Recent Results for the Discrepancy of Polynomial Lattice Point Sets

Friedrich Pillichshammer[1]

Linz/Austria

Joint work with

Josef Dick, Aicke Hinrichs, Peter Kritzer, Gunther Leobacher and Wolfgang Ch. Schmid

[1]Supported by the Austrian Science Foundation (FWF), Project S9609.

Multivariate integration – QMC

We want to approximate an integral

$$I_s(f) = \int_{[0,1]^s} f(\mathbf{x}) \, \mathrm{d}\mathbf{x}$$

using a QMC rule

$$Q_{N,s}(f) = \frac{1}{N} \sum_{n=0}^{N-1} f(\mathbf{x}_n)$$

where $\mathbf{x}_0, \ldots, \mathbf{x}_{N-1} \in [0,1)^s$.

Multivariate integration – QMC

We want to approximate an integral

$$I_s(f) = \int_{[0,1]^s} f(\mathbf{x}) \,\mathrm{d}\mathbf{x}$$

using a QMC rule

$$Q_{N,s}(f) = \frac{1}{N} \sum_{n=0}^{N-1} f(\mathbf{x}_n)$$

where $\mathbf{x}_0, \ldots, \mathbf{x}_{N-1} \in [0,1)^s$.

Point sets with good distribution properties yield a small error.

Multivariate integration – QMC

Koksma-Hlawka inequality

$$|I_s(f) - Q_{N,s}(f)| \le V(f) D_N^*(\mathcal{P}),$$

where

- $V(f)$ is a measure for the variation of f;
- D_N^* is the star discrepancy of the nodes $\mathcal{P} = \{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}$.

Multivariate integration – QMC

Koksma-Hlawka inequality

$$|I_s(f) - Q_{N,s}(f)| \leq V(f) D_N^*(\mathcal{P}),$$

where

- $V(f)$ is a measure for the variation of f;
- D_N^* is the star discrepancy of the nodes $\mathcal{P} = \{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}$.

For $\mathbf{z} \in [0,1]^s$ let

$$\Delta_{\mathcal{P}}(\mathbf{z}) := \frac{|\{0 \leq n < N : \mathbf{x}_n \in [\mathbf{0}, \mathbf{z})\}|}{N} - \lambda_s([\mathbf{0}, \mathbf{z})).$$

Multivariate integration – QMC

Koksma-Hlawka inequality

$$|I_s(f) - Q_{N,s}(f)| \leq V(f) D_N^*(\mathcal{P}),$$

where

- $V(f)$ is a measure for the variation of f;
- D_N^* is the star discrepancy of the nodes $\mathcal{P} = \{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}$.

For $\mathbf{z} \in [0,1]^s$ let

$$\Delta_{\mathcal{P}}(\mathbf{z}) := \frac{|\{0 \leq n < N \, : \, \mathbf{x}_n \in [\mathbf{0}, \mathbf{z})\}|}{N} - \lambda_s([\mathbf{0}, \mathbf{z})).$$

Definition (star discrepancy)

$$D_N^*(\mathcal{P}) := \sup_{\mathbf{z} \in [0,1]^s} |\Delta_{\mathcal{P}}(\mathbf{z})|$$

Multivariate integration – QMC

Lower bound on D_N^* (Roth 1954; Schmidt 1972; Bilyk, Lacey, Vagharshakyan 2008)

For $s \in \mathbb{N}$ there exists $c_s > 0$ and $0 < \delta_s < \frac{1}{2}$ such that for all $\mathcal{P} \subseteq [0,1)^s$

$$D_N^*(\mathcal{P}) \geq c_s \frac{(\log N)^{\kappa_s}}{N} \quad \text{where} \quad N = |\mathcal{P}|$$

and where $\kappa_1 = 0$, $\kappa_2 = 1$ and $\kappa_s \geq \frac{s-1}{2} + \delta_s$ for $s > 2$.

Multivariate integration – QMC

Lower bound on D_N^* (Roth 1954; Schmidt 1972; Bilyk, Lacey, Vagharshakyan 2008)

For $s \in \mathbb{N}$ there exists $c_s > 0$ and $0 < \delta_s < \frac{1}{2}$ such that for all $\mathcal{P} \subseteq [0,1)^s$

$$D_N^*(\mathcal{P}) \geq c_s \frac{(\log N)^{\kappa_s}}{N} \quad \text{where} \quad N = |\mathcal{P}|$$

and where $\kappa_1 = 0$, $\kappa_2 = 1$ and $\kappa_s \geq \frac{s-1}{2} + \delta_s$ for $s > 2$.

Point sets with $D_N^*(\mathcal{P}) \ll_s \frac{(\log N)^{\beta_s}}{N}$ are called **low discrepancy point sets**.

Multivariate integration – QMC

Lower bound on D_N^* (Roth 1954; Schmidt 1972; Bilyk, Lacey, Vagharshakyan 2008)

For $s \in \mathbb{N}$ there exists $c_s > 0$ and $0 < \delta_s < \frac{1}{2}$ such that for all $\mathcal{P} \subseteq [0,1)^s$

$$D_N^*(\mathcal{P}) \geq c_s \frac{(\log N)^{\kappa_s}}{N} \quad \text{where} \quad N = |\mathcal{P}|$$

and where $\kappa_1 = 0$, $\kappa_2 = 1$ and $\kappa_s \geq \frac{s-1}{2} + \delta_s$ for $s > 2$.

Point sets with $D_N^*(\mathcal{P}) \ll_s \frac{(\log N)^{\beta_s}}{N}$ are called **low discrepancy point sets**.

- van der Corput-Halton sequences and Hammersley point sets;
- good lattice point sets (Korobov and Hlawka);
- (t, m, s)-nets, polynomial lattice point sets (Niederreiter).

(t, m, s)-net in base b

Definition (Niederreiter 1987)

A point set $\mathcal{P}$ consisting of b^m points in $[0,1)^s$ is called (t,m,s)-**net in base** b if every interval of the form

$$\prod_{i=1}^{s} \left[\frac{a_i}{b^{d_i}}, \frac{a_i+1}{b^{d_i}}\right) \subseteq [0,1)^s$$

of volume b^{t-m} contains exactly b^t points of $\mathcal{P}$.

(t, m, s)-net in base b

Definition (Niederreiter 1987)

A point set $\mathcal{P}$ consisting of b^m points in $[0,1)^s$ is called (t,m,s)**-net in base** b if every interval of the form

$$\prod_{i=1}^{s} \left[\frac{a_i}{b^{d_i}}, \frac{a_i+1}{b^{d_i}}\right) \subseteq [0,1)^s$$

of volume b^{t-m} contains exactly b^t points of $\mathcal{P}$.

