Asymptotic behavior of average L_p-discrepancies

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Outline

Definition (discrepancy function)

Let $d, n \in \mathbb{N}$. For a point set $\{\mathbf{t}_1, \ldots, \mathbf{t}_n\} \subset [0,1)^d$ and $\mathbf{x} \in [0,1]^d$ we define the **discrepancy function**

$$D(\{\mathbf{t}_1, \ldots, \mathbf{t}_n\})(\mathbf{x}) = \lambda^d([\mathbf{0}, \mathbf{x})) - \frac{1}{n} \sum_{i=1}^{n} \chi_{[\mathbf{0},\mathbf{x})}(\mathbf{t}_i), \qquad (1)$$

where $\lambda^d([\mathbf{0}, \mathbf{x}))$ is the d-dimensional Lebesgue measure of the box

$$[\mathbf{0}, \mathbf{x}) = \left\{ \mathbf{y} \in [0,1)^d : 0 \leq y_i < x_i, i = 1, \ldots, d \right\}.$$

Definition (L_p-star discrepancy)

Let $d, n \in \mathbb{N}$. For $0 < p < \infty$ and a point set $\{\mathbf{t}_1, \ldots, \mathbf{t}_n\} \subset [0,1)^d$ we define the **L_p-star discrepancy** as the L_p-norm of the discrepancy function

$$\mathrm{disc}_p^*(\mathbf{t}_1, \ldots, \mathbf{t}_n) = \|D(\{\mathbf{t}_1, \ldots, \mathbf{t}_n\})\|_p = \left(\int_{[0,1]^d} \left| \lambda^d([\mathbf{0}, \mathbf{x})) - \frac{1}{n} \sum_{i=1}^{n} \chi_{[\mathbf{0},\mathbf{x})}(\mathbf{t}_i) \right|^p \mathrm{d}\mathbf{x} \right)^{1/p}. \quad (2)$$

Definition (average L_p-star discrepancy)

Let $d, n \in \mathbb{N}$. We define the **average L_p-star discrepancy** as the expected value of the L_p-star discrepancy

$$\mathrm{av}_p^*(n,d) = \left(\mathbb{E}\left(\mathrm{disc}_p^*(\mathbf{t}_1,\ldots,\mathbf{t}_n)^p\right)\right)^{1/p} = \left(\int_{[0,1]^{nd}} \mathrm{disc}_p^*(\mathbf{t}_1,\ldots,\mathbf{t}_n)^p \,\mathrm{d}\mathbf{t}\right)^{1/p}, \tag{3}$$

with $\mathbf{t} = (\mathbf{t}_1,\ldots,\mathbf{t}_n)$. The points $\mathbf{t}_1,\ldots,\mathbf{t}_n \in [0,1)^d$ are independent and uniformly distributed.

Theorem (Heinrich, Novak, Wasilkowski, Woźniakowski [2])

Assume that p is an even integer and $d \in \mathbb{N}$. Then

$$\mathrm{av}_p^*(n,d)^p = \sum_{r=p/2}^{p-1} C(r,p,d) n^{-r}, \tag{4}$$

where the $C(r,p,d)$ are known constants which depend on Stirling numbers of the first and second kind.

In particular we have

$$\lim_{n\to\infty} \mathrm{av}_p^*(n,d)^p n^{p/2} = C(p/2,p,d).$$

Theorem (Steinerberger [5])

Let $p > 0, d \in \mathbb{N}$. Then

$$\lim_{n\to\infty} \mathrm{av}_p^*(n,d)^p n^{p/2} = C(p) \int\limits_{[0,1]^d} \Big[\lambda^d([\mathbf{0},\mathbf{x}))\big(1-\lambda^d([\mathbf{0},\mathbf{x}))\big)\Big]^{p/2} \mathrm{d}\mathbf{x}, \tag{5}$$

where

$$C(p) = \frac{2^{p/2}}{\pi^{1/2}} \Gamma\left(\frac{1+p}{2}\right).$$

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Goal

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- Obtain the Theorem of Steinerberger for different types of discrepancies.
- Estimate the integral above for different types of discrepancies.

