Some unsolved problems in the theory of distribution functions

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Definitions

x_n, $n = 1, 2, \dots, x_n \in [0,1)$.

$\frac{\#\{n \le N_k; x_n \in [0,x)\}}{N_k} \to g(x)$ as $k \to \infty$ is a distribution function (d.f.) of x_n.

$G(x_n)$ is the set of all d.f.s of x_n.

$G(x_n)$ is non-empty, closed and connected in the metric

$d(g_1, g_2) = \sqrt{\int_0^1 (g_1(x) - g_2(x)^2 \mathrm{d}x}$.

Given a non–empty set H of distribution functions, there exists a sequence x_n in $[0,1)$ such that $G(x_n) = H$ if and only if H is closed and connected.

Example. For $x_n = \log n \bmod 1$ we have

$$G(x_n) = \left\{ g_u(x) = \frac{e^{\min(x,u)} - 1}{e^u} + \frac{1}{e^u}\frac{e^x - 1}{e - 1}; u \in [0,1] \right\}.$$

Method

Let $f : [0,1] \to [0,1]$ be given Jordan measurable function. To compute $g(x)$ we use a map
$x_n \to f(x_n)$ which induces
$g(x) \to g_f(x)$, where

$$g_f(x) = \int_{f^{-1}([0,x))} 1.\mathrm{d}g(x).$$

Also, if $f^{-1}([0,x))$ is a sum of finitely many pairwise disjoint subintervals $I_i(x)$ of $[0,1]$ with endpoints $\alpha_i(x) \leq \beta_i(x)$, then $g_f(x) = \sum_i g(\beta_i(x)) - g(\alpha_i(x))$.

Method

In order to find d.f. $g(x)$ of x_n by using $g_f(x)$ of the transforming sequence $f(x_n)$, we can solve e.g.
- $g_f(x) = g_h(x)$ for $x \in [0,1]$ if $f(x_n)$ and $h(x_n)$ coincide.
- $g(x) = g_f(x)$ for $x \in [0,1]$, if x_n and $f(x_n)$, $n = 1, 2, \dots$ have the same a.d.f.

Note that the d.f. $g_f(x)$ is characterized by: For every continuous $F : [0,1] \to \mathbb{R}$ and every d.f. $g(x)$ we have

$$\int_0^1 F(f(x))\mathrm{d}g(x) = \int_0^1 F(x)\mathrm{d}g_f(x).$$

Functional equations

In this note we consider:

$$
\begin{aligned}
& g(x/2) + g((x+1)/2) - g(1/2) \\
& = g(x/3) + g((x+1)/3) + g((x+2)/3) - g(1/3) - g(2/3), \qquad (1)
\end{aligned}
$$

$$
g(x) = \sum_{n=1}^{\infty} g\left(\frac{1}{n}\right) - g\left(\frac{1}{n+x}\right), \qquad (2)
$$

$$
x = \sum_{i=0}^{\infty} \left(g\left(\frac{1}{b^i}\right) - g\left(\frac{1}{b^{i+x}}\right) \right). \qquad (3)
$$

(1) is connected with the sequence $x_n = \xi(3/2)^n \bmod 1$, $n = 1, 2, \ldots$
(2) is connected with Gauss-Kuzmin theorem.
(3) is connected with Benford law.
In all cases it is **not known** the general solution.

Equation (1)

Simple equations

$$\left\{3\xi\left(\frac{3}{2}\right)^n\right\} = \left\{2\xi\left(\frac{3}{2}\right)^{n+1}\right\} = \left\{3\left\{\xi\left(\frac{3}{2}\right)^n\right\}\right\} = \left\{2\left\{\xi\left(\frac{3}{2}\right)^{n+1}\right\}\right\}$$

Implies:

(I) Any distribution function $g(x)$ of $\xi(3/2)^n \bmod 1$ satisfies $g_f(x) = g_h(x)$ for all $x \in [0,1]$, where $f(x) = 2x \bmod 1$ and $h(x) = 3x \bmod 1$. Here

$g_f(x) = g(x/2) + g((x+1)/2) - g(1/2)$,

$g_h(x) = g(x/3) + g((x+1)/3) + g((x+2)/3) - g(1/3) - g(2/3)$

Motivation

(II) The study of d.f.s in (I) is motivated by K. Mahler's conjecture (1968):
There exists no $\xi \in \mathbb{R}^+$ such that $0 \leq \{\xi(3/2)^n\} < 1/2$ for $n = 0, 1, 2, \ldots$.

(III) Mahler conjecture follows from the conjecture:
If $g(x) =$ constant for all $x \in I$, where I is a subinterval of $[0, 1]$, then the length $|I| < 1/2$.

