Approximation of the maximal commutative subgroups of $GL(n, R)$

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28 June 2012

Diophantine approximations

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Classical problem of Diophantine approximations

How to define good approximations

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Definition

A rational number a/b (where $b > 0$) is a *best approximation* of a real α if for any other fraction c/d with $0 < d \leq b$ it holds

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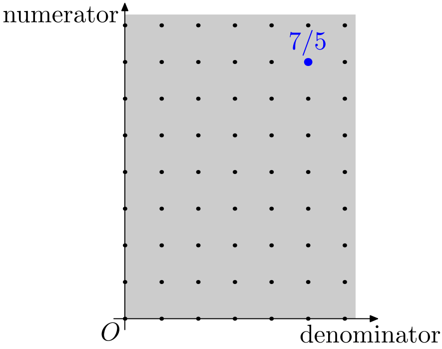

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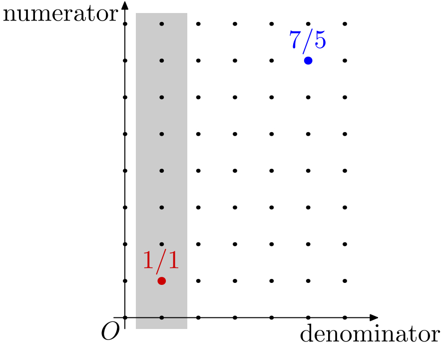

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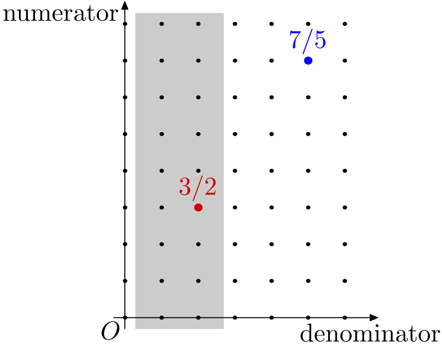

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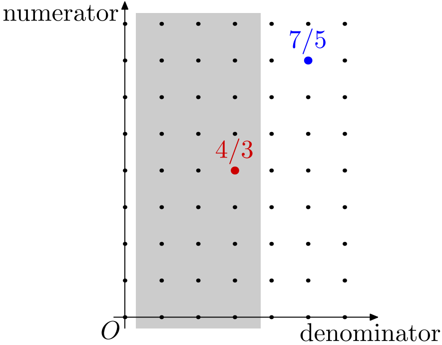

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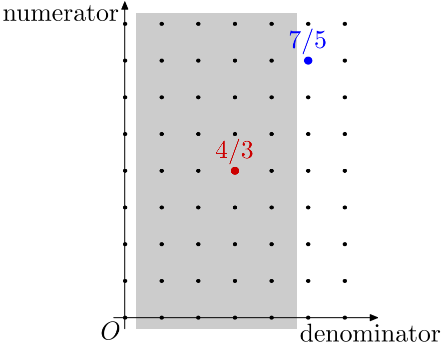

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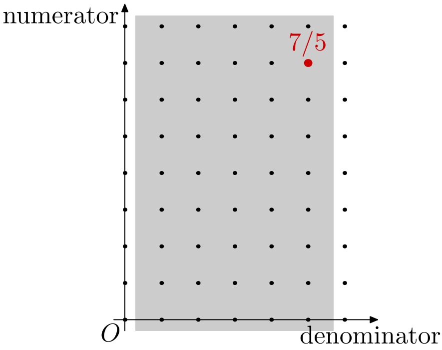

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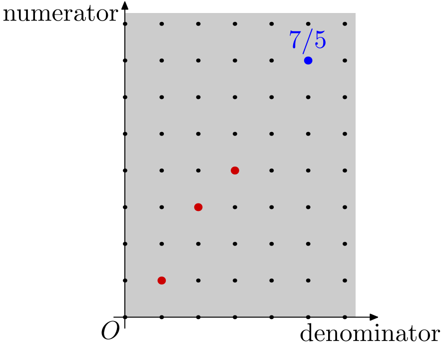

Ordinary continued fractions

The expression (finite or infinite)

$$a_0 + 1/(a_1 + 1/(a_2 + \ldots) \ldots))$$

is an *ordinary continued fraction* if $a_0 \in \mathbb{Z}$, $a_k \in \mathbb{Z}_+$ for $k > 0$.
Denote it $[a_0 : a_1; \ldots]$ (or $[a_0 : a_1; \ldots; a_n]$).

