Group extensions : from old to new results with various applications

Pierre LIARDET
(Université d'Aix-Marseille)

Group extensions :
from old to new results with various applications

I. Basic definitions and examples

I.1 Skew product construction

- A set X
- A group $(\Gamma, .)$
- A map $T : X \to X$
- A map $\varphi : X \to \Gamma$

$\to$ The (φ, Γ)-group extension of base T, or a skew product of T, with *cocycle* φ, is :

$$T_\varphi : X \times \Gamma \to X \times \Gamma$$

$$\boxed{T_\varphi(x, \gamma) = (Tx, \gamma.\varphi(x))}$$

In topological dynamics :

- X is a compact metrizable space,
- T is continuous,
- Γ is a locally compact metrizable group,
- φ is continuous.

In metric theory of dynamical systems :

- X is a standard Borel space $(X, \mathcal{B})$, equipped with a probability measure μ,
- T is an endomorphism of $(X, \mathcal{B}, \mu)$,
- Γ is a compact (eventually locally compact) metrizable group,
- ν is the Haar measure of Γ (eventually a right invariant measure),
- φ is measurable,

and the metric skew product is

$$(T_\varphi, X \times \Gamma, \mu \otimes \nu)$$

I.2 Basic examples

(1) Group extension of a one point map : $X := \{p\}$, $\varphi(p) = \alpha$.

T_φ is naturally identified to the translation $\gamma \mapsto \gamma + \alpha$ on Γ.

(2) Anzai skew product (the father of skew products (Anzai, 1951)) :

$$X := \mathbf{T} \text{ (1-torus)},\ \Gamma = \mathbf{T}\,,\ \varphi \text{ is a measurable map.}$$

(3) The Furstenberg tower extensions (related to polynomial sequences $n \mapsto P(n)$ (mod 1) with dominant coefficient α)

$$X := \mathbf{T}^{s-1},\ \Gamma = \mathbf{T},\ \varphi(x_1, \ldots, x_{s-1}) = x_{s-1}$$

$$T(x_1, \ldots, x_s) = (x_1 + s!\alpha, x_2 + x_1, \ldots, x_s + x_{s-1})$$

In fact, set $P_{s-i}(X) = \Delta^i P(X)$, $i = 1, \ldots, s-1$, with $\Delta Q(X) = Q(X+1) - Q(X)$, then

$$T^n(P_1(0), \ldots, P_{s-1}(0), P(0)) = (P_1(n), \ldots, P_{s-1}(n), P(n))\,.$$

We may replace $\mathbf{T}$ by any compact abelian group Γ.

I.3 Skew products from odometers

Let G be a scale $1 = G_0 < G_1 < G_2 < \cdots$.

The greedy extension of integer $n \in \mathbf{N}_0$ $(= \mathbf{N} \cup \{0\})$ is given by

$$n = e_0(n)G_0 + e_1(n)G_1 + \cdots + e_{h(n)}(n)G_{h(n)} \quad \text{and} \quad e_{h(n)+i}(n) = 0 \quad (i = 1, 2, 3, \ldots)$$

with the constraints

$$(C) \qquad e_0(n)G_0 + e_1(n)G_1 + \cdots + e_k(n)G_k < G_{k+1} \quad (k = 0, 1, 2, \ldots)\,.$$

• G-compactification of $\mathbf{N}_0$

$$\mathcal{K}_G = \{e \in \mathbf{N}_0^\infty\,;\; e_0G_0 + e_1G_1 + \cdots + e_kG_k < G_{k+1} \text{ for all } k = 0, 1, 2, \ldots\}$$

endowed with the product topology.

$\rightarrow$ $\mathbf{N}_0$ is identified into a part of $\mathcal{K}_G$ by setting

$$\tilde{n} = e_0(n)e_1(n)e_2(n)\cdots\,.$$

We also use $\hat{n} = e_0(n)e_1(n)e_2(n)\ldots e_{h(n)}(n)$ called G-word of n, $(\hat{0} = \wedge)$.

$\rightarrow$ By construction, $\widetilde{\mathbf{N}_0}$ is dense in $\mathcal{K}_G$.