- Any $\mathcal{P}$ consisting of b^m points in $[0,1)^s$ is a (m,m,s)-net in base b;

(t, m, s)-net in base b

Definition (Niederreiter 1987)

A point set $\mathcal{P}$ consisting of b^m points in $[0,1)^s$ is called (t, m, s)-**net in base** b if every interval of the form

$$\prod_{i=1}^{s} \left[\frac{a_i}{b^{d_i}}, \frac{a_i+1}{b^{d_i}}\right) \subseteq [0,1)^s$$

of volume b^{t-m} contains exactly b^t points of $\mathcal{P}$.

- Any $\mathcal{P}$ consisting of b^m points in $[0,1)^s$ is a (m, m, s)-net in base b;
- small **quality parameter** $0 \leq t \leq m$ implies good distribution properties;

(t, m, s)-net in base b

Definition (Niederreiter 1987)

A point set $\mathcal{P}$ consisting of b^m points in $[0,1)^s$ is called (t, m, s)-**net in base** b if every interval of the form

$$\prod_{i=1}^{s} \left[\frac{a_i}{b^{d_i}}, \frac{a_i+1}{b^{d_i}}\right) \subseteq [0,1)^s$$

of volume b^{t-m} contains exactly b^t points of $\mathcal{P}$.

- Any $\mathcal{P}$ consisting of b^m points in $[0,1)^s$ is a (m, m, s)-net in base b;
- small **quality parameter** $0 \leq t \leq m$ implies good distribution properties;
- **Niederreiter:** if $\mathcal{P}$ is a (t, m, s)-net in base b, then

$$D^*_{b^m}(\mathcal{P}) \ll_{s,b} b^t \frac{m^{s-1}}{b^m} \quad \left(\ll_{s,b} b^t \frac{(\log N)^{s-1}}{N}\right).$$

Digital (t, m, s)-net over $\mathbb{Z}_b$

Let $b \in \mathbb{P}$ and let $\mathbb{Z}_b$ be the finite field with b elements.

Digital (t, m, s)-net over $\mathbb{Z}_b$

Let $b \in \mathbb{P}$ and let $\mathbb{Z}_b$ be the finite field with b elements.

- choose $C_1, \ldots, C_s \in \mathbb{Z}_b^{m \times m}$;

Digital (t, m, s)-net over $\mathbb{Z}_b$

Let $b \in \mathbb{P}$ and let $\mathbb{Z}_b$ be the finite field with b elements.

- choose $C_1, \ldots, C_s \in \mathbb{Z}_b^{m \times m}$;
- for $n \in \{0, \ldots, b^m - 1\}$ and $i \in \{1, \ldots, s\}$:

Digital (t, m, s)-net over $\mathbb{Z}_b$

Let $b \in \mathbb{P}$ and let $\mathbb{Z}_b$ be the finite field with b elements.

- choose $C_1, \ldots, C_s \in \mathbb{Z}_b^{m \times m}$;
- for $n \in \{0, \ldots, b^m - 1\}$ and $i \in \{1, \ldots, s\}$:

$$n \mapsto \vec{n}$$

Digital (t, m, s)-net over $\mathbb{Z}_b$

Let $b \in \mathbb{P}$ and let $\mathbb{Z}_b$ be the finite field with b elements.

- choose $C_1, \ldots, C_s \in \mathbb{Z}_b^{m \times m}$;
- for $n \in \{0, \ldots, b^m - 1\}$ and $i \in \{1, \ldots, s\}$:

$$n \mapsto \vec{n} \mapsto \vec{y}_i := C_i \vec{n}$$

Digital (t, m, s)-net over $\mathbb{Z}_b$

Let $b \in \mathbb{P}$ and let $\mathbb{Z}_b$ be the finite field with b elements.

- choose $C_1, \ldots, C_s \in \mathbb{Z}_b^{m \times m}$;
- for $n \in \{0, \ldots, b^m - 1\}$ and $i \in \{1, \ldots, s\}$:

$$n \mapsto \vec{n} \mapsto \vec{y}_i := C_i \vec{n} \mapsto x_n^{(i)} := \frac{y_{i,1}}{b} + \cdots + \frac{y_{i,m}}{b^m}$$

Digital (t, m, s)-net over $\mathbb{Z}_b$

Let $b \in \mathbb{P}$ and let $\mathbb{Z}_b$ be the finite field with b elements.

- choose $C_1, \ldots, C_s \in \mathbb{Z}_b^{m \times m}$;
- for $n \in \{0, \ldots, b^m - 1\}$ and $i \in \{1, \ldots, s\}$:

$$n \mapsto \vec{n} \mapsto \vec{y}_i := C_i \vec{n} \mapsto x_n^{(i)} := \frac{y_{i,1}}{b} + \cdots + \frac{y_{i,m}}{b^m}$$

- $\mathbf{x}_n = (x_n^{(1)}, \ldots, x_n^{(s)})$.

Digital (t, m, s)-net over $\mathbb{Z}_b$

Let $b \in \mathbb{P}$ and let $\mathbb{Z}_b$ be the finite field with b elements.

- choose $C_1, \ldots, C_s \in \mathbb{Z}_b^{m \times m}$;
- for $n \in \{0, \ldots, b^m - 1\}$ and $i \in \{1, \ldots, s\}$:

$$n \mapsto \vec{n} \mapsto \vec{y}_i := C_i \vec{n} \mapsto x_n^{(i)} := \frac{y_{i,1}}{b} + \cdots + \frac{y_{i,m}}{b^m}$$

- $\mathbf{x}_n = (x_n^{(1)}, \ldots, x_n^{(s)})$.

$\mathcal{P} = \{\mathbf{x}_0, \ldots, \mathbf{x}_{b^m-1}\}$ is a (t, m, s)-net in base b for some $0 \leq t \leq m$.

Digital (t, m, s)-net over $\mathbb{Z}_b$

Let $b \in \mathbb{P}$ and let $\mathbb{Z}_b$ be the finite field with b elements.

- choose $C_1, \ldots, C_s \in \mathbb{Z}_b^{m \times m}$;
- for $n \in \{0, \ldots, b^m - 1\}$ and $i \in \{1, \ldots, s\}$:

$$n \mapsto \vec{n} \mapsto \vec{y}_i := C_i \vec{n} \mapsto x_n^{(i)} := \frac{y_{i,1}}{b} + \cdots + \frac{y_{i,m}}{b^m}$$

- $\mathbf{x}_n = (x_n^{(1)}, \ldots, x_n^{(s)})$.

$\mathcal{P} = \{\mathbf{x}_0, \ldots, \mathbf{x}_{b^m-1}\}$ is a (t, m, s)-net in base b for some $0 \leq t \leq m$.

Definition

$\mathcal{P}$ is called a **digital** (t, m, s)**-net over** $\mathbb{Z}_b$.

Digital (t, m, s)-net over $\mathbb{Z}_b$

Let $b \in \mathbb{P}$ and let $\mathbb{Z}_b$ be the finite field with b elements.