Definition (L_p-B-discrepancy)

Let $d, n \in \mathbb{N}$. For a probability space (Ω_d, μ_d) and for abstract sets $B(\mathbf{x}) \subset [0,1]^d$, which are measurable for each $\mathbf{x} \in \Omega_d$, we define the ***L_p-B*-discrepancy**

$$\mathrm{disc}_p^B(\mathbf{t}_1, \ldots, \mathbf{t}_n) = \left(\int_{\Omega_d} \left| \lambda^d(B(\mathbf{x})) - \frac{1}{n} \sum_{i=1}^{n} \chi_{B(\mathbf{x})}(\mathbf{t}_i) \right|^p \mathrm{d}\mu_d(\mathbf{x}) \right)^{1/p}. \tag{6}$$

This definition is similar to the L_p-B-discrepancy of Novak and Woźniakowski [4]. While they use densities we use measures.

Definition (average L_p-B-discrepancy)

Let $d, n \in \mathbb{N}$. We define the **average L_p-B-discrepancy** as the expected value of the L_p-B discrepancy

$$\mathrm{av}_p^B(n,d) = \left(\mathbb{E}\left(\mathrm{disc}_p^B(\mathbf{t}_1,\dots,\mathbf{t}_n)^p\right)\right)^{1/p} = \left(\int_{[0,1]^{nd}} \mathrm{disc}_p^B(\mathbf{t}_1,\dots,\mathbf{t}_n)^p \,\mathrm{d}\mathbf{t}\right)^{1/p}, \qquad (7)$$

with $\mathbf{t} = (\mathbf{t}_1,\dots,\mathbf{t}_n)$. The points $\mathbf{t}_1,\dots,\mathbf{t}_n \in [0,1)^d$ are independent and uniformly distributed.

Theorem (Hinrichs, W.)

Let $p > 0, d \in \mathbb{N}$, let further (Ω_d, μ_d) be a probability space and $\{B(\mathbf{x}) : \mathbf{x} \in \Omega_d\}$ the allowed sets. Then

$$\lim_{n\to\infty} n^{p/2}\mathrm{av}_p^B(n,d)^p = C(p) \int_{\Omega_d} \left[\lambda^d(B(\mathbf{x}))\big(1-\lambda^d(B(\mathbf{x}))\big)\right]^{p/2} \mathrm{d}\mu_d(\mathbf{x}). \tag{8}$$

where

$$C(p) = \frac{2^{p/2}}{\pi^{1/2}}\Gamma\left(\frac{1+p}{2}\right).$$

How to proof the main result? - A helpful lemma

Lemma

Let (Ω_d, μ_d) be a probability space and p even. We have

$$\mathrm{av}_p^B(n,d)^p \le c_p n^{-p/2}. \tag{9}$$

To proof the lemma we need

- Idea of symmetrization, first used in this context by Hinrichs and Novak [3],
- Estimation of $\mathrm{av}_p^B(n,d)^p$, idea of Gnewuch [1].

How to proof the main result? - sketch of the proof

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Theorem (Steinerberger)

Let $p > 0, d \in \mathbb{N}$, let further (Ω_d, μ_d) be a probability space and $\{B(\mathbf{x}) : \mathbf{x} \in \Omega_d\} \subset 2^{[0,1]^d}$ the allowed sets. Then we have for

$$I = \int_{\Omega_d} \left[\lambda^d(B(\mathbf{x}))\left(1 - \lambda^d(B(\mathbf{x}))\right)\right]^{p/2} \mathrm{d}\mu_d(\mathbf{x})$$

the estimates

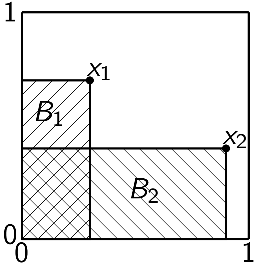
$$I \leq \int_{\Omega_d} \left[\lambda^d(B(\mathbf{x}))\right]^{p/2} \mathrm{d}\mu_d(\mathbf{x}), \tag{10}$$

$$I \geq \int_{\Omega_d} \left[\lambda^d(B(\mathbf{x}))\right]^{p/2} - (2^{p/2} - 1)\left[\lambda^d(B(\mathbf{x}))\right]^{p/2+1} \mathrm{d}\mu_d(\mathbf{x}). \tag{11}$$