Chain of solutions

(IV) Let g_1 be an absolutely continuous d.f. satisfying $g_{1_h}(x) = g_{1_f}(x)$ for $x \in [0,1]$. Then the absolutely continuous d.f. $g(x)$ satisfies $g_f(x) = g_1(x)$ and $g_h(x) = g_1(x)$ for $x \in [0,1]$ if and only if $g(x) =$

$$\begin{cases} \Psi(x), & \text{for } x \in [0,1/6], \\ \Psi(1/6) + \Phi(x-1/6), & \text{for } x \in [1/6,2/6], \\ \Psi(1/6) + \Phi(1/6) + g_1(1/3) - \Psi(x-2/6) & \\ +\Phi(x-2/6) - g_1(2x-1/3) + g_1(3x-1), & \text{for } x \in [2/6,3/6], \\ 2\Phi(1/6) + g_1(1/3) - g_1(2/3) + g_1(1/2) & \\ -\Psi(x-3/6) + g_1(2x-1), & \text{for } x \in [3/6,4/6], \\ -\Psi(1/6) + 2\Phi(1/6) + g_1(1/3) - g_1(2/3) + g_1(1/2) & \\ -\Phi(x-4/6) + g_1(2x-1), & \text{for } x \in [4/6,5/6], \\ -\Psi(1/6) + \Phi(1/6) + g_1(1/3) + \Psi(x-5/6) & \\ -\Phi(x-5/6) - g_1(2x-5/3) + g_1(3x-2), & \text{for } x \in [5/6,1], \end{cases}$$

Cont.

where $\Psi(x) = \int_0^x \psi(t)\mathrm{d}t$, $\Phi(x) = \int_0^x \phi(t)\mathrm{d}t$, for $x \in [0, 1/6]$, and $\psi(t)$, $\phi(t)$ are Lebesgue integrable functions on $[0, 1/6]$ satisfying
$0 \le \psi(t) \le 2g_1'(2t)$, $0 \le \phi(t) \le 2g_1'(2t + 1/3)$,
$2g_1'(2t) - 3g_1'(3t + 1/2) \le \psi(t) - \phi(t) \le -2g_1'(2t + 1/3) + 3g_1'(3t)$,
for almost all $t \in [0, 1/6]$.

Putting $g_1(x) = x$ in (IV) we find a new solution $g_2(x)$ of $g_f(x) = g_h(x)$

$$g_2(x) = \begin{cases} 0 & \text{for } x \in [0, 2/6], \\ x - 1/3 & \text{for } x \in [2/6, 3/6], \\ 2x - 5/6 & \text{for } x \in [3/6, 5/6], \\ x & \text{for } x \in [5/6, 1]. \end{cases}$$