- choose $C_1, \ldots, C_s \in \mathbb{Z}_b^{m \times m}$;
- for $n \in \{0, \ldots, b^m - 1\}$ and $i \in \{1, \ldots, s\}$:

$$n \mapsto \vec{n} \mapsto \vec{y}_i := C_i \vec{n} \mapsto x_n^{(i)} := \frac{y_{i,1}}{b} + \cdots + \frac{y_{i,m}}{b^m}$$

- $\mathbf{x}_n = (x_n^{(1)}, \ldots, x_n^{(s)})$.

$\mathcal{P} = \{\mathbf{x}_0, \ldots, \mathbf{x}_{b^m - 1}\}$ is a (t, m, s)-net in base b for some $0 \leq t \leq m$.

Definition

$\mathcal{P}$ is called a **digital** (t, m, s)**-net over** $\mathbb{Z}_b$.

t depends only on the generating matrices $C_1, \ldots, C_s$.

Digital nets: an example

Let $b = 2$, $s = 2$ and $m = 4$.

Digital nets: an example

Let $b=2$, $s=2$ and $m=4$. Choose the matrices

$$C_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } C_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Digital nets: an example

We construct $\mathbf{x}_5$.

Digital nets: an example

We construct $\mathbf{x}_5$. In base 2 we have

$$5 = 1 + 0 \cdot 2^1 + 1 \cdot 2^2 + 0 \cdot 2^3.$$

Digital nets: an example

We construct $\mathbf{x}_5$. In base 2 we have

$$5 = 1 + 0 \cdot 2^1 + 1 \cdot 2^2 + 0 \cdot 2^3.$$

Now

$$C_1 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } C_2 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

Digital nets: an example

We construct $\mathbf{x}_5$. In base 2 we have

$$5 = 1 + 0 \cdot 2^1 + 1 \cdot 2^2 + 0 \cdot 2^3.$$

Now

$$C_1 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } C_2 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

Hence

$$x_5^{(1)} = \frac{1}{2} + \frac{1}{2^3} = \frac{5}{8} \text{ and } x_5^{(2)} = \frac{1}{2^2} + \frac{1}{2^4} = \frac{5}{16}$$

Digital nets: an example

We construct $\mathbf{x}_5$. In base 2 we have

$$5 = 1 + 0 \cdot 2^1 + 1 \cdot 2^2 + 0 \cdot 2^3.$$

Now

$$C_1 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } C_2 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

Hence

$$x_5^{(1)} = \frac{1}{2} + \frac{1}{2^3} = \frac{5}{8} \text{ and } x_5^{(2)} = \frac{1}{2^2} + \frac{1}{2^4} = \frac{5}{16}$$

and

$$\mathbf{x}_5 = \left(\frac{5}{8}, \frac{5}{16}\right).$$

Digital nets: an example

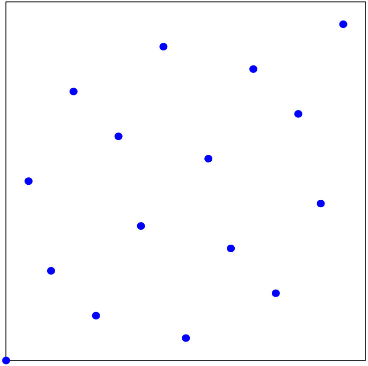


Figure: $(0, 4, 2)$-net over $\mathbb{Z}_2$ generated from C_1 and C_2.

Digital nets: an example

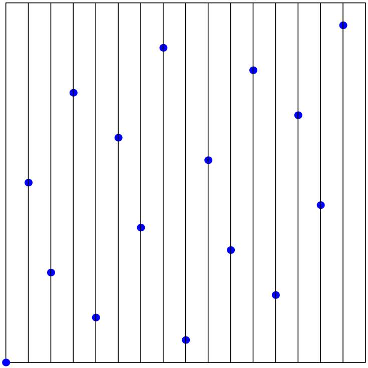


Figure: $(0,4,2)$-net over $\mathbb{Z}_2$ generated from C_1 and C_2.

Digital nets: an example

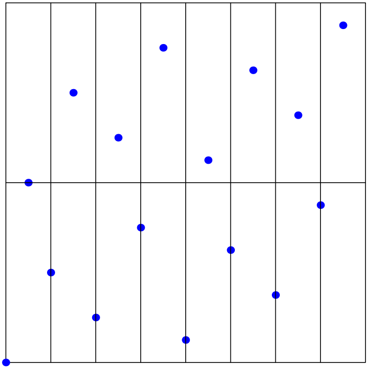


Figure: $(0, 4, 2)$-net over $\mathbb{Z}_2$ generated from C_1 and C_2.

Digital nets: an example

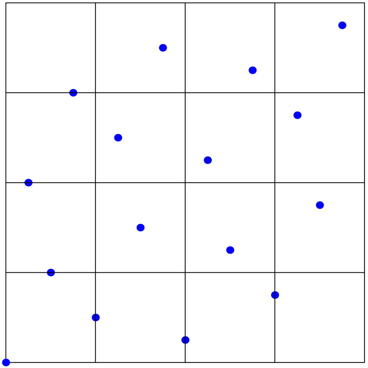


Figure: $(0, 4, 2)$-net over $\mathbb{Z}_2$ generated from C_1 and C_2.

Digital nets: an example

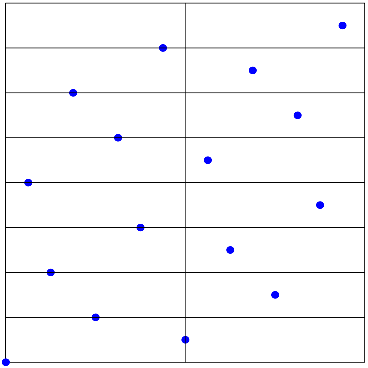


Figure: $(0, 4, 2)$-net over $\mathbb{Z}_2$ generated from C_1 and C_2.

Digital nets: an example

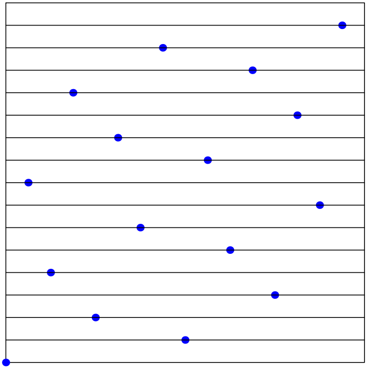


Figure: $(0, 4, 2)$-net over $\mathbb{Z}_2$ generated from C_1 and C_2.

