# DISTRIBUTION OF SEQUENCES: A THEORY

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**Abstract**

In this book we develop the theory of distribution of sequences which we shall identify it with the theory of distribution functions of sequences.

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# 1 Preface

Let $f(x)$ be a measurable function defined on $[0,\infty)$. In the probabilistic theory $f(x)$ is called random variable and $g(x) = |f^{-1}([0,x))|$ is a uniquely defined distribution function of $f(x)$. Here $|X|$ is the Lebesgue measure of the set $X$. In the uniform distribution theory a random variable is replaced by a sequence $x_n$, $n = 1, 2, \dots$, $x_n \in [0,1)$. The sequence $x_n$ can have infinitely many distribution functions defined as all possible limits

$$\frac{\#\{n \le N_k; x_n \in [0,x)\}}{N_k} \to g(x)$$

as $k \to \infty$. The set of all such $g(x)$ we shall denote by $G(x_n)$ and the notion of the distribution of $x_n$ we shall identify with $G(x_n)$, i.e. the distribution of $x_n$ is known if we known the set $G(x_n)$. The importance of the set $G(x_n)$ is reflected in the fact that most properties of a sequence $x_n$ expressed in terms of limiting processes may be characterized using $G(x_n)$. For example, the fundamental Weyl's limit relation

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)\mathrm{d}x,$$

holding for any continuous function $f(x)$ defined on $[0,1]$ and any uniformly distributed sequence $x_n$, can be generalized to the relation

$$\lim_{k\to\infty} \frac{1}{N_k} \sum_{n=1}^{N_k} f(x_n) = \int_0^1 f(x)\mathrm{d}g(x)$$

which is true for every $g(x) \in G(x_n)$ if an appropriate index sequence $N_k$ is used. For multi–dimensional sequences $\mathbf{x}_n$ the set $G(\mathbf{x}_n)$ can be used in the correlation analysis of co–ordinate sequences of $\mathbf{x}_n$ which yields different results from those obtained by the statistical analysis.

Thus the main objects in the present monograph are;

a) Infinite sequence $x_n$, $n = 1, 2, \dots$, in $[0, 1)$.

b) The set $G(x_n)$ of distribution functions of $x_n$.

The study of the set of distribution functions of a sequence, still unsatisfactory today, was initiated by J.G. van der Corput [185]. The one-element set of $G(x_n)$ corresponds to the notion of asymptotic distribution function of a sequence mod1 was introduced by I.J. Schoenberg [147]. In the monograph L. Kuipers and H. Niederreiter [92] to distribution functions is devoted Chapter 7, pp. 53–68, in M. Drmota and R.F. Tichy Part 1.5, pp. 138–153. Some authors, see R. Winkler [193], instead of distribution functions $g(x)$ use measures $\mu$ induced by the interval $(x, y)$ measure $\mu((x, y)) = g(y) - g(x)$. [1]

This book is meant as a supplement to the previous monograph of the author and Š. Porubský [171] which is an encyclopedia of distribution of sequences but it does not contain proofs of theorems. The basis of the book are results published in [155], [158], [156], [159], [161], [162], [166], [173], [174], [164], [130], [105] and in Doctor Thesis [163].

This book is only attempt to the theory of distribution functions.

The outline of our conception is as follows.

In the Section 2 we remember definitions and notations and summarize results for singleton $G(x_n)$, i.e., for asymptotic distribution function. The list of the applications of the theory of distribution functions will be discussed in following sections.

In the Section 4 we summarize search methods for distribution functions (d.f.s) of sequences. We shall see that the main part is solving of some functional equations.

The main part of the Section 5 is a statistical independence of sequences $x_n$ and $y_n$, the uniform distribution of $x_n + y_n \bmod 1$, $x_n + \log p_n \bmod 1$, where $p_n$ are primes.

The Section 6 details results of the applications in the Section 3, specially functional equations for d.f.s of $\xi(3/2)^n \bmod 1$, ratio block sequences, the sequence $\varphi(n)/n$, generalized Benford's law, multidimensional Benford's law

[1] For a sequence $\mathbf{x} = (x_n)_{n\geq 1}$ in $[0, 1)$ they define the measure $\mu_{\mathbf{x},n}$ by $\mu_{\mathbf{x},n} = \frac{1}{N}\sum_{n=1}^{N} \delta_{x_n}$, where $\delta_x$ denotes the point measure in $x$. The set of accumulation points of the sequence $\mu_{\mathbf{x},n}$, $n = 1, 2, \dots$, is denoted by $M(\mathbf{x})$ an is called the set of limit measures of the sequence $\mathbf{x} = (x_n)_{n\geq 1}$.

and Gauss-Kuzmin theorem.

The main part of the Section 7 is a study of d.f.s $g(x,y)$, $g(x,1) = x$, $g(1,y) = y$ called copulas.

In the Section 9 we study extremes of $\int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y)$ over copulas.

In the Section 10 we start to study the solution $\int_0^1\int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y) = 0$ over d.f.s $g(x)$. It is the main parts of this book.

Note that we shall not cover metric aspects of the theory of distribution functions.

**Notes 1.** From a technical point of view in this book we only use Theorems, Examples and Notes. Numbering of Figures is only local, from case to case. Throughout this book we shall use the shorthand notation $x_n$, $n = 1,2,\dots$, for sequences, instead of the more common ones $(x_n)$, or $(x_n)_{n\geq 1}$. Consequently, the symbol $G(x_n)$ will stand for $G((x_n))$, while $f(x_n)$ may denote either the value of the function $f(x)$ at $x = x_n$, or the sequence $f(x_n)$, $n = 1,2,\dots$. The meaning will be clear from the context.

# 2 One-dimensional case

## 2.1 Basic notations and properties, distribution functions (d.f.s)

We follow the monograph [92], [73], [38] and [171].

Let $x_n$, $n = 1,2,\dots$ be a sequence from the unit interval $[0,1)$. Denote:

- $\#\{n \leq N; x_n \in [0,x)\}$ is the number $n \leq N$ for which $x_n \in [0,x)$. It is occasionally writing as the *counting function* $A([0,x);N;x_n)$.
- $$F_N(x) = \frac{\#\{n \leq N; x_n \in [0,x)\}}{N} \tag{1}$$

is the *step distribution function* (step d.f.) of the finite sequence $x_1,\dots,x_N$ in $[0,1)$, while $F_N(1) = 1$, e.g.

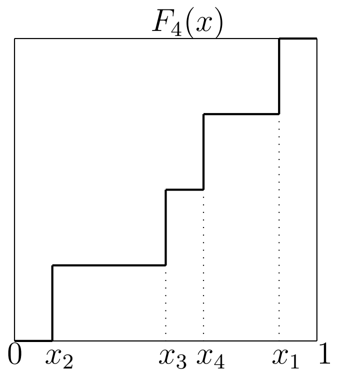


Figure: Step d.f. of $x_1, x_2, x_3, x_4$.

If $c_A(x)$ is the characteristic function of the set $A$, we can write

$$F_N(x) = \frac{1}{N}\sum_{n=1}^{N} c_{[0,x)}(x_n) = \frac{1}{N}\sum_{n=1}^{N} c_{(x_n,1]}(x). \tag{2}$$

If $f : [0,1] \to \mathbb{R}$ is continuous then by Riemann-Stieltjes integration

$$\frac{1}{N}\sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)\mathrm{d}F_N(x). \tag{3}$$

• A $g : [0,1] \to [0,1]$ is a distribution function (d.f.) if
(i) $g(x)$ is nondecreasing;
(ii) $g(0) = 0$ and $g(1) = 1$.

We shall identify any two d.f.s $g_1$, $g_2$ satisfying $g_1(x) = g_2(x)$ for every common continuity point $x \in [0,1]$, or equivalently, if $g_1(x) = g_2(x)$ a.e. on $[0,1]$.

• D.f. $g(x)$ is called a d.f. of the sequence $x_n$, $n = 1, 2, \dots$ if an increasing sequence of positive integers $N_1, N_2, \dots$ exists such that $\lim_{k\to\infty} F_{N_k}(x) = g(x)$ a.e. on $[0,1]$. [2]

• D.f. $g(x)$ is called an asymptotic d.f. (a.d.f.) of the sequence $x_n$, $n = 1, 2, \dots$ if $\lim_{N\to\infty} F_N(x) = g(x)$ a.e. on $[0,1]$.

• The existence of the limit $\lim_{k\to\infty} F_{N_k}(x) = g(x)$ a.e. on $[0,1]$ for a given sequence $N_k$ is equivalent to the existence of the limit

$$\lim_{k\to\infty} \frac{1}{N_k}\sum_{n=1}^{N_k} f(x_n) = \int_0^1 f(x)\mathrm{d}g(x) \tag{4}$$

[2] Note that the a.e. convergence (*almost everywhere conference*) is the same as the *weak convergence* since every monotone function is almost everywhere continuous.

for every continuous $f : [0,1] \to \mathbb{R}$. This is called the Weyl limit relation. [3]

- $G(x_n)$ denotes the set of all d.f.s of the given sequence $x_n$, $n = 1, 2, \dots$.
- The lover $\underline{g}(x)$ and the upper $\overline{g}(x)$ d.f.s are

$$\liminf_{N\to\infty} F_N(x) = \underline{g}(x), \quad \limsup_{N\to\infty} F_N(x) = \overline{g}(x). \tag{5}$$

It is equivalent to

$$\underline{g}(x) = \inf_{g\in G(x_n)} g(x), \quad \overline{g}(x) = \sup_{g\in G(x_n)} g(x).$$

Note that either the lower or the upper d.f. assigned to a given sequence $x_n$ need not be a d.f. of $x_n$ in general, i.e. they do not necessarily belong to $G(x_n)$. [4]

- Define two-dimensional set $\Omega(x_n)$ in the unit square $[0,1]^2$ as

$$\Omega(x_n) = \{(x,y) \in [0,1]^2; \exists g \in G(x_n)(y = g(x) \text{ or } y \in [g(x-0), g(x+0)])\}. \tag{6}$$

- For any non-zero set $H$ of d.f.s define lower $\underline{g}_H$ and upper $\overline{g}_H$ d.f.s of $H$ as

$$\underline{g}_H(x) = \inf_{g\in H} g(x), \quad \overline{g}_H(x) = \sup_{g\in H} g(x).$$

- For given d.f. $g(x)$ define the Graph(g) as a continuous curve in $[0,1]^2$ formed by all the points $(x, g(x))$, $x \in [0,1]$, and the all line segments connecting the points of discontinuity $(x, g(x-0))$ and $(x, g(x+0))$. [5]
- $c_\alpha(x)$ is one-step d.f. for which
  (i) $c_\alpha(x) = 0$ for $x \in [0, \alpha]$;
  (ii) $c_\alpha(x) = 1$ for $x \in (\alpha, 1]$.
- $h_\alpha(x)$ is a constant d.f. for which
  (i) $h_\alpha(x) = \alpha$ for $x \in (0,1)$;
  (ii) $h_\alpha(0) = 0$ and $h_\alpha(1) = 1$.
- If $A \subset \mathbb{N}$, then we denote by $\#A$ the cardinality of $A$. The numbers

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{\#\{k \le n; k \in A\}}{n}, \quad \overline{d}(A) = \limsup_{n\to\infty} \frac{\#\{k \le n; k \in A\}}{n} \tag{7}$$

[3]The Riemann – Stieltjes integration with limits $\int_{0-0}^{1+0}$ is understood in this case, see Section 2.1.1.

[4]Using notation in footnote in page 8, then for $\mathbf{x} = (x_n)_{n\ge1}$ and any interval $I \subset [0,1]$ we define $\overline{\mu}_{\mathbf{x}}(I) = \sup\{\mu(I); \mu \in M(\mathbf{x})\}$. It cannot be no defined by $\overline{g}(x)$.

[5]Here $g(x-0) = \liminf_{x'\to x} g(x')$ and $g(x+0) = \limsup_{x'\to x} g(x')$.

are called the *lower and upper asymptotic density* of $A$, respectively. If there exists the limit $\lim_{n\to\infty}\frac{\#\{k\le n; k\in A\}}{n}=d(A)$, then $d(A)=\underline{d}(A)=\overline{d}(A)$ is said to be the *asymptotic density of* $A$. It is easy to see that if $A$ is ordered to the increasing sequence $k_1<k_2<\dots$, then

$$\underline{d}(A)=\liminf_{n\to\infty}\frac{n}{k_n},\quad \overline{d}(A)=\limsup_{n\to\infty}\frac{n}{k_n}$$

and we write $\underline{d}(A)=\underline{d}(k_n)$ and $\overline{d}(A)=\overline{d}(k_n)$.

