On the distribution properties of hybrid point sets

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(partly joint work with P. Hellekalek and F. Pillichshammer)

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Outline

Hybrid quasi-Monte Carlo point sets

In many applications of mathematics (e.g., financial mathematics, computer graphics) we use quasi-Monte Carlo (QMC) algorithms of the form

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for numerically approximating the integrals of functions f.

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It is a good idea to choose the points such that they are evenly distributed in $[0,1)^s$.

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- Third option (in this talk): mix different sorts of QMC point sets, first proposed by Faure, related to Owen's latin supercube sampling. Yields "hybrid" point sets.
- Many results on hybrid point sets (Hellekalek, Hofer, Larcher, Niederreiter, Pillichshammer, K.,. . .).

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As a way of measuring uniformity of distribution, we consider diaphony.

Diaphony

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Definition

$$F_N^{\mathrm{trig}}(\omega_1) := \left(c_{s_1} \sum_{\boldsymbol{k} \in \mathbb{Z}^{s_1} \setminus \boldsymbol{0}} \rho_1(\boldsymbol{k}) \left| \frac{1}{N} \sum_{n=0}^{N-1} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{x}_n \cdot \boldsymbol{k}} \right|^2 \right)^{1/2}$$

for the first N terms of a sequence $\omega_1 = (\boldsymbol{x}_n)_{n \geq 0}$ in $[0,1)^{s_1}$.

$\rho_1(\boldsymbol{k})$ is a weight function depending on s_1 and $\boldsymbol{k}$, and c_{s_1} a constant depending on s_1.

F_N^{trig} is well suited to study distribution properties of lattice points.

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Definition

$$F_N^{\mathrm{walsh}}(\omega_2) := \left(c_{s_2} \sum_{\boldsymbol{k} \in \mathbb{N}_0^{s_2} \setminus \boldsymbol{0}} \rho_2(\boldsymbol{k}) \left| \frac{1}{N} \sum_{n=0}^{N-1} {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_n) \right|^2 \right)^{1/2}$$

for the first N terms of a sequence $\omega_2 = (\boldsymbol{x}_n)_{n \geq 0}$ *in* $[0,1)^{s_2}$*.*

$\rho_2(\boldsymbol{k})$ *is a weight function depending on* s_2 *and* $\boldsymbol{k}$*, and* c_{s_2} *a constant depending on* s_2*.*

F_N^{walsh} is well suited to study distribution properties of digital sequences.

Carried over to the $\boldsymbol{p}$-adic function system $\{\gamma_{\boldsymbol{p},\boldsymbol{k}} : \boldsymbol{k} \in \mathbb{N}_0^{s_3}\}$ by Hellekalek. $\boldsymbol{p} = (p_1, \ldots, p_s)$ is a vector of primes.

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Definition

$$F_N^{\boldsymbol{p}-\mathrm{adic}}(\omega_3) := \left(c_{s_3} \sum_{\boldsymbol{k} \in \mathbb{N}_0^{s_3} \setminus \boldsymbol{0}} \rho_3(\boldsymbol{k}) \left| \frac{1}{N} \sum_{n=0}^{N-1} \gamma_{\boldsymbol{p},\boldsymbol{k}}(\boldsymbol{x}_n) \right|^2 \right)^{1/2}$$

for the first N terms of a sequence $\omega_3 = (\boldsymbol{x}_n)_{n \geq 0}$ in $[0,1)^{s_3}$.

$\rho_3(\boldsymbol{k})$ is a weight function depending on s_3 and $\boldsymbol{k}$, and c_{s_3} a constant depending on s_3.

$F_N^{\boldsymbol{p}-\mathrm{adic}}$ is well suited to study distribution properties of, e.g., Halton sequences.

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We need hybrid diaphony.

Hybrid diaphony

Hybrid diaphony, based on three function systems (Hellekalek).

Definition

$$F_N^{\text{hybrid}}(\omega) := \Bigg(c \underbrace{\sum_{\boldsymbol{k}_1 \in \mathbb{Z}^{s_1}} \sum_{\boldsymbol{k}_2 \in \mathbb{N}_0^{s_2}} \sum_{\boldsymbol{k}_3 \in \mathbb{N}_0^{s_3}}}_{(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) \neq \boldsymbol{0}} \rho_1(\boldsymbol{k}_1) \rho_2(\boldsymbol{k}_2) \rho_3(\boldsymbol{k}_3) \times$$

$$\times \left| \frac{1}{N} \sum_{n=0}^{N-1} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{x}_{n,1} \cdot \boldsymbol{k}_1} \, {}_b\mathrm{wal}_{\boldsymbol{k}_2}(\boldsymbol{x}_{n,2}) \gamma_{\boldsymbol{p}, \boldsymbol{k}_3}(\boldsymbol{x}_{n,3}) \right|^2 \Bigg)^{1/2}$$

for the first N terms of a sequence $\omega = (\boldsymbol{x}_{n,1}, \boldsymbol{x}_{n,2}, \boldsymbol{x}_{n,3})_{n \geq 0}$ in $[0,1)^{s_1+s_2+s_3}$. c a constant depending on s_1, s_2, s_3.

