Quasi-Monte Carlo methods for integration of functions with dominating mixed smoothness

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Discrepancy function

N positive integer, $\mathcal{P} \in [0,1)^d$ point set with N points

$$D_{\mathcal{P}}(x) = \frac{\#\mathcal{P} \cap [0,x)}{N} - |[0,x)|$$

$|[0,x)| = x_1 \cdot \ldots \cdot x_d$ volume of $[0,x) = [0,x_1) \times \ldots \times [0,x_d)$

Discrepancy function

- *Measure the deviation between the number of points of $\mathcal{P}$ in $[0, x)$ and $N|[0, x)|$ (perfect distribution)*
- *Task 1: calculation of the norm of $D_{\mathcal{P}}$ in $L_\infty([0, 1)^2)$, $L_2([0, 1)^2)$, also $L_p([0, 1)^d)$ and $S^r_{pq}B([0, 1)^d)$*
- *Task 2: search for best possible distributions*
- *Purpose: quadrature formulas (quasi-Monte Carlo), number theory*

Star discrepancy

$$D^*(N) = \inf_{\#\mathcal{P}=N} \|D_{\mathcal{P}}|L_\infty\|$$

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- $d = 2$: (Schmidt; van der Corput)

$$D^*(N) \asymp \frac{\log N}{N}$$

- $d > 2$: (Bilyk, Lacey, Vagharshakyan; Halton)

$$\frac{(\log N)^{\frac{d-1}{2}+\eta_d}}{N} \preceq D^*(N) \preceq \frac{(\log N)^{d-1}}{N}$$

L_p-discrepancy

$$D^p(N) = \inf_{\#\mathcal{P}=N} \|D_{\mathcal{P}}|L_p\|$$

L_p-discrepancy

$$D^p(N) = \inf_{\#\mathcal{P}=N} \|D_{\mathcal{P}}|L_p\|$$

- $1 < p < \infty$: (Schmidt; Chen)

$$D^p(N) \asymp \frac{(\log N)^{\frac{d-1}{2}}}{N}$$

L_p-discrepancy

- $p = 1$: (Halász; Chen)

$$\frac{(\log N)^{\frac{1}{2}}}{N} \preceq D^p(N) \preceq \frac{(\log N)^{\frac{d-1}{2}}}{N}$$

$S^r_{pq}B$-spaces

$$D^r_{pq}(N) = \inf_{\#\mathcal{P}=N} \left\| D_{\mathcal{P}} | S^r_{pq}B \right\|$$

$S^r_{pq}B$-spaces

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$\varphi_0 \in \mathcal{S}(\mathbb{R})$, $\varphi_0(t) = 1$ for $|t| \leq 1$ and $\varphi_0(t) = 0$ for $|t| > \frac{3}{2}$

$\varphi_k(t) = \varphi_0(2^{-k}t) - \varphi_0(2^{-k+1}t)$ for $t \in \mathbb{R}$, $k \in \mathbb{N}$

$$\varphi_k(t) = \varphi_{k_1}(t_1) \dots \varphi_{k_d}(t_d)$$

$k = (k_1, \dots, k_d) \in \mathbb{N}_0^d$, $t = (t_1, \dots, t_d) \in \mathbb{R}^d$

φ_k a dyadic resolution of unity

$$\sum_{k \in \mathbb{N}_0^d} \varphi_k(x) = 1$$

$S^r_{pq}B$-spaces

$$\left\| f | S^r_{pq}B(\mathbb{R}^d) \right\| = \left(\sum_{k \in \mathbb{N}_0^d} 2^{r(k_1+\ldots+k_d)q} \left\| \mathcal{F}^{-1}(\varphi_k \mathcal{F} f) | L_p(\mathbb{R}^d) \right\|^q \right)^{\frac{1}{q}}$$

$$\left\| f | S^r_{pq}B([0,1)^d) \right\| = \inf \left\{ \left\| g | S^r_{pq}B(\mathbb{R}^d) \right\| : \; g \in S^r_{pq}B(\mathbb{R}^d), \; g|_{[0,1)^d} = f \right\}$$

Besov space with dominating mixed smoothness

$$S^r_{pq}B([0,1)^d) \hookrightarrow L_p([0,1)^d)$$

$S^r_{pq}B$-discrepancy

- $1 \le p, q \le \infty$, $\frac{1}{p} - 1 < r < \frac{1}{p}$; $q < \infty$ if $p = 1$ and $q > 1$ if $p = \infty$: (Triebel)

$$N^{r-1}(\log N)^{\frac{d-1}{q}} \preceq D^r_{pq}(N) \preceq N^{r-1}(\log N)^{(d-1)(\frac{1}{q}+1-r)}$$

