On Normal Numbers

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Let $E := \{e_1, e_2, \ldots, e_q\}$, $E_q^* =$ words over E_q, $E_q^{\mathbb{N}} = E_q \times E_q \times \cdots =$ infinite sequences over E_q.

Let $\alpha \in E^*$, $\underline{\varepsilon} = \varepsilon_1 \varepsilon_2 \ldots \in E_q^{\mathbb{N}}$, $\underline{\varepsilon}^T = \varepsilon_1 \ldots \varepsilon_T$.

Let

$$M_T(\underline{\varepsilon}|\alpha) = \#\{\beta_1, \beta_2 | \underline{\varepsilon}^T = \beta_1 \alpha \beta_2\}.$$

Definition. *$\underline{\varepsilon}$ is a normal sequence if for every $\alpha \in E^*$,*

$$\frac{q^{\lambda(\alpha)} M_T(\underline{\varepsilon}|\alpha)}{q^T} \to 1 \quad (T \to \infty),$$

$\lambda(\alpha) =$ *length of α.*

Clear: If $E = A_q = \{0, 1, \ldots, q-1\}$, then $\underline{\varepsilon}$ is a normal sequence if and only if

$$\xi = \sum \frac{\varepsilon_n}{q^n}$$

is a q-normal number, i.e. if $\{q^n \xi\}$ is UD sequence.

I. Let $R(n) = n(n+1)\cdots(n+q-1)$.

$$R(n) = \vartheta \pi_1^{\alpha_1} \ldots \pi_r^{\alpha_r}$$
$$q+1 \leq \pi_1 < \ldots < \pi_r, \quad \pi_j \in \mathcal{P}$$
$$P(\vartheta) \leq q.$$

Let

$$h_n(\pi^a) = h_n(\pi) = \begin{cases} \Lambda & \text{if } \pi | \vartheta \\ l & \text{if } \pi^a | n + l \end{cases}$$

$$\alpha_n = h_n(\pi_1^{\alpha_1}) \dots h_n(\pi_r^{\alpha_r}).$$

Let

$$\underline{\eta} = \alpha_1 \alpha_2 \dots;$$
$$\underline{\xi} = \alpha_3 \alpha_4 \alpha_6 \dots \alpha_{p+1} \dots$$

Theorem 1. $\underline{\eta}$ *and* $\underline{\xi}$ *are normal sequences.*

II. $\mathcal{B} \subset \mathbb{N},\ B(x) = \#\{b \le x;\ b \in \mathcal{B}\}$

$$F : \mathcal{B} \to \mathbb{N}, \quad (0 <) c_1 \le \frac{F(b)}{b^r} \le c_2 \quad (b \in \mathcal{B})$$

$$n = \sum \varepsilon_j(n) q^j, \quad \varepsilon_j(n) \in A_q$$

$$\overline{n} = \varepsilon_0(n) \dots \varepsilon_t(n), \quad \varepsilon_t(n) \ne 0$$

$$N = \left[\frac{\log x}{\log q}\right]$$

Let $0 \le l_1 < \cdots < l_h (\le rN),\ a_1, \dots, a_h \in A_q$.

Let

$$B_F\left(x \,\middle|\, \begin{matrix} l_1, \dots l_h \\ a_1, \dots, a_h \end{matrix}\right) = \#\{b \le x,\ b \in \mathbb{B},\ \varepsilon_{l_j}(F(b)) = a_j\ j = 1, \dots, h\},$$

$$B_F(x) = B(x).$$

We say: $F(\mathcal{B})$ is a q-ary smooth sequence, if there exists $0 < \alpha < 1$, $\varepsilon(x) \to 0$, such that for every fixed $h \geq 1$:

$$\sup_{N^\alpha \leq l_1 < \dots < l_h \leq rN - N^\alpha} \left| \frac{q^h B_F\left(x \,\Big|\, {l_1,\dots l_h \atop a_1,\dots,a_h}\right)}{B_F(x)} - 1 \right| \leq c(h)\varepsilon(x)$$

$$\xi_n = \overline{F(b_n)} = \varepsilon_0(F(b_n)) \dots \varepsilon_t(F(b_n)).$$