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for the first N terms of a sequence $\omega = (\boldsymbol{x}_{n,1}, \boldsymbol{x}_{n,2}, \boldsymbol{x}_{n,3})_{n\geq 0}$ in $[0,1)^{s_1+s_2+s_3}$. c a constant depending on s_1, s_2, s_3.

If s_1, s_2 or s_3 is zero $\rightarrow$ obvious adaptions.

Analyzing hybrid diaphony is challenging due to the expression

$$\left| \frac{1}{N} \sum_{n=0}^{N-1} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{x}_{n,1} \cdot \boldsymbol{k}_1} \ {}_b\mathrm{wal}_{\boldsymbol{k}_2}(\boldsymbol{x}_{n,2}) \gamma_{\boldsymbol{p},\boldsymbol{k}_3}(\boldsymbol{x}_{n,3}) \right|^2 ,$$

involving three different function systems.

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- Low values mean good distribution properties,
- high values mean bad distribution properties.
- A sequence is uniformly distributed if and only if its diaphony converges to zero.

Results on the diaphony of hybrid point sets

(A) Hybrid point sets based on digital sequences and lattice points (joint work with F. Pillichshammer).

For this case, we consider the special case where $s_3 = 0$,

$$F_N^{\text{hybrid}}(\omega) := \Bigg(c \underbrace{\sum_{\boldsymbol{k}_1 \in \mathbb{Z}^{s_1}} \sum_{\boldsymbol{k}_2 \in \mathbb{N}_0^{s_2}}}_{(\boldsymbol{k}_1, \boldsymbol{k}_2) \neq \boldsymbol{0}} \rho_1(\boldsymbol{k}_1) \rho_2(\boldsymbol{k}_2) \times \times \left| \frac{1}{N} \sum_{n=0}^{N-1} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{x}_{n,1} \cdot \boldsymbol{k}_1} \, {}_b\mathrm{wal}_{\boldsymbol{k}_2}(\boldsymbol{x}_{n,2}) \right|^2 \Bigg)^{1/2}$$

Theorem

Let $(\boldsymbol{x}_{n,2})_{n\geq 0}$ *be a digital* (t, s_2)*-sequence over* $\mathbb{Z}_b$ *(b prime). For any* $\alpha \in [0, 1)$*, and any odd prime* $N \neq b$*,*

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Let $(\boldsymbol{x}_{n,2})_{n\geq 0}$ be a digital (t,s_2)-sequence over $\mathbb{Z}_b$ (b prime). For any $\alpha \in [0,1)$, and any odd prime $N \neq b$,

one can find more than αN^{s_1} generating vectors $\boldsymbol{g} \in \{0,1,\ldots,N-1\}^{s_1}$ of an s_1-dimensional lattice point set $(\boldsymbol{x}_{n,1})_{n=0}^{N-1}$, such that the hybrid point set $\boldsymbol{\omega} = ((\boldsymbol{x}_{n,1},\boldsymbol{x}_{n,2}))_{n=0}^{N-1}$ satisfies

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$$\left(F_N^{\mathrm{hybrid}}(\omega)\right)^2 \leq \frac{c}{1-\alpha}\left(\frac{4}{N} + b^{2t}\frac{(\log_b N)^{s_2}}{N^2}\right),$$

where $c > 0$ is a constant independent of N.

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Theorem

Let $(\boldsymbol{x}_{n,3})_{n\geq 0}$ *be an* s_3*-dimensional Halton sequence in bases* $p_1,\ldots,p_{s_3}$*, where* $p_1,\ldots,p_{s_3}$ *are* s_3 *distinct primes. Let* N *be a prime different from* $p_1,\ldots,p_{s_3}$*.*

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Then there exists a generating vector $\boldsymbol{g}\in\{1,\ldots,N-1\}^{s_1}$ of an s_1-dimensional lattice point set $(\boldsymbol{x}_{n,1})_{n=0}^{N-1}$, such that the hybrid point set $\omega=((\boldsymbol{x}_{n,1},\boldsymbol{x}_{n,3}))_{n=0}^{N-1}$ satisfies

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$$F_N^{\mathrm{hybrid}}(\omega)\leq c\frac{(\log N)^{s_1+s_3+1}}{N},$$

where c is a positive constant that is independent of N.

Conclusion

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- Theorems 1 and 2 are existence results.
- **Open problem 1:** Component by component constructions, in particular, for (A)?
- **Open problem 2:** Explicit constructions?

Thank you for your attention.