**Example 1.** The sequence

$$x_n=\log n \bmod 1,\quad n=1,2,\dots,$$

has the set of d.f.s

$$G(x_n)=\left\{g_u(x)=\frac{1}{e^u}\frac{e^x-1}{e-1}+\frac{e^{\min(x,u)}-1}{e^u};u\in[0,1]\right\},\tag{8}$$

and $\{\log N_k\}\to u$ implies $F_{N_k}(x)\to g_u(x)$.
The lower and the upper d.f. of $\log n$ mod 1 are

$$\underline{g}(x)=\frac{e^x-1}{e-1},\qquad \overline{g}(x)=\frac{1-e^{-x}}{1-e^{-1}},$$

and $\underline{g}\in G(x_n)$ but $\overline{g}\notin G(x_n)$. This set $G(x_n)$ was found by A. Wintner

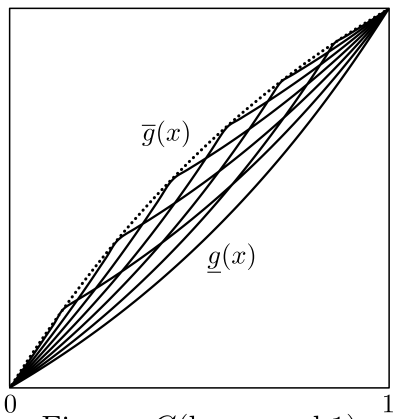



Figure: $G(\log n \bmod 1)$

[194] and it also follows from Theorem 48.

**Example 2.** The following sequences $x_n$, $n = 1, 2, \dots,$

(I) $x_n = \log(n \log^{(i)} n) \bmod 1$,

(II) $x_n = \log p_n \bmod 1$ (Y. Ohkubo [122]),

(III) $x_n = \log(p_n \log p_n) \bmod 1$ (Y. Ohkubo [122]),

(IV) $x_n = \frac{p_n}{n} \bmod 1$ ([169]),

have the same d.f.s as $\log n \bmod 1$. Here $p_n$ by the $n$th prime.

In the theory of d.f.s the following well-known Helly theorems are systematically used:

**Theorem 1** (First Helly theorem)**.** *Every sequence $g_n(x)$ of d.f.s contains a subsequence $g_{k_n}(x)$ such that $\lim_{n\to\infty} g_{k_n}(x) = g(x)$ for every $x \in [0, 1]$. Furthermore, the point limit $g(x)$ is d.f. again.*

**Theorem 2** (Second Helly theorem)**.** *Let $g_n(x)$, $n = 1, 2, \dots$ be a sequence of d.f.s for which $\lim_{n\to\infty} g_n(x) = g(x)$ a.e. on $[0, 1]$. Then for every $s$–dimensional continuous function $f : [0, 1]^s \to \mathbb{R}$ we have*

$$\lim_{n\to\infty} \int_0^1 \dots \int_0^1 f(t_1, \dots, t_s) \mathrm{d}g_n(t_1) \dots \mathrm{d}g_n(t_s) =$$
$$= \int_0^1 \dots \int_0^1 f(t_1, \dots, t_s) \mathrm{d}g(t_1) \dots \mathrm{d}g(t_s).$$

Occasionally, the First Helly theorem is also called Helly selection principle and the Second Helly theorem is also called the Helly – Bray theorem (see [96, p. 135, Th. 3.1.3., ]). One of the most important applications of this theorem is a following result:

**Theorem 3.** *Let $f : [0, 1]^s \to \mathbb{R}$ be a continuous function and let $x_n$ be a sequence in $[0, 1)$. If for an increasing sequence of indices $N_k$, $k = 1, 2, \dots,$ we have $\lim_{k\to\infty} F_{N_k}(x) = g(x)$ a.e., then*

$$\lim_{k\infty} \frac{1}{N_k^s} \sum_{i_1,\dots,i_s=1}^{N_k} f(x_{i_1}, \dots, x_{i_s}) = \int_0^1 \dots \int_0^1 f(t_1, \dots, t_s) \mathrm{d}g(t_1) \dots \mathrm{d}g(t_s).$$

*Proof.* By Riemman-Stieltjes integration we have

$$\frac{1}{N_k^s} \sum_{i_1,\dots,i_s=1}^{N_k} f(x_{i_1}, \dots, x_{i_s}) = \int_0^1 \dots \int_0^1 f(t_1, \dots, t_s) \mathrm{d}F_{N_k}(t_1) \dots \mathrm{d}F_{N_k}(t_s)$$

and the rest follows from the Second Helly theorem. □

Also we need

**Theorem 4** (Lebesgue's Dominated Convergence Theorem)**.** *Let $f_n(x)$ be a sequence of measurable functions on $[0,1]$. Suppose that the sequence converges pointwise to a function $f(x)$ and is dominated by some integrable function $g(x)$ in the sense that*

$$|f_n(x)| \leq g(x)$$

*for all $n$ and all $x \in [0,1]$. Then $f(x)$ is integrable and*

$$\lim_{n\to\infty} \int_0^1 f_n(x)\mathrm{d}x = \int_0^1 f(x)\mathrm{d}x.$$

**Theorem 5** (Lebesgue's Decomposition Theorem)**.** *Any d.f. $g(x)$ can be uniquely expressed as*

$$g(x) = \alpha_1 g_d(x) + \alpha_2 g_s(x) + \alpha_3 g_{ac}(x),$$

*where $\alpha_1$,$\alpha_2$,$\alpha_3$ are non–negative constants, $\alpha_1 + \alpha_2 + \alpha_3 = 1$, and*

- *$g_d(x)$ is a discrete (or atomic) d.f., i.e. $g_d(x) = \sum_{t_n<x} h_n$, where $t_n$ is a sequence of points of discontinuity of $g(x)$ with jumps $h_n$ at these points,*
- *$g_s(x)$ is a purely singular d.f. (also called singular continuous d.f.), i.e. continuous, strictly increasing and having zero derivative a.e.,*
- *and $g_{ac}(x)$ is an absolutely continuous d.f., i.e. $g_{ac}(x) = \int_0^x h(t)\mathrm{d}t$ for some non–negative Lebesgue integrable function $h(t)$ such that $\int_0^1 h(t)\mathrm{d}t = 1$. Function $h(t)$ is called the density of $g_{ac}(x)$.*

### 2.1.1 Riemann-Stieltjes integral

Let $f(x)$ be a function defined on $[0,1]$, $0 = u_0 < u_1 < \cdots < u_N = 1$ be a division of $[0,1]$ and let $g(x)$ be a d.f. If for every choice $\xi_n \in [u_{n-1}, u_n]$, $n = 0, 1, \ldots, N$ the Riemann-Stieltjes sum

$$\sum_{n=1}^{N} f(\xi_n)(g(u_n) - g(u_{n-1})$$

has a limit as $(u_n - u_{n-1}) \to 0$, $N \to \infty$, then this limit is unique and denoted $\int_0^1 f(x)\mathrm{d}g(x)$. If $f(x)$ is continuous and

$$g(x) = \alpha_1 g_d(x) + \alpha_2 g_s(x) + \alpha_3 g_{ac}(x),$$

is the Lebesgue decomposition of $g(x)$ (se Theorem 5) then

$$\int_0^1 f(x)\mathrm{d}g_d(x) = \sum_n f(t_n)h_n, \int_0^1 f(x)\mathrm{d}g_{ac}(x) = \int_0^1 f(x)g'_{ac}(x)\mathrm{d}x,$$
$$\int_0^1 f(x)\mathrm{d}g_s(x) = 0, \tag{9}$$

where $t_n$ is a sequence of points of discontinuity of $g(x)$ with jumps $h_n$ at these points and $g_s(x)$ is a singular part of $g(x)$ having $(g_s(x))' = 0$ a.e.

• In one-dimensional case in the uniform distribution theory the main application of Riemann-Stieltjes integration is the fundamental

$$\frac{1}{N}\sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)\mathrm{d}F_N(x).$$

In H. Riesel [136, Appendix 9, p. 358] there are other interesting examples

(i) $\sum_{p-\mathrm{prime}\leq \mathrm{N}} f(p) = \int_1^N f(x)\mathrm{d}\pi(x)$.

• The familiar formula for integration by parts of an ordinary Rieman integral is valid without any change also for Riemann-Stieltjes integrals

$$\int_a^b f(x)\mathrm{d}g(x) = [f(x)g(x)]_a^b - \int_a^b g(x)\mathrm{d}f(x).$$

• The Mean Value Theorem for Riemann-Stieltjes integrals: If $f(x)$ is a continuous and $g(x)$ a non-decreasing function, then

$$\int_a^b f(x)\mathrm{d}g(x) = f(\xi)\int_a^b \mathrm{d}g(x) = f(\xi)(g(b) - g(a))$$

where $\xi$ is a number between $a$ and $b$.

See [71]:

• If $\int f\mathrm{d}g$ exists for every $f(x)$ continuous on $a \leq x \leq b$, then $g(x)$ must be of bounded variation.

• The necessary and sufficient condition that $F(x, y)$ should possess a Riemann-Stieltjes integral with respect to a function $g(x, y)$ of bounded variation, is that the variation of $g(x, y)$ over the set of points at which $F(x, y)$ is discontinuous must be zero.

## 2.2 Basic properties of the set $G(x_n)$ of all d.f.s of $x_n$

**Theorem 6.** *For every sequence $x_n \in [0,1)$ we have*

(i) *$G(x_n)$ is non–empty, and it is either a singleton or has uncountable many elements.*

(ii) *$G(x_n)$ is closed.*

(iii) *$G(x_n)$ is connected in the weak topology defined by the metric*

$$d(g_1,g_2) = \sqrt{\int_0^1 (g_1(x) - g_2(x)^2 \mathrm{d}x}. \tag{10}$$

(iv) *For given a non–empty set $H$ of d.f.s there exists a sequence $x_n$ in $[0,1)$ such that $G(x_n) = H$ if and only if $H$ is closed and connected.*

*Proof.* The (i) follows from the first Helly theorem 1 (cf. [92, p. 54, Th. 7.1]).

The (ii) was proved by van der Corput:

**Theorem 7** (J.G. van der Corput (1935–36, Satz 10)[185]). *If*

$$g_1(x), g_2(x), g_3(x), \dots$$

*are d.f.s of $x_n \bmod 1$ and $\lim_{n\to\infty} g_n(x)$ exists at every common point $x$ of continuity, then the corresponding limit function is also a d.f. of $x_n \bmod 1$.*

The (iii) follows from the theorem of H.G. Barone

**Theorem 8** (H.G. Barone (1939)[16]). *If $t_n$, $n = 1,2,\dots$ is a sequence in a metric space $(X,\rho)$ and*
- *any subsequence of $t_n$ contains a convergent subsequence;*
- *$\lim_{n\to\infty} \rho(t_{n+1}, t_n) = 0$;*

*then the set of all limit points of $t_n$ is connected in $(X,\rho)$.*

Now we put $X=$ the set of all d.f.s defined on $[0,1]$, $t_N = F_N(x)$ and $\rho(t_{N+1}, t_N) = d(F_{N+1}(x), F_N(x))$. The limit $d(F_{N+1}(x), F_N(x)) \to 0$ follows directly from the definition $F_N(x)$ by using the identity (see (33))

$$\int_0^1 (g_1(x) - g_2(x))^2 \mathrm{d}x = \int_0^1 \int_0^1 |x-y| \mathrm{d}g_1(x) \mathrm{d}g_2(y) -$$

$$-\frac{1}{2}\int_0^1\int_0^1|x-y|\mathrm{d}g_1(x)\mathrm{d}g_1(y)-\frac{1}{2}\int_0^1\int_0^1|x-y|\mathrm{d}g_2(x)\mathrm{d}g_2(y)$$

which holds for every d.f.s $g_1(x)$ and $g_2(x)$ and putting $g_1(x)=F_{N+1}(x)$ and $g_2(x)=F_N(x)$ we find exactly

$$\int_0^1 (F_{N+1}(x)-F_N(x))^2\mathrm{d}x =$$
$$=-\frac{1}{2(N+1)^2N^2}\sum_{m,n=1}^N|x_m-x_n|+\frac{1}{(N+1)^2N}\sum_{n=1}^N|x_{N+1}-x_n|, \qquad (11)$$

where the right-hand side of (11) tends to zero as $N\to\infty$. The proof of (33) is given in Theorem 23.

Now, we return to (iv) of Theorem 6. It was proved by van der Corput:

**Theorem 9** (J.G. van der Corput (1935–36, Satz 5))**.** *Let $H$ be a non–empty set of d.f.s. Then there exists a sequence $x_n\in[0,1)$ with $G(x_n)=H$ if and only if there exists a sequence of d.f.s $g_n(x)$, $n=1,2,\dots,$ in $H$ such that*

(i) *If $\lim_{k\to\infty} g_{n_k}(x)=g(x)$ at common points $x$ of continuity, then $g\in H$, and conversely, there is such a subsequence for any $g\in H$.*

(ii) *$\lim_{n\to\infty} g_{n+1}(x)-g_n(x)=0$ at any common point $x$ of continuity of $g_n(x)$, $n=1,2,\dots$.*

□

A purely topological characterization of $G(x_n)$ with a short history can be found in R. Winkler (1997). Since the weak topology is metrisable by the metric $d$ in (10)

• a non–empty closed set $H$ is connected if and only if, for any two $g,\widetilde{g}\in H$ and every $\varepsilon>0$ there exist finitely many $g_1,\dots,g_n\in H$ such that $g_1=g$, $g_n=\widetilde{g}$ and $d(g_i,g_{i+1})<\varepsilon$ for $i=1,\dots,n-1$.