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- $d = 2$, $r \ge 0$: (Hinrichs)

$$D^r_{pq}(N) \preceq N^{r-1}(\log N)^{\frac{1}{q}}$$

$S^r_{pq}B$-discrepancy

Question:

$$D^r_{pq}(N) \preceq N^{r-1}(\log N)^{\frac{d-1}{q}} \ ?$$

$S^r_{pq}B$-discrepancy

Question:

$$D^r_{pq}(N) \preceq N^{r-1}(\log N)^{\frac{d-1}{q}} \ ?$$

The answer is YES for $r \geq 0$!

1 Introduction

2 Known results
- Star discrepancy
- L_p-discrepancy
- $S^r_{pq}B$-discrepancy

3 The new result
- *b*-adic Haar bases
- Characterization of the spaces
- Constructions

b-adic intervals

b-adic interval of length b^{-j}, $j \in \mathbb{N}_0$

$$I_{jm} = [b^{-j}m, b^{-j}(m+1))$$

for $m = 0, 1, \ldots, b^j - 1$

divide I_{jm} into b intervals of length b^{-j-1} (notation I_{jm}^k)

$j = (j_1, \ldots, j_d)$, $m = (m_1, \ldots, m_d)$

$$I_{jm} = I_{j_1 m_1} \times \ldots \times I_{j_d m_d}$$

Haar basis

$h_{jm\ell}$ function with support in I_{jm} and value $e^{\frac{2\pi i}{b}\ell k}$ in I_{jm}^k for $\ell = 1, \ldots, b-1$

$h_{jm\ell},\ h_{-1,0,1} = \mathbb{1}_{[0,1)} \longrightarrow$ orthogonal basis of $L_2([0,1))$

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$j = (j_1, \ldots, j_d), m = (m_1, \ldots, m_d), \ell = (\ell_1, \ldots, \ell_d), x = (x_1, \ldots, x_d)$

$h_{jm\ell}(x) = h_{j_1 m_1 \ell_1}(x_1) \ldots h_{j_d m_d \ell_d}(x_d)$

$h_{jm\ell} \longrightarrow$ orthogonal basis of $L_2([0,1)^d)$

Haar coefficients

$f \in L_2([0,1))^d$ by Parseval's equation

$$\|f|L_2\|^2 = \sum_{j \in \mathbb{N}_{-1}^d} b^{\max(0,j_1)+\ldots+\max(0,j_d)} \sum_{m,\ell} |\mu_{jm\ell}|^2$$

where

$$\mu_{jm\ell} = \int_{[0,1)^d} f(x) h_{jm\ell}(x) \, \mathrm{d}x$$

Characterization

$0 < p, q \leq \infty$, $\frac{1}{p} - 1 < r < \min(\frac{1}{p}, 1)$, then $f \in S^r_{pq}B([0,1)^d)$ if and only if it can be represented as

$$f = \sum_{j \in \mathbb{N}^d_{-1}} \sum_{m,\ell} \mu_{jm\ell} \, b^{\max(0,j_1)+\ldots+\max(0,j_d)} h_{jm\ell}$$

for some sequence $(\mu_{jm\ell})$ satisfying

$$\left(\sum_{j \in \mathbb{N}^d_{-1}} b^{(j_1+\ldots+j_d)(r-\frac{1}{p}+1)q} \left(\sum_{m,\ell} |\mu_{jm\ell}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} < \infty$$

Generalized Hammersley point sets

$$\mathcal{R}_n = \left\{ \left(\frac{t_n}{b} + \frac{t_{n-1}}{b^2} + \ldots + \frac{t_1}{b^n}, \frac{s_1}{b} + \frac{s_2}{b^2} + \ldots + \frac{s_n}{b^n} \right) \mid t_1, \ldots, t_n \in \{0, 1, \ldots, b-1\} \right\}$$

for any $i = 1, \ldots, n$ either $s_i = t_i$ (half of them) or $s_i = b - 1 - t_i$

$N = b^n$ points

Chen-Skriganov point sets

$$f(z) = f_0 + f_1 z + \ldots + f_{h-1} z^{h-1}$$

polynomial in $\mathbb{F}_b[z]$

$\lambda \in \mathbb{N}$, λ-th hyper-derivative

$$\partial^\lambda f(z) = \sum_{i=0}^{h-1} \binom{i}{\lambda} f_\lambda z^{i-\lambda}$$

