Uniform distribution of generalized Kakutani's sequences of partitions

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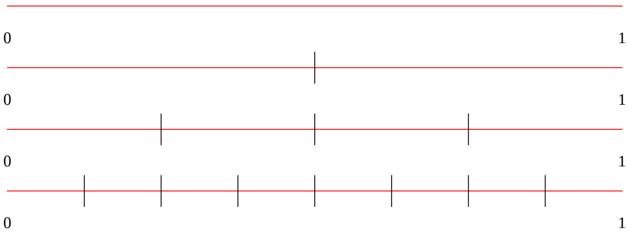
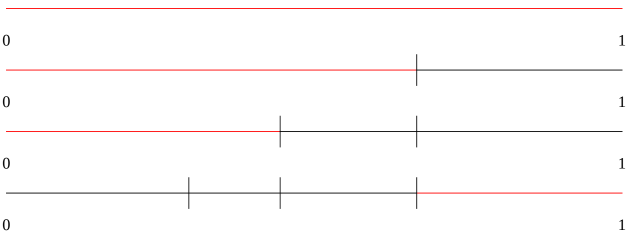
Definition

If $\alpha \in (0,1)$ and $\pi = \{[t_{i-1}, t_i] : 1 \leq i \leq k\}$ is any partition of $[0,1]$, then $\alpha\pi$ denotes its α -refinement which is obtained by subdividing all intervals of π having maximal length in two parts, proportional to α and $1-\alpha$, respectively.
The so-called Kakutani's sequence of partitions $(\alpha^n \omega)_{n \in \mathbb{N}}$ is obtained as the successive α -refinement of the trivial partition $\omega = \{[0,1]\}$.

Kakutani splitting with $\alpha = 1/2$



Kakutani splitting with $\alpha = 1/3$



Definition

Let $(\pi_n)_{n\in\mathbb{N}}$ be a sequence of partitions of $[0,1]$, with

$$\pi_n = \{[t^n_{i-1}, t^n_i] : 1 \leq i \leq k(n)\}.$$

Then π_n is uniformly distributed (u.d.), if for any continuous function f on $[0,1]$

$$\lim_{n\to\infty} \frac{1}{k(n)} \sum_{i=1}^{k(n)} f(t^n_i) = \int_0^1 f(t)dt.$$

. . . and equivalently . . .

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{k(n)} \mathbf{1}_{[a,b]}(t_i^n)}{k(n)} = b - a.$$

Important former results

- Kakutani (1976): $(\alpha^n \omega)_{n \in \mathbb{N}}$ is u.d. for every $\alpha \in (0,1)$.
- Brennan and Durrett (1986) und van Zwet (1978): modification of $(\alpha^n \omega)_{n \in \mathbb{N}}$ where the intervals of maximal length are split at a random position.
- Carbone and Volčič (2007): multi-dimensional case.
- Volčič (2011): $(\rho^n \omega)_{n \in \mathbb{N}}$ is u.d., where ρ is an arbitrary finite partition of $[0,1]$.
  Kakutani splitting: $\rho = [0,\alpha],[\alpha,1]$.
- Volčič (2011), Carbone (2011) and Drmota and Infusino (2011): discrepancy of partitions and related point sequences.

Question

Let π be an arbitrary finite partition of $[0, 1]$. Is $(\rho^n \pi)_{n\in\mathbb{N}}$ u.d. ?

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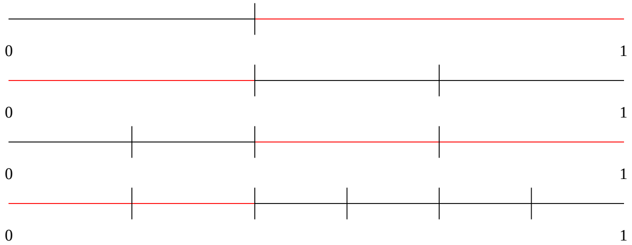
Answer No! Choose for example: $\pi = \left\{ \left[0, \frac{2}{5}\right], \left[\frac{2}{5}, 1\right] \right\}$ and $\rho = \left\{ \left[0, \frac{1}{2}\right], \left[\frac{1}{2}, 1\right] \right\}$

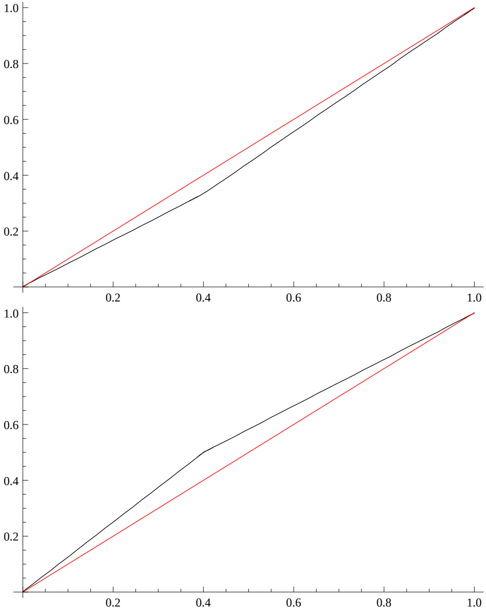
Bad case

$$\pi = \left\{ \left[0, \frac{2}{5}\right], \left[\frac{2}{5}, 1\right] \right\}$$

$$\rho = \left\{ \left[0, \frac{1}{2}\right], \left[\frac{1}{2}, 1\right] \right\}$$