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Ordinary continued fraction is *odd* (*even*) if it has odd (even) number of elements.

$$\frac{7}{5} = 1 + \frac{1}{2 + \frac{1}{2}} = 1 + \frac{1}{2 + \frac{1}{1+1/1}}$$

$$\frac{7}{5} = [1 : 2; 2] = [1 : 2; 1; 1]$$

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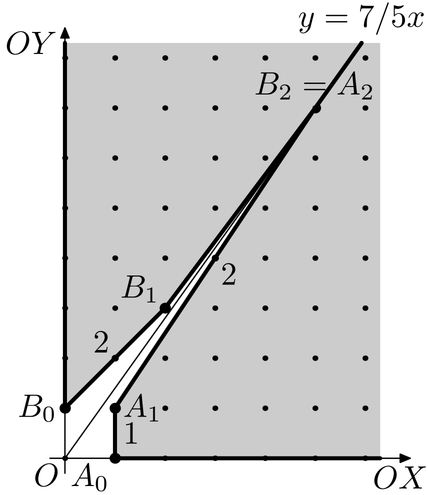
Proposition

Any rational number has a unique odd and even ordinary continued fractions.
Any irrational number has a unique infinite ordinary continued fraction

Geometry of continued fractions



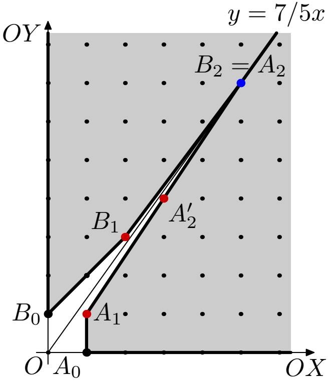
$$a_0 = \mathrm{l}\ell(A_0A_1) = 1;$$

$$a_1 = \mathrm{l}\ell(B_0B_1) = 2;$$

$$a_2 = \mathrm{l}\ell(A_1A_2) = 2.$$

$$7/5 = [1; 2 : 2].$$

Description of best approximations



$A_1 = (1,1) \quad 1/1 = [1];$

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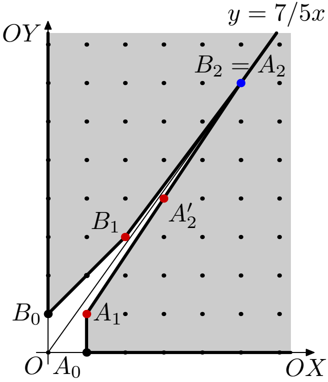

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Theorem

$\alpha = [a_0; a_1 : \ldots : a_n]$. *Then the best approximations are*

- $p_k/q_k = [a_0; a_1 : \ldots : a_k]$;
- *extra:* $[a_0; a_1 : \ldots : a_{n-1} : a_n - 1]$.

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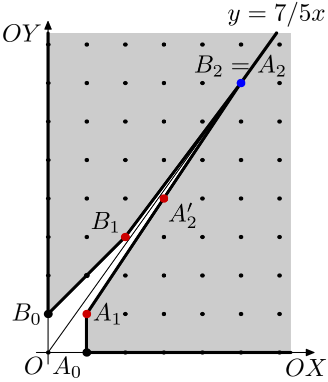

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- Algorithms: Jacobi-Perron, matrix method, etc.

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- Generators $E, A, A^2, \ldots, A^{n-1}$
- MCRS-group $\xleftrightarrow{1-1}$ a collection of conjugate n planes in $\mathbb{C}^n$

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$$\Phi_{\mathcal{A}}(x) = \frac{\prod\limits_{k=1}^{n} \left(L_k(x_1, \ldots, x_n)\right)}{\Delta(L_1, \ldots, L_n)}$$

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The Markoff-Davenport form of Fibonacci operator is

$$\frac{(y+\theta x)(y-\theta^{-1}x)}{\theta-\theta^{-1}} = \frac{1}{\sqrt{5}}(-x^2+xy+y^2).$$

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Examples: $\nu(\mathcal{A}_i)=1, \qquad \nu(\mathcal{A}_{ii})=2.$

Discrepancy functional

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For MCRF-groups $\mathcal{A}_1$ and $\mathcal{A}_2$ consider two symmetric quadratic forms

$$\Phi_{\mathcal{A}_1}(v)+\Phi_{\mathcal{A}_2}(v) \quad \text{and} \quad \Phi_{\mathcal{A}_1}(v)-\Phi_{\mathcal{A}_2}(v)$$

Let a_i, b_i – the coefficients of them.

