Omega results for a class of arithmetic functions (On a question of A. Schinzel)

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3rd Conference on Uniform Distribution Theory,

Smolenice, June 2012

An asymmetric divisor problem with weights

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$$f_1(n) = d(1,1,2;0,0,1;n) = \sum_{m_1 m_2 m_3^2 = n} m_3$$

$$= \sum_{m \mid n} \sigma\left(\gcd\left(m, \frac{n}{m}\right)\right)$$

A class of arithmetic functions

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- More generally, consider the class $\mathfrak{C}$ of arithmetic functions $f_H(n)$ with a generating Dirichlet series

$$\sum_{n=1}^{\infty} f_H(n) n^{-s} = \zeta^2(s)\zeta(2s-1)(\zeta(2s))^M H(s) \qquad (\Re(s) > 1)\,,$$

where $M \in \mathbb{Z}$ fixed and $H(s)$ has an Euler product absolutely convergent in a half-plane $\Re(s) \geq \sigma_0$ for some $\sigma_0 < \frac{1}{2}$.

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- If n is square-free, $\quad f^*(n) = d(n) = d(1,1;0,0;n)$.
- Thus $f^*(n) - d(n)$ measures the "aberration" of n from the property to be square-free.
- The function $f^*(n)$ is multiplicative, with a generating Dirichlet series

$$\sum_{n=1}^{\infty} \frac{f^*(n)}{n^s} = \frac{\zeta^2(s)\zeta(2s-1)}{\zeta(2s)} \qquad (\Re(s) > 1)$$

Example II

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where $H_{p_2}(s)$ converges absolutely in $\Re(s) > \frac{1}{3}$:

$$H_{p_2}(s) := \prod_{p \text{ prime}} \left(1 - \frac{1}{p^{s+1}} - \frac{2}{p^{2s}} + \frac{1}{p^{2s+1}} + \frac{1}{p^{3s}}\right)(1 - p^{-2s})^{-2}$$

$$= \prod_{p \text{ prime}} \left(1 + p^{-3s} - p^{-s-1} + p^{-2s-1} - p^{-4s} - 2p^{-3s-1} + \dots\right)$$

Upper estimates

- For every arithmetic function $f_H \in \mathfrak{C}$,

$$\sum_{0<n\le x} f_H(n) = \operatorname*{Res}_{s=1}\left(\zeta^2(s)\zeta(2s-1)(\zeta(2s))^M H(s)\,\frac{x^s}{s}\right) + \mathcal{R}_H(x)$$
$$= x\,P_H(\log x) + \mathcal{R}_H(x)\,,$$

where P_H is a quadratic polynomial whose coefficients depend on H and M.

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- $\mathcal{R}_H(x) \ll x^{5/8}(\log x)^4$ by the whole toolkit of multiple exponential sums including monomials (Kolesnik, Sargos, Bhowmik & Wu).

Schinzel's question

- On the basis of the article by E. Krätzel, L. Tóth & N. entitled *"On certain arithmetic functions involving the greatest common divisor"*, Cent. Eur. J. Math. **10** (2012),
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- We are now able to give a fairly general answer, based on a method which can be traced back to work by R. Balasubramanian, K. Ramachandra & M.V. Subbarao (1988) and A. Schinzel (1991) (!).

Lower estimate for the error term

- **Theorem.** *For every arithmetic function* $f_H \in \mathfrak{C}^*$ *with a generating Dirichlet series*

$$F_H(s) = \zeta^2(s)\zeta(2s-1)(\zeta(2s))^M H(s)\,,$$

where $M \in \mathbb{Z}$ *fixed and* $H(s)$ *has an Euler product absolutely convergent in a half-plane* $\Re(s) \geq \sigma_0$ *for some* $\sigma_0 < \frac{1}{2}$, *it follows that, as* $x \to \infty$,

$$\sum_{n\leq x} f_H(n) = \operatorname*{Res}_{s=1}\left(F_H(s)\frac{x^s}{s}\right) + \Omega\left(\frac{\sqrt{x}\,(\log x)^2}{(\log\log x)^{|M+1|}}\right).$$

A very special case

The special case $M = 3$

- G. Bhowmik & J. Wu, *On the asymptotic behaviour of the number of subgroups of finite abelian groups,* Arch. Math (Basel) **69** (1997), 95-104:

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For $M = 3\,,$ $$\mathcal{R}_H(x) = \Omega\left(\sqrt{x}\,(\log x)^2\right)$$

- Observe that, for $M = 3$, $F(s)$ has a triple pole at $s = \frac{1}{2}$, hence

$$\mathcal{R}_H(x) \asymp \operatorname*{Res}_{s=1/2}\left(F(s)\frac{x^s}{s}\right) \asymp x^{1/2}(\log x)^2$$

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- Of course this is not true for general M.