L_p-star discrepancy

$$\Omega_d = [0,1]^d, \mu_d = \lambda^d, B(\mathbf{x}) = [\mathbf{0}, \mathbf{x})$$

$$L^* = \lim_{n\to\infty} n^{p/2} \mathrm{av}_p^*(n,d)^p$$



Using our main result we get

$$C(p)\left[\left(\frac{1}{\frac{p}{2}+1}\right)^d - \left(2^{\frac{p}{2}}-1\right)\left(\frac{1}{\frac{p}{2}+2}\right)^d\right] \leq L^* \leq C(p)\left(\frac{1}{\frac{p}{2}+1}\right)^d$$

$$\implies L^* \in \Theta\left(\left(\frac{1}{p/2+1}\right)^d\right).$$

Extreme L_p-discrepancy

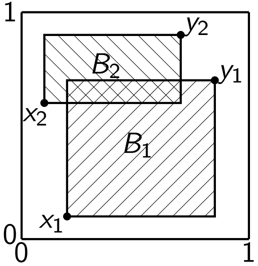
$$\Omega_d = \left\{ (\mathbf{x}, \mathbf{y}) \in [0,1]^d \times [0,1]^d : \mathbf{x} < \mathbf{y} \right\}$$

$$\mu_d = 2^d \lambda^{2d}, B((\mathbf{x}, \mathbf{y})) = [\mathbf{x}, \mathbf{y})$$

$$L = \lim_{n \to \infty} n^{p/2} \mathrm{av}_p(n,d)^p$$

Using our main result we get

$$L \in \Theta\left(\left(\frac{2}{\left(\frac{p}{2}+1\right)\left(\frac{p}{2}+2\right)} \right)^d \right).$$



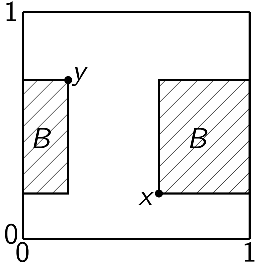
Periodic L_p-discrepancy

$$\Omega_d = [0,1]^d \times [0,1]^d, \mu_d = \lambda^{2d}$$

$$L^\circ = \lim_{n\to\infty} n^{p/2} \mathrm{av}_p^\circ(n,d)^p$$

Using our main result we get

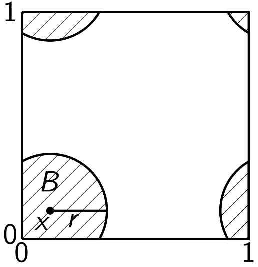
$$L^\circ \in \Theta\left(\left(\frac{p}{2}+1\right)^d\right).$$



Periodic Ball L_p-discrepancy

$$\Omega_d = [0,1]^d \times [0,1/2]$$
$$\mu_d = 2 \cdot \lambda^{d+1}$$

$$L^\bullet = \lim_{n\to\infty} n^{p/2} \mathrm{av}_p^\bullet(n,d)^p$$



Using our main result we get

$$L^\bullet \in \Theta\left(\frac{\pi^{dp/4}(1/2)^{dp/2}}{\left(\Gamma\left(d/2+1\right)\right)^{p/2}\left(dp/2+1\right)}\right).$$

- Our result gives an expression for arbitrary $p \in (0, \infty)$ instead of known results for even $p \in \mathbb{N}$.

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- We can constitute, why the sum known from the work of Heinrich, Novak , Wasilkowski and Woźniakowski

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has to start from $r = p/2$ instead of $r = 1$.
- It would be interesting to find convergence estimates for our main result.

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