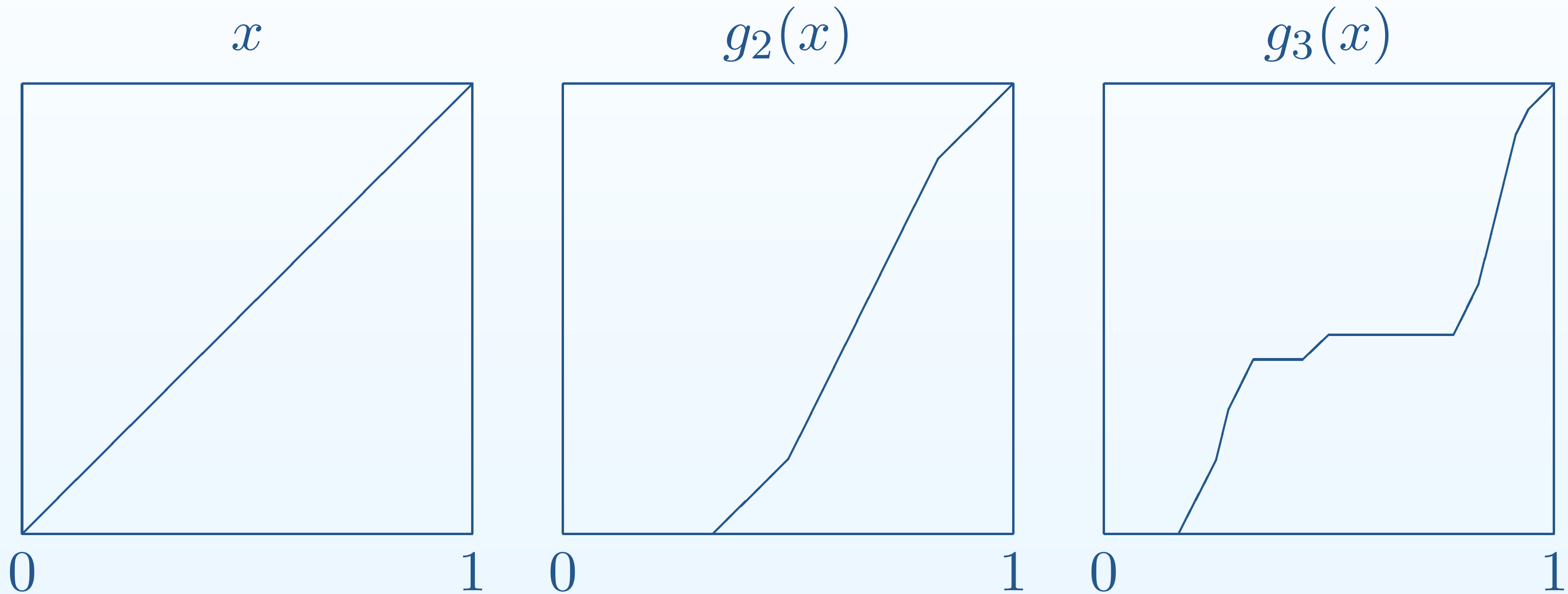
Example

Putting in (IV) $g_1(x) = g_2(x)$ we find the solution $g_3(x)$ of $g_f(x) = g_h(x)$:

$$g_3(x) = \begin{cases} 0 & \text{for } x \in [0, 1/6], \\ 2x - 1/3 & \text{for } x \in [1/6, 3/12], \\ 4x - 5/6 & \text{for } x \in [3/12, 5/18], \\ 2x - 5/18 & \text{for } x \in [5/18, 2/6], \\ 7/18 & \text{for } x \in [2/6, 8/18], \\ x - 1/18 & \text{for } x \in [8/18, 3/6], \\ 8/18 & \text{for } x \in [3/6, 7/9], \\ 2x - 20/18 & \text{for } x \in [7/9, 5/6], \\ 4x - 50/18 & \text{for } x \in [5/6, 11/12], \\ 2x - 17/18 & \text{for } x \in [11/12, 17/18], \\ x & \text{for } x \in [17/18, 1]. \end{cases}$$

Graphs

Their graphs are



Sets of uniqueness

(V) Let g_1, g_2 be any two distribution functions satisfying $g_{i_f}(x) = g_{i_h}(x)$ for $i = 1, 2$ and $x \in [0, 1]$. Denote

$$I_1 = [0, 1/3], \quad I_2 = [1/3, 2/3], \quad I_3 = [2/3, 1].$$

If $g_1(x) = g_2(x)$ for $x \in I_i \cup I_j$, $1 \leq i \neq j \leq 3$, then $g_1(x) = g_2(x)$ for all $x \in [0, 1]$.
(IV) Denote

$$F(x, y) = |\{2x\} - \{3y\}| + |\{2y\} - \{3x\}| - |\{2x\} - \{2y\}| - |\{3x\} - \{3y\}|.$$

The continuous distribution function $g(x)$ satisfies $g_f(x) = g_h(x)$ on $[0, 1]$ if and only if

$$\int_0^1 \int_0^1 F(x, y) \mathrm{d}g(x) \mathrm{d}g(y) = 0.$$

New equation

$$\left\{3\left(\frac{3}{2}\right)^{n}\right\} = \left\{2\left(\frac{3}{2}\right)^{n+1}\right\} = \left\{\frac{9}{2}\left(\frac{3}{2}\right)^{n-1}\right\}$$

which gives

$$\left\{3\left\{\left(\frac{3}{2}\right)^{n}\right\}\right\} = \left\{2\left\{\left(\frac{3}{2}\right)^{n+1}\right\}\right\} = \left\{\frac{9}{2}\left\{\left(\frac{3}{2}\right)^{n-1}\right\} + \frac{9}{2}\left[\left(\frac{3}{2}\right)^{n-1}\right]\right\},$$

where

$$\left\{\frac{9}{2}\left(\frac{3}{2}\right)^{n-1}\right\} = \begin{cases} \left\{\frac{9}{2}\left\{\left(\frac{3}{2}\right)^{n-1}\right\}\right\} & \text{if } \left[\left(\frac{3}{2}\right)^{n-1}\right] \text{ is even,} \\ \left\{\frac{9}{2}\left\{\left(\frac{3}{2}\right)^{n-1}\right\} + \frac{1}{2}\right\} & \text{if } \left[\left(\frac{3}{2}\right)^{n-1}\right] \text{ is odd.} \end{cases}$$

Functional equation

Put $f(x) = 2x \bmod 1$, $h(x) = 3x \bmod 1$, $v(x) = \frac{9}{2}x \bmod 1$, $u(x) = \frac{9}{2}x + \frac{1}{2} \bmod 1$.
For some sequence $N = N_k$, $k \to \infty$, we have

$$\frac{\#\{n \le N; \{(3/2)^n\} \in [0,x)\}}{N} \to g(x),$$
$$\frac{\#\{n \le N; \{(3/2)^n\} \in [0,x), [(3/2)^n] \text{ is odd}\}}{N} \to g^{(1)}(x)$$
$$\frac{\#\{n \le N; \{(3/2)^n\} \in [0,x), [(3/2)^n] \text{ is even}\}}{N} \to g^{(2)}(x)$$