Lattice point sets

Lattice point set (Korobov, Hlawka)

For $N \geq 2$ and $\mathbf{g} \in \mathbb{Z}^s$ let

$$\mathbf{x}_n = \left\{ \frac{n}{N} \mathbf{g} \right\} \quad \text{for} \quad 0 \leq n < N.$$

Lattice point sets

Lattice point set (Korobov, Hlawka)

For $N \geq 2$ and $\mathbf{g} \in \mathbb{Z}^s$ let

$$\mathbf{x}_n = \left\{ \frac{n}{N} \mathbf{g} \right\} \quad \text{for} \quad 0 \leq n < N.$$

Polynomial lattice point sets are in their over all structure very similar to (classical) lattice point sets.

lattice point set	**polynomial lattice point set**
based on number theoretic concepts	based on algebraic methods (polynomial arithmetic)

Polynomial lattice point sets

Let $\mathbb{Z}_b((x^{-1}))$ be the field of **formal Laurent series** over $\mathbb{Z}_b$ (b prime). Its elements are of the form

$$L = \sum_{\ell=w}^{\infty} t_\ell x^{-\ell} \text{ where } \quad w \in \mathbb{Z} \text{ and } \quad t_\ell \in \mathbb{Z}_b.$$

Polynomial lattice point sets

Let $\mathbb{Z}_b((x^{-1}))$ be the field of **formal Laurent series** over $\mathbb{Z}_b$ (b prime). Its elements are of the form

$$L = \sum_{\ell=w}^{\infty} t_\ell x^{-\ell} \text{ where } \; w \in \mathbb{Z} \text{ and } \; t_\ell \in \mathbb{Z}_b.$$

For $m \in \mathbb{N}$ let

$$\nu_m : \mathbb{Z}_b((x^{-1})) \to [0,1)$$

$$\nu_m\left(\sum_{\ell=w}^{\infty} t_\ell x^{-\ell}\right) = \sum_{\ell=\max(1,w)}^{m} t_\ell b^{-\ell}.$$

Polynomial lattice point sets

Let $\mathbb{Z}_b((x^{-1}))$ be the field of **formal Laurent series** over $\mathbb{Z}_b$ (b prime). Its elements are of the form

$$L = \sum_{\ell=w}^{\infty} t_\ell x^{-\ell} \text{ where } \; w \in \mathbb{Z} \text{ and } \; t_\ell \in \mathbb{Z}_b.$$

For $m \in \mathbb{N}$ let

$$\nu_m : \mathbb{Z}_b((x^{-1})) \to [0,1)$$

$$\nu_m\left(\sum_{\ell=w}^{\infty} t_\ell x^{-\ell}\right) = \sum_{\ell=\max(1,w)}^{m} t_\ell b^{-\ell}.$$

Let $G_{b,m} := \{h \in \mathbb{Z}_b[x] \, : \, \deg(h) < m\}$. We have $|G_{b,m}| = b^m$.

Polynomial lattice point sets

Definition (Niederreiter 1992)

For $s, m \in \mathbb{N}$, choose $p \in \mathbb{Z}_b[x]$ with $\deg(p) = m$, and $\mathbf{q} \in \mathbb{Z}_b[x]^s$.
The point set consisting of the b^m points

$$\mathbf{x}_h = \nu_m\left(\frac{h(x)}{p(x)}\mathbf{q}(x)\right) \quad \text{where } h \in G_{b,m}$$

is called a **polynomial lattice point set** $\mathcal{P}(\mathbf{q}, p)$.

Polynomial lattice point sets

Definition (Niederreiter 1992)

For $s, m \in \mathbb{N}$, choose $p \in \mathbb{Z}_b[x]$ with $\deg(p) = m$, and $\mathbf{q} \in \mathbb{Z}_b[x]^s$.
The point set consisting of the b^m points

$$\mathbf{x}_h = \nu_m\left(\frac{h(x)}{p(x)}\mathbf{q}(x)\right) \quad \text{where } h \in G_{b,m}$$

is called a **polynomial lattice point set** $\mathcal{P}(\mathbf{q}, p)$.

Dual polynomial lattice

$$\mathcal{D} = \{\mathbf{h} \in G_{b,m}^s \,:\, \mathbf{q} \cdot \mathbf{h} \equiv 0 \pmod{p}\}$$

Polynomial lattice point sets

Definition (Niederreiter 1992)

For $s, m \in \mathbb{N}$, choose $p \in \mathbb{Z}_b[x]$ with $\deg(p) = m$, and $\mathbf{q} \in \mathbb{Z}_b[x]^s$.
The point set consisting of the b^m points

$$\mathbf{x}_h = \nu_m \left(\frac{h(x)}{p(x)} \mathbf{q}(x) \right) \quad \text{where } h \in G_{b,m}$$

is called a **polynomial lattice point set** $\mathcal{P}(\mathbf{q}, p)$.

Dual polynomial lattice

$$\mathcal{D} = \{\mathbf{h} \in G_{b,m}^s \,:\, \mathbf{q} \cdot \mathbf{h} \equiv 0 \pmod{p}\}$$

$\mathcal{P}(\mathbf{q}, p)$ are the polynomial versions of **good lattice point sets**.

Polynomial lattice point sets

Theorem (Niederreiter 1992)

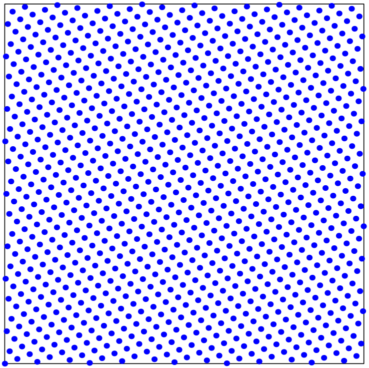
$\mathcal{P}(\mathbf{q}, p)$ is a digital (t, m, s)-net over $\mathbb{Z}_b$ with generating matrices $C_1, \ldots, C_s$, where

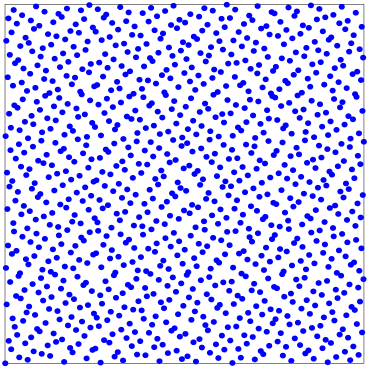
$$C_i = \begin{pmatrix} u_1^{(i)} & u_2^{(i)} & \ldots & u_m^{(i)} \\ u_2^{(i)} & u_3^{(i)} & \ldots & u_{m+1}^{(i)} \\ \vdots & \vdots & & \vdots \\ u_m^{(i)} & u_{m+1}^{(i)} & \ldots & u_{2m-1}^{(i)} \end{pmatrix} \quad \text{for} \quad \frac{q_i(x)}{p(x)} = \sum_{\ell=w_i}^{\infty} \frac{u_\ell^{(i)}}{x^\ell}$$

and

$$t = m - s + 1 - \min_{\mathbf{h} \in \mathcal{D} \setminus \{\mathbf{0}\}} \sum_{i=1}^{s} \deg(h_i).$$

(Polynomial) lattice point sets: an example





left: $N = 987$, $\mathbf{g} = (1, 610)$
right: $p(x) = x^{10} + x^8 + x^4 + x^2 + 1$, $\mathbf{q} = (1, x^9 + x^5 + x)$ in $\mathbb{Z}_2[x]$

Discrepancy estimates

Three methods for estimating the star discrepancy of $\mathcal{P}(\mathbf{q}, p)$:

Discrepancy estimates

Three methods for estimating the star discrepancy of $\mathcal{P}(\mathbf{q}, p)$:
Discrepancy estimate based on

- the quality parameter t;

Discrepancy estimates

Three methods for estimating the star discrepancy of $\mathcal{P}(\mathbf{q}, p)$:
Discrepancy estimate based on

- the quality parameter t;
- a Erdős-Turan-Koksma type inequality for Walsh functions (Niederreiter 1986, Hellekalek 1994);

Discrepancy estimates

Three methods for estimating the star discrepancy of $\mathcal{P}(\mathbf{q}, p)$:
Discrepancy estimate based on

- the quality parameter t;
- a Erdős-Turan-Koksma type inequality for Walsh functions (Niederreiter 1986, Hellekalek 1994);
- the digital construction.