Definition. The G-odometer $\tau : \mathcal{K}_G \to \mathcal{K}_G$ is built from the addition of 1 in $\mathbf{N}_0$, computed in the scale G :

$$\tau(e) = \lim_{k\to\infty} (e[k]+1)^{\sim} \quad \text{where} \quad e[k] = e_0G_0 + \cdots + e_kG_k.$$

The set $\tau^{-1}(0^\infty)$ plays an important role.

Theorem (G-L-T (1995)) *τ is continuous if and only if for all $k \geq 1$ the G-word W_k of $G_k - 1$ is the prefix of at most finitely many G-words W_ℓ of $G_\ell - 1$ with $\ell > k$ but $|W_\ell| = |W_k| + 1$.*

Classical examples.

(CE1) **q-adic scale :**

$G_k = q^k$. One gets the classical q-adic odometer (adding 1 in the group of q-adic integers $\mathbf{Z}_q$). Here τ is continuous and $\tau^{-1}(0^\infty) = \{(q-1)^\infty\}$.

(CE2) α-**Ostrowski scale :**

α is an irrational number in $[1/2, 1]$. The regular continued fraction expansion

$$\alpha = [0; a_1, a_2, a_3, \ldots]$$

leads to convergents p_n/q_n ; we take $G_n = q_n$.

Here

$$\tau^{-1}(0^\infty) = \{0a_2 0a_4 0a_6 \cdots, (a_1 - 1)0a_3 0a_5 \cdots\},$$

τ is continuous and $(\mathcal{K}_G, \tau)$ is minimal, uniquely ergodic, quasi-conjugate to the translation $x \to x + \alpha \bmod 1$ with conjugate map $\varphi : \mathcal{K}_G \to [0, 1[$ given by

$$\varphi(e) = \alpha + \sum_{n=0}^{\infty} e_n(q_n\alpha - p_n)\,.$$

• For general results about G-odometers we refer to the following seminal papers :

P. Grabner, P.L. and R. Tichy : Odometers and systems of numeration, AA, **70** (1995), 103-123.

G. Barat, T. Downarovicz, A. Iwanik. and P. L. : Propriétés topologiques et combinatoires des échelles de numération, Colloq. Math., **84-85**, part 2 (2000), 285–306.

G. Barat, T. Downarovicz, and P. L. : *Dynamiques associées à des échelles de numération*, AA, **103**(1) (2002), 281-290.

•• Many papers investigate more or less exotic scales of less or more interests and more or less untractable :=(

• • ⋄ The first goal is usually to decide if the G-odometer is uniquely ergodic.

• • ⋄⋄ If the answer is positive, the next goal should be a part of the subject of this talk...

G-multiplicative group extensions

Let G be a scale.

Definition. $f : \mathbf{N}_0 \to \mathbf{U}$ (resp. $f : \mathbf{N}_0 \to \mathbf{R}$ is G-multiplicative (resp. additive) if

$$f(n) = \prod_{k=0}^{h(n)} f(e_k(n)G_k) \quad \Big(\text{resp. } f(n) = \sum_{k=0}^{h(n)} f(e_k(n)G_k\Big);$$

in the additive case, f is called a weighted sum-of-digits if $f(e_k(n)G_k) = e_k(n)f(G_k)$.

Usual multiplicative cases :

- For q-adic scale, we speak about q-multiplicative sequence.
- For α-Ostrowski scale, we speak about α-multiplicative sequence.
- Eventually replace $\mathbf{U}$ or $\mathbf{R}$ by Γ (locally compact metrizable group).

For other cases we also assume that the odometer is uniquely ergodic with invariant measure μ such that $\mu(\tau^{-1}(0^\infty)) = 0$.

Definition. Discrete derivative Δf of $f : \mathbf{N}_0 \to \mathbf{U}$ is defined on $\mathcal{K}_G \setminus \tau^{-1}(0^\infty)$ by

$$\Delta f(x) = \lim_{\substack{\tilde{n} \to x \\ n \in \mathbf{N}}} f(n+1)/f(n)$$

On $\tau^{-1}(0^\infty)$ put your favorite values if you cannot extend Δf by continuity.