**Theorem 10.** *Let $x_n$ and $y_n$ be two real sequences. Assume that all d.f.s in $G(x_n \bmod 1)$ are continuous at 0 and 1. Then the zero limit of fractional parts*

$$\lim_{n\to\infty}(x_n-y_n)\bmod 1=0$$

*implies $G(x_n \bmod 1)=G(y_n \bmod 1)$. The same implication follows from the continuity of d.f.s in $G(y_n \bmod 1)$ at 0 and 1.*

*Proof.* Since $\{x_n - y_n\} = x_n - y_n - [x_n - y_n] = (x_n - [x_n - y_n]) - y_n$ and $x_n - [x_n - y_n] \equiv x_n \bmod 1$, we can assume that $|x_n - y_n| \to 0$. For common increasing sequence of positive integers $N_k$, $k = 1, 2, \dots$, let

$$F_{N_k}(x) = \frac{1}{N_k}\sum_{n=1}^{N_k} c_{[0,x)}(\{x_n\}) \to g(x), \quad \tilde{F}_{N_k}(x) = \frac{1}{N_k}\sum_{n=1}^{N_k} c_{[0,x)}(\{y_n\}) \to \tilde{g}(x)$$

for every continuity point $x \in [0,1]$. Because, for every $h = \pm 1, \pm 2, \dots$, the limit $|x_n - y_n| \to 0$ implies $\lim_{k\to\infty} \frac{1}{N_k}\sum_{n=1}^{N_k} e^{2\pi i h x_n} = \lim_{k\to\infty} \frac{1}{N_k}\sum_{n=1}^{N_k} e^{2\pi i h y_n}$ and since

$$\frac{1}{N_k}\sum_{n=1}^{N_k} e^{2\pi i h x_n} = \int_0^1 e^{2\pi i h x}\mathrm{d}F_{N_k}(x), \frac{1}{N_k}\sum_{n=1}^{N_k} e^{2\pi i h y_n} = \int_0^1 e^{2\pi i h x}\mathrm{d}\tilde{F}_{N_k}(x),$$

the well-known second Helly theorem implies $\int_0^1 e^{2\pi i h x}\mathrm{d}g(x) = \int_0^1 e^{2\pi i h x}\mathrm{d}\tilde{g}(x)$ for every $h = \pm 1, \pm 2, \dots$. This gives

$$\int_0^1 f(x)\mathrm{d}g(x) = \int_0^1 f(x)\mathrm{d}\tilde{g}(x), \text{ i.e. } \int_0^1 g(x)\mathrm{d}f(x) = \int_0^1 \tilde{g}(x)\mathrm{d}f(x)$$

for every continuous $f : [0,1] \to \mathbb{R}$, $f(0) = f(1)$.

For two common points $0 < x_1 < x_2 < 1$ of continuity of $g(x)$ and $\tilde{g}(x)$ and for sufficiently small $\Delta > 0$, define

$f(x) = 0$ for $x \in [0, x_1 - \Delta]$,
$f'(x) = 1/\Delta$ for $x \in (x_1 - \Delta, x_1)$,
$f(x) = 1$ for $x \in [x_1, x_2 - \Delta]$,
$f'(x) = -1/\Delta$ for $x \in (x_2 - \Delta, x_2)$, and
$f(x) = 0$ for $x \in [x_2, 1]$. Then

$$\int_0^1 g(x)\mathrm{d}f(x) \approx \frac{1}{\Delta}g(x_1)\Delta - \frac{1}{\Delta}g(x_2)\Delta, \int_0^1 \tilde{g}(x)\mathrm{d}f(x) \approx \frac{1}{\Delta}\tilde{g}(x_1)\Delta - \frac{1}{\Delta}\tilde{g}(x_2)\Delta,$$

and $\Delta \to 0$ gives

$$g(x_1) - \tilde{g}(x_1) = g(x_2) - \tilde{g}(x_2).$$

Now, we assume that $g(x_1) \neq \tilde{g}(x_1)$ and $g(x_2) \neq \tilde{g}(x_2)$. Fixing $x_1$ and letting $x_2 \to 1$ we have that one of $g, \tilde{g}$ must be discontinuous at 1, and fixing $x_2$, letting $x_1 \to 0$, one of $g, \tilde{g}$ must be discontinuous at 0. This gives Theorem 10. □

Theorem 10 has the following modifications:

**Theorem 11.** (i) *If all d.f.s in $G(\{x_n\})$ and $G(\{y_n\})$ are continuous at 0, then*

$$\{x_n - y_n\} \to 0 \Longrightarrow G(\{x_n\}) = G(\{y_n\}).$$

*The same follows from the continuity at 1.*

(ii) *The limit $\{x_n - y_n\} \to 0$ also implies*

$$\{g \in G(\{x_n\}); g \text{ is continuous at } 0,1\} = \{\tilde{g} \in G(\{y_n\}); \tilde{g} \text{ is continuous at } 0,1\}.$$

(iii) *Assume that all d.f.s in $G(\{x_n\})$ are continuous at 0. Then*

$$\{x_n - y_n\} \to 0 \Longrightarrow \{\tilde{g} \in G(\{y_n\}); \tilde{g} \text{ is continuous at } 0\} \subset G(\{x_n\}).$$

(iv) *If $x_n, y_n \in [0,1)$, $n = 1, 2, \dots,$ then*

$$|x_n - y_n| \to 0 \Longrightarrow G(x_n) = G(y_n)$$

*i.e. the continuity assumption can be omitted.*

(v) *If $x_n, y_n \in [0,1)$, $n = 1, 2, \dots,$ then*

$$\frac{1}{N}\sum_{n=1}^{N} |x_n - y_n| \to 0 \Longrightarrow G(x_n) = G(y_n)$$

*i.e. the continuity assumption can be omitted.*

*Proof.* (v). Put $F_N^{(1)}(x) = \frac{1}{N}\sum_{n=1}^{N} c_{[0,x)}(x_n)$ and $F_N^{(2)}(x) = \frac{1}{N}\sum_{n=1}^{N} c_{[0,x)}(y_n)$ and applying (33) we find

$$\int_0^1 (F_N^{(1)}(x) - F_N^{(2)}(x))^2 \mathrm{d}x = \frac{1}{N^2}\sum_{m,n=1}^{N} |x_m - y_n| - \\ - \frac{1}{2}\frac{1}{N^2}\sum_{m,n=1}^{N} |x_m - x_n| - \frac{1}{2}\frac{1}{N^2}\sum_{m,n=1}^{N} |y_m - y_n|. \quad (12)$$

From $x_m - y_n = x_m - x_n + x_n - y_n$ and $x_m - y_n = y_m - y_n + x_m - y_n$ it follows

$$|x_m - y_n| \le \frac{1}{2}|x_m - x_n| + \frac{1}{2}|y_m - y_n| + \frac{1}{2}|x_n - y_n| + \frac{1}{2}|x_m - y_m|. \quad (13)$$

Substituting (13) to (12) we find

$$\int_0^1 (F_N^{(1)}(x) - F_N^{(2)}(x))^2 \mathrm{d}x \leq \frac{1}{N} \sum_{n=1}^{N} |y_n - x_n|. \tag{14}$$

□

**Notes 2.** Putt $\gamma_n = |e^{2\pi i h(x_n - y_n)}|$, $h$ is an integer. Then

$$\begin{aligned}
&\gamma_n \leq 2\pi |h(x_n - y_n)|,\\
&\gamma_n \leq 2\pi(|h|\{x_n - y_n\}),\\
&\gamma_n \leq 2\pi(|h||\{x_n\} - \{y_n\}|),\\
&\gamma_n \leq 2\pi(|h|\|x_n - y_n\|),\\
&\gamma_n \leq 2\pi(|h||x_n - y_n - A|),
\end{aligned} \tag{15}$$

for $\gamma_n \to 0$ it can be used not only assumptions from Theorems 10 but also arbitrary zero-convergence of right-hands of (15). (Compare with (507).)

**Example 3.** [124, p. 257, Ex. 5.37]: For any sequence $x_n \in [0, 1)$, $n = 0, 1, 2, \ldots,$ we can associate a real number $\alpha$ such that $\lim_{n\to\infty}(\{n!\alpha\} - x_n) = 0$. Thus by Theorem 11 $G(\{n!\alpha\}) = G(x_n)$, if every $g(x) \in G(x_n)$ is continuous at 0 and 1.

*Proof.* [124, p. 297, Solution 5.37]: We require $\alpha$ in the form

$$\alpha = \sum_{n=0}^{\infty} \frac{a_n}{n!}, \text{ where } |a_n| \leq 1, n = 0, 1, 2, \ldots$$

Then $|\alpha - \sum_{k=0}^{n} \frac{a_k}{k!}| \leq \frac{1}{n(n!)}$ and thus

$$\left| n!\alpha - \sum_{k=0}^{n} a_k \frac{n!}{k!} \right| \leq \frac{1}{n}. \tag{16}$$

Now we compute sequences $a_n$ and $A_n$, $n = 0, 1, 2, \ldots$ by recurrences
(i) $a_0 = x_0$, $A_0 = 0$,
(ii) $a_n = x_n - \{A_n\}$, $A_{n+1} = (n + 1)(A_n + a_n)$, $n = 1, 2, \ldots$
Then by induction we can see

$$\sum_{n=0}^{n} a_k \frac{n!}{k!} = a_n + A_n, \quad n = 0, 1, 2, \ldots \tag{17}$$

From (ii) we have $a_n + A_n = x_n + [A_n]$ and (16) gives

$$\bigl|n!\alpha - [A_n] - x_n\bigr| \leq \frac{1}{n}, \quad n = 1, 2, \dots \tag{18}$$

If $\varepsilon > \frac{1}{n}$, the inequality (18) implies

$$x_n \in (\varepsilon, 1 - \varepsilon) \Longrightarrow n!\alpha - [A_n] = \{n!\alpha\}$$

and thus the limit $\lim_{n\to\infty}(\{n!\alpha\} - x_n) = 0$ need not hold over full sequence $n = 0, 1, 2, \dots$. But for proof of $G(\{n!\alpha\}) = G(x_n)$ we can use Theorem 10 with respect to Notes 2 and (15). □

## 2.3 Continuity points of $g(x) \in G(x_n)$

The Wiener-Schoenberg theorem (see [92, p. 55, Th. 7.5]) has a following generalization.

**Theorem 12.** *For given sequence $x_n \in [0, 1)$ and any integer $h$ define $\omega_h$ as $\omega_h = \limsup_{N\to\infty} \left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i h x_n}\right|^2$. If*

$$\lim_{H\to\infty} \frac{1}{H}\sum_{n=1}^{H} \omega_h = 0$$

*then every d.f. $g(x) \in G(x_n)$ is continuous on $[0, 1]$.*

*Proof.* We adapt [92, pp. 55–56]. For $g(x) \in G(x_n)$, let $\lim_{k\to\infty} F_{N_k}(x) = g(x)$ for every point $x \in [0, 1]$ of continuity of $g(x)$. By the second Helly theorem the following limit $\omega_h(g)$ exists

$$\begin{aligned}\omega_h(g) &= \lim_{k\to\infty}\left|\frac{1}{N_k}\sum_{n=1}^{N_k} e^{2\pi i h x_n}\right|^2 = \lim_{k\to\infty}\frac{1}{N_k^2}\sum_{m,n=1}^{N_k} e^{2\pi i h(x_m - x_n)} \\ &= \int_0^1\int_0^1 e^{2\pi i h(x-y)}\mathrm{d}g(x)\mathrm{d}g(y).\end{aligned} \tag{19}$$

Since

$$\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H} e^{2\pi i h(x-y)} = \begin{cases} 1 & \text{if } x - y \in \mathbb{Z}, \\ 0 & \text{others}\end{cases}$$

by applying Lebesgue dominance theorem we have

$$\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}\omega_h(g)=\int_0^1\int_0^1\left(\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}e^{2\pi ih(x-y)}\right)\mathrm{d}g(x)\mathrm{d}g(y)$$
$$=\iint_X 1.\mathrm{d}g(x)\mathrm{d}g(y), \tag{20}$$

where $X=\{(x,y)\in[0,1]^2;x-y\in\mathbb{Z}\}$. Since $X$ has zero measure the continuity of $g(x)$ implies zero value of (20) and if $x_0\in[0,1]$ is discontinuity point of $g(x)$ then (20) is bounded by $(g(x_0+0)-g(x_0-0))^2$ from below. From the other hand (19) gives $\omega_h\geq\omega_h(g)$. $\square$

**Example 4.** For $x_n=t\log n \bmod 1$, $t\neq 0$ we have (see [124, p. 281, Solution 5.18])

$$\omega_h=\frac{1}{4\pi^2h^2t^2+1}.$$

Theorem 12 implies that every $g(x)\in G(t\log n \bmod 1)$ is continuous on $[0,1]$.

Discontinuity points of d.f. have the following characterization.

**Theorem 13.** *The $x_0\in[0,1]$ is a discontinuity point for some $g(x)\in G(x_n)$ if and only if there exists a subsequence $x_{k_n}$ of $x_n$ such that*
(i) $\lim_{k\to\infty}x_{k_n}=x_0$,
(ii) $\overline{d}(k_n)>0$,
*where $\overline{d}(k_n)$ is the upper asymptotic density of $k_n$, $n=1,2,\dots$.*

For proof cf. Theorem 37.