Induced measures



Open problem

Volčič (2011):

What are sufficient conditions on π *to obtain uniform distribution of* $(\rho^n \pi)_{n \in \mathbb{N}}$*?*

Definition

Rationally related

The numbers $\log\left(\frac{1}{p_1}\right), \dots, \log\left(\frac{1}{p_m}\right)$ are called rationally related if there exists a positive real number Λ such that

$$\log\left(\frac{1}{p_j}\right) = \nu_j \Lambda, \quad \nu_j \in \mathbb{Z}, j = 1, \dots, m.$$

If the numbers $\log\left(\frac{1}{p_1}\right), \dots, \log\left(\frac{1}{p_m}\right)$ are not rationally related, they are called irrationally related.

Remark

Note that the numbers $\log\left(\frac{1}{p_1}\right), \ldots, \log\left(\frac{1}{p_m}\right)$ are rationally related if and only if all fractions

$$\frac{\log p_i}{\log p_j}, \quad i, j = 1, \ldots, m,$$

are rational, i.e.,

$$p_i^{c_{ij}} = p_j^{d_{ij}}, \quad i, j = 1, \ldots, m, \; c_{ij}, d_{ij} \in \mathbb{N}.$$

Theorem (Aistleitner, H.)

Let $\alpha_j, j = 1, \ldots, l$ denote the lengths of the intervals of the starting partition π and let $p_i,\ i = 1, \ldots, m$ be the length of the intervals in ρ. Then the sequence $(\rho^n \pi)_{n \in \mathbb{N}}$ is uniformly distributed if and only if one of the following conditions is satisfied:

1. the real numbers $\log\left(\frac{1}{p_1}\right), \ldots, \log\left(\frac{1}{p_m}\right)$ are irrationally related or
2. the real numbers $\log\left(\frac{1}{p_1}\right), \ldots, \log\left(\frac{1}{p_m}\right)$ are rationally related with parameter Λ and the lengths of the intervals of π can be written in the form

$$\alpha_i = c e^{v_i \Lambda}, \quad c \in \mathbb{R}^+, v_i \in \mathbb{Z},$$

for $i = 1, \ldots, l$.

Remark

Note that if

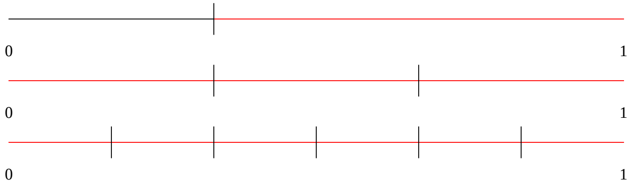
$$p_i^{c_{ij}} = p_j^{d_{ij}}, \quad i,j = 1,\dots,m,\ c_{ij}, d_{ij} \in \mathbb{N}$$

then the number of intervals which are split at the k-th step increases. "Worst case": $c_{ij} = 1,\ d_{ij} = 1$.

Nice case

$\pi = \{[0, 1/3], [1/3, 1]\}$

$\rho = \{[0, 1/2], [1/2, 1]\}$



About the proof . . .

- We show that the induced measure converges to the Lebesgue measure under the above conditions and that in all other cases we can at least find two different limit measures.
- A central incredient is a result of Drmota and Infusino (2011). They give an asymptotic formula for the number of intervals when all intervals with length greater than ϵ have been split and $\pi = \omega$.

Corollary Let the sequence of partitions $(\rho^n\pi)_{n\in\mathbb{N}}$ be defined as a ρ-refinement with
$\rho = [0, p], [p, 1]$ and $\pi = [0, \alpha], [\alpha, 1]$. Then $(\rho^n\pi)_{n\in\mathbb{N}}$ is u.d. if and only if one of the following conditions is satisfied:

- $\log(p)/\log(1-p)$ is irrational, or
- $\log\left(\frac{1}{p}\right)$ and $\log\left(\frac{1}{1-p}\right)$ are rationally related with parameter Λ and $\alpha = \frac{1}{e^{k\Lambda}+1}$ for $k \in \mathbb{Z}$.

Theorem (Aistleitner, H.)

Assume that neither condition 1 nor condition 2 is satisfied. Then for any interval $A = [a, b] \subset [0, 1]$ which is completely contained in the i-th interval of the starting partition π for some $i, 1 \leq i \leq l$, we have

$$\limsup_{n\to\infty} \frac{\sum_{j=1}^{k(n)} \mathbf{1}_{[a,b]}(t_j^n)}{k(n)} = c_1(b-a),$$
$$\liminf_{n\to\infty} \frac{\sum_{j=1}^{k(n)} \mathbf{1}_{[a,b]}(t_j^n)}{k(n)} = c_2(b-a),$$

where c_1, c_2 are explicit constants depending on i.