Now we have at least two options :

- Study the G-multiplicative group extension

$$(\tau_{\Delta f}, \mathcal{K}_G \times \mathbf{U}, \mu \otimes \nu)$$

$$\boxed{\tau_{\Delta f}(x, \zeta) = (\tau(x), \zeta \Delta f(x))}$$

- Look at f as a point in $\Omega := \mathbf{U}^{\mathbf{N}_0}$, then consider the shift $\sigma : \Omega \to \Omega$, the orbit closure

$$K(f) := \overline{\{\sigma^n f\, ;\, n \in \mathbf{N}_0\}}$$

and the continuous dynamical system $(K(f), \sigma')$ where σ' is the restriction of σ on $K(f)$.

$\to$ Find topological, ergodical properties (including spectral analysis) of such dynamical systems.

I.4 Skew products from Bernouilli shifts

With finite number q of states :

X is the compact space

$$\Omega_q = \{0, 1, \ldots, q-1\}^{\mathbf{N}_0}$$

equipped with the infinite product ν_q^∞ probability of the uniform distribution ν_q on $\{0, 1, \ldots, q-1\}$ and the shift σ.

Eventually, replace $(\Omega_q, \sigma, \nu_q^\infty)$ by $([0,1), b^*, \lambda)$:

$$b^*(x) := bx - \lfloor bx \rfloor \,, \quad \lambda \text{ is the Lebesgue probability}\,.$$

$\Gamma = \mathbf{T}$ and the cocycle φ is measurable.

With infinite number of states :

Here we are concerned with $X := \Omega(\mathbf{T})$ $(= \mathbf{T}^{\mathbf{N}_0})$ and the shift is still noted by σ.

II. Minimality and distality

In this part we assume

- X compact metrizable
- Γ abelian, compact and metrizable
- $T : X \to X$ and $\varphi : X \to \Gamma$ continuous.

Theorem (Furstenberg). *Assume that (T, X) is minimal, then T_φ is minimal if and only if for all non trivial character χ of γ, there is no continuous map $F : X \to \mathbf{C}$, except the null map, such that*

$$(*) \qquad \boxed{F \circ Tx = (\chi \circ \varphi).F} \, .$$

Notice that the set of characters χ such that $(*)$ has a non trivial continuous solution F form a subgroup H of $\hat{\Gamma}$.

Fact : the map $\Theta_g : (x, \gamma) \mapsto (x, \gamma.g)$ is an automorphisme of T_φ.

Theorem (from Furstenberg). *Assume T is minimal, then T_φ is distal (i.e. for all $(u, v) \in (X \times \Gamma)^2$, $\mathcal{O}_{T_\varphi \times T_\varphi}(u, v)$ is disjoint of the diagonal).*

$\to\to$ In particular, for all $(x, \gamma) \in X \times \Gamma$, T_φ **restreint to** $\mathcal{O}_{T_\varphi}(x, \gamma)$ **is minimal**.

III. Criteria for ergodicity and unique ergodicity

Suppose (T, X, μ) ergodic and let Γ be a compact metrizable group.

III.1 Ergodicity of T_φ

The cocycle $\varphi : X \to \Gamma$ is said to be ergodic (uniquely ergodic) if $(T_\varphi, X \times \Gamma, h \otimes \nu)$ is ergodic (uniquely ergodic).

- There are essentially 2 criteria for the ergodicity of φ.

First Criterium :

Theorem. *The following assertions are equivalent :*

(i) φ is ergodic ;

(ii) for all irreducible nontrivial representation π of Γ in a Hilbert space H_π of unit ball B_π, there exists no measurable map $F : X \to B_\pi$, except the null map, such that

$$F(Tx) = \pi(\varphi(x))(F(x)) \quad \mu - p.p.$$

Second criterium (with X non atomic, T inversible, Γ Abelian metrizable and compact) :

Define φ_n for $n \in \mathbf{Z}$ by

$$T_\varphi^n(x,\gamma) = (T^n x, \gamma\varphi_n(x)).$$

We need to introduce the notion of essential value of K. Schmidt (1977) :

Definition. $\alpha \in \Gamma$ is an essential value of φ with respect to T if for any Borel set B of X with $\mu(B) > 0$ and any neighborhood V of α in Γ, one has

$$\mu\left(\bigcup_{n\in Z}\left(B \cap T^{-n}(B) \cap \varphi_n^{-1}(V)\right)\right) > 0\,.$$

Fracts :

(j) the set $E(\varphi)$ of essential values of φ is a compact sub-group of Γ ;

(jj) multiplying φ by a **coboundary**, *i.e.* a measurable map of the form $x \mapsto g(Tx)g(x)^{-1}$, does not change the essential values ;

(jjj) φ is itself a coboundary if and only if $E(\varphi) = \{1_\Gamma\}$.