A generalization of the van der Corput Difference Theorem (cf. [92, Th. 3.1, p. 26]) is a difficult problem. We present here only the following partial result.

**Theorem 14.** *If for every $k=1,2,\dots$ the set $G(x_{n+k}-x_n)$ contains only continuous d.f., then the same holds also for $G(x_n)$.*

*Proof.* Let $F_N^{(k)}(x)$ be $F_N(x)$ defined for $x_{n+k}-x_n \bmod 1$. Let $N_n$ be a sequence of indices such that $\lim_{n\to\infty}F_{N_n}(x)=g(x)$. From $N_n$ we can select a subsequence $N_{i_n}$ such that $\lim_{n\to\infty}F_{N_{i_n}}^{(1)}(x)=g_1(x)$. Step by step we can find the same for $k=2,3,\dots$ and then by the diagonal method we can find a sequence $M_n$ for which the following limits exists:

$$\lim_{n\to\infty}F_{M_n}(x)=g(x) \text{ and } \lim_{n\to\infty}F_{M_n}^{(k)}(x)=g_k(x) \text{ for every } k=1,2,\dots.$$

Wiener-Schoenberg Theorem [92, Th. 7.5, p. 55] yields

$$g_k(x) \text{ is continuous } \Longleftrightarrow \lim_{H\to\infty} \frac{1}{H}\sum_{n=1}^{H} |\omega_h^{(k)}|^2 = 0,$$

where

$$\omega_h^{(k)} = \int_0^1 e^{2\pi\imath hx}\mathrm{d}g_k(x) = \lim_{i\to\infty}\int_0^1 e^{2\pi\imath hx}\mathrm{d}F_{M_i}^{(k)}(x) = \lim_{i\to\infty}\frac{1}{M_i}\sum_{n=1}^{M_i} e^{2\pi\imath h(x_{n+k}-x_n)}.$$

In the next step we can use the van der Corput inequality (cf. [92, Th. 3.1, p. 26])

$$\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi\imath hx_n}\right|^2 \le \frac{N+M-1}{NM}$$
$$+ 2\sum_{k=1}^{M-1}\frac{(N+M-1)(M-k)(N-k)}{N^2M^2}\left|\frac{1}{N-k}\sum_{n=1}^{N-k} e^{2\pi\imath h(x_{n+k}-x_n)}\right| \qquad (21)$$

for every $x_n$ and $0 \le M \le N$. Putting $N = M_i$ and keeping $M$ fixed and then getting $i \to \infty$, we find

$$|\omega_h|^2 \le \frac{1}{M} + 2\sum_{k=1}^{M-1}\frac{M-k}{M^2}|\omega_h^{(k)}|,$$

where again $\omega_h = \int_0^1 e^{2\pi\imath hx}\mathrm{d}g(x)$. Summing up this inequalities for $h = 1,\dots,H$ and then getting $H \to \infty$ and using the fact [92, Ex. 7.17, p. 68] that

$$\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}|\omega_h^{(k)}|^2 = 0 \Longleftrightarrow \lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}|\omega_h^{(k)}| = 0$$

we find that

$$\limsup_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}|\omega_h|^2 \le \frac{1}{M}$$

for every positive $M$ which implies the continuity of $g(x)$. $\square$

## 2.4 Lower and upper d.f. of $x_n$

**Theorem 15.** *For every sequence $x_n \in [0,1)$ we have $\underline{g}(x), \overline{g}(x) \in G(x_n)$ if and only if*

$$\int_0^1 (\overline{g}(x) - \underline{g}(x))\mathrm{d}x = \limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^N x_n - \liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^N x_n.$$

*Proof.* Helly's theorems imply

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^N x_n - \liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^N x_n = \max_{g_1,g_2\in G(x_n)} \int_0^1 (g_1(x) - g_2(x))\mathrm{d}x.$$

Assume that this maximum is appeared in $g_1 = g^*$ and $g_2 = g_*$. Since $g^*(x) \le \overline{g}(x)$ and $\underline{g}(x) \le g_*(x)$ we have

$$\int_0^1 (\overline{g}(x) - \underline{g}(x))\mathrm{d}x \ge \int_0^1 (g^*(x) - g_*(x))\mathrm{d}x. \tag{22}$$

If $\underline{g}, \overline{g} \in G(x_n)$, then in (22) we have an equality, and if e.g. $\overline{g} \notin G(x_n)$, then $\overline{g}(x) > g^*(x)$ for some common continuity point $x \in (0,1)$ which implies strong inequality in (22). □

**Theorem 16.** *Assume that for the sequence $x_n \in [0,1)$ there exists the first moment*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^n x_n = \alpha \in [0,1].$$

*Then either $G(x_n)$ is singleton or $\underline{g} \notin G(x_n)$ or $\overline{g} \notin G(x_n)$.*

*Proof.* We have $\int_0^1 x\mathrm{d}F_N(x) = \frac{1}{N}\sum_{n=1}^N x_n$ and by Helly Theorem 2, if the first moment is constant, then for every $g(x) \in G(x_n)$, we have $\int_0^1 x\mathrm{d}g(x) = \alpha$. Thus if $\underline{g}(x), \overline{g}(x) \in G(x_n)$, then $\int_0^1 x\mathrm{d}\underline{g}(x) = \int_0^1 x\mathrm{d}\overline{g}(x) = \alpha$ and from $\underline{g}(x) \le \overline{g}(x)$ for all $x \in [0,1]$ it follows $\underline{g}(x) = \overline{g}(x)$ for common continuity points $x \in [0,1]$. □

**Theorem 17.** *Assume that for every nonzero closed $H \subset G(x_n)$ we have $\underline{g}_H, \overline{g}_H \in H$. Then every d.f. $g \in G(x_n)$ is characterized by the first moment, i.e. if $\int_0^1 x\mathrm{d}g_1(x) = \int_0^1 x\mathrm{d}g_2(x)$ then $g_1(x) = g_2(x)$ a.e.* [6]

[6] Another $\lim_{k\to\infty} F_{N_k}(x) = g(x)$ a.e. $\iff \lim_{k\to\infty} \frac{1}{N_k}\sum_{n=1}^{N_k} x_n = \int_0^1 x\mathrm{d}g(x)$, for every sequence $N_1 < N_2 < \dots$.

*Proof.* Let $H$ is the set of all $g \in G(X_n)$ such that $\int_0^1 x\mathrm{d}g(x) =$ const. Then $\int_0^1 x\mathrm{d}\underline{g}_H(x) = \int_0^1 x\mathrm{d}\overline{g}_H(x) =$ const. implies that $H$ is a singleton. □

**Theorem 18.** (i) *If the lower $\underline{g}$ and upper $\overline{g}$ d.f.s of the sequence $x_n \in [0,1]$ are of the type $c_\alpha(x)$ then $\underline{g}, \overline{g} \in G(x_n)$.*

(ii) *If $\overline{g}(x)$ has a point $x_0 \in [0,1]$ of discontinuity, then there exists $g(x) \in G(x_n)$ with the same point of discontinuity and $\overline{g}(x_0 - 0) = g(x_0 - 0)$ and $\overline{g}(x_0 + 0) = g(x_0 + 0)$. Similarly to $\underline{g}(x)$.*

(iii) *The following holds*

$$\limsup_{M,N\to\infty} \frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} |x_m - x_n| = 1 \Longleftrightarrow (\underline{g}(x) = c_1(x) \text{ and } \overline{g}(x) = c_0(x)).$$

*Proof.* (iii) Assume that $\lim_{k\to\infty} \frac{1}{M_k N_k} \sum_{m=1}^{M_k} \sum_{n=1}^{N_k} |x_m - x_n| = 1$ and that $F_{M_k}(x) \to g_1(x)$ and $F_{N_k}(x) \to g_2(x)$ for $x \in (0,1)$. By the second Helly theorem (Theorem 2) we have $\int_0^1 \int_0^1 |x-y|\mathrm{d}g_1(x)\mathrm{d}g_2(y) = 1$ and the equation (33) implies $\int_0^1 \int_0^1 |x-y|\mathrm{d}g_1(x)\mathrm{d}g_1(y) = 0$ and $\int_0^1 \int_0^1 |x-y|\mathrm{d}g_2(x)\mathrm{d}g_2(y) = 0$. This is possible only if $\{g_1(x), g_2(x)\} = \{c_0(x), c_1(x)\}$.

The opposite implication in (iii) follows from (i). □

## 2.5 Everywhere dense sequence $x_n$ in $[0,1]$

**Theorem 19.** *The sequence $x_n \in [0,1)$, $n = 1, 2, \dots$, is everywhere dense in $[0,1]$ if*

(i) $\lim_{N\to\infty} \frac{1}{N^2} \sum_{m,n=1}^{N} |x_m - x_n| = 0$;

(ii) $\liminf_{n\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_n = 0$;

(iii) $\limsup_{n\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_n = 1$;

*Proof.* From (i) it follows
$G(x_n) \subset \{c_\alpha(x), \alpha \in [0,1]\}$ and
(ii) and (iii) implies $c_0(x), c_1(x) \in G(x_n)$.
Finally we use the connectivity of $G(x_n)$. □

The connectivity of $G(x_n)$ also implies

**Theorem 20.** *For every sequence $x_n \in [0,1)$, $n = 1,2,\dots$ and an arbitrary Riemann integrable $f : [0,1] \to \mathbb{R}$ the sequence*

$$\frac{1}{N}\sum_{n=1}^{N} f(x_n), \quad N = 1,2,\dots$$

*is everywhere dense in the interval*

$$\left[m = \min_{g\in G(x_n)} \int_0^1 f(x)\mathrm{d}g(x), M = \max_{g\in G(x_n)} \int_0^1 f(x)\mathrm{d}g(x)\right].$$

## 2.6 Singleton $G(x_n) = \{g(x)\}$

In this case $g(x)$ is also called *asymptotic distribution function* (a.d.f.) of the sequence $x_n \in [0,1)$.

**Theorem 21.** (I) *The sequence $x_n \bmod 1$ has a given a.d.f. $g(x)$ if and only if for every continuous function $f : [0,1] \to \mathbb{R}$ we have*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(\{x_n\}) = \int_0^1 f(x)\mathrm{d}g(x),$$

*and $x_n \bmod 1$ has an a.d.f. if and only if the limit on the left hand side exists for every continuous $f$. Note that it is sufficient to take the polynomials $x, x^2, x^3, \dots$ for $f(x)$.*
(II) *In order to $x_n \bmod 1$ has an a.d.f., it is both necessary and sufficient that the limit*

$$\beta_k = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k x_n} \tag{23}$$

*exists for every integer $k$. This a.d.f. then will be continuous if and only if*

$$\liminf_{N\to\infty} \frac{1}{2N+1}\sum_{k=-N}^{N} |\beta_k|^2 = 0, \tag{24}$$

*and absolutely continuous with the derivative belonging to $L^2(0,1)$ if and only if*

$$\sum_{k=-\infty}^{\infty} |\beta_k|^2 < \infty. \tag{25}$$

(III) *Let $g(x)$ be continuous at $x = 0$ and $x = 1$. Then the sequence $x_n$* mod 1 *has a.d.f. $g(x)$ if and only if*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}e^{2\pi i h x_n} = \int_0^1 e^{2\pi i h x}\mathrm{d}g(x) \quad \textit{for all integers } h \neq 0.$$

(IV) For a given a d.f. $g(x)$ the sequence $x_n$ mod 1 has a.d.f. $g(x)$ if and only if

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}e^{2\pi i \frac{h}{2}\{x_n\}} = \int_0^1 e^{2\pi i \frac{h}{2}x}\mathrm{d}g(x) \quad \text{for all integers } h \neq 0.$$

(V) In order to the sequence $x_n \in [0,1)$ has an a.d.f., it is both necessary and sufficient that the limit

$$s_k = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}x_n^k \tag{26}$$

exists for every integer $k$.

*Proof.* (I) This is a modification of the Weyl's limit relation. The second Helly theorem 2 $\Big($saying that $\frac{1}{N}\sum_{n=1}^{N}f(\{x_n\}) = \int_0^1 f(x)\mathrm{d}F_N(x) \to \int_0^1 f(x)\mathrm{d}g(x)\Big)$ implies the necessary condition in (I). The sufficiency follows from the first Helly theorem 1.

The reduction to $f(x) = x, x^2, x^3, \dots$ is clear, it follows, for instance, from the approximation of $f(x)$ by Berstein polynomial of degree $n$

$B_n(x;f) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}$.

(II) The assertion (24) involving the continuity of the a.d.f. is due to N. Wiener (1924)[192] and I.J. Schoenberg (1928)[145], and is called Wiener – Schoenberg theorem (see Section 2.3, Theorem 12) (cf. [KN, p. 55, Th. 7.5]). The case (25) involving the absolute continuity is due to R.E. Edwards (1967), cf. P.D.T.A. Elliott [1979, Vol. 1, p. 67, Lemma 1.46][43]. Note that the conditions in (II) are equivalent to the following ones: (1) the coefficients $\beta_k$ exist for $k = 1, 2, \dots$, (2) $\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}|\beta_k|^2 = 0$, and (3) $\sum_{k=1}^{\infty}|\beta_k|^2 < \infty$.