Every d.f. $g(x)$ of the sequence $(3/2)^n \bmod 1$ can be expressed as $g(x) = g^{(1)}(x) + g^{(2)}(x)$ for every $x \in [0,1]$ such that

$$g_u^{(1)}(x)+g_v^{(2)}(x)=g_f(x)=g_h(x)=g_f^{(1)}(x)+g_f^{(2)}(x)=g_h^{(1)}(x)+g_h^{(2)}(x). \quad (5)$$

example

For $x \leq \frac{1}{2}$ we have

$$
\begin{aligned}
g_v^{(2)}(x) = g^{(2)}\left(\frac{2}{9}x\right) &+ g^{(2)}\left(\frac{2}{9} + \frac{2}{9}x\right) - g^{(2)}\left(\frac{2}{9}\right) \\
&+ g^{(2)}\left(\frac{4}{9} + \frac{2}{9}x\right) - g^{(2)}\left(\frac{4}{9}\right) \\
&+ g^{(2)}\left(\frac{6}{9} + \frac{2}{9}x\right) - g^{(2)}\left(\frac{6}{9}\right) \\
&+ g^{(2)}\left(\frac{8}{9} + \frac{2}{9}x\right) - g^{(2)}\left(\frac{8}{9}\right).
\end{aligned}
$$

Generalization

The result (5) can extend to: Every d.f. $g(x)$ of the sequence $(3/2)^n \bmod 1$ can be expressed as

$$g(x) = \sum_{i=0}^{2^k-1} g_i(x),$$

$$\sum_{i=0}^{2^k-1} (g_i)_{u_i}(x) = g_f(x) = g_h(x), \text{ where}$$

$$u_i(x) = 3\left(\frac{3}{2}\right)^k x + \frac{i}{2^k} \bmod 1.$$

Gauss-Kuzmin theorem

Denote

$$
\begin{aligned}
f(x) &= 1/x \bmod 1, \\
g_f(x) &= \int_{f^{-1}([0,x))} 1.\mathrm{d}g(x), \\
(g_{f^n})_f(x) &= g_{f^{n+1}}(x), \\
g_0(x) &= \frac{\log(1+x)}{\log 2}.
\end{aligned}
$$

The problem is to find all solutions $g(x)$ of the functional equation $g(x) = g_f(x)$ for $x \in [0,1]$ which is equivalent to

$$
g(x) = \sum_{n=1}^{\infty} g\left(\frac{1}{n}\right) - g\left(\frac{1}{n+x}\right). \tag{2}
$$

Results

- $g_0(x)$ satisfies (3).

- Gauss-Kuzmin theorem: If $g(x) = x$, then $g_{f^n}(x) \to g_0(x)$ and the rate of convergence is $O(q^{\sqrt{n}})$, $0 < q < 1$.

- This Theorem was proved by R. Kuzmin (1928) and for a starting function $g(x)$ for which
(i) $0 < g'(x) < M$ and
(ii) $|g''(x)| < \mu$.
Thus, if $g(x)$ satisfies (i), (ii), and $g_f(x) = g(x)$, then $g(x) = g_0(x)$.

- This Theorem was inspired by Gauss. He conjectured $m_n(x) \to g_0(x)$, where $m_n(x) = |\{\alpha \in [0,1]; 1/r_n(\alpha) < x\}|$ and for continued fraction expansion $\alpha = [a_0(\alpha); a_1(\alpha), a_2(\alpha), \ldots]$, $r_n(\alpha) = [a_{n+1}(\alpha); a_{n+2}(\alpha), \ldots,]$. In this case $m_n(x) = g_{f^n}(x)$ for $g(x) = x$, since $f(1/r_n(\alpha)) = 1/r_{n+1}(\alpha)$.

Example

- For starting point $x_0 \in [0, 1]$ we define the iterate sequence x_n as

$$x_1 = f(x_0), x_2 = f(f(x_0)), x_3 = f(f(f(x_0))), \ldots$$

If $\{g(x)\} = G(x_n)$ then $g(x)$ solve (2) for example, the sequence $x_1 = 1/r_1, x_2 = 1/r_2, \ldots$ for $\frac{\sqrt{5}-1}{2} = [0; 1, 1, 1, \ldots]$ produces solution $g(x) = c_{\frac{\sqrt{5}-1}{2}}(x)$.