Theorem 2. *Assume that* $B(x) \geq c\frac{x}{\log x}$. *Let* $F(\mathcal{B})$ *be a q-ary smooth sequence,* $\mathcal{B} = \{b_1, b_2, \dots\}$, $b_n < b_{n+1}$, $(n = 1, 2, \dots)$

$$\eta = 0, \xi_1 \xi_2 \dots$$

Then η *is a q-normal number.*

Remarks. 1. If $P \in \mathbb{Z}[x]$, $r = \deg P(\geq 1)$, $P(x) \to \infty$ ($x \to \infty$), then for $\mathcal{B} = \mathbb{N}$, and $\mathcal{B} = \mathcal{P}$, the corresponding η is normal.

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2. **L. Germán - I. Kátai**: Let $\tilde{\mathcal{P}}$ be a subset of the primes such that $\pi(x|\tilde{\mathcal{P}}) = \#\{p \leq x,\ p \in \tilde{\mathcal{P}}\} > c\frac{x}{\log x}$ if $x > x_0$, $c > 0$. Let $N(\tilde{\mathcal{P}})$ = subgroup generated by $\tilde{\mathcal{P}}$. $P \in \mathbb{Z}[x]$, $P(x) \to \infty$ ($x \to \infty$), $N(\tilde{\mathcal{P}}) = \{n_1 < n_2 < \ldots\}$. Then $\eta = 0, \overline{P(n_1)}\ \overline{P(n_2)} \ldots$ is a normal number. *(Annales Sectio Computatorica ELTE,* ***38*** *2012))*

III. $q \geq 2$, $n = p_1^{e_1} \cdots p_{k+1}^{e_{k+1}}$, $p_1 < \ldots < p_{k+1}$

$H(n) = c_1(n) \cdots c_k(n)$, if $k \geq 1$

$$c_j(n) := \left[\frac{q \log p_j}{\log p_{j+1}} \right] \in A_q$$

$H(p^\alpha) = \Lambda$.

$$\xi = 0, H(1)H(2) \ldots$$

Theorem 3. *ξ is a q-normal number.*

Theorem 4. *Let* $R(x) \in \mathbb{Z}[x]$, $R(m) > 0$ *if* $m \geq m_0$ $H(R(n))$ *as earlier,*

$$\xi = 0, H(R(m_0))H(R(m_0 + 1)) \ldots$$

Then ξ *is a q-normal number. Assume further that* $R(0) \neq 0$*.*

Let

$$\eta = 0, H(R(p_1))H(R(p_2)) \ldots$$

where $p_1 < p_2 < \ldots$ *is the whole sequence of the primes. Then* η *is a q-normal number.*

IV. $\mathcal{P} = \mathcal{R} \cup \mathcal{P}_0 \cup \mathcal{P}_1 \cup \ldots \cup \mathcal{P}_{d-1}$ disjoint

$$\pi([u, u+v] | \mathcal{P}_j) = \frac{1}{d}\pi([u, u+v]) + \mathcal{O}\left(\frac{u}{(\log u)^c}\right)$$

$c \geq 5$, $2 \leq u \leq v$. Let

$$H(p) = \begin{cases} j & \text{if } p \in \mathcal{P}_j \quad j = 0, \ldots, d-1 \\ \Lambda & \text{if } p \in \mathcal{R}. \end{cases}$$

If $n = p_1^{a_1} \cdots p_r^{a_r}$, then let

$$R(n) = H(p_1) \cdots H(p_r).$$

Let

$$\xi = 0, R(1)R(2)\ldots$$
$$\eta_a = 0, R(2+a)R(3+a)\ldots R(p+a)\ldots$$

Theorem 5. *We have ξ is a d-normal number, furthermore η_a is a d-normal number, if $a \neq 0$.*