Discrepancy estimate based on t

Theorem (Larcher, Lauss, Niederreiter, Schmid 1996)

Let $m \in \mathbb{N}$ large enough and let $p \in \mathbb{Z}_b[x]$ be irreducible with $\deg(p) = m$. Then there exists $\mathbf{q} \in G_{b,m}^s$ such that

$$t \leq 1 + \log_b \frac{m^{s-1}}{(s-1)!}.$$

Discrepancy estimate based on t

Theorem (Larcher, Lauss, Niederreiter, Schmid 1996)

Let $m \in \mathbb{N}$ large enough and let $p \in \mathbb{Z}_b[x]$ be irreducible with $\deg(p) = m$. Then there exists $\mathbf{q} \in G_{b,m}^s$ such that

$$t \leq 1 + \log_b \frac{m^{s-1}}{(s-1)!}.$$

Theorem (Niederreiter 1987)

If $\mathcal{P}$ is a (t, m, s)-net in base b, then

$$D_{b^m}^*(\mathcal{P}) \ll_{s,b} b^t \frac{m^{s-1}}{b^m}.$$

Discrepancy estimate based on t

Hence

$$D^*_{b^m}(\mathcal{P}(\mathbf{q},p)) \ll_{s,b} \frac{m^{2s-2}}{b^m} \qquad \left(= \frac{(\log N)^{2s-2}}{N}\right).$$

Discrepancy estimate based on t

Hence

$$D_{b^m}^*(\mathcal{P}(\mathbf{q},p)) \ll_{s,b} \frac{m^{2s-2}}{b^m} \qquad \left(= \frac{(\log N)^{2s-2}}{N}\right).$$

For dimension $s = b = 2$ and for any $m \in \mathbb{N}$ let

$$g_m(x) = \sum_{j=0}^{\lfloor \log_2 m \rfloor + 1} x^{m - \lfloor m/2^j \rfloor} \in \mathbb{Z}_2[x] \text{ and } \mathbf{g}_m = (1, g_m).$$

Discrepancy estimate based on t

Hence

$$D^*_{b^m}(\mathcal{P}(\mathbf{q},p)) \ll_{s,b} \frac{m^{2s-2}}{b^m} \quad \left(= \frac{(\log N)^{2s-2}}{N}\right).$$

For dimension $s = b = 2$ and for any $m \in \mathbb{N}$ let

$$g_m(x) = \sum_{j=0}^{\lfloor \log_2 m \rfloor + 1} x^{m - \lfloor m/2^j \rfloor} \in \mathbb{Z}_2[x] \text{ and } \mathbf{g}_m = (1, g_m).$$

Theorem (Niederreiter 1987, 1992; Larcher, P. 2002)

For any $m \in \mathbb{N}$, $\mathcal{P}(\mathbf{g}_m, x^m)$ is a digital $(0, m, 2)$-net over $\mathbb{Z}_2$ and hence

$$D^*_{2^m}(\mathcal{P}(\mathbf{g}_m, x^m)) \le \left(\frac{m}{3} + \frac{9}{19}\right)\frac{1}{2^m} \quad \left(\ll \frac{\log N}{N}\right).$$

Discrepancy estimate based on R_b

Theorem (Niederreiter 1992; Dick, Leobacher, P. 2005; Dick, Kritzer, Leobacher, P. 2007)

$$D^*_{b^m}(\mathcal{P}(\mathbf{q},p)) \le \frac{s}{b^m} + R_b(\mathbf{q},p),$$

where

$$R_b(\mathbf{q},p) = \sum_{\mathbf{h}\in\mathcal{D}\setminus\{\mathbf{0}\}} \prod_{i=1}^{s} r_b(h_i),$$

where for $h \in G_{b,m}$

$$r_b(h) := \begin{cases} 1 & \text{if } h = 0, \\ \frac{1}{b^{r+1}\sin(\pi\kappa_r/b)} & \text{if } h = \kappa_0 + \kappa_1 x + \cdots + \kappa_r x^r,\ \kappa_r \ne 0. \end{cases}$$

Discrepancy estimate based on R_b

Theorem (Niederreiter 1992; Dick, Leobacher, P. 2005; Dick, Kritzer, Leobacher, P. 2007)

For (irreducible) $p \in \mathbb{Z}_b[x]$ with $\deg(p) = m$ there exists a vector $\mathbf{q} \in G_{b,m}^s$ with

$$R_b(\mathbf{q}, p) \le \frac{1}{b^m - 1}\left(1 + m\frac{b^2 - 1}{3b}\right)^s$$

and hence

$$D^*_{b^m}(\mathcal{P}(\mathbf{q}, p)) \ll_{s,b} \frac{m^s}{b^m} \quad \left(= \frac{(\log N)^s}{N}\right).$$

Such a $\mathbf{q}$ can be constructed with a component-by-component algorithm.

Discrepancy estimate based on R_b

Theorem (Niederreiter 1992; Dick, Leobacher, P. 2005; Dick, Kritzer, Leobacher, P. 2007)

For (irreducible) $p \in \mathbb{Z}_b[x]$ with $\deg(p) = m$ there exists a vector $\mathbf{q} \in G_{b,m}^s$ with

$$R_b(\mathbf{q}, p) \leq \frac{1}{b^m - 1}\left(1 + m\frac{b^2-1}{3b}\right)^s$$

and hence

$$D_{b^m}^*(\mathcal{P}(\mathbf{q}, p)) \ll_{s,b} \frac{m^s}{b^m} \qquad \left(= \frac{(\log N)^s}{N}\right).$$

Such a $\mathbf{q}$ can be constructed with a component-by-component algorithm.

The fast cbc-approach requires only $O(smb^m)$ operations (Nuyens and Cools 2006).