(jv) There is a measurable coboundary $\varphi_0 : x \mapsto g(Tx)g(x)^{-1}$ such that $\varphi' = g \circ T.\varphi.(g)^{-1}$ is $E(\varphi)$-valued. In that case T_φ is metrically isomorphic to $T_{\varphi'}$.

Theorem (K. Schmidt). *$(T_\varphi, X \times \Gamma, \mu \otimes \nu)$ is ergodic if and only if $E(\varphi) = \Gamma$.*

Note : In the Abelian case the set of characters π of Γ such that the corresponding functional equation

$$F(Tx) = \pi(\varphi(x))(F(x)) \quad \mu - \text{p.p.}$$

has a non-trivial measurable solution form a subgroup H of $\hat{\Gamma}$ and

$E(\varphi)$ is the annihilator of H.

III.2 Weakly mixing

The cocycle $\varphi : X \to \Gamma$ is said to be weakly mixing if it is ergodic and the eigenvalues of the unitary operator U_{T_φ} of $L^2(X \times \Gamma, \mu)$ associated to T_φ are only those coming from of the unitary operator U_T.

Theorem. *If φ is ergodic, the following assertions are equivalent :*

(i) φ is weakly mixing ;

(ii) for all irreducible nontrivial representation π of Γ in $\mathbf{U}$ and for all complex numbers ζ of modulus 1, there exists no measurable map $F : X \to \mathbf{U}$, except the null map, such that

$$F(Tx) = \zeta\pi(\varphi(x))(F(x)) \quad \mu - p.p.$$

III.2 Genericity and unique ergodicity

We assume :

- X is metrizable compact, μ is a Borel measure of probability.
- $T : X \to X$ is measurable μ-continuous.
- $\varphi : X \to \Gamma$ is measurable, μ-continuous.

With these assumptions, the skew product T_φ will be said regular.

Definition. $x \in X$ is said to be T-generic if for all continuous functions $f : X \to \mathbf{C}$, one has

$$\lim_N \frac{1}{N} \sum_{n=0}^{N-1} f(T^n(x)) = \int_X f d\mu \,.$$

This definition has also a meaning if we replace T by T_φ.

Theorem (P.L. 1977). *In the regular case, φ is ergodic if and only if for all T-generic point x and all $\gamma \in \Gamma$, the point (x, γ) is T_φ-generic.*

In particular, this theorem says that if $(T_\varphi, X \times G, \mu \otimes \nu)$ is ergodic and x is T-generic, then the sequence

$$n \mapsto \varphi_n(x)$$

is ν-distributed in Γ.

The following theorem is well known in the continuous case.

Theorem (P.L. 1977). *In the regular case, T_φ is uniquely ergodic if and only if T is uniquely ergodic and T_φ ergodic.*

To obtain uniform distribution of $n \mapsto \varphi_n(x)$ uniformly in x, we need additional regularity on T and φ.

IV. A worked example and a bit more

Take

$X = \Gamma = [0,1)$ (identified to the torus $\mathbf{T} = \mathbf{R}/\mathbf{Z}$).

$\varphi : [0,1) \to [0,1)$, $\varphi(x) = \frac{1}{x+1}$,

$u = (u_n)_{n\geq 0}$ a sequence in $[0,1)$

and

$$\Sigma_u : n \mapsto \frac{1}{u_0+1} + \frac{1}{u_1+1} + \cdots + \frac{1}{u_{n-1}+1} \pmod 1$$

IV.1 Case of completely u.d. sequences

Theorem. *If u is completely uniformly distributed the sequence Σ_u is uniformly distributed.*

A glance at the proof :

View φ as a map defined on $\Omega([0,1))$ by setting

$$\varphi(x_0, x_1, \ldots) = \frac{1}{x_0 + 1}$$

and consider the skew product

$$(\sigma_\varphi, \Omega([0,1)) \times [0,1), \lambda^\infty \otimes \lambda)\,.$$

Ergodicity comes from a general result where we can replace this special cocycle φ by any regular cocycle f depending on the first coordinate and its essential values generate a dense subgroup of $[0,1)$ (P.L. (1981)).