Special case: $\{h_0, \ldots, h_{\varphi(D)-1}\}$ set of reduced residues mod D,

$$\mathcal{P}_j = \{p | p \equiv h_j \pmod{D}\}$$
$$j = 0, \ldots, \varphi(D) - 1$$
$$\mathcal{R} = \{p;\ p|D\}.$$

*Functiones et Approximatio **45**, 2(2011), 231-253.*

$$P(x) = e_k x^k + \cdots + e_1 x \in \mathbb{R}[x], \quad P(x) \notin \mathbb{Q}[x].$$

$$I_0, I_1 \subset [0,1), \lambda(I_0) = \lambda(I_1) > 0,$$

intervals

$$\mathcal{P}_0 = \{p : \{P(p)\} \in I_0\}; \quad \mathcal{P}_1 = \{p : \{P(p)\} \in I_1\}$$

$$\mathcal{R} = \mathcal{P} \setminus (\mathcal{P}_0 \cup \mathcal{P}_1).$$

$$H(p) = \begin{cases} 0 & \text{if } p \in \mathcal{P}_0 \\ 1 & \text{if } p \in \mathcal{P}_1 \\ \Lambda & \text{if } p \in \mathcal{R}. \end{cases}$$

$$n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}, \; H(n) = H(p_1) \dots H(p_r)$$

$$\xi = 0, H(2)H(3) \dots$$

V.

Theorem 6. *Let* $F \in \mathbb{Z}[x]$, $r = \deg F$, *the leading coefficient of* F *is positive. For some* $n \in \mathbb{N}$ *let* $n = \sum_{j=0}^{t} \epsilon_j(n) q^j$, $\epsilon_t(n) \neq 0$, $\overline{n} = \epsilon_0(n) \dots \epsilon_t(n)$. *For some nonpositive integer* m, *let* $\overline{m} = \Lambda$. *Let*

$$\eta = 0, \overline{F(P(2+1))}\,\overline{F(P(3+1))} \dots \overline{F(P(p+1))} \dots$$
$$\xi = 0, \overline{F(P(2))}\,\overline{F(P(3))} \dots \overline{F(P(n))} \dots$$

Then η *and* ξ *are q-normal numbers.*

J-M. D.K. - I.K.: *On a problem of normal numbers raised by Igor Shparlinski, Bull. Austr. Math. Soc.* ***84*** *(2011), 337-349.*

Remark. The assertion remains true by changing $P \to P_k =$ the k'th largest prime divisor.

VI. Problems on normal numbers

Let $q \geq 2$, $E_q := \{l_0, l_1, \ldots, l_{\varphi(q)-1}\}$, $l_\nu \pmod q$ are the reduced residues mod q. $A_{\varphi(q)} := \{0, 1, \ldots, \varphi(q) - 1\}$. Let $\mathcal{P} = \{p_1, p_2, \ldots\}$ be the whole sequence of primes. $f_q : \mathcal{P} \to \{\Lambda, A_{\varphi(q)}\}$.

Let

$$f_q(p) = \begin{cases} j & \text{if } p \equiv l_j \pmod q \\ \Lambda & \text{if } (q, p) > 1. \end{cases}$$

Let

$$\mathcal{B}_q = \{f_q(p_1), f_q(p_2), \ldots\} \in A_{\varphi(q)}^{\mathcal{N}}$$
$$\xi_q = 0, f_q(p_1) f_q(p_2) \ldots$$

Conjecture 1. ξ_q *is a* $\varphi(q)$*-normal number, i.e.*

$$x_n = \{\varphi(q)^n \xi_q\}$$

is UD $\mod 1$*.*

Conjecture 2. *Let* $q \geq 3$*,* $k_1, k_2, \ldots, k_R \in A_{\varphi(q)}$*. Then*

$$p_{n+j} \equiv l_{k_j} \pmod{q} \quad (j = 1, \ldots R)$$

holds for infinitely many n*.*

This seems to be open for $R \geq 2$ and for $R \geq 3$ in the case $q = 3$.