Proof of the Theorem

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- By an elementary convolution argument, it suffices to consider the case of $H(s) = \mathbf{1}(s) = 1$ throughout.
- For this case, the Omega bound is an easy consequence of the following (localized) mean value estimate:
- **Proposition.** *There exist positive constants B and C_0, such that*

$$\int_T^\infty \frac{|\mathcal{R}_{\mathbf{1}}(u)|^2}{u^2} e^{-u/T^B} \,\mathrm{d}u \geq \frac{C_0 (\log T)^5}{(\log \log T)^{|2M+2|}}$$

for all T sufficiently large.

Proof of the Theorem

- Starting from Perron's formula in the shape

$$\sum_{n=1}^{\infty} \frac{f_1(n)}{n^s} e^{-n/y} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} F_1(s+w) y^w \Gamma(w) \, \mathrm{d}w \,,$$

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a suitable modification of the classic method gives

$$\int_T^{\infty} \frac{|\mathcal{R}_{\mathbf{1}}(u)|^2}{u^2} e^{-u/T^B} \, \mathrm{d}u \gg \int_{\mathcal{U}(T) \subseteq [T^{9/10}, T]} \left| \frac{F_{\mathbf{1}}(\frac{1}{2}+it)}{\frac{1}{2}+it} \right|^2 \mathrm{d}t \,.$$

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- Here the exceptional set $[T^{9/10}, T] \setminus \mathcal{U}(T)$ takes care of possible zeros of $\zeta(2s)$ close to $\Re(s) = \frac{1}{2}$.

Proof of the Theorem

- By the functional equation of the ζ-function,

$$\begin{aligned} \left|F_1\left(\tfrac{1}{2}+it\right)\right| &= \left|\zeta^2\left(\tfrac{1}{2}+it\right)\zeta(-2it)\left(\zeta\left(1+2it\right)\right)^M\right| \\ &\asymp \left|\zeta^2\left(\tfrac{1}{2}+it\right)\left(\zeta\left(1+2it\right)\right)^{M+1}\right| |t|^{1/2} \end{aligned}$$

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- Therefore,

$$\int\limits_{\mathcal{U}(T)} \left|\frac{F(\frac{1}{2}+it)}{\frac{1}{2}+it}\right|^2 \mathrm{d}t \gg (\log\log T)^{-2|M+1|} \int\limits_{\mathcal{U}(T)} \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^4 \frac{\mathrm{d}t}{t}.$$

Proof of the Theorem

- For the whole interval $[T^{9/10}, T]$ the fourth power moment is used:

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- On the exceptional set $[T^{9/10}, T] \setminus \mathcal{U}(T)$, which is of measure $O(T^{\epsilon})$, the pointwise bound

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$$\int\limits_{\mathcal{U}(T)} \left|\frac{F(\frac{1}{2}+it)}{\frac{1}{2}+it}\right|^2 \mathrm{d}t \gg \frac{(\log T)^5}{(\log\log T)^{|2M+2|}}$$

Special arithmetic functions

For the examples mentioned the result reads in particular:

- $$\sum_{m_1 m_2 m_3^2 \le x} m_3 = \operatorname*{Res}_{s=1}\left(F_{\mathbf{1}}(s)\frac{x^s}{s}\right) + \Omega\left(\frac{\sqrt{x}\,(\log x)^2}{\log\log x}\right),$$

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- $$\sum_{n\le x}\sum_{m\,|\,n} \gcd\left(m,\frac{n}{m}\right) = \operatorname*{Res}_{s=1}\left(F_{H_0}(s)\frac{x^s}{s}\right) + \Omega\left(\sqrt{x}\,(\log x)^2\right),$$

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- $$\sum_{n\le x}\frac{1}{n}\sum_{k=1}^{n} \gcd(k^2, n) = \operatorname*{Res}_{s=1}\left(F_{H^*}(s)\frac{x^s}{s}\right) + \Omega\left(\frac{\sqrt{x}\,(\log x)^2}{\log\log x}\right).$$