IV. Sequences $n\alpha$

Theorem. *Let* $\alpha = [0; a_1, a_2, \ldots]$ *be an irrational number in* $[0, 1)$ *and set* $u_n = n\alpha - \lfloor n\alpha \rfloor$. *Then the sequence* Σ_u *is uniformly distributed.*

Key step in the proof (derived from P. Gabriel, M. Lemanczyk and P.L. (1991)).

We look at the n discontinuity points of

$$\varphi_m(x) = \varphi(x) + \ldots + \varphi(x + (m-1)\alpha)$$

namely the points $-k\alpha \pmod 1$ $k = 0, 1, \ldots, m_n - 1$ where φ_{m_n} has a jump $1/2$.

For $m_n = q_n + q_{n+1}$, the points $-k\alpha \pmod 1$, $k = 0, \ldots, m_n - 1$ built a subdivision of $[0, 1)$ made of q_{n+1} intervals of length $||q_n\alpha||$ and q_n intervals of length $||q_{n+1}\alpha||$.

The absolute value of the derivative of φ_n is bounded by m_n. Classically $q_{n+1}||q_n\alpha|| + q_n||q_{n+1}\alpha|| = 1$ and

$$\frac{1}{2q_{n+1}} \leq ||q_n\alpha|| \leq \frac{1}{q_{n+1}}, \quad ||q_{n-2}\alpha|| = a_n||q_{n-1}\alpha|| + ||q_n\alpha||\,.$$

This analysis leads to the key lemma :

Lemma. *There exists two constants* $\varepsilon_0 > 0$ *and* $\eta_0 > 0$ *such that for* $m_n := q_n + q_{n+1}$ *one has for all* $c \in [0,1)$ *:*

$$\lambda(\{x \in [0,1)\,;\ ||\varphi_{m_n} - c|| \geq \varepsilon_0\}) \geq \eta_0\,.$$

End of the proof.

→ If k is even the cocycle $\psi_k(\cdot) = k\varphi(\cdot)$ is absolutely continuous with degré $k/2$ and according to a theorem of Gabriel-Lemanczyk-Liardet (1991) $\psi_k(\cdot)$ is weak mixing.

→ If k is odd we can replace φ in the above lemma by ψ_k and look at the functional equation

$$F(x+\alpha) = \zeta e^{2i\pi k\varphi(x)} F(x) \quad \lambda - p.p.$$

with $|F| = 1 = |\zeta|$; that gives

$$\int_{[0,1)} F(x + m_n\alpha)\overline{F(x)}dx = \int_{[0,1)} \zeta^{m_n} e^{2i\pi k\varphi_{m_n}(x)}dx\,.$$

The left member of this equality tends to 1 hence by convexity argument, $\zeta^{m_n} e^{2i\pi k\varphi_{m_n}(\cdot)}$ tend to 1 in measure. Contradiction with the lemma.

Some open problems.

– Show that the sequence Σ_u is uniformly distributed mod 1 if $u_n = q^n\theta - \lfloor q^n\theta \rfloor$ where θ is q-normal.

– Use for u your favorite sequence, like the Van der Corput sequence.

– A lot…

V. Results involving continued fractions

$X = [0, 1)$

T is the Gauss transformation $T(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor$ $(T(0) = 0)$

$\mu(dx) = \frac{1}{\log 2} \frac{dx}{1+x}$

set :

$a(x) = \left\lfloor \frac{1}{x} \right\rfloor, \;\; a(0) = \infty, \;\; M_a = \begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix}, \; J := M_0,$

then

$$x = \frac{1}{a(x) + Tx} = \frac{1}{a(x) + \cfrac{1}{a(Tx) + T^2(x)}} = [0; a(x), a(Tx), a(T^2(x), \ldots]$$

if x is irrational

$$x := [0; a_1, a_2, a_3, \ldots];$$

if x is rational :

$$x = [0, a_1, \ldots, a_k, \infty, \infty, \ldots] \quad \text{with } a_k > 1.$$

