Hausdorff Dimension of Sets of Numbers with Prescribed Digit Densities

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1 Definitions

- Class of infinite subsets $$\mathcal{P}_\infty = \{A \subseteq \mathbb{N} \mid \operatorname{card} A = \infty\}$$

- Dyadic mapping $\varrho\colon \mathcal{P}_\infty \to (0,1]$ $$\varrho(A) = \sum_{n=1}^{\infty} \frac{\chi_A(n)}{2^n}$$

- Asymptotic density of $A \in \mathcal{P}_\infty$ $$\overline{\underline{d}}(A) = \overline{\underline{\lim}}_{N\to\infty} \frac{\operatorname{card}\{n \in A \mid n \le N\}}{N}$$

- Borel (1909) $$\lambda\varrho\left\{A \in \mathcal{P}_\infty \,\middle|\, d(A) = \frac{1}{2}\right\} = 1$$

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- Question $$\dim \varrho\{A \in \mathcal{P}_\infty \mid \underline{d}(A) = \alpha, \overline{d}(A) = \beta\} = ?$$

- For $0 \leq \alpha \leq \beta \leq 1$ define $\mathcal{G}(\alpha,\beta) = \left\{ A \in \mathcal{P}_\infty \;\middle|\; \underline{d}(A) = \alpha, \overline{d}(A) = \beta \right\}$

$$\dim \varrho\mathcal{G}(\alpha,\beta) =$$

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- Entropy function $\delta\colon [0,1] \to [0,1]$ $\delta(x) = -x \log x - (1-x) \log(1-x)$
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- Gap density of $A = \{a_1 < a_2 < \dots\} \in \mathcal{P}_\infty$ $g(A) = \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}$
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- For $0 \le \alpha \le \beta \le 1 \le \gamma \le \dfrac{\beta}{\alpha}$ define

$$\mathcal{G}(\alpha, \beta, \gamma) = \{A \in \mathcal{P}_\infty \mid \underline{d}(A) = \alpha, \overline{d}(A) = \beta, g(A) = \gamma\}$$

- Question $\dim \varrho\mathcal{G}(\alpha, \beta, \gamma) = ?$

2 Segments and Descendants

- Initial segment of $A \in \mathcal{P}_\infty$ $A^n = A \cap \{1, \ldots, n\}$
- Initial segments of $\mathcal{S} \subseteq \mathcal{P}_\infty$ $\mathcal{S}^n = \{A^n \mid A \in \mathcal{S}\}$

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$$
\begin{aligned}
\mathcal{S} = \{&000100101011010101 1\ldots, \\
&001100101001100100 1\ldots, \\
&001100100111010001 1\ldots, \\
&001111010101100101 0\ldots, \\
&100010010101001010 1\ldots, \\
&100100011100010101 1\ldots, \\
&100101010110010001 0\ldots, \\
&100101011001001110 1\ldots, \\
&100110010101100110 0\ldots, \\
&110011101001001110 1\ldots, \\
&110011110010100010 1\ldots\}
\end{aligned}
$$

$\operatorname{card} \mathcal{S} = 11$

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$$
\begin{aligned}
\mathcal{S}^4 = \{&0001001010110101011\dots,\\
&0011001010011001001\dots,\\
&0011001001110100011\dots,\\
&0011110101011001010\dots,\\
&1000100101010010101\dots,\\
&1001000111000101011\dots,\\
&1001010101100100010\dots,\\
&1001010110010011101\dots,\\
&1001100101011001100\dots,\\
&1100111010010011101\dots,\\
&1100111100101000101\dots\}
\end{aligned}
$$

$\operatorname{card} \mathcal{S}^4 = 5$

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$$\mathcal{S}^8 = \{\begin{array}{l} 00010010\,10110101011\ldots, \\ 00110010\,10011001001\ldots, \\ 0011001001110100011\ldots, \\ 00111101\,01011001010\ldots, \\ 10001001\,01010010101\ldots, \\ 10010001\,11000101011\ldots, \\ 10010101\,01100100010\ldots, \\ 1001010110010011101\ldots, \\ 10011001\,01011001100\ldots, \\ 11001110\,10010011101\ldots, \\ 11001111\,00101000101\ldots \end{array}\}$$