Discrepancy: $\rho(\mathcal{A}_1,\mathcal{A}_2)=\min(\max_i(a_i),\max_i(b_i)).$

Discrepancy is a natural distance between MCRF-groups.

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Therefore, $\rho(\mathcal{A}_i, \mathcal{A}_{ii}) = \frac{\sqrt{3}}{2}$.

General approximation model

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A rational MCRS-group $\mathcal{A}$ is a *best approximation* of $\mathcal{A}$ if for any other $\mathcal{A}'$ with $\nu(\mathcal{A}') \leq \nu(A)$ it holds

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Here $\nu(a/b) = b$, and $\rho(\alpha_1, \alpha_2) = |\alpha_1 - \alpha_2|$.

Special case 1: Diophantine approximations

For a real α denote by $\mathcal{A}[\alpha]$ an MCRS-group of $GL(2,\mathbb{R})$ defined by the two spaces $x = 0$ and $y = \alpha x$.

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Diophantine approximations $\subset$ MCRS-group approximations.

Special case 2: Simultaneous approximations

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Then

- For Gaussian simple $[a,b,c]$ we have
$$\nu\big(\mathcal{A}[a,b,c]\big)=\big|[a,b,c]\big|.$$
- Denote $\mathcal{A}=\mathcal{A}[a,b,c]$ and $\mathcal{A}'=\mathcal{A}[a',b',c']$, then
$$\rho(\mathcal{A},\mathcal{A}')=\min\left(\max\left(\left|\frac{b}{a}-\frac{b'}{a'}\right|,\left|\frac{c}{a}-\frac{c'}{a'}\right|,\left|\frac{b^2+c^2}{2a^2}-\frac{b'^2+c'^2}{2a'^2}\right|\right),\right.$$
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Simultaneous approximations "$\subset$" MCRS-group approximations.

General approximation in 2D

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- Hyperbolic (totally real) case
- Nonhyperbolic (complex) case

Hyperbolic case in 2D

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Theorem

Let α_1 and α_2 be real numbers having infinite continued fractions with bounded elements. Then there exist $C_1, C_2 > 0$ such that for any $N > 0$ we have

$$\frac{C_1}{N^2} < \rho(\mathcal{A}[\alpha_1, \alpha_2], \mathcal{A}_N[\alpha_1, \alpha_2]) < \frac{C_2}{N^2}.$$

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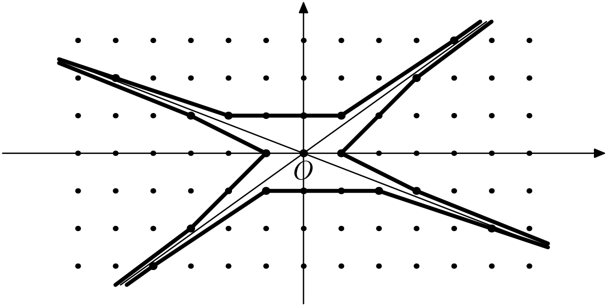
$$\frac{C_1}{N^2} < \rho(\mathcal{A}[\alpha_1, \alpha_2], \mathcal{A}_N[\alpha_1, \alpha_2]) < \frac{C_2}{N^2}.$$

Example

Let $\varepsilon > 0$. Consider $\alpha_1 = [a_0, a_1, \ldots]$, such that $a_0 = 1$, $a_n = (n_{k-1})^{M-1}$. Denote $\frac{m_k}{n_k} = [a_0, \ldots, a_k]$. Let $\alpha_2 = 0$. Take $N_k = \frac{n_k + n_{k+1}}{2}$. Then there exists $C > 0$ such that for any integer i we have

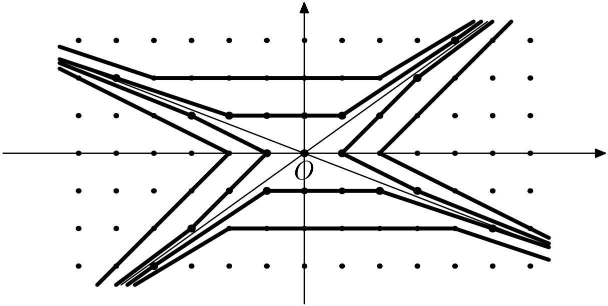
$$\rho(\mathcal{A}, \mathcal{A}_{N_i}) \geq \frac{C}{N_i^{1+\varepsilon}}.$$

Hyperbolic case in 2D



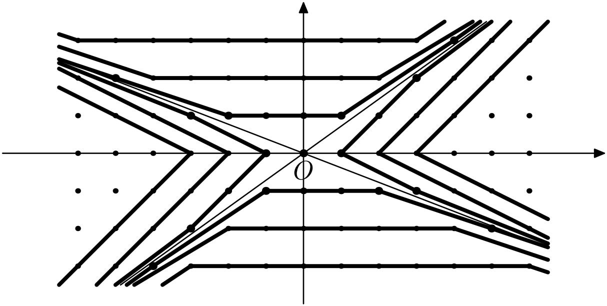
The set of all sails is called *geometric continued fraction* (in the sense of Klein).