Fondamental formula for x irrational :

Notation : $a_k(x) = a(T^{k-1}(x))$.

$$\boxed{\begin{pmatrix} p_{n-1}(x) & p_n(x) \\ q_{n-1}(x) & q_n(x) \end{pmatrix} = M_{a_1(x)} \cdots M_{a_n(x)}\,, \quad (p_0 = 0,\; q_0 = 1)}$$

For integer $m \geq 2$, set $\Gamma_m = GL_2(\mathbf{Z}/m\mathbf{Z})$ (compact but not commutative). One has also

$$\Gamma_m = GL_2(\mathbf{Z})/G_{2,m}$$

where

$$G_{2,m} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbf{Z}\,;\; a \equiv d \equiv 1 \pmod{m} \;\&\; b \equiv c \equiv 0 \pmod{m} \right\}$$

Example with $m = 2$:

$$\Gamma_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}.$$

Cocycle : $\varphi(x) = M_{\bar{a}(x)}$ and $T_\varphi : [0,1) \times \Gamma_m \to [0,1) \times \Gamma_m$ defined by

$$T_\varphi(x, \gamma M_{\bar{a}(x)})$$

where $M_{\bar{a}}$ is the class of M_a mod $G_{2,m}$.

Theorem (H. Jager and P.L. (1988)) *The skew product $(T_\varphi, [0,1) \times \Gamma_m, \lambda \otimes \nu_m)$ is ergodic.*

Notice that $\#\Gamma_m = 2m^3 \prod_{p|m} \left(1 - \frac{1}{p^2}\right)$ and T_φ is regular, hence for any matrix g in $GL_2(\mathbf{Z})$ and all T-generic point x :

$$\lim_n \frac{1}{N} \# \left\{ n \leq N \, ; \, \begin{pmatrix} p_{n-1}(x) & p_n(x) \\ q_{n-1}(x) & q_n(x) \end{pmatrix} \equiv g \bmod G_{2,m} \right\} = \frac{1}{\#\Gamma_m}$$

The above result has been extended to various continued fraction expansions, in particular for the S-continued fractions introduced by C. Kraaikamp (K. Dajani and C. Kraaikamp (1998)). It has been also adapted in multidimensional continued fraction expansions (V. Berthé, H. Nakada and R. Natsui (2006)).

More arithmetical structures of denominators $q_n(x)$ of converges $p_n(x)/q_n(x)$

Let π_n denote the n-th prime number, consider a sequence $c : \mathbf{N} \to \mathbf{N}_0 \cup \{\infty\}$ and define

$$E_c = \{n \in \mathbf{N} \, ; \, \forall \, k \, , \pi_k^{c_k} \not| \; n\}$$

For example $E_{2,2,2,\ldots}$ is the set of square-free integers.

Theorem. *For any T-generic point x (here, T is the Gauss transformation or a suitable analogous transformation) the limit*

$$\lim_N \frac{1}{N} \#\{n \leq N \, ; q_n(x) \in E_c\}$$

exist (and is computable).

Example, for the Gauss transformation and $\mathcal{Q} := E_{2,2,2,\ldots}$ the limit is

$$\prod_p \Big(1 - \frac{2}{p(p-1)}\Big)$$

About the proof (in the case of square free integers for example)

The group Γ_m is replaced by the projective limit

$$\Gamma_{\mathcal{Q}} := \varprojlim_{q \in \mathcal{Q}} \Gamma_{q^2} \,.$$

and let $c : GL_2(\mathbf{Z}) \to \Gamma_{\mathcal{Q}}$ be the canonical morphism.

Now, identify in $\Gamma_{\mathcal{Q}}$ a compact $K_{\mathcal{Q}}$ such that

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix}) \in GL_2(\mathbf{Z}) \;\;\&\;\; u \in \mathcal{Q} \iff c(\begin{pmatrix} r & s \\ t & u \end{pmatrix}) \in K_{\mathcal{Q}}$$

then the cocycle $y \mapsto c(M_x)$ is ergodic.

Unfortunately, the interior of $K_{\mathcal{Q}}$ is empty, so we cannot apply directly the theorem on generic points. We need extra probabilistic arguments.