(III) The Weyl criterion cannot be modified immediately, for instance because $c_0(x)$, $c_1(x)$, $h_\beta(x)$ cannot be distinguished by $e^{2\pi i h x}$ with $h = 0, \pm1, \pm2, \dots$.

(IV) Note that every continuous $f : [0,1] \to \mathbb{R}$ can be approximated by polynomials in $e^{2\pi i \frac{h}{2}x}$ with $h = 0, \pm1, \pm2, \dots$ (cf. [p. 34, Th. 1,2][163]),

because such polynomials approximate continuous $f:[0,2]\to\mathbb{R}$, $f(0)=f(2)$ by arbitrary accurate.
(V) By the Hausdorff moment problem: Let $s_0=1,s_2,s_3,\dots$ be a given sequence in $[0,1]$. Then there exists a d.f. $g(x)$ such that

(i) $s_n=\int\limits_0^1 x^n\mathrm{d}g(x),\quad n=0,1,2,\dots,$ if and only if

(ii) $\sum\limits_{i=0}^{m}(-1)^i\binom{m}{i}s_{i+k}\ge 0\quad$ for $m,k=0,1,2,\dots,$

and the solution function $g(x)$ is unique (cf. N.I. Achyeser (1961), J.A. Shohan and J.D. Tamarkin (1943) and see also Section 4.9).

Since $G(x_n)$ is non-empty, for a $g(x)\in G(x_n)$ there exists $N_1<N_2<\dots$ such that

$$s_k=\lim_{i\to\infty}\frac{1}{N_i}\sum_{n=1}^{N_i}x_n^k=\int_0^1 x^k\mathrm{d}g(x).$$

Thus $s_k$ in (26) satisfies (ii) and thus $G(x_n)=\{g(x)\}$. □

**Theorem 22.** *The sequence $x_n$ in $[0,1)$ has an a.d.f. $g(x)$ if and only if*

$$\lim_{N\to\infty}\Bigg(1+\int_0^1 g^2(x)\mathrm{d}x-2\int_0^1 g(x)\mathrm{d}x+\frac{2}{N}\sum_{n=1}^{N}\int_0^{x_n}g(x)\mathrm{d}x-\\ -\frac{1}{N}\sum_{n=1}^{N}x_n-\frac{1}{2N^2}\sum_{m,n=1}^{N}|x_m-x_n|\Bigg)=0,\qquad(27)$$

*or equivalently, if and only if*

(i) $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}x_n=\int_0^1 x\mathrm{d}g(x)$,

(ii) $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\int_0^{x_n}g(x)\mathrm{d}x=\int_0^1\left(\int_0^x g(t)\mathrm{d}t\right)\mathrm{d}g(x)$,

(iii) $\lim_{N\to\infty}\frac{1}{N^2}\sum_{m,n=1}^{N}|x_m-x_n|=\int_0^1\int_0^1|x-y|\mathrm{d}g(x)\mathrm{d}g(y)$.

*Proof.* This is an $L^2$ discrepancy criterion in [158, p. 176, Th. 1]. For proof we use:

**Theorem 23.** *For every polynomial $\psi(y)=a(x)y^2+b(x)y+c(x)$ with continuous coefficients the one-dimensional integral of the form $\int_0^1\psi(F_N(x))\mathrm{d}x$*

*can be expressed as the double integral*

$$\int_0^1 \psi(F_N(x))\mathrm{d}x = \int_0^1\int_0^1 F(x,y)\mathrm{d}F_N(x)\mathrm{d}F_N(y) = \frac{1}{N^2}\sum_{m,n=1}^{N} F(x_m,x_n) \tag{28}$$

*where*

$$F(x,y) = \int_{\max(x,y)}^1 a(t)\mathrm{d}t + \frac{1}{2}\int_x^1 b(t)\mathrm{d}t + \frac{1}{2}\int_y^1 b(t)\mathrm{d}t + \int_0^1 c(t)\mathrm{d}t, \tag{29}$$

*and furthermore*

$$\int_0^1 \psi(g(x))\mathrm{d}x = \int_0^1\int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y) \tag{30}$$

*for every d.f.* $g$.

*Proof.* We compute

$$\begin{aligned}
&\int_0^1 \psi(F_N(x))\mathrm{d}x \\
&=\int_0^1 \left( a(x)\left(\frac{1}{N}\sum_{n=1}^{N} c_{(x_n,1]}(x)\right)^2 + b(x)\left(\frac{1}{N}\sum_{n=1}^{N} c_{(x_n,1]}(x)\right) + c(x)\right)\mathrm{d}x \\
&=\frac{1}{N^2}\sum_{m,n=1}^{N}\int_{\max(x_m,x_n)}^1 a(x)\mathrm{d}x + \frac{1}{N}\sum_{n=1}^{N}\int_{x_n}^1 b(x)\mathrm{d}x + \int_0^1 c(x)\mathrm{d}x
\end{aligned}$$

and thus we find $F(x,y)$ of the form (29). [7] Then using Lebesgue theorem in the left and the Helly theorem in the middle of (28) we find (30). □

New, putting $\psi(F_N(x)) = (F_N(x) - g(x))^2$ we find $F(x,y) = F_g(x,y)$, where

$$F_g(x,y) = 1 - \max(x,y) - \int_x^1 g(t)\mathrm{d}t - \int_y^1 g(t)\mathrm{d}t + \int_0^1 g^2(t)\mathrm{d}t. \tag{31}$$

Bearing in mind $\max(x,y) = \frac{x+y+|x-y|}{2}$ and (28) we find $\lim_{N\to\infty}\int_0^1 (F_N(x) - g(x))^2\mathrm{d}x = 0$ in the form (27).

Finally, from (i), (ii), (iii) follows (27), and in opposite case, if $\lim_{N\to\infty} F_N(x) = g(x)$ a.e. then the second Helly theorem implies (i), (ii), and (iii). □

[7] Similarly, for any polynomial $\psi(y)$ of the $n$th order there exists $F(x_1,\dots,x_n)$ such that $\int_0^1 \psi(g(x))\mathrm{d}x = \int_0^1\dots\int_0^1 F(x_1,\dots,x_n)\mathrm{d}g(x_1)\dots\mathrm{d}g(x_n)$.

**Theorem 24.** *The sequence $x_n \in [0,1)$ possesses an a.d.f. if and only if* [8]

$$\lim_{M,N\to\infty}\Bigg(\frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}|x_m-x_n|$$
$$-\frac{1}{2M^2}\sum_{m,n=1}^{M}|x_m-x_n|-\frac{1}{2N^2}\sum_{m,n=1}^{N}|x_m-x_n|\Bigg)=0. \quad (32)$$

*Proof.* We prove generally

$$\int_0^1(g_1(x)-g_2(x))^2\mathrm{d}x=\int_0^1\int_0^1|x-y|\mathrm{d}g_1(x)\mathrm{d}g_2(y)-$$
$$-\frac{1}{2}\int_0^1\int_0^1|x-y|\mathrm{d}g_1(x)\mathrm{d}g_1(y)-\frac{1}{2}\int_0^1\int_0^1|x-y|\mathrm{d}g_2(x)\mathrm{d}g_2(y) \quad (33)$$

for every two d.f.s $g_1(x)$ and $g_2(x)$.

For given $g_1(x)$ and $g_2(x)$, let $x_n\in[0,1)$, $n=1,2,\dots,$ be a sequence such that for index sequence $N_1<N_2<\dots$ and $M_1<M_2<\dots$ we have $\lim_{k\to\infty}F_{N_k}(x)=g_1(x)$ and $\lim_{k\to\infty}F_{M_k}(x)=g_2(x)$. Such sequence $x_n$ exists by Theorem 6. Putting $N_k=N$ and $M_k=M$, express step d.f.s $F_N(x)$ and $F_M(x)$ by (2) and compute

$$\int_0^1(F_N(x)-F_M(x))^2\mathrm{d}x$$
$$=\int_0^1\left(\frac{1}{N}\sum_{n=1}^{N}c_{(x_n,1]}(x)-\frac{1}{M}\sum_{m=1}^{M}c_{(x_m,1]}(x)\right)^2\mathrm{d}x$$
$$=\frac{1}{N^2}\sum_{m,n=1}^{N}\int_0^1c_{(x_n,1]}(x)c_{(x_m,1]}(x)\mathrm{d}x+\frac{1}{M^2}\sum_{m,n=1}^{M}\int_0^1c_{(x_n,1]}(x)c_{(x_m,1]}(x)\mathrm{d}x$$
$$-2\frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{M}\int_0^1c_{(x_n,1]}(x)c_{(x_m,1]}(x)\mathrm{d}x. \quad (34)$$

Since
$$\int_0^1c_{(x_n,1]}(x)c_{(x_m,1]}(x)\mathrm{d}x=1-\max(x_m,x_n)$$

[8]The multi-dimensional case is in (532).

from (34) we find

$$\int_0^1 (F_N(x) - F_M(x))^2 \mathrm{d}x \tag{35}$$

$$= \int_0^1 \int_0^1 (1 - \max(x,y)) \mathrm{d}F_N(x) \mathrm{d}F_N(y)$$

$$+ \int_0^1 \int_0^1 (1 - \max(x,y)) \mathrm{d}F_M(x) \mathrm{d}F_M(y)$$

$$- 2\int_0^1 \int_0^1 (1 - \max(x,y)) \mathrm{d}F_M(x) \mathrm{d}F_N(y)$$

$$= \int_0^1 \int_0^1 (1 - \max(x,y)) \mathrm{d}(F_N(x) - F_M(x)) \mathrm{d}(F_N(y) - F_M(y)). \tag{36}$$

Computing the limit of (35) by Lebesgue theorem of dominant convergence and the limit of (36) by the second Helly theorem (see Theorem 2) and for $M = M_k$, $N = N_k$, $k \to \infty$ we find

$$\int_0^1 (g_1(x) - g_2(x))^2 \mathrm{d}x = \int_0^1 \int_0^1 (1 - \max(x,y)) \mathrm{d}(g_1(x) - g_2(x)) \mathrm{d}(g_1(y) - g_2(y)). \tag{37}$$

Since

$$\max(x,y) = \frac{x + y + |x - y|}{2},$$

$$\int_0^1 \int_0^1 1.\mathrm{d}(g_1(x) - g_2(x)) \mathrm{d}(g_1(y) - g_2(y)) = 0,$$

$$\int_0^1 \int_0^1 (x + y) \mathrm{d}(g_1(x) - g_2(x)) \mathrm{d}(g_1(y) - g_2(y)) = 0,$$

we find (33) in the form

$$\int_0^1 (g_1(x) - g_2(x))^2 \mathrm{d}x = \int_0^1 \int_0^1 -\frac{|x - y|}{2} \mathrm{d}(g_1(x) - g_2(x)) \mathrm{d}(g_1(y) - g_2(y)). \tag{38}$$

□

**Example 5.** Put $x_n = \frac{1}{n}$, $n = 1, 2, \dots$. Then $|1/m - 1/n| = |m - n|mn$ and
$\frac{1}{M^2} \sum_{m,n=1}^{M} |x_m - x_n| \leq c \frac{\log^2 M}{M} \to 0$,
$\frac{1}{N^2} \sum_{m,n=1}^{N} |x_m - x_n| \leq c \frac{\log^2 M}{M} \to 0$.

This implies
$\frac{1}{M^2}\sum_{m,n=1}^{M}|x_m-x_n|\to\int_0^1\int_0^1|x-y|\mathrm{d}g_1(x)\mathrm{d}g_1(y)=0\to g_1(x)=c_\alpha(x)$,
$\frac{1}{N^2}\sum_{m,n=1}^{N}|x_m-x_n|\to\int_0^1\int_0^1|x-y|\mathrm{d}g_2(x)\mathrm{d}g_2(y)=0\to g_2(x)=c_\beta(x)$.
Assuming (32), then
$\frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}|x_m-x_n|\to 0$
and then
$\frac{1}{N^2}\sum_{m,n=1}^{N}|x_m-x_n|\to\int_0^1\int_0^1|x-y|\mathrm{d}g_1(x)\mathrm{d}g_2(y)=0$
which gives $\alpha=\beta$. Then $x_n$ has an a.d.f., concretely $c_0(x)$.

Another criterion for singleton $G(x_n)$.

