k}^{(1)})\leq H, \\ \mathbf{k}^{(2)}\in\Delta_{\mathbf{b}}(\mathbf{g}), \\ \mathbf{k}^{(3)}\in\Delta_{\mathbf{p}}(\mathbf{h})}} c_{J^{(1)}}(\mathbf{k}^{(1)})\hat{\mathbf{1}}_{J^{(2)}}(\mathbf{k}^{(2)})\hat{\mathbf{1}}_{J^{(3)}}(\mathbf{k}^{(3)})S_N(\xi_{\mathbf{k}},\omega).$$

Our assumption implies

$$\lim_{N\to\infty}\Sigma_2 = 0.$$

As

$$S_N(\mathbf{1}_J - \lambda_s(J),\omega) = \Sigma_1 + \Sigma_2,$$

we obtain

$$\lim_{N\to\infty} S_N(\mathbf{1}_J - \lambda_s(J),\omega) = 0.$$

This finishes the proof of the hybrid Weyl criterion. □

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Some weight functions

Weight functions

We define two types of weight functions:

$$k \in \mathbb{Z}: \qquad r(k) = \begin{cases} 1 & \text{if} \quad k = 0 \\ 1/k & \text{if} \quad k \neq 0, \end{cases}$$

$$\mathbf{k} = (k_1, \dots, k_s) \in \mathbb{Z}^s: \qquad r(\mathbf{k}) = \prod_{i=1}^{s} r(k_i),$$

$$\sigma(r, s) = \sum_{\mathbf{k}} r(\mathbf{k})^2 = (1 + \pi^2/3)^s.$$

$\mathbf{b} = (b_1, \dots, b_s) \in \mathbb{N}^s$, $b_i \geq 2$

$$\mathbf{k} = (k_1, \dots, k_s) \in \mathbb{N}_0^s: \qquad \rho_{\mathbf{b}}(\mathbf{k}) = \prod_{i=1}^{s} \rho_{b_i}(k_i).$$

$$\sigma(\rho_{\mathbf{b}}, s) = \sum_{\mathbf{k}} \rho_{\mathbf{b}}(\mathbf{k})^2 = \prod_{i=1}^{s} (b_i + 1).$$

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Hybrid diaphony

Definition (Hybrid diaphony)

s, $\mathbf{b}$, and $\mathbf{b}'$ as in hybrid Weyl criterion
ω a sequence in $[0,1[^s$

$$F_N(\omega) = \frac{1}{(\sigma-1)^{1/2}}\left(\sum_{\mathbf{k}\neq\mathbf{0}} \rho(\mathbf{k})^2 \left|S_N(\xi_{\mathbf{k}},\omega)\right|^2\right)^{1/2}$$

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Results

Theorem (Inequality of Erdös-Turán-Koksma)

$s = s_1 + s_2 + s_3$, $s_i \in \mathbb{N}_0$,
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$H \in \mathbb{N}$, $\mathbf{g} \in \mathbb{N}^{s_2}$ arbitrary, $\mathbf{h} \in \mathbb{N}^{s_3}$ arbitrary.
ω a sequence in $[0,1[^s$:

$$F_N^2(\omega) \leq \frac{\sigma}{\sigma - 1} s\delta + \frac{1}{\sigma - 1} \sum_{\mathbf{k} \in \Delta^*} \varrho(\mathbf{k}) \left| S_N(\xi_{\mathbf{k}}, \omega) \right|^2 ,$$

$$\Delta = \Delta(H, \mathbf{g}, \mathbf{h}) = \{(\mathbf{k}^{(1)}, \mathbf{k}^{(2)}, \mathbf{k}^{(3)}) : 0 \leq ||\mathbf{k}^{(1)}||_\infty \leq H, \mathbf{k}^{(2)} \in \Delta_{\mathbf{b}}(\mathbf{g}), \mathbf{k}^{(3)} \in \Delta_{\mathbf{b}'}(\mathbf{h})\},$$

$$\Delta^* = \Delta \setminus \{\mathbf{0}\},$$

$$\delta = \max\left\{(2/(1+\pi^2/3))/H, \max_i (b_i + 1)^{-1} b_i^{-g_i+1}, \max_i (b'_i + 1)^{-1} b_i'^{-h_i+1}\right\}.$$

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