Example

Define a non-increasing $g(x)$ on $[0,1/2)$ such that derivatives

$$g'\left(\frac{1}{1+x}\right) = g'(x)(1+x)^2 - \sum_{n=2}^{\infty} g'\left(\frac{1}{m+x}\right)\left(\frac{1+x}{n+x}\right)^2 \geq 0$$

and then complete $g(x)$ on $[1/2,1]$ by

$$g\left(\frac{1}{x+1}\right) = 1 - g(x) + \sum_{n=2}^{\infty} g\left(\frac{1}{n}\right) - g\left(\frac{1}{n+x}\right).$$

Using this method and putting $g(x) = x$ on $[0,1/2]$ the following d.f.

$$g(x) = \begin{cases} x & \text{if } x \in [0,1/2], \\ 2 - \frac{1}{x} + \sum_{n=2}^{\infty} \frac{1-x}{n(nx-x+1)} & \text{if } x \in [1/2,1] \end{cases}$$

solve $g_f(x) = g(x)$ for $x \in [0,1]$.

Sets of uniqueness

If d.f. $g_1(x)$ and $g_2(x)$ satisfy (1) and $g_1(x) = g_2(x)$ for $x \in [0, 1/2)$, then $g_1(x)$ and $g_2(x)$ coincide on the whole interval $[0, 1]$. Other sets of uniqueness are $[0, \frac{1}{n+1}) \cup (\frac{1}{n}, 1]$ for arbitrary positive integer n.

Other application. If $x_n \in [0, 1)$ has a.d.f. $g_1(x)$, then $\left\{\frac{1}{x_n}\right\}$ has a.d.f. $g_2(x) = \sum_{n=1}^{\infty} g_1\left(\frac{1}{n}\right) - g_1\left(\frac{1}{n+x}\right)$.

R.O. Kuzmin: *A problem of Gauss*, Dokl. Akad. Nauk, Ser. A, 375–380, 1928.

Benford law

Let x_n, $n = 1, 2, \dots,$ be a sequence in $(0, 1)$ and $G(x_n)$ be the set of all d.f.s of x_n.

• Assume that every d.f. $g(x) \in G(x_n)$ is continuous at $x = 0$. Then the sequence x_n satisfies strong Benford law in the base b if and only if for every $g(x) \in G(x_n)$ we have

$$x = \sum_{i=0}^{\infty} \left(g\left(\frac{1}{b^i}\right) - g\left(\frac{1}{b^{i+x}}\right) \right) \text{ for } x \in [0, 1]. \tag{3}$$

Examples of (3)

$$g(x) = \begin{cases} x & \text{if } x \in \left[0, \frac{1}{b}\right], \\ 1 + \log_b x + (1-x)\frac{1}{b-1} & \text{if } x \in \left[\frac{1}{b}, 1\right]. \end{cases}$$

$$\tilde{g}(x) = \begin{cases} 0 & \text{if } x \in \left[0, \frac{1}{b}\right], \\ 1 + \frac{\log x}{\log b} & \text{if } x \in \left[\frac{1}{b}, 1\right], \end{cases}$$

$$g^*(x) = \begin{cases} 0 & \text{if } x \in \left[0, \frac{1}{b^2}\right], \\ 2 + \frac{\log x}{\log b} & \text{if } x \in \left[\frac{1}{b^2}, \frac{1}{b}\right], \\ 1 & \text{if } x \in \left[\frac{1}{b}, 1\right] \end{cases}$$

$$g^{**}(x) = \begin{cases} 0 & \text{if } x \in \left[0, \frac{1}{b^3}\right], \\ 3 + \frac{\log x}{\log b} & \text{if } x \in \left[\frac{1}{b^3}, \frac{1}{b^2}\right], \\ 1 & \text{if } x \in \left[\frac{1}{b^2}, 1\right]. \end{cases}$$

Definition

Let $b \geq 2$ be an integer considered as a base, $x > 0$ and $x = M_b(x) \times b^{n(x)}$ with the mantissa $1 \leq M_b(x) < b$.

A sequence x_n, $n = 1, 2, \ldots$, of positive real numbers satisfies Benford law (abbreviated to B.L.) of order r if for every r-digits number $K = k_1 k_2 \ldots k_r$ we have

$$\lim_{N \to \infty} \frac{\#\{n \leq N;\ \text{first } r \text{ digits of } M_b(x_n) \text{ are equal to } K\}}{N} = \log_b(K+1) - \log_b K.$$

If a sequence x_n, $n = 1, 2, \ldots$, satisfies B.L. of order r, for every $r = 1, 2, \ldots$, then it is called that x_n satisfies strong B.L.

Criterion

A sequence x_n, $x_n > 0$, $n = 1, 2, \dots,$ satisfies strong B.L. if and only if the sequence $\log_b x_n \bmod 1$ is u.d. in $[0, 1)$.

V. Baláž – K. Nagasaka – O. Strauch: *Benford's law and distribution functions of sequences in* $(0, 1)$ (Russian), Mat. Zametki **88** (2010), no. 4, 485–501; (Anglish translation) Math. Notes **88** (2010), no. 4, 449–463.

