Probabilistic star discrepancy bounds for double infinite random matrices

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Overview

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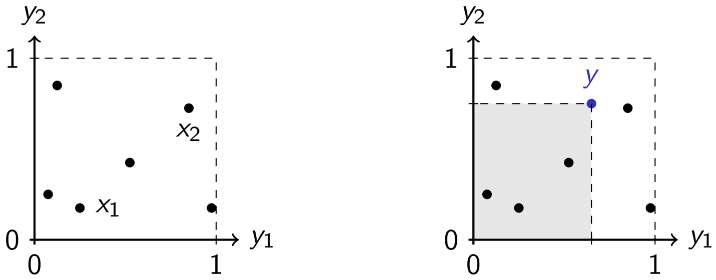
Basics

For **fixed** $s \in \mathbb{N}$ and $(x_n)_{n\in\mathbb{N}} \subset [0,1]^s$ let

$$D_N^{*s}(x_1,\ldots,x_N) = \sup_{y\in[0,1]^s} \left| \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{[0,y)}(x_n) - \lambda\left([0,y)\right) \right|, \quad N \in \mathbb{N},$$

denote the *star discrepancy* of the point sequence $(x_n)_{n\in\mathbb{N}}$.



- $(x_n)_{n\in\mathbb{N}} \subset [0,1]^s$ is called *uniformly distributed (modulo 1)* iff

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{[0,y)}(x_n) = \lambda([0,y)) \qquad \forall y \in [0,1]^s$$

iff

$$\lim_{N\to\infty} D_N^{*s}(x_1,\ldots,x_N) = 0$$

iff

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_{[0,1]^s} f(x)\,\mathrm{d}x \qquad \forall f\colon [0,1]^s \to \mathbb{R} \text{ cont.}$$

- The decay of D_N^{*s} controls the error of certain numerical integration schemes (see Koksma-Hlawka inequality, Quasi-Monte Carlo (QMC) methods).

- $(x_n)_{n\in\mathbb{N}} \subset [0,1]$ is called *completely uniformly distributed (c.u.d.)* iff for all $s \in \mathbb{N}$

$$((x_n, \dots, x_{n+s-1}))_{n\in\mathbb{N}} \subset [0,1]^s \text{ is uniformly distributed,}$$

or equivalently iff for all $s \in \mathbb{N}$

$$((x_{(n-1)s+1}, \dots, x_{ns}))_{n\in\mathbb{N}} \subset [0,1]^s \text{ is uniformly distributed.}$$

- example for $s = 3$:

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, \dots) \subset [0,1]$$

$$(x_1, x_2, x_3),$$
$$(x_2, x_3, x_4),$$
$$(x_3, x_4, x_5), \dots$$

$$(x_1, x_2, x_3), (x_4, x_5, x_6), (x_7, x_8, x_9), \dots \in [0,1]^3$$

Survey of known results

- Heinrich, Novak, Wasilkowski, Woźniakowski 2001:

$$n^*(\varepsilon, s) \leq C_{\mathrm{abs}}\, s\, \varepsilon^{-2}$$

  - $n^*(\varepsilon, s)$ is the minimal cardinality for a point set in $[0,1]^s$ having star discrepancy not exceeding $\varepsilon > 0$.
    (*inverse of the discrepancy*)
  - Hinrichs 2004: $n^*(\varepsilon, s) \geq c_{\mathrm{abs}}\, s\, \varepsilon^{-1}$
    ($\Rightarrow$ dependence on s is sharp)

- equivalent: For all N and s there exists a point set such that

$$D_N^{*s} \leq C_{\mathrm{abs}} \frac{\sqrt{s}}{\sqrt{N}}.$$

  - probabilistic approach; proof uses deep theoretical results
  - constant unknown

- Aistleitner 2011:
  For all N and s there exists a N-point set in $[0,1]^s$ such that

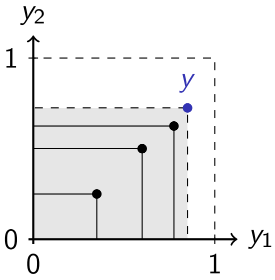
$$D_N^{*s} \leq 10 \frac{\sqrt{s}}{\sqrt{N}}$$

  - more direct, probabilistic proof based on a discretization via δ-(bracketing-) covers
  - probability estimates for exceptional sets using Hoeffding-type inequalities, e.g.

$$\mathbb{P}\left(\left| \sum_{n=1}^{N} Z_n \right| > t \right) \leq 2 \exp(-2t^2/N),$$

    where $Z_n = \mathbb{1}_I(X_n) - \lambda(I)$

  - unknown final success probability



- Dick 2007:
  If the discrepancy bound is slightly enlarged the probabilities of the exceptional sets in the proof of Heinrich *et al.* are summable over s and N.