**Theorem 25.** *The sequence $x_n\in[0,1)$ possesses an a.d.f. $g(x)$ if and only if the sequence of functions*

$$\phi_N(t)=\frac{1}{N}\sum_{n=1}^{N}e^{itx_n}$$

*converges point-wise on $\mathbb{R}$ to a function $\phi(t)$ which is continuous at $0$. In this case $\phi(t)$ is a characteristic function of $g(x)$.*

*Proof.* See [177, p. 287, Th. 4]. □

[177, p. 283, Th. 2]: [9]

**Theorem 26.** *Let $x_n$, $n=1,\dots,$ be a sequence in $\in[0,1)$. Suppose that for any $\varepsilon>0$ there exists a function $f_\varepsilon:\mathbb{N}\to\mathbb{N}$ having the following properties:*
(i) $\lim_{\varepsilon\to 0}\overline{d}(\{n;|x_n-y_n|\ge\varepsilon\})=0$, *where $y_n=x_{f_\varepsilon(n)}$ and $\overline{d}(X)$ denotes the upper asymptotic density of $X$.*
(ii) *For each $a\in\mathbb{N}$ the asymptotic density $d(\{n;f_\epsilon(n)=a\})$ exists.*
*Then $x_n$ has an a.d.f.*

*Proof.* Denote
$A(a)=\{n\in\mathbb{N};f_\varepsilon(n)=a\}$;
$A(a)_N=\#\{n\le N;f_\varepsilon(n)=a\}$.
By assumption (ii) there exists limit

$$\lim_{N\to\infty}\frac{A_N(a)}{N}=d(a,\varepsilon). \tag{39}$$

[9] In original theorem there is also assumption (i) $\lim_{\varepsilon\to 0}\limsup_{T\to\infty}\overline{d}\{n;f_\varepsilon(n)>T\}=0$; which is by our proof superfluous.

For

$$\frac{1}{N}\#\{n \leq N; y_n < x\} \tag{40}$$

we have

$$\frac{1}{N}\#\{n \leq N; y_n < x\} = \frac{1}{N}\#\{n \leq N; x_{f_\varepsilon(n)} < x\} = \frac{1}{N}\sum_{x_a < x} A_N(a). \tag{41}$$

By (39) and since the sum in (41) is finite, then the following limit exists

$$g(x,\varepsilon) := \lim_{N\to\infty} \frac{1}{N}\#\{n \leq N; y_n < x\} = \lim_{N\to\infty} \frac{1}{N}\sum_{x_a < x} A_N(a) = \sum_{x_a < x} d(a,\varepsilon) \tag{42}$$

In the following we compare d.f.s. of $y_n$ and $x_n$.

$1^0$.

$$\begin{aligned} &\frac{1}{N}\#\{n \leq N; y_n < x\} \\ &= \frac{1}{N}\#\{n \leq N; |x_n - y_n| < \varepsilon, y_n < x\} \qquad (43) \\ &+ \frac{1}{N}\#\{n \leq N; |x_n - y_n| \geq \varepsilon, y_n < x\}. \qquad (44) \end{aligned}$$

Since

$$|x_n - y_n| < \varepsilon \text{ and } y_n < x \Longrightarrow |x_n - y_n| + y_n < x + \varepsilon \Longrightarrow x_n < x + \varepsilon,$$

then (43) can be bounded by

$$\frac{1}{N}\#\{n \leq N; |x_n - y_n| < \varepsilon, y_n < x\} \leq \frac{1}{N}\#\{n \leq N; x_n < x + \varepsilon\}. \tag{45}$$

Furthermore (45) can be bounded by upper d.f. $\overline{g}(x+\varepsilon)$ of $x_n$ defined as

$$\overline{g}(x+\varepsilon) = \limsup_{N\to\infty} \frac{1}{N}\#\{n \leq N; x_n < x + \varepsilon\}.$$

The part (44) is bounded by

$$\frac{1}{N}\#\{n \leq N; |x_n - y_n| \geq \varepsilon, y_n < x\} \leq \frac{1}{N}\#\{n \leq N; |x_n - y_n| \geq \varepsilon\}. \tag{46}$$

If $N \to \infty$ in (45) and (46) and using (42) we have

$$g(x,\varepsilon) \le \overline{g}(x+\varepsilon) + \overline{d}(\{n \in \mathbb{N}; |x_n - y_n| \ge \varepsilon\}) \tag{47}$$

for every $\varepsilon > 0$ and $x \in [0,1)$.

$2^0$.

$$\begin{aligned}
&\frac{1}{N}\#\{n \le N; x_n < x\} \\
&= \frac{1}{N}\#\{n \le N; |x_n - y_n| < \varepsilon, x_n < x\} & (48)\\
&+ \frac{1}{N}\#\{n \le N; |x_n - y_n| \ge \varepsilon, x_n < x\}. & (49)
\end{aligned}$$

Similarly as in (45) and (47) we have

$$= \frac{1}{N}\#\{n \le N; |x_n - y_n| < \varepsilon, x_n < x\} \le \frac{1}{N}\#\{n \le N; y_n < x + \varepsilon\} \tag{50}$$

$$= \frac{1}{N}\#\{n \le N; |x_n - y_n| \ge \varepsilon, x_n < x\} \le \frac{1}{N}\#\{n \le N; |x_n - y_n| \ge \varepsilon\}. \tag{51}$$

Let $N \to \infty$ in (50) and (51) and bearing in mind that

$$\begin{aligned}
&\liminf_{N\to\infty} \frac{1}{N}\#\{n \le N; y_n < x\} = \underline{g}(x);\\
&\lim_{N\to\infty} \frac{1}{N}\#\{n \le N; y_n < x+\varepsilon\} = g(x+\varepsilon,\varepsilon)\\
&\limsup_{N\to\infty} \frac{1}{N}\#\{n \le N; |x_n - y_n| \ge \varepsilon\} = \overline{d}(\{n; |x_n - y_n| \ge \varepsilon\})
\end{aligned}$$

then we have

$$\underline{g}(x) \le g(x+\varepsilon,\varepsilon) + \overline{d}(\{n; |x_n - y_n| \ge \varepsilon\}) \tag{52}$$

for every $x \in [0,1)$ and every $\varepsilon > 0$. Putting $x = x - \varepsilon$ then (52) is transformed to

$$\underline{g}(x-\varepsilon) \le g(x,\varepsilon) + \overline{d}(\{n; |x_n - y_n| \ge \varepsilon\}) \tag{53}$$

for every $x \ge \varepsilon$ and every $\varepsilon > 0$. Combining (47) and (53) we have

$$\underline{g}(x-\varepsilon) \le \overline{g}(x+\varepsilon) + \overline{d}(\{n; |x_n - y_n| \ge \varepsilon\}) \tag{54}$$

for every $x \ge \varepsilon$ and every $\varepsilon > 0$. Then the upper and the lower d.f. of the sequence $x_n$ are arbitrary close, thus are identical. $\square$

## 2.7 Uniform distribution (u.d.) of $x_n$

For completeness, we add a part of the $G(x_n) = \{g(x)\}$, where $g(x) = x$.
• The sequence $x_n$ is said to be *uniformly distributed modulo one* (abbreviated u.d. mod1) if for every subinterval $[x, y) \subset [0, 1]$ we have

$$\lim_{N\to\infty}(F_N(y) - F_N(x)) = y - x.$$

This convergence is uniformly with respect to $N$.

**Theorem 27** (Weyl limit relation). *The sequence $x_n$ mod 1 is u.d. if and only if for every continuous $f : [0, 1] \to \mathbb{R}$ we have*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(\{x_n\}) = \int_0^1 f(x)\mathrm{d}x.$$

**Theorem 28** (Weyl criterion). *The sequence $x_n$ mod 1 is u.d. if and only if*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i h x_n} = 0 \textit{ for all integers } h > 0.$$

**Theorem 29** (van der Corput's difference theorem). *Let $x_n$ be a sequence of real numbers. If for every positive integer $h$ the sequence $x_{n+h} - x_n$ mod 1 is u.d., then $x_n$ mod 1 is also u.d.*

**Theorem 30** ($L^2$ discrepancy criterion). *The sequence $x_n$ in $[0, 1)$ is u.d. if and only if*

$$\lim_{N\to\infty}\left(\frac{1}{3} + \frac{1}{N}\sum_{n=1}^{N} x_n^2 - \frac{1}{N}\sum_{n=1}^{N} x_n - \frac{1}{2N^2}\sum_{m,n=1}^{N}|x_m - x_n|\right) = 0,$$

*or equivalently, if and only if*

(i) $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} x_n = \frac{1}{2}$, *and*

(ii) $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} x_n^2 = \frac{1}{3}$, *and*

(iii) $\lim_{N\to\infty}\frac{1}{N^2}\sum_{m,n=1}^{N}|x_m - x_n| = \frac{1}{3}$.

The notion of discrepancy was introduced to measure the distribution deviation of sequences from the expected ideal one, cf. [92, Chap. 2], [38, Chap. 1].

• Let $x_1,\dots,x_N$ be a given sequence of real numbers from the unit interval $[0,1)$. Then the number

$$D_N = D_N(x_1,\dots,x_N) = \sup_{0\le\alpha<\beta\le1}\left|\frac{A([\alpha,\beta);N;x_n)}{N}-(\beta-\alpha)\right| \tag{55}$$

is called the *(extremal) discrepancy* of this sequence. The number

$$D_N^* = \sup_{x\in[0,1]}\left|\frac{A([0,x);N;x_n)}{N}-x\right|$$

is called *star discrepancy*, and the number

$$D_N^{(2)} = \int_0^1\left(\frac{A([0,x);N;x_n)}{N}-x\right)^2\mathrm{d}x$$

is called its $L^2$ *discrepancy*. The above discrepancies are mutually related by the following inequalities

$$\begin{gathered}D_N^*\le D_N\le 2D_N^*,\quad [92,\text{p.}91],\\ (D_N^*)^3\le 3D_N^{(2)}\le (D_N^*)^2,\quad [117].\end{gathered}$$

These inequalities hold for the arbitrary sequence $x_1,\dots,x_N$ in $[0,1)$ having $N$ terms. The following theorem demonstrate the role of the discrepancy notions:

**Theorem 31** (H. Weyl (1916))**.** *A sequence $x_n\in[0,1)$ is u.d. if and only if*

$$\lim_{N\to\infty}D_N(x_n)=0.$$

**Theorem 32** (P. Erdős and P. Turn (1948))**.** *If* $x_1,\dots,x_N$ mod 1 *is a finite sequence and $m$ a positive integer, then*

$$D_N\le\frac{6}{m+1}+\frac{4}{\pi}\sum_{h=1}^{m}\left(\frac{1}{h}-\frac{1}{m+1}\right)\left|\frac{1}{N}\sum_{n=1}^{N}e^{2\pi ihx_n}\right|.$$

**Theorem 33** (Fejér's difference theorem)**.** *Let $x_n$ be a sequence such that*

(i) $x_n \to \infty$, *and*

(ii) $\Delta x_n \downarrow 0$, *where* $\Delta x_n = x_{n+1} - x_n$.

*Then* $x_n \bmod 1$ *is u.d. if and only if*

(iii) $\lim_{n\to\infty} n\Delta x_n = \infty$.

**Theorem 34** (H. Weyl (1916)). *An s–dimensional sequence* $\mathbf{x}_n \bmod 1$ *is u.d. if and only if for every integral vector* $(h_1, \ldots, h_s) \neq (0, \ldots, 0)$ *the one–dimensional sequence*

$$h_1 x_{n,1} + \cdots + h_s x_{n,s} \bmod 1, \quad n = 1, 2, \ldots,$$

*is u.d.*

The following briefly lists some various results:

(I) If $x_n$, $n = 1, 2, \ldots$, is a monotone sequence that is u.d.mod1, then

$$\lim_{n\to\infty} \frac{|x_n|}{\log n} = \infty.$$

See H. Niederreiter [120].

(II) If $x_n$ is a sequence that is u.d.mod1, then

$$\limsup_{n\to\infty} n|x_{n+1} - x_n| = \infty.$$

See P.B. Kennedy [84], cf. [92, p. 15, Th. 2.6].

(III) Let $P$ be a set of primes such that $\sum_{p\in P} 1/p$ diverges. If the sequence

$$x_{hn} \bmod 1, \quad n = 1, 2, \ldots,$$

is u.d. for every $h$ composed only from primes taken from $P$ then

$$x_n \bmod 1, \quad n = 1, 2, \ldots,$$

is u.d. See G. Myerson and A.D. Pollington [110].

(IV) Define spectrum of $x_n$, $\mathrm{sp}(x_n)$ as

$$\mathrm{sp}(x_n) = \{\alpha \in [0, 1]; x_n - n\alpha \bmod 1 \text{ is not u.d.}\}.$$

M. Mendès France [103] proved: Necessary and sufficient condition that every subsequence $x_{an+b}$ will be u.d. for every integer $a \geq 1$ and $b \geq 0$ is that $\mathrm{sp}(x_n) \cap \mathbb{Q} = \emptyset$.

(V) Let $x_n$, $n = 1, 2, \dots$, be a sequence in $[0, 1)$ such that

$$\frac{\#\{n \leq N; x_n \in I\}}{N} \to |I|$$

for all intervals $I \subset [0, 1]$ of the fixed length $|I| = a$. Then $x_n$ may not be u.d., see following example.

**Example 6.** Assume that all $g(x) \in G(x_n)$ satisfy

$$g(x + a) = g(x) + a \text{ for } x \in [0, 1 - a]. \tag{56}$$

The following d.f. $g(x)$ satisfies (56) but $g(x) \neq x$.