Discrepancy estimate based on R_b

Theorem (Kritzer, P. 2010)

For any $p \in \mathbb{Z}_b[x]$ with $\deg(p) = m$ and any $\mathbf{q} \in (G_{b,m} \setminus \{0\})^s$ we have

$$R_b(\mathbf{q}, p) \gg_{s,b} b^{\deg(\delta_s)} \frac{(m - \deg(\delta_s))^s}{b^m},$$

where $\delta_s := \gcd(q_1, \ldots, q_s, p)$.

Discrepancy estimate based on R_b

Theorem (Kritzer, P. 2010)

For any $p \in \mathbb{Z}_b[x]$ with $\deg(p) = m$ and any $\mathbf{q} \in (G_{b,m} \setminus \{0\})^s$ we have

$$R_b(\mathbf{q}, p) \gg_{s,b} b^{\deg(\delta_s)} \frac{(m - \deg(\delta_s))^s}{b^m},$$

where $\delta_s := \gcd(q_1, \ldots, q_s, p)$.

Hence

$$R_b(\mathbf{q}, p) \asymp_{s,b} \frac{m^s}{b^m} \quad \left(= \frac{(\log N)^s}{N} \right).$$

Discrepancy estimate based on the digital construction

Theorem (Larcher 1993; Kritzer, P. 2011)

For any $p \in \mathbb{Z}_b[x]$ with the property

- $p(x) = x^m$; or
- $\gcd(p, x) = 1$ and $\deg(p) = m$

there exists a generating vector $\mathbf{q} \in G_{b,m}^s$ such that

$$D_{b^m}^*(\mathcal{P}(\mathbf{q}, p)) \ll_{s,b} \frac{m^{s-1} \log m}{b^m} \qquad \left(= \frac{(\log N)^{s-1} \log \log N}{N}\right).$$

Discrepancy estimate based on the digital construction

Sketch of Proof (rather technical): Let $C_1, \ldots, C_s$ be the generating matrices of $\mathcal{P}(\mathbf{q}, p)$.

Discrepancy estimate based on the digital construction

Sketch of Proof (rather technical): Let $C_1, \ldots, C_s$ be the generating matrices of $\mathcal{P}(\mathbf{q}, p)$.

- Show that

$$b^m D^*_{b^m}(\mathcal{P}(\mathbf{q}, p)) \ll_{s,b} \sum_{n=0}^{m-1} \sum_{w=0}^{s-1} \sum_{\substack{\mathbf{p} \in \mathbb{N}^w \\ +\text{conditions depending on} \\ C_1,\ldots,C_w}} \frac{b^n}{b^{|\mathbf{p}|+h(w+1)}}$$

  where $h(w+1)$ depends on $\mathbf{p} \in \mathbb{N}^w$ and on C_{w+1}.

Discrepancy estimate based on the digital construction

Sketch of Proof (rather technical): Let $C_1, \ldots, C_s$ be the generating matrices of $\mathcal{P}(\mathbf{q}, p)$.

- Show that

$$b^m D^*_{b^m}(\mathcal{P}(\mathbf{q}, p)) \ll_{s,b} \sum_{n=0}^{m-1} \sum_{w=0}^{s-1} \sum_{\substack{\mathbf{p} \in \mathbb{N}^w \\ +\text{conditions depending on} \\ C_1, \ldots, C_w}} \frac{b^n}{b^{|\mathbf{p}|+h(w+1)}}$$

  where $h(w+1)$ depends on $\mathbf{p} \in \mathbb{N}^w$ and on C_{w+1}.
- Define a set H of $\mathbf{q}$ such that $h(w+1)$ is "large" for all w.

Discrepancy estimate based on the digital construction

Sketch of Proof (rather technical): Let $C_1, \ldots, C_s$ be the generating matrices of $\mathcal{P}(\mathbf{q}, p)$.

- Show that

$$b^m D^*_{b^m}(\mathcal{P}(\mathbf{q}, p)) \ll_{s,b} \sum_{n=0}^{m-1} \sum_{w=0}^{s-1} \sum_{\substack{\mathbf{p} \in \mathbb{N}^w \\ +\text{conditions depending on} \\ C_1,\ldots,C_w}} \frac{b^n}{b^{|\mathbf{p}|+h(w+1)}}$$

  where $h(w+1)$ depends on $\mathbf{p} \in \mathbb{N}^w$ and on C_{w+1}.
- Define a set H of $\mathbf{q}$ such that $h(w+1)$ is "large" for all w.
- Show that H contains enough polynomials; $|H| \geq b^{s(m-1)}/2$.

Discrepancy estimate based on the digital construction

Sketch of Proof (rather technical): Let $C_1, \ldots, C_s$ be the generating matrices of $\mathcal{P}(\mathbf{q}, p)$.

- Show that

$$b^m D^*_{b^m}(\mathcal{P}(\mathbf{q}, p)) \ll_{s,b} \sum_{n=0}^{m-1} \sum_{w=0}^{s-1} \sum_{\substack{\mathbf{p} \in \mathbb{N}^w \\ +\text{conditions depending on} \\ C_1, \ldots, C_w}} \frac{b^n}{b^{|\mathbf{p}|+h(w+1)}}$$

  where $h(w+1)$ depends on $\mathbf{p} \in \mathbb{N}^w$ and on C_{w+1}.
- Define a set H of $\mathbf{q}$ such that $h(w+1)$ is "large" for all w.
- Show that H contains enough polynomials; $|H| \geq b^{s(m-1)}/2$.
- Average the discrepancy bound over all $\mathbf{q} \in H$.

Summary of the three methods:

Method 1: based on t yields optimal discrepancy bound and a construction for $s = 2$, but not for $s > 2$;

Summary of the three methods:

Method 1: based on t yields optimal discrepancy bound and a construction for $s = 2$, but not for $s > 2$;

Method 2: based on R_b yields a good discrepancy bound for any $s \geq 2$ together with a reasonable fast construction;

Summary of the three methods:

Method 1: based on t yields optimal discrepancy bound and a construction for $s = 2$, but not for $s > 2$;

Method 2: based on R_b yields a good discrepancy bound for any $s \geq 2$ together with a reasonable fast construction;

Method 3: based on $C_1, \ldots, C_s$ yields the currently best discrepancy bounds for $s \geq 3$, but no construction.

Open question

Open question

For $s \geq 3$, do there exist $\mathcal{P}(\mathbf{q}, p)$ such that

$$D^*_{b^m}(\mathcal{P}(\mathbf{q}, p)) \ll_{s,b} \frac{m^{s-1}}{b^m} \qquad \left(= \frac{(\log N)^{s-1}}{N}\right) ?$$

Open question

Open question

For $s \geq 3$, do there exist $\mathcal{P}(\mathbf{q}, p)$ such that

$$D^*_{b^m}(\mathcal{P}(\mathbf{q}, p)) \ll_{s,b} \frac{m^{s-1}}{b^m} \qquad \left(= \frac{(\log N)^{s-1}}{N}\right)?$$

Computer experiments ($s = 3$): $\mathbf{g}_m = (1, g_m)$ can be extended to $\mathbf{g}'_m = (1, g_m, f_m)$ such that $\mathcal{P}(\mathbf{g}'_m, x^m)$ is a digital $(2, m, 3)$-net over $\mathbb{Z}_2$. This would mean that

$$D^*_{2^m}(\mathcal{P}(\mathbf{g}'_m, x^m)) \leq \frac{m^2 - 2m + 9}{2^m} \qquad \left(\ll \frac{(\log N)^2}{N}\right).$$

The weighted star discrepancy

Let $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots) \in \mathbb{R}_+^\infty$ (weights). For $\mathfrak{u} \subseteq \{1, \ldots, s\}$ let $\gamma_{\mathfrak{u}} = \prod_{i \in \mathfrak{u}} \gamma_i$.