Hyperbolic case in 2D



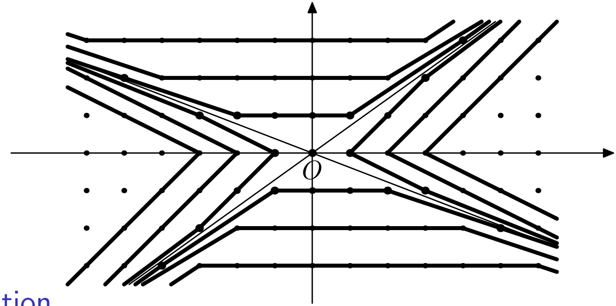
2-sails.

Hyperbolic case in 2D



3-sails.

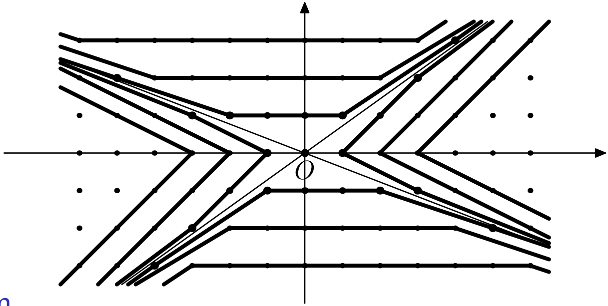
Hyperbolic case in 2D



Proposition

The k-sail is homothetic to the 1-sail with coefficient k.

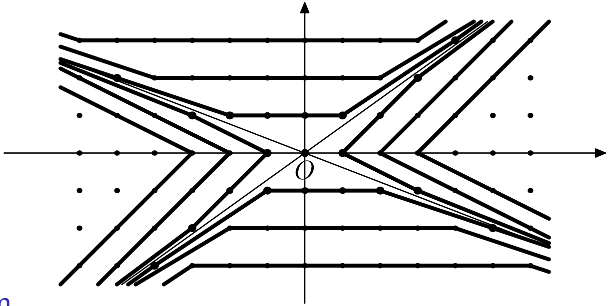
Hyperbolic case in 2D



Theorem

Let $\mathcal{A}$ be an algebraic (with periodic sail) MCRS-group. Then there exists $C > 0$ such that for any $N > 0$ the following holds. Let $\mathcal{A}_N$ be defined by primitive vectors v_1 and v_2 contained in k_1- and k_2-geometric continued fractions respectively, then $k_1, k_2 \leq C$.

Hyperbolic case in 2D



Theorem

Let $\mathcal{A}$ be an algebraic (with periodic sail) MCRS-group. Then there exists $C > 0$ such that for any $N > 0$ the following holds. Let $\mathcal{A}_N$ be defined by primitive vectors v_1 and v_2 contained in k_1- and k_2-geometric continued fractions respectively, then $k_1, k_2 \leq C$.

Conjecture

The constant C can be chosen to be equal to 1.

Non-Hyperbolic case in 2D

Denote by $\mathcal{A}[\alpha{+}I\beta]$ an MCRS-group with eigenspaces $y=(\alpha{+}I\beta)x$ and $y=(\alpha-I\beta)x$.

Non-Hyperbolic case in 2D

Denote by $\mathcal{A}[\alpha+I\beta]$ an MCRS-group with eigenspaces $y=(\alpha+I\beta)x$ and $y=(\alpha-I\beta)x$.

Theorem

Let α and β be real numbers having infinite continued fractions with bounded elements. Then there exist $C_1, C_2>0$ such that for any $N>0$ we have

$$\frac{C_1}{N^2}<\rho(\mathcal{A}[\alpha+I\beta],\mathcal{A}_N[\alpha+I\beta])<\frac{C_2}{N^2}.$$

What to do next?

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How do the best approximations of MCRS-groups in $\mathbb{R}^3$ related to
— vertices of Klein polyhedra;
— Minkovski minima;
— etc.