Chen-Skriganov point sets

$b \geq 2d^2$ prime, $w \in \mathbb{N}$, $n = 2dw$

$\beta_{i,\nu} \in \mathbb{F}_b$ for $1 \leq i \leq d$ and $1 \leq \nu \leq 2d$ (distinct)

$$a_i(f) = \left(\left(\partial^{\lambda-1} f(\beta_{i,\nu}) \right)_{\lambda=1}^{w} \right)_{\nu=1}^{2d} \in \mathbb{F}_b^n$$

$$\mathcal{C}_n = \{A(f) = (a_1(f), \ldots, a_d(f)) : f \in \mathbb{F}_b[z],\ \deg(f) < n\} \subset \mathbb{F}_b^{dn}$$

Chen-Skriganov point sets

$a = (\alpha_1, \ldots, \alpha_n) \in \mathbb{F}_b^n$

$$\Phi_n(a) = \frac{\alpha_1}{b} + \ldots + \frac{\alpha_n}{b^n}$$

$A = (a_1, \ldots, a_d) \in \mathbb{F}_b^{dn}$

$$\Phi_n^d(A) = (\Phi_n(a_1), \ldots, \Phi_n(a_d))$$

$$\mathcal{CS}_n = \Phi_n^d(\mathcal{C}_n)$$

$N = b^n$ points

Property of the point sets

b-adic interval I_{jm}, with $|I_{jm}| = b^{-n}$

$$\#\left(I_{jm} \cap \mathcal{CS}_n\right) = 1$$

$$\left(\sum_{j \in \mathbb{N}_{-1}^d} b^{(j_1+\ldots+j_d)(r-\frac{1}{p}+1)q} \left(\sum_{m,\ell} |\mu_{jm\ell}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \preceq b^{n(r-1)} n^{\frac{d-1}{q}}$$

Calculation of μ_{jml}

(i) $j = (-1, \ldots, -1)$ then $\left|\mu_{jm\ell}\right| \leq c\, b^{-n}$

(ii) $j \neq (-1, \ldots, -1)$, $j_1 + \ldots + j_d \leq n$ then

$$\left|\mu_{jm\ell}\right| \leq c\, b^{-2j_1 - \ldots - 2j_d},$$

(iii) $j \neq (-1, \ldots, -1)$, $j_1 + \ldots + j_d > n$, $j_1, \ldots, j_d < n$ then

$$\left|\mu_{jm\ell}\right| \leq c\, b^{-j_1 - \ldots - j_d - n}$$

and

$$\left|\mu_{jm\ell}\right| \leq c\, b^{-2j_1 - \ldots - 2j_d}$$

for all but b^n coefficients $\mu_{jm\ell}$

(iv) $j \neq (-1, \ldots, -1)$, $j_1 \geq n$ or $\ldots$ or $j_d \geq n$ then

$$\left|\mu_{jm\ell}\right| \leq c\, b^{-2j_1 - \ldots - 2j_d},$$

Results

Integration error

$$\mathrm{err}_N(M([0,1)^d)) = \inf_{\{x_1,\ldots,x_N\}\subset[0,1)^d} \sup_{f\in M_0^1([0,1)^d)} \left| \int_{[0,1)^d} f(x)\,\mathrm{d}x - \frac{1}{N}\sum_{i=1}^{N} f(x_i) \right|.$$

Let $1 \leq p, q \leq \infty$ *and* $0 \leq r < \frac{1}{p}$*. Then for any integer* $N \geq 2$ *we have*

(i)

$$D^r_{pq}(N) \asymp N^{r-1} (\log N)^{\frac{d-1}{q}},$$

(ii)

$$err_N(S^{1-r}_{pq}B([0,1)^d)) \asymp N^{r-1} (\log N)^{\frac{(q-1)(d-1)}{q}},$$

References

- W. W. L. Chen, M. M. Skriganov, *Explicit constructions in the classical mean squares problem in irregularities of point distribution.* J. Reine Angew. Math. **545** (2002), 67–95.
- J. Dick, F. Pillichshammer, Digital nets and sequences. Discrepancy theory and quasi-Monte Carlo integration. Cambridge Univ. Press, Cambridge 2010.
- A. Hinrichs, *Discrepancy of Hammersley points in Besov spaces of dominating mixed smoothness.* Math. Nachr. 283 (2010), 478–488.
- H. Triebel, Bases in function spaces, sampling, discrepancy, numerical integration. European Math. Soc. Publishing House, Zürich, 2010.

Thank you!