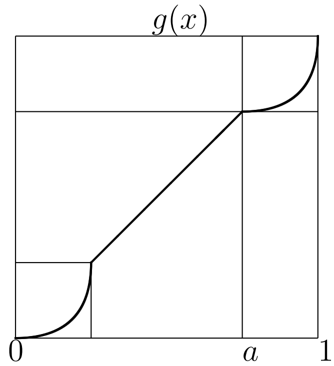



Fig.: Solution of (56).

Note that A. Volčič [189] proved that the limits

$$\frac{\#\{n \leq N; x_n \in I\}}{N} \to \frac{l}{2\pi}$$

for family of all arcs $C$ on the unit circle of a constant length $l$ implies u.d. of $x_n$ in the unit circle if $\frac{l}{2\pi}$ is irrational.

(VI) If $x_n$ is a sequence that is u.d. in $[0, 1)$, then

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |x_{n+1} - x_n| \leq \frac{1}{2}.$$

See F. Pillichshammer and S. Steinerberger [127] and our Theorem 214.

(VII) In [124, Ex. 5.11, p. 249–250] is proved:

$$\begin{aligned}&x_n - y_n = \log(n+1), n = 1, 2, \dots\\ &\implies ((x_n \bmod 1 \text{ is u.d. }) \Leftrightarrow (y_n \bmod 1 \text{ is u.d. })).\end{aligned}$$

See also Theorem 92 in this book.

### 2.7.1 Examples of u.d. sequences

(1) The sequence

$$[\alpha n]\beta n \bmod 1, \quad n = 1, 2, \dots \tag{57}$$

is u.d. in $[0,1]$ if and only if either $\alpha^2, \beta \notin \mathbb{Q}$ or $\alpha^2 \in \mathbb{Q}$ and $1, \alpha, \beta$ are linear independent over $\mathbb{Q}$. This proved I.J. Høaland [76, Prop. 5.3]. He gave similar examples: $[\sqrt{2n}]^2\sqrt{2}$ mod 1 is u.d. but the sequence $[\sqrt{2n}]^2\sqrt{2}n$ mod 1 is not.

(2) The sequence

$$\log n! \bmod 1, \quad n = 1, 2, \dots \tag{58}$$

is u.d. in $[0,1]$. This proved K. Goto and T. Kano [62, Th. 3].

(3) Let $F_n$, $n = 1, 2, \dots$ be the sequence of Fibonacci numbers. Then the sequence

$$\log F_n, \quad n = 1, 2, \dots \tag{59}$$

is u.d.mod1 (see [92, Ex. 3.3, p. 31]). A similar result for $\log_b F_n$ mod 1 is not known, but it is everywhere dense in $[0,1]$.

(4) Let $\alpha \neq 0$, $\beta$, $\gamma$, $\delta$ be real numbers and denote

$$x_n = \alpha n^\beta \log^\gamma n \log^\delta(\log n) \bmod 1 \tag{60}$$

for $n = 2, 3, \dots$. Then the sequence $x_n$ is u.d. if and only if it satisfies one of the following conditions:

(i) $\beta$ is a positive non-integer;

(ii) $\beta$ is a positive integer and either $\alpha$ is irrational, or $\gamma \neq 0$, or $\delta \neq 0$;

(iii) $\beta = 0$ and $\gamma > 1$;

(iv) $\beta = 0$, $\gamma = 1$ and $\delta > 0$;

This proved M.D. Boshernitzan [23].

(5) Denote the $n$th nontrivial root of Riemann's dzeta funkcion $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ in critical zone $0 \leq \Re s \leq 1$ as $\varrho(n) = \beta(n) + \imath\gamma(n)$. Then for every $x \neq 0$ the sequence

$$x\gamma(n), \quad n = 1, 2, \dots \tag{61}$$

is u.d mod1. After partial results of H. Rademacher and E. Hlawka this general result proved J. Kaczorowski [82].

(6) Let $p_n$, $n = 1, 2, \dots$ be an increasing sequence of all primes and $\alpha > 0$ an arbitrary non-integer.Then

$$p_n^\alpha, \quad n = 1, 2, \dots$$

is u.d. mod1. For $\alpha > 1$ this proved I.M. Vinogradov [188].

# 3 Examples of applications of d.f.s

The purpose of this section is a preliminary illustration of applications of $G(x_n)$. Some known classes of sequences $x_n$ originally defined by properties of $x_n$, we characterize by using the set $G(x_n)$ of all d.f.s of $x_n$.
For example:

- $(\lambda, \lambda')$-distribution;
- Statistically independent sequences;
- Statistically convergent sequences;
- Statistical limit points;
- Uniform maldistributed sequences;
- $\xi(3/2)^n$ mod 1, $n = 1, 2, \dots$;
- Benford's law;
- Copulas;
- Ratio sequences.

This section lists basic characteristics of these sequences. Additional features with proofs will be given in Section 6.

## 3.1 $(\lambda, \lambda')$–distribution

• J. Chauvineau (1967/68) [26]: Let $\lambda$ and $\lambda'$ be two real numbers such that $0 < \lambda \leq 1 \leq \lambda'$. The sequence $x_n$ mod 1 is said to be $(\lambda, \lambda')$–*distributed* if for every non–empty proper subinterval $I \subset [0,1]$ we have both

(i) $\liminf_{N\to\infty} \frac{\#\{n\le N; x_n \bmod 1\in I\}}{N} \ge \lambda|I|$, and

(ii) $\limsup_{N\to\infty} \frac{\#\{n\le N; x_n \bmod 1\in I\}}{N} \le \lambda'|I|$.

If only (i) is satisfied, the sequence $x_n \bmod 1$ is said to be $(\lambda,\infty)$*–distributed* or *positively distributed* (cf. [153, p. 234]). On the other hand, if only (ii) is true the sequence $x_n \bmod 1$ is said to be $(0,\lambda')$*–distributed.* These distributions can be characterized using d.f.s as follows (cf. [161]):

**Theorem 35.** *A sequence* $x_n \bmod 1$ *is* $(\lambda,\lambda')$*–distributed if and only if every* $g(x) \in G(x_n \bmod 1)$ *has the lower derivative* $\ge \lambda$ *and the upper derivative* $\le \lambda'$ *at every point* $x \in (0,1)$.

**Example 7.** G. Pólya and G. Szegő (1964, Part 2, Ex. 179)[129] proved that the derivative (density) $g'(x)$ of any $g(x) \in G(c\log n \bmod 1)$, $c > 0$, has the form

$$g'(x) = \begin{cases} \dfrac{\log q}{q-1}\, q^{x-\alpha+1}, & \text{if } 0 \le x < \alpha, \\ \dfrac{\log q}{q-1}\, q^{x-\alpha}, & \text{if } \alpha < x \le 1, \end{cases}$$

where $q = e^{1/c}$ and $\alpha \in (0,1)$. If $\alpha = 0$ or $\alpha = 1$ then

$$g'(x) = \frac{\log q}{q-1}\, q^x$$

and $c\log n \bmod 1$ is $(\lambda,\lambda')$–distributed with $\lambda = \frac{\log q}{q-1}$ and $\lambda' = q\frac{\log q}{q-1}$. Also see Example 21, p. 75.

## 3.2 Statistically independent sequences

Preliminary definitions to Section 5.1:

• Let $(\Omega, X, P)$ be a probability space. For the random vector $(X,Y)$ we define:

distribution function $F(x,y) = P(X < x, Y < y)$;
marginal distribution functions $F_1(x) = P(X < x)$, $F_2(y) = P(Y < y)$;
density function $f(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y)$;
marginal density functions $f_1(x) = F_1(x)'$, $f_2(y) = F_2(y)'$.

• The pair $X$ and $Y$ of random variables are independent if $F(x,y) = F_1(x)F_2(y)$ or $f(x,y) = f_1(x)f_2(y)$ for every relevant $x, y$. This definition for random variables is Theorem 36 for sequences.

**Definition 1.** G. Rauzy [135, p. 91, 4.1. Def.]: Let $x_n$ and $y_n$ be two infinite sequences from the unit interval $[0,1)$. The pair of sequences $(x_n, y_n)$ is called *statistically independent* if

$$\lim_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N} f_1(x_n)f_2(y_n) - \left(\frac{1}{N}\sum_{n=1}^{N} f_1(x_n)\right)\left(\frac{1}{N}\sum_{n=1}^{N} f_2(y_n)\right)\right) = 0 \quad (62)$$

for all continuous real functions $f_1$, $f_2$ defined on $[0,1]$. In other words, the double sequence $(x_n, y_n)$ is called *statistically independent* if the coordinate sequences $x_n$ and $y_n$ are statistically independent.

**Theorem 36** ([135, p. 92, 4.2. par.])**.** *For an arbitrary $(x_n, y_n) \in [0,1)^2$, $n = 1, 2, \dots$, we have $(x_n, y_n)$ statistically independent if and only if*

$$\forall_{g\in G(x_n,y_n)} \quad g(x,y) = g(x,1)g(1,y) \quad a.e.\ on\ [0,1]^2. \quad (63)$$

*Proof.* For given two-dimensional sequence $(x_n, y_n)$ put

$$F_N(x,y) = \frac{\#\{n \le N; (x_n, y_n) \in [0,x) \times [0,y)\}}{N}.$$

By Riemann-Stieltjes integration and by Helly theorem, there exists a sequence of indices $N_1 < N_2 < \dots$ and d.f. $g(x,y)$ such that

$$\begin{aligned}
\frac{1}{N_k}\sum_{n=1}^{N_k} f_1(x_n)f_2(y_n) &= \int_0^1\int_0^1 f_1(x)f_2(y)\mathrm{d}F_{N_k}(x,y)\\
&\to \int_0^1\int_0^1 f_1(x)f_2(x)\mathrm{d}g(x,y),\\
\frac{1}{N_k}\sum_{n=1}^{N_k} f_1(x_n) &= \int_0^1 f_1(x)\mathrm{d}F_{N_k}(x,1) \to \int_0^1 f_1(x)\mathrm{d}g(x,1),\\
\frac{1}{N_k}\sum_{n=1}^{N_k} f_2(y_n) &= \int_0^1 f_2(y)\mathrm{d}F_{N_k}(1,y) \to \int_0^1 f_2(y)\mathrm{d}g(1,y)
\end{aligned}$$

as $k \to \infty$.

$1^0$. Assuming statistical independence (62) we have

$$\int_0^1\int_0^1 f_1(x)f_2(x)\mathrm{d}g(x,y) = \left(\int_0^1 f_1(x)\mathrm{d}g(x,1)\right)\left(\int_0^1 f_2(y)\mathrm{d}g(1,y)\right). \quad (64)$$

Applying on the left (64) the (570) we find

$$\begin{aligned}\int_0^1\int_0^1 f_1(x)f_2(x)\mathrm{d}g(x,y) =& f_1(1)f_2(1) - \int_0^1 g(x,1)\mathrm{d}f_1(x) - \int_0^1 g(1,y)\mathrm{d}f_2(y)\\ &+ \int_0^1\int_0^1 g(x,y)\mathrm{d}f_1(x)\mathrm{d}f_2(y)\end{aligned}$$

and integration by parts
$\int_0^1 f_1(x)\mathrm{d}g(x,1) = f_1(1) - \int_0^1 g(x,1)\mathrm{d}f_1(x)$,
$\int_0^1 f_1(x)\mathrm{d}g(x,1) = f_2(1) - \int_0^1 g(1,y)\mathrm{d}f_2(y)$.
Then from (64) follows

$$\int_0^1\int_0^1 g(x,y)\mathrm{d}f_1(x)\mathrm{d}f_2(y) = \left(\int_0^1 g(x,1)\mathrm{d}f_1(x)\right)\left(\int_0^1 g(1,y)\mathrm{d}f_2(y)\right) \quad (65)$$

for arbitrary differentiable functions $f_1(x)$ and $f_2(y)$. Now for a continuity point $(x_0,y_0)$ of $g(x,y)$ we select $f_1(x)$ and $f_2(y)$ such that $[\mathrm{d}f_1(x)]_{x=x_0} = 1$ and $[\mathrm{d}f_2(y)]_{y=y_0} = 1$ and others $\mathrm{d}f_1(x) = 0$ and $\mathrm{d}f_2(y) = 0$. Then (65) implies $g(x_0,y_0) = g(x_0,1)g(1,y_0)$.