Knapowski-Turán wrote several papers on $\Omega_\pm$ estimations on

$$\pi(x, k, l_1) - \pi(x, k, l_2).$$

A. Rényi formulated a conjecture, namely that for every permutation π of $A_{\varphi(q)}$: $\pi : A_{\varphi(q)} \to A_{\varphi(q)}$ there exists a sequence $x_1, x_2 \ldots, x_n \to \infty$ such that

$$\pi(x_\nu, q, \pi(0)) > \pi(x_\nu, q, \pi(1)) > \cdots > \pi(x_\nu, q, \pi(\varphi(q) - 1))$$

This seems to be very hard also.

Assume that the Hardy-Littlewood conjecture holds, namely that if $0 < b < q$, $(b, q) = 1$, $a_1, \ldots, a_{k-1}$ be such positive numbers for which the number of solutions $\rho(p)$ of $F(m) \equiv 0 \pmod{p}$ is smaller than p, where

$$F(m) = (mq + b)(mq + b + a_1) \cdots (mq + b + a_1 + \cdots + a_{k-1}).$$

Then the number of $p_0 = p \le x$, for which $p \equiv b \pmod q$

$$p_j = p_0 + a_1 + \cdots + a_j \quad (j = 1, \ldots, k-1)$$

are primes satisfies

$$N_b(x, a_1, \ldots, a_{k-1}) = (1 + o_x(1)) \times \prod_{\substack{\pi \le x \\ \pi \in \mathcal{P}}} \left(1 - \frac{\rho(\pi)}{\pi}\right).$$

From the Hardy-Littlewood conjecture one can obtain that large part of the primes $(p_0, \ldots, p_{k-1})$ counted in $N_b(x, a_1, \ldots, a_k)$ are consecutive primes, i.e. no primes exists in the intervals (p_j, p_{j+1}) $(j = 1, \ldots k-1)$.

From this not hard to deduce that for every $h_0, \ldots, h_{k-1} \in E_q$ there are infinitely many consecutive primes $p_0, \ldots p_{k-1}$, such that

$$p_j \equiv h_j \pmod q \quad (j = 0, \ldots, k-1).$$

Problem. *Construct explicitly such a sequence of integers: $n_1 < n_2 \dots$ for which in the notation $n_j \equiv \delta_j \pmod q$, $0 \le \delta_j < q$, the number*

$$\eta = 0, \delta_1 \delta_2 \dots \quad (\text{ } q\text{-ary expansion})$$

is a q-ary normal number.

I can imagine that one can prove that it is impossible to define such a sequence explicitly, if we would like to use only "simple functions". The following theorem seems to be interesting.

Let $\xi_1, \xi_2, \dots$ be independent random variables in $(\Omega, \mathcal{A}, P)$ taking values 0 and 1, $P(\xi_n = 1) = \lambda_n$, $P(\xi_n = 0) = 1 - \lambda_n$.

Assume that $\lambda_1 = \lambda_2 = 0,\ \lambda_n = \frac{1}{\log n} \quad (n \geq 3)$.

For some $\omega \in \Omega$ let $S(\omega) := \{n_1, n_2, \ldots\}$ where $n_1, n_2, \ldots$ are those indices for which $\xi_{n_j} = 1$.

Let $S(\omega)\mid_q = \{l_1, l_2, \ldots\}$, where $n_j \equiv l_j \pmod q$, $l_j \in A_q$.

Theorem 7. *Let*

$$\theta_q := 0, l_1 l_2 \ldots \quad (\textit{ q-ary exp.})$$

Then θ_q is a q-ary normal number with probability 1.