  That is, all the $(N \times s)$-dim. projections $X^{(1)}, \dots, X^{(N)}$ of the random double infinite matrix $(X_{n,i})_{n,i \in \mathbb{N}}$ satisfy

$$D_N^{*s} \leq c_{\mathrm{abs}} \sqrt{\ln N} \frac{\sqrt{s}}{\sqrt{N}} \qquad \text{with positive probability.}$$

  (here all $X_{n,i}$ are independent Uniform$[0,1]$ distributed)

$$\begin{pmatrix} X_{1,1} & \dots & X_{1,s} & \dots \\ X_{2,1} & \dots & X_{2,s} & \\ \vdots & \ddots & \vdots & \\ X_{N,1} & \dots & X_{N,s} & \\ \vdots & & & \ddots \end{pmatrix} \Rightarrow \begin{matrix} X^{(1)} = (X_{1,1}, \dots, X_{1,s}) \in [0,1]^s \\ X^{(2)} = (X_{2,1}, \dots, X_{2,s}) \\ \vdots \\ X^{(N)} = (X_{N,1}, \dots, X_{N,s}) \end{matrix}$$

- Doerr, Gnewuch, Kritzer and Pillichshammer 2008: improvement of Dick's result for matrices (essentially $\ln N$ is replaced by $\ln(c + N/s)$)
  - point set extendable with respect to N and s
  - again completely probabilistic
  - unknown constants and/or success probabilities
- Chen, Dick, Owen 2011: Dick's proof can also be used for c.u.d. sequences (applications for Markov Chain Quasi-Monte Carlo)
- Aistleitner 2012: hybrid construction that satisfies the bound of Heinrich *et al.*

Main result

Theorem (Aistleitner, W. 2012)

Let $\gamma \geq \zeta^{-1}(2) \approx 1.73$ be fixed and $X^{(n)}$ as before. Then with probability strictly larger than $1-(\zeta(\gamma)-1)^2 \geq 0$ we have for all $s \in \mathbb{N}$ and every $N \geq 2$

$$D_N^{*s}\left(X^{(1)},\ldots,X^{(N)}\right) \leq \sqrt{\gamma}\cdot\sqrt{1165+178\frac{\ln\log_2 N}{s}}\cdot\sqrt{\frac{s}{N}}.$$

In particular, there exists a positive probability that the $(N \times s)$-dim. projections of a random matrix $(X_{n,i})_{n,i\in\mathbb{N}}$ satisfy

$$D_N^{*s} \leq \sqrt{2130+308\frac{\ln\ln N}{s}}\cdot\sqrt{\frac{s}{N}}$$

for all $s \in \mathbb{N}$ and every $N \geq 2$.

Advantages:

- simple structure: i.i.d. Uniform[0,1] random variables
- explicit constants and probability estimate
- $\sqrt{c_{\mathrm{abs}} + \frac{\ln\ln N}{s}}\frac{\sqrt{s}}{\sqrt{N}}$ is close to $c_{\mathrm{abs}}\frac{\sqrt{s}}{\sqrt{N}}$ (Heinrich *et al.*) and improves $c_{\mathrm{abs}}\sqrt{\ln N}\frac{\sqrt{s}}{\sqrt{N}}$ (Dick)
- arguments also applicable for c.u.d. sequences

Optimality

- *Law of the iterated logarithm* yields:
  For any sequence $(X^{(n)})_{n\in\mathbb{N}} \subset [0,1]^s$ of independent uniformly distributed vectors

  $$\limsup_{N\to\infty} \frac{\sqrt{N}\, D_N^{*s}(X^{(1)},\dots,X^{(N)})}{\sqrt{\ln\ln N}} = \frac{1}{\sqrt{2}} \quad \text{almost surely.}$$

  ($\Rightarrow$ factor $\sqrt{\ln\ln N}$ is necessary for random constructions)
  For fixed s and large N our bound reads

  $$D_N^{*s} \leq c(s)\frac{\sqrt{\ln\ln N}}{\sqrt{N}}.$$

- On the other hand, if $N \leq \exp(\exp(s))$ then we obtain

  $$D_N^{*s} \leq c_{\mathrm{abs}}\frac{\sqrt{s}}{\sqrt{N}}.$$

  (Otherwise known *low-discrepancy sets* do much better)

Ideas of the proof

- probabilistic approach (see Heinrich *et al.* '01)
- summable exception probabilities (see Dick '07)
- δ-(bracketing-) covers (see Gnewuch '08 and Aistleitner '11)
- additional parameter γ to control the success probability (inspired by Doerr *et al.* '08)
- maximal Bernstein inequality to replace $\sqrt{\ln N}$ by $\sqrt{\ln \ln N}$
- connection to c.u.d. sequences (see Chen *et al.* '11)
- straightforward calculation

References (in the order of appearance)

Thank you for your attention!

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