$2^0$. On the other hand (63) implies that in every sequence $N_1 < N_2 < \dots$ there exists subsequence $N_1' < N_2' < \dots$ such that such that the limit (62) holds for $N = N_k'$ and thus it holds for arbitrary $N$. □

(I) Theorem 36 can also be found in P.J. Grabner, O. Strauch and R.F. Tichy (1999) [64] where it is used (p. 109) to give the following $s$–dimensional generalization of statistical independence: Let $\mathbf{x}_n = (x_{n,1},\dots,x_{n,s})$ $n = 1,2,\dots,$ be an $s$–dimensional sequence in $[0,1)^s$ formed from $s$ sequences $x_{n,1}, x_{n,2},\dots,x_{n,s}$. Then $\mathbf{x}_n$ is called *statistically independent* (or that $\mathbf{x}_n$ has *statistically independent co–ordinates* $x_{n,1},\dots,x_{n,s}$) if every d.f. $g(\mathbf{x}) \in G(\mathbf{x}_n)$ can be written as a product $g(\mathbf{x}) = g_1(x_1)\dots g_s(x_s)$ of one–dimensional d.f.s. Here $g_i$, $i = 1,\dots,s$, can depend on $g$. Another

$$g(x,y,z,\dots) = g(x,1,1,\dots)g(1,y,1,\dots)g(1,1,z,\dots)\dots.$$

As an example in [64] (see Example 191 in this book):

$$\mathbf{x}_n = \left((-1)^{[[\log^{(j)} n]^{1/p_1}]}[\log^{(j)} n]^{1/p_1}, \dots, (-1)^{[[\log^{(j)} n]^{1/p_s}]}[\log^{(j)} n]^{1/p_s}\right) \bmod 1,$$

where $\log^{(j)} n$ denotes $j$th iterate logarithm $\log \dots \log n$ and $p_1, \dots, p_s$ are pairwise coprime positive integers. Then for $j > 1$ the $G(\mathbf{x}_n)$ is the set of all $c_{\boldsymbol{\alpha}}(\mathbf{x})$ where

$$c_{\boldsymbol{\alpha}}(\mathbf{x}) = \begin{cases} 1 & \text{for } \mathbf{x} \in [\boldsymbol{\alpha}, \mathbf{1}] \\ 0 & \text{otherwise.} \end{cases} \tag{66}$$

Thus $\mathbf{x}_n$ has statistically independent coordinates.
(II) Grabner and Tichy (1994) [31] proved that the extremal discrepancy does not characterize statistical independence, but the limit $\lim_{N\to\infty} D_N^{(2)} = 0$ of the $L^2$ discrepancy provides a characterization.
(III) J. Coquet and P. Liardet (1987) call two multi–dimensional sequences $\mathbf{x}_n$ and $\mathbf{y}_n$ *statistically independent* if for every (complex valued) continuous $f_1$, $f_2$

$$\lim_{N\to\infty} \left( \frac{1}{N} \sum_{n=1}^{N} f_1(\mathbf{x}_n) f_2(\mathbf{y}_n) - \left( \frac{1}{N} \sum_{n=1}^{N} f_1(\mathbf{x}_n) \right) \left( \frac{1}{N} \sum_{n=1}^{N} f_2(\mathbf{y}_n) \right) \right) = 0.$$

If for an integer $s \geq 1$ the $s$–dimensional sequences $\mathbf{x}_n = (x_{n+1}, \dots, x_{n+s})$ and $\mathbf{y}_n = (y_{n+1}, \dots, y_{n+s})$ are statistically independent, then the one–dimensional sequences $x_n$ and $y_n$ are said to be *statistically independent at rank* $s$. If they are statistically independent at rank $s$ for all integers $s$, they are called *completely statistically independent.*

For an example in [30]:

**Example 8.** For a given integer $q \geq 2$, a real number $\theta$ and a real polynomial $p(x)$, let

(i) $x_n = \theta q^n \bmod 1$,

(ii) $y_n = p(n) \bmod 1$,

(iii) $\mathbf{x}_n = (x_{n+1}, \dots, x_{n+s})$ and $\mathbf{y}_n = (y_{n+1}, \dots, y_{n+s})$.

If $x_n$ is u.d. (i.e. $\theta$ is normal in the base $q$), then for every $s = 1, 2, \dots$ the sequence

$$(\mathbf{x}_n, \mathbf{y}_n), \quad n = 1, 2, \dots,$$

has d.f.s $g(\mathbf{x},\mathbf{y}) \in G((\mathbf{x}_n,\mathbf{y}_n))$ only of the form $g(\mathbf{x},\mathbf{y}) = g_1(\mathbf{x})g_2(\mathbf{y})$ for some $g_1(\mathbf{x}) \in G(\mathbf{x}_n)$ and $g_2(\mathbf{y}) \in G(\mathbf{y}_n)$, i.e. the sequences $\mathbf{x}_n$ and $\mathbf{y}_n$ are *completely statistically independent.*

(IV) Coquet and Liardet (1987) [30] defined (following Rauzy (1976) [135]) the *statistical independence for a family of sequences* $H$ using the limit

$$\lim_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N} f_1(\mathbf{x}_{n,1})\dots f_k(\mathbf{x}_{n,k}) - \left(\frac{1}{N}\sum_{n=1}^{N} f_1(\mathbf{x}_{n,1})\right)\dots\left(\frac{1}{N}\sum_{n=1}^{N} f_k(\mathbf{x}_{n,k})\right)\right),$$

provided that this limit vanishes for any subfamily $\mathbf{x}_{n,1},\dots,\mathbf{x}_{n,k}$ of $H$ and for every continuous $f_1,\dots,f_k$. The equivalent reformulation in terms of the decomposition of any $g \in G(\mathbf{x}_{n,1},\dots,\mathbf{x}_{n,k})$ into the product of d.f.s from $G(\mathbf{x}_{n,1}),\dots,G(\mathbf{x}_{n,k})$ they call *independence criterion.* Cf. also Coquet and Liardet (1984) [29].
(V) Liardet (1990) [100] also defined the statistical independence of a sequence $x_n$ with respect to a set $\Psi$ of mappings $\psi : \mathbb{N} \to \mathbb{N}$ such that $\lim_{n\to\infty}\psi(n) = \infty$ provided the family of sequences $x_{\psi(n)}$, $\psi \in \Psi$, is statistically independent. Along parallel lines to those of the previous note he defined the notion of $\Psi$–independence at rank $s$ and the complete $\Psi$–independence.
(VI) Examples of statistical dependence are copulas in Section 8. For further properties see also Section 5.1.

## 3.3 Statistical limit points

Following the concept of a statistically convergent sequence, J. A. Fridy [58] introduced the notion of a statistical limit point of a given sequence $x_n$, $n = 1, 2, \dots$ of real numbers: A real number $x$ is said to be *a statistical limit point* of the sequence $x_n$ if there exists a subsequence $x_{k_n}$, $n = 1, 2, \dots$, such that $\lim_{n\to\infty} x_{k_n} = x$ and the set of indices $k_n$ has a positive upper asymptotic density (see Definitions and Notations). Fridy studied the set $\Lambda(x_n)$ of all such points. In the paper [90] is proved the following Theorem 37 that the set $\Lambda(x_n)$, for $x_n \in [0, 1)$ $n = 1, 2, \dots$, coincides with the set of all discontinuity points of d.f.s of $x_n$. This is inspired by I.J. Schoenberg [147] who noted that the sequence $x_n$ is statistically convergent to $\alpha$ if and only if $x_n$ admits a.d.f. $c_{[\alpha,1]}(x)(= c_\alpha(x)$ see p. 11).

**Theorem 37.** *For every sequence* $x_n \in [0,1)$ *we have*

$$\Lambda(x_n) = \{\alpha \in \mathbb{R}; \exists (g(x) \in G(x_n))\ g(x) \text{ is discontinuous at } \alpha\}.$$

*Proof.* 1°. Let $\alpha$ be a discontinuity point of $g(x) \in G(x_n)$ with a jump of size $h$. Let $\alpha_i$ and $\beta_i$ be two sequences satisfying $\beta_i - \alpha_i \to 0$ as $i \to \infty$, $\alpha_i < \alpha < \beta_i$ and $\alpha_i$ and $\beta_i$ are continuity points of $g(x)$ for every $i$. From $N_k$ (using the definition of $g(x)$) we can select a subsequence $N_{k_i}$ such that

$$\frac{1}{N_{k_i}} \sum_{n=N_{k_{i-1}}+1}^{N_{k_i}} c_{[\alpha_i,\beta_i)}(x_n) > h - \varepsilon$$

for some $\varepsilon \in (0,h)$. Ordering elements of $\cup_{i=1}^{\infty}\{n \in \mathbb{N}, N_{k_{i-1}} < n \leq N_{k_i}\}$ to an increasing sequence $n_i$, then we have $x_{n_i} \to \alpha$ and $\overline{d}(n_i) \geq h - \varepsilon$.

2°. Assume that $x_{n_i} \to \alpha$ for $i \to \infty$, $\overline{d}(n_i) = h > 0$, and $\varepsilon \in (0,h)$. Then there exists a sequence $N_k$ such that for every $k = 1, 2, \dots$,

$$\frac{|\{i \in \mathbb{N}; n_i \leq N_k\}|}{N_k} \geq h - \varepsilon.$$

By Helly selection principle from $N_k$ we can select a subsequence $N_{k_i}$ such that

$$\lim_{i\to\infty} \frac{1}{N_{k_i}} \sum_{n=1}^{N_{k_i}} c_{[0,x)}(x_n) = g(x)$$

for every point $x$ of continuity of $g(x)$. Clearly, $g(x)$ has at $\alpha$ a jump of size $\geq h - \varepsilon > 0$ and $g(x) \in G(x_n)$. □

**Theorem 38.** *Let* $x_n$ *be a sequence of real numbers. If for every* $k = 1, 2, \dots$ *the difference sequence* $x_{n+k} - x_n$, $n = 1, 2, \dots$ *has* $\Lambda(x_{n+k} - x_n) = \emptyset$, *then* $\Lambda(x_n) = \emptyset$.

*Proof.* Van der Corput difference theorem in the form Theorem 14 gives that if $G((x_{n+k} - x_n) \bmod 1)$, $k = 1, 2, \dots$, contains only continuous d.f.s, then the same holds for $G(x_n \bmod 1)$. Thus we have the implication

$$\Lambda((x_{n+k} - x_n) \bmod 1) = \emptyset, k = 1, 2, \dots \Longrightarrow \Lambda(x_n \bmod 1) = \emptyset$$

. □

**Theorem 39.** *For any two sequences $x_n$ and $y_n$ we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |x_n - y_n| = 0 \Longrightarrow \Lambda(x_n) = \Lambda(y_n).$$

*Proof.* For $x_n, y_n \in [0,1]$ in Theorem 11 we have
$\lim_{N\to\infty} N^{-1} \sum_{n=1}^{N} |x_n - y_n| = 0 \Longrightarrow G(x_n) = G(y_n)$
and thus $\Lambda(x_n) = \Lambda(y_n)$ is applying Theorem 37. $\square$

### 3.3.1 Examples

**Example 9.** Assume $G(x_n \bmod 1) = \{g(x)\}$ and $g(x) = x$, thus the sequence $x_n$ is u.d. mod1. Applying Theorem 37 and the continuity of $g(x) = x$ we find

$$\Lambda(x_n \bmod 1) = \emptyset.$$

E.g. $\Lambda$-set is empty for the following sequences:

$n\theta \bmod 1$ with irrational $\theta$ (cf. [171, 2.8.1]);

$n^2\theta + \sin(2\pi\sqrt{n}) \bmod 1$ with irrational $\theta$ (cf. [171, 2.13.2]);

$\log F_n \bmod 1$ with Fibonacci numbers $F_{n+1} = F_n + F_{n-1}$,
$F_1 = F_2 = 1$ (cf. [171, 2.12.21]);

$n \log\log \dots \log n \bmod 1$ (cf. [171, 2.12.5]);

etc.

**Example 10.** By Example 1 (cf. [171, 2.12.1])

$$G(\log n \bmod 1) = \left\{ g_u(x) = \frac{1}{e^u}\frac{e^x - 1}{e - 1} + \frac{e^{\min(x,u)} - 1}{e^u}; u \in [0,1] \right\},$$

Since all distribution functions in $G(\log n \bmod 1)$ are continuous, we have

$$\Lambda(\log n \bmod 1) = \emptyset.$$

Another proof follows from Theorem 12 because for $x_n = \log n \bmod 1$, we have

$$\omega_h = \frac{1}{4\pi^2 h^2 + 1}.$$

More generally, for $x_n = t\log n \bmod 1$, $t \neq 0$, we have

$$\omega_h = \frac{1}{4\pi^2h^2t^2 + 1}$$

which implies $\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}\omega_h = 0$ and thus by Theorem 12 we have $\Lambda(t\log n \bmod 1) = \emptyset$ for any $t \neq 0$. For computing $\omega_h$ we have used a method described in [124, Solution 5.18, p. 281].

**Example 11.** It is proved in Example 20 (cf. [160], [171, 2.12.2]) that starting with $\log\log n \bmod 1$ all the sequences $\log\log\dots\log n \bmod 1$ have

$$G(\log\log\dots\log n \bmod 1) = \{c_\alpha(x); \alpha\in[0,1]\}\cup\{h_\alpha(x); \alpha\in[0,1]\}.$$

Here $c_\alpha : [0,1]\to[0,1]$ is a one-jump distribution function

$$c_\alpha(x) = \begin{cases} 0, & \text{if } x\in[0,\alpha),\\ 1, & \text{if } x\in(\alpha,1] \end{cases}$$

$c_\alpha(0) = 0$, $c_\alpha(1) = 1$, and $c_\alpha(\alpha) = 0$ if $0<\alpha<1$; $h_\alpha : [0,1]\to[0,1]$ is constant distribution function, where $h_\alpha(0) = 0$, $h_\alpha(1) = 1$, and $h_\alpha(x) = \alpha$ if $x\in(0,1)$.

Applying Theorem 37 we have

$$\Lambda(\log\log\dots\log n \bmod 1) = [