The weighted star discrepancy

Let $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots) \in \mathbb{R}_+^\infty$ (weights). For $\mathfrak{u} \subseteq \{1, \ldots, s\}$ let $\gamma_{\mathfrak{u}} = \prod_{i \in \mathfrak{u}} \gamma_i$.

Weighted star discrepancy (Sloan and Woźniakowski 1998)

The **weighted star discrepancy** is given by

$$D_{N,\boldsymbol{\gamma}}^*(\mathcal{P}) = \sup_{\mathbf{z} \in [0,1]^s} \max_{\emptyset \neq \mathfrak{u} \subseteq \{1,\ldots,s\}} \gamma_{\mathfrak{u}} |\Delta_{\mathcal{P}_{\mathfrak{u}}}(\mathbf{z})|.$$

The weighted star discrepancy

Let $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots) \in \mathbb{R}_+^{\infty}$ (weights). For $\mathfrak{u} \subseteq \{1, \ldots, s\}$ let $\gamma_{\mathfrak{u}} = \prod_{i \in \mathfrak{u}} \gamma_i$.

Weighted star discrepancy (Sloan and Woźniakowski 1998)

The **weighted star discrepancy** is given by

$$D_{N,\boldsymbol{\gamma}}^*(\mathcal{P}) = \sup_{\mathbf{z} \in [0,1]^s} \max_{\emptyset \neq \mathfrak{u} \subseteq \{1,\ldots,s\}} \gamma_{\mathfrak{u}} |\Delta_{\mathcal{P}_{\mathfrak{u}}}(\mathbf{z})|.$$

Weighted Koksma-Hlawka inequality

$$|I_s(f) - Q_{N,s}(f)| \leq D_{N,\boldsymbol{\gamma}}^*(\mathcal{P}) \|f\|_{s,\boldsymbol{\gamma}}.$$

Tractability

Discrepancy bounds of the form

$$D_N^*, D_{N,\gamma}^* \ll_s \frac{(\log N)^s}{N} \ll_{s,\varepsilon} \frac{1}{N^{1-\varepsilon}}.$$

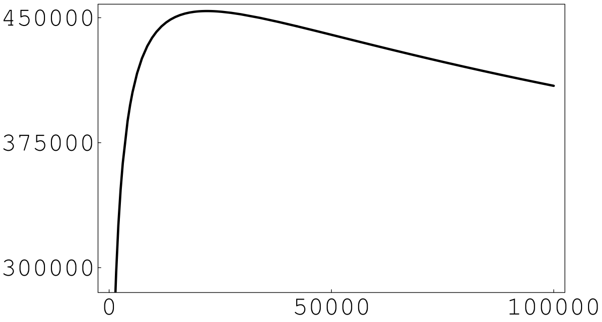


Figure: The function $N \mapsto (\log N)^s/N$ is increasing for $N \leq \mathrm{e}^s$ (here $s = 10$).

Tractability

For $s \in \mathbb{N}$ and $\varepsilon > 0$ define

$$N_\gamma(s,\varepsilon) = \min\{N \in \mathbb{N} \,:\, \exists \mathcal{P} \subseteq [0,1)^s, |\mathcal{P}| = N \text{ and } D^*_{N,\gamma}(\mathcal{P}) \le \varepsilon\}.$$

Tractability

For $s \in \mathbb{N}$ and $\varepsilon > 0$ define

$$N_\gamma(s,\varepsilon) = \min\{N \in \mathbb{N} : \exists \mathcal{P} \subseteq [0,1)^s, |\mathcal{P}| = N \text{ and } D^*_{N,\gamma}(\mathcal{P}) \le \varepsilon\}.$$

Tractability

The weighted star discrepancy is called **polynomially tractable**, if $\exists C, \alpha, \beta > 0$ such that

$$N_\gamma(s,\varepsilon) \le C s^\alpha \varepsilon^{-\beta} \quad \forall s \in \mathbb{N}, \quad \forall \varepsilon \in (0,1).$$

If $\alpha = 0$, then $D^*_{N,\gamma}$ is said to be **strongly tractable**. The infimum of β for which $\exists C > 0$ such that $N_\gamma(s,\varepsilon) \le C\varepsilon^{-\beta}$ is called the ε-**exponent** of strong tractability.

The weighted star discrepancy

Theorem (Dick, Leobacher, P. 2005 and Dick, Leobacher, Kritzer, P. 2007)

$$D^*_{b^m,\boldsymbol{\gamma}}(\mathcal{P}(\mathbf{q},p)) \leq \sum_{\emptyset \neq \mathfrak{u} \subseteq \{1,\ldots,s\}} \gamma_{\mathfrak{u}} \left(1 - \left(1 - \frac{1}{b^m}\right)^{|\mathfrak{u}|}\right) + R_{b,\boldsymbol{\gamma}}(\mathbf{q},p),$$

where

$$R_{b,\boldsymbol{\gamma}}(\mathbf{q},p) = \sum_{\mathbf{h} \in \mathcal{D} \setminus \{\mathbf{0}\}} \prod_{i=1}^{s} r_b(h_i,\gamma_i).$$

Here for $h \in G_{b,m}$ we put

$$r_b(h,\gamma) = \begin{cases} 1+\gamma & \text{if } h = 0, \\ \gamma r_b(h) & \text{if } h \neq 0. \end{cases}$$

$R_{b,\boldsymbol{\gamma}}(\mathbf{q},p)$ can be computed in $O(b^m s)$ operations.

The weighted star discrepancy – CBC-construction

CBC-algorithm

Given a prime b, $s, m \in \mathbb{N}$, $p \in \mathbb{Z}_b[x]$, with $\deg(p) = m$, and weights $\gamma = (\gamma_i)_{i \geq 1}$.

The weighted star discrepancy – CBC-construction

CBC-algorithm

Given a prime b, $s, m \in \mathbb{N}$, $p \in \mathbb{Z}_b[x]$, with $\deg(p) = m$, and weights $\gamma = (\gamma_i)_{i \geq 1}$.

1. Choose $q_1 = 1$.

The weighted star discrepancy – CBC-construction

CBC-algorithm

Given a prime b, $s, m \in \mathbb{N}$, $p \in \mathbb{Z}_b[x]$, with $\deg(p) = m$, and weights $\gamma = (\gamma_i)_{i \geq 1}$.