Integral equations

Let $F(x, y)$ be a real continuous function defined on $[0, 1]^2$. We discuss the following Riemann-Stieltjes integrals:

$$\int_0^1 \int_0^1 F(x, y) \mathrm{d}g(x) \mathrm{d}g(y) = 0, \quad (4)$$

$$\int_0^1 \int_0^1 F(x, y) \mathrm{d}g(x, y), \ g(x, y) \text{ take copulas }. \quad (5)$$

(4) is used to estimate the set $G(x_n)$ of the set of all d.f.s of the sequence x_n, $n = 1, 2, \ldots$.

(5) we use to find extreme limit points of the sequence $\frac{1}{N} \sum_{n=1}^{N} F(x_n, y_n), \quad N = 1, 2, \ldots$, where x_n and y_n both u.d. in $[0, 1)$.

Here d.f. $g(x, y)$ is a copula, if the face d.f.s $g(x, 1) = x$ and $g(1, y) = y$.

Cont.

Let $F(x,y)$ be a real continuous function defined on $[0,1]^2$ and let $G(F)$ be a set of all d.f.s $g(x)$ satisfying

$$\int_0^1 \int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y) = 0.$$

The study of $G(F)$ is motivated by the fact that for every sequence $x_n \in [0,1)$ we have

$$G(x_n) \subset G(F) \Longleftrightarrow \lim_{N\to\infty} \frac{1}{N^2} \sum_{m,n=1}^{N} F(x_m, x_n) = 0,$$

Application

For any sequence x_n in $[0,1]$ we have

$$G((x_n)) \subset \{c_\alpha(x); \alpha \in [0,1]\} \iff \lim_{N\to\infty} \frac{1}{N^2} \sum_{m,n=1}^{N} |x_m - x_n| = 0.$$

Moreover, if $G((x_n)) \subset \{c_\alpha(x); \alpha \in [0,1]\}$, then $G((x_n)) = \{c_\alpha(x); \alpha \in I\}$, where I is a closed subinterval of $[0,1]$ which can be found as

$$I = \left[\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} x_n,\ \limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} x_n\right],$$

and the length $|I|$ of I $|I| = \limsup_{M,N\to\infty} \frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} |x_m - x_n|$.

$G(x_n) = G(F)$

Define
$\underline{g}_{G(F)}(x) = \inf_{g \in G(F)} g(x)$ - lower d.f. with respect to $G(F)$,
$\overline{g}_{G(F)}(x) = \sup_{g \in G(F)} g(x)$ - upper d.f. with respect to $G(F)$.
Assume that
(i) $\underline{g}_{G(F)}(x), \overline{g}_{G(F)}(x) \in G(F)$.
(ii) $\underline{g}_{G(F)}(x) = \underline{g}_{G(x_n)}(x)$, $\overline{g}_{G(F)}(x) = \overline{g}_{G(x_n)}(x)$.
(iii) For every $g \in G(F)$ there exists a point $(x, y) \in \mathrm{Graph(g)}$ such that $(x, y) \notin \mathrm{Graph}(\widetilde{\mathrm{g}})$ for any $\widetilde{g} \in G(F)$ with $\widetilde{g} \neq g$.
(iv) $G(x_n) \subset G(F)$.
Then $G(x_n) = G(F)$.

Extension

Extension of (I): Define that $G(A \leq F \leq B)$ is the set of all d.f.s $g(x)$ for which $A \leq \int_0^1 \int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y) \leq B$. Then as in (I) again $G(x_n) \subset G(A \leq F \leq B)$ is equivalent to $A \leq \liminf_{N\to\infty} \frac{1}{N^2} \sum_{m,n=1}^{N} F(x_m, x_n)$ and $\limsup_{N\to\infty} \frac{1}{N^2} \sum_{m,n=1}^{N} F(x_m, x_n) \leq B$.

Result

Put

$$F_{\tilde{g}}(x,y) = \int_0^1 \tilde{g}^2(t)\mathrm{d}t - \int_x^1 \tilde{g}(t)\mathrm{d}t - \int_y^1 \tilde{g}(t)\mathrm{d}t + 1 - \max(x,y).$$

Then

$$\int_0^1 (g(x) - \tilde{g}(x))^2 \mathrm{d}x = \int_0^1 \int_0^1 F_{\tilde{g}}(x,y)\mathrm{d}g(x)\mathrm{d}g(y),$$

and we see that $\int_0^1 \int_0^1 F_{\tilde{g}}(x,y)\mathrm{d}g(x)\mathrm{d}g(y) = 0$ has the unique solution $g(x) = \tilde{g}(x)$.