$\operatorname{card} \mathcal{S}^8 = 9$

- Descendants of $\mathcal{T} \subseteq \mathcal{S}^k$ in $\mathcal{S}^n$ $$\mathcal{S}^n(\mathcal{T}) = \bigcup_{T \in \mathcal{T}} \{S \in \mathcal{S}^n \mid S^k = T^k\}$$

$$\mathcal{S}^4 \supset \mathcal{T} = \{0001001010110101011\ldots, \; \mathbf{0011}001010011001001\ldots, \; 0011001001110100011\ldots, \; 0011110101011001010\ldots, \; 1000100101010010101\ldots, \; \mathbf{1001}000111000101011\ldots, \; 1001010101100100010\ldots, \; 1001010110010011101\ldots, \; 1001100101011001100\ldots, \; 1100111010010011101\ldots, \; 1100111100101000101\ldots\}$$

$$\mathcal{S}^8(\mathcal{T}) = \{0001001010110101011\ldots, \; \mathbf{00110010}10011001001\ldots, \; 0011001001110100011\ldots, \; \mathbf{00111101}01011001010\ldots, \; 1000100101010010101\ldots, \; \mathbf{10010001}11000101011\ldots, \; \mathbf{10010101}01100100010\ldots, \; 1001010110010011101\ldots, \; \mathbf{10011001}01011001100\ldots, \; 1100111010010011101\ldots, \; 1100111100101000101\ldots\}$$

$\operatorname{card} \mathcal{T} = 2$

$\operatorname{card} \mathcal{S}^8(\mathcal{T}) = 5$

3 Homogenity

- Homogeneous class $\dfrac{\operatorname{card}\mathcal{S}^n(S)}{\operatorname{card}\mathcal{S}^n(T)} = 2^{o(k)}$ for every $n_0 \le k < n \le ck$ and $S, T \in \mathcal{S}^k$

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- $\mathcal{S} = \left\{ A \subseteq \mathcal{P}_\infty \;\middle|\; \dfrac{A(n_0)}{n_0} \ge \dfrac{1}{2} \Rightarrow n \in A \,\forall n > n_0 \right\}$ not homogeneous

$$\begin{aligned}
\mathcal{S}^4 = \{&00000010101101010111\ldots,\\
S = &00000011010110110 01\ldots,\\
&00000110011101000 11\ldots,\\
&00001101010110010 10\ldots,\\
&00001101011101111 11\ldots,\\
&00010001110001010 11\ldots,\\
&00010101011001000 10\ldots,\\
&00100001000011000 10\ldots,\\
&00111111111111111 11\ldots,\\
T = &01111111111111111 11\ldots,\\
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4 Saturation

- For every $\mathcal{S} \subseteq \mathcal{P}_\infty$ $$\varrho\mathcal{S} \subseteq \bigcap_{n=1}^{\infty} \underbrace{\bigcup_{S\in\mathcal{S}} \{\varrho(T) \mid T \in \mathcal{P}_\infty \text{ and } S^n = T^n\}}_{\text{covering of } \varrho\mathcal{S} \text{ with } \left(\frac{a}{2^n}, \frac{a+1}{2^n}\right]}$$

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- $\varrho^{-1}\big(\mathbb{Q} \cap (0,1]\big)$ not saturated, RHS $= (0,1]$

- $\mathcal{G}(\alpha,\beta,\gamma) = \big\{A \in \mathcal{P}_\infty \mid \underline{d}(A) = \alpha, \overline{d}(A) = \beta, g(A) = \gamma\big\}$
not saturated, RHS $= (0,1]$

5 Theorems

- Mišík, Volkmann, Šustek (201?)

For homogeneous and saturated $\mathcal{S} \subseteq \mathcal{P}_\infty$

$$\dim \varrho\mathcal{S} = \liminf_{n\to\infty} \frac{\log \operatorname{card} \mathcal{S}^n}{n}$$

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- Volkmann (1953) For $A \in \mathcal{P}_\infty$ $\qquad \dim \varrho\{B \subseteq A\} = \underline{d}(A)$

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$$\dim \varrho\mathcal{G}(\alpha, \beta, \gamma) = \min\left\{\delta(\alpha), \delta(\beta), \frac{1}{\gamma} \max_{\sigma\in[\alpha\gamma,\beta]} \delta(\sigma)\right\}$$

Thank You for Your Attention!