1. Choose $q_1 = 1$.
2. For $d > 1$, assume we have already constructed $q_1, \ldots, q_{d-1} \in G_{b,m}^*$. Then find $q_d \in G_{b,m}^*$ which minimises the quantity

$$R_{b,\gamma}((q_1, \ldots, q_{d-1}, q_d), p)$$

   as a function of q_d.

The weighted star discrepancy – CBC-construction

CBC-algorithm

Given a prime b, $s, m \in \mathbb{N}$, $p \in \mathbb{Z}_b[x]$, with $\deg(p) = m$, and weights $\gamma = (\gamma_i)_{i \geq 1}$.

1. Choose $q_1 = 1$.
2. For $d > 1$, assume we have already constructed $q_1, \ldots, q_{d-1} \in G^*_{b,m}$. Then find $q_d \in G^*_{b,m}$ which minimises the quantity

$$R_{b,\gamma}((q_1, \ldots, q_{d-1}, q_d), p)$$

as a function of q_d.

- The cost for the CBC-algorithm is of $O(b^{2m} s^2)$ operations.

The weighted star discrepancy – CBC-construction

CBC-algorithm

Given a prime b, $s, m \in \mathbb{N}$, $p \in \mathbb{Z}_b[x]$, with $\deg(p) = m$, and weights $\gamma = (\gamma_i)_{i \geq 1}$.

1. Choose $q_1 = 1$.
2. For $d > 1$, assume we have already constructed $q_1, \ldots, q_{d-1} \in G_{b,m}^*$. Then find $q_d \in G_{b,m}^*$ which minimises the quantity

$$R_{b,\gamma}((q_1, \ldots, q_{d-1}, q_d), p)$$

as a function of q_d.

- The cost for the CBC-algorithm is of $O(b^{2m}s^2)$ operations.
- Nuyens and Cools (2006) introduced the Fast CBC-construction with a significant reduction of cost to $O(smb^m)$ operations with $O(b^m)$ memory space.

The weighted star discrepancy – CBC-construction

Theorem (Dick, Leobacher, P. 2005; Dick, Kritzer, Leobacher, P. 2007)

If $\mathbf{q} \in (G_{b,m}^*)^s$ is constructed with the CBC-algorithm. Then

$$R_{b,\gamma}(\mathbf{q}, p) \leq \frac{1}{b^m - 1} \prod_{i=1}^{s} \left(1 + \gamma_i \left(1 + m \frac{b^2 - 1}{3b}\right)\right),$$

The weighted star discrepancy – CBC-construction

Theorem (Dick, Leobacher, P. 2005; Dick, Kritzer, Leobacher, P. 2007)

If $\mathbf{q} \in (G_{b,m}^*)^s$ is constructed with the CBC-algorithm. Then

$$R_{b,\gamma}(\mathbf{q}, p) \leq \frac{1}{b^m - 1} \prod_{i=1}^{s} \left(1 + \gamma_i \left(1 + m\frac{b^2 - 1}{3b}\right)\right),$$

Corollary

If $\sum_{i=0}^{\infty} \gamma_i < \infty$, then for any $\delta > 0$ there exists $c_{\gamma,\delta} > 0$, such that

$$D_{b^m,\gamma}^*(\mathcal{P}(\mathbf{q}, p)) \leq \frac{c_{\gamma,\delta}}{b^{m(1-\delta)}}.$$

The weighted star discrepancy – strong tractability

- Let $N = 2^{m_1} + \cdots + 2^{m_k}$, where $0 \leq m_1 < m_2 < \ldots < m_k$.

The weighted star discrepancy – strong tractability

- Let $N = 2^{m_1} + \cdots + 2^{m_k}$, where $0 \leq m_1 < m_2 < \ldots < m_k$.
- For $1 \leq j \leq k$ choose $p^{(j)} \in \mathbb{Z}_2[x]$ with $\deg(p^{(j)}) = m_j$ and construct $\mathbf{q}^{(j)} \in G_{2,m_j}^s$ with the CBC-algorithm.

The weighted star discrepancy – strong tractability

- Let $N = 2^{m_1} + \cdots + 2^{m_k}$, where $0 \leq m_1 < m_2 < \ldots < m_k$.
- For $1 \leq j \leq k$ choose $p^{(j)} \in \mathbb{Z}_2[x]$ with $\deg(p^{(j)}) = m_j$ and construct $\mathbf{q}^{(j)} \in G_{2,m_j}^s$ with the CBC-algorithm.
- Set $\mathcal{P}_N = \mathcal{P}(\mathbf{q}^{(1)}, p^{(1)}) \cup \ldots \cup \mathcal{P}(\mathbf{q}^{(k)}, p^{(k)})$.

The weighted star discrepancy – strong tractability

- Let $N = 2^{m_1} + \cdots + 2^{m_k}$, where $0 \leq m_1 < m_2 < \ldots < m_k$.
- For $1 \leq j \leq k$ choose $p^{(j)} \in \mathbb{Z}_2[x]$ with $\deg(p^{(j)}) = m_j$ and construct $\mathbf{q}^{(j)} \in G_{2,m_j}^s$ with the CBC-algorithm.
- Set $\mathcal{P}_N = \mathcal{P}(\mathbf{q}^{(1)}, p^{(1)}) \cup \ldots \cup \mathcal{P}(\mathbf{q}^{(k)}, p^{(k)})$.

Theorem (Hinrichs, P., Schmid 2008)

If $\sum_{i=0}^{\infty} \gamma_i < \infty$, then for any $\delta > 0$ there exists $C_{\gamma,\delta} > 0$, such that

$$D_{N,\gamma}^*(\mathcal{P}_N) \leq \frac{C_{\gamma,\delta}}{N^{1-\delta}}.$$

The weighted star discrepancy – strong tractability

- Let $N = 2^{m_1} + \cdots + 2^{m_k}$, where $0 \leq m_1 < m_2 < \ldots < m_k$.
- For $1 \leq j \leq k$ choose $p^{(j)} \in \mathbb{Z}_2[x]$ with $\deg(p^{(j)}) = m_j$ and construct $\mathbf{q}^{(j)} \in G_{2,m_j}^s$ with the CBC-algorithm.
- Set $\mathcal{P}_N = \mathcal{P}(\mathbf{q}^{(1)}, p^{(1)}) \cup \ldots \cup \mathcal{P}(\mathbf{q}^{(k)}, p^{(k)})$.

Theorem (Hinrichs, P., Schmid 2008)

If $\sum_{i=0}^{\infty} \gamma_i < \infty$, then for any $\delta > 0$ there exists $C_{\gamma,\delta} > 0$, such that

$$D_{N,\gamma}^*(\mathcal{P}_N) \leq \frac{C_{\gamma,\delta}}{N^{1-\delta}}.$$

The weighted star discrepancy is **strongly tractable** with ε-exponent equal to one.