Result

Let $F:[0,1]^2\to\mathbb{R}$ be a continuous and symmetric function. For every distribution functions $g(x)$, $\tilde{g}(x)$ we have

$$\int_0^1\int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y)=0\Longleftrightarrow\int_0^1\int_0^1 F(x,y)\mathrm{d}\tilde{g}(x)\mathrm{d}\tilde{g}(y)$$

$$=\int_0^1(g(x)-\tilde{g}(x))\left(2\mathrm{d}_xF(x,1)-\int_0^1(g(y)+\tilde{g}(y))\mathrm{d}_y\mathrm{d}_xF(x,y)\right).$$

- Especially, putting $\tilde{g}(x)=c_0(x)$, we have

$$\int_0^1\int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y)=0\Longleftrightarrow$$

$$F(0,0)=\int_0^1(g(x)-1)\left(2\mathrm{d}_xF(x,1)-\int_0^1(g(y)+1)\mathrm{d}_y\mathrm{d}_xF(x,y)\right).$$

Result

Putting

$$F(x,y) = \frac{1}{3} + \frac{x^2+y^2}{2} - \frac{x+y}{2} - \frac{|x-y|}{2}$$

then we have

$$g(x) = x \Longleftrightarrow \frac{1}{3} = \int_0^1 g(x)(2x - g(x))\,\mathrm{d}x.$$

Copulas

D.f. $g(x, y)$ is a copula, if the face d.f.s $g(x, 1) = x$ and $g(1, y) = y$.
A problem is to find extreme values of

$$\int_0^1 \int_0^1 F(x, y) \mathrm{d}_x \mathrm{d}_y g(x, y),$$

where

$$\mathrm{d}_x \mathrm{d}_y g(x, y) = g(x + \mathrm{d}x, y + \mathrm{d}y) + g(x, y) - g(x + \mathrm{d}x, y) - g(x, y + \mathrm{d}y).$$

Motivation

Let $x_n, y_n \in [0,1)$, $n = 1, 2, \dots$, both u.d.sequences. For every continuous $F(x,y)$ the set of limit points of $\frac{1}{N}\sum_{n=1}^{N} F(x_n, y_n)$, $N = 1, 2, \dots$ coincides with

$$\left\{ \int_0^1 \int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y); g(x,y) \in G((x_n, y_n)) \right\},$$

where d.f. $g(x,y)$ of (x_n, y_n) satisfies $g(x,1) = x$, $g(1,y) = y$ and it is called copula. Here we using Riemann-Stiltjes integration

$$\frac{1}{N}\sum_{n=1}^{N} F(x_n, y_n) = \int_0^1 \int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y F_N(x,y)$$

and Helly theorem.

Result

Let $F(x, y)$ be a Riemann integrable function defined on $[0, 1]^2$.
Assume that $\mathrm{d}_x \mathrm{d}_y F(x, y) > 0$ for $(x, y) \in (0, 1)^2$. Then

$$\max_{g(x,y)\text{-copula}} \int_0^1 \int_0^1 F(x, y) \mathrm{d}_x \mathrm{d}_y g(x, y) = \int_0^1 F(x, x) \mathrm{d}x,$$

$$\min_{g(x,y)\text{-copula}} \int_0^1 \int_0^1 F(x, y) \mathrm{d}_x \mathrm{d}_y g(x, y) = \int_0^1 F(x, 1-x) \mathrm{d}x,$$

where the $\max$ is attained in $g(x, y) = \min(x, y)$ and $\min$ in $g(x, y) = \max(x + y - 1, 0)$, uniquely.
Here we have used the Fréchet-Hoeffding bounds

$$\max(x + y - 1, 0) \leq g(x, y) \leq \min(x, y)$$

which holds for every $(x, y) \in [0, 1]^2$ and for every copula $g(x, y)$.

Open problem

Find extremes of $\int_0^1 \int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y)$, for $F(x,y)$, where differential $\mathrm{d}_x\mathrm{d}_y F(x,y)$ has no a constant signum on $[0,1]^2$. For example, if

$$F(x,y) = \begin{cases} 2\left(y-\frac{1}{2}\right)x, & \text{if } y \in \left[\frac{1}{2},1\right], \\ 2\left(\frac{1}{2}-y\right)x, & \text{if } y \in \left[0,\frac{1}{2}\right]. \end{cases}$$

then

$$\max_{g(x,y)\text{-copula}} \int_0^1 \int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y) = \frac{1}{3}.$$

[1] PILLICHSHAMMER, F. – STEINERBERGER, S.: *Average distance between consecutive points of uniformly distributed sequences*, Uniform Distribution Theory **4** (2009), no. 1, 51–67.
[2] J. FIALOVÁ – O. STRAUCH *On two-dimensional sequences composed by one-dimensional uniformly distributed seqiuences*, Uniform Distribution Theory **6** (2011), no. 1, 101–125.

Matrix game

Payoff matrix $\mathbf{A}$

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ \hdotsfor{4} \\ a_{m,1} & a_{m,2} & \cdots & a_{m,m} \end{pmatrix}$$

Player I chooses the pure strategy as a row i represented as the vector $\mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$,
Similarly, the pure strategy for Player II is the j-th column may be represented by $\mathbf{e}_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ and the payoff to I is

$$\mathbf{e}_i \mathbf{A} \mathbf{e}_j^T = a_{i,j}.$$

Mixed strategy

Now, let Player I use a sequence $\mathbf{e}_n^{(I)}$, $n = 1, 2, \ldots,$ of pure strategy and Player II a sequence $\mathbf{e}_n^{(II)}$, $n = 1, 2, \ldots,$ of pure strategy. Then the mean-value of the payoff of I after N games is

$$\frac{1}{N}\sum_{n=1}^{N} \mathbf{e}_n^{(I)} \mathbf{A} \left(\mathbf{e}_n^{(II)}\right)^T.$$

Assume that there exist densities
$\lim_{N\to\infty} \frac{1}{N} \#\{n \leq N; \mathbf{e}_n^{(I)} = \mathbf{e}_i\} = p_i$, $i = 1, 2, \ldots, m$
$\lim_{N\to\infty} \frac{1}{N} \#\{n \leq N; \mathbf{e}_n^{(II)} = \mathbf{e}_i\} = q_i$, $i = 1, 2, \ldots, m$
The vector $\mathbf{p} = (p_1, p_2, \ldots, p_m)$ is called a mixed strategy for Player I.
Similarly $\mathbf{q} = (q_1, q_2, \ldots, q_n)$ is a mixed strategy for Player II.

Mean-value of the payoff

If the sequences $\mathbf{e}_n^{(I)}$ and $\mathbf{e}_n^{(II)}$ are statistically independent, then we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbf{e}_n^{(I)} \mathbf{A} (\mathbf{e}_n^{(II)})^T = \mathbf{pAq}^T = \sum_{i,j=1}^{m} p_i a_{i,j} q_j .$$

For zero-sum matrix game the mixed strategies $\mathbf{p}$ and $\mathbf{q}$ can be computed optimally such that

$$\mathbf{pAq}^T = 0$$

, but independence of $\mathbf{e}_n^{(I)}$ and $\mathbf{e}_n^{(II)}$ is a **problem**. Player with better sequence can be found payoff positive.

Transform of matrix game

$$I_{i,j} = [p_1+p_2+\cdots+p_{i-1}, p_1+p_2+\cdots+p_i)\times[q_1+q_2+\cdots+q_{j-1}, q_1+q_2+\cdots+q_j$$

Define $F(x,y)$ on $[0,1]^2$ such that

$$F(x,y) = a_{i,j} \text{ if } (x,y) \in I_{i,j}, i, j, = 1, 2, \dots, m.$$

Let Player I be use u.d. sequence x_n and Player II u.d. sequence y_n, $n = 1, 2, \dots$ If $(x_n, y_n) \in I_{i,j}$ the Player I choice pure strategy $\mathbf{e}_i$ and Player II pure strategy $\mathbf{e}_j$. Then mean value of the payoff of the Player I is

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} F(x_n, y_n) = \int_0^1\int_0^1 F(x,y)\mathrm{d}g(x,y),$$

where $g(x,y)$ is a d.f. of the sequence (x_n, y_n), $n = 1, 2, \dots$.

Example: Odd or Even

Players I and II simultaneously call out one of the numbers one or two. Player I wins if the sum of the numbers is odd. Player II wins if the sum of the numbers is even. The payoff matrix $\mathbf{A}$ is

$$\mathbf{A} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

and the function $F(x, y)$ corresponding to (1) is

$F(x, y)$

1	-1
-1	1

Examples

Let x_n and y_n, $n = 1, 2, \dots$ be u.d. sequences such that (x_n, y_n) has a.d.f. $g(x, y)$.

(i) If $g(x, y) = xy$ then $\int_0^1 \int_0^1 F(x, y) \mathrm{d}x \mathrm{d}y = 0$.

(ii) If $g(x, y) = \min(x, y)$ then
$\int_0^1 \int_0^1 F(x, y) \mathrm{d} \min(x, y) = \int_0^1 F(x, x) \mathrm{d}x = -1$.

(iii) If $g(x, y) = \max(x + y - 1, 0)$ then
$\int_0^1 \int_0^1 F(x, y) \mathrm{d} \max(x + y - 1, 0) = \int_0^1 F(x, 1 - x) \mathrm{d}x = 1$.

(iv) If $(x_n, y_n) = (\gamma_q(n), \gamma_q(n + 1))$, where $\gamma_q(n)$ is the van der Corput sequence in the base q, then
$\int_0^1 \int_0^1 F(x, y) \mathrm{d}g(x, y) = \frac{4}{q} - 1$. Thus for $q = 2$ Player I wins prize 1 and in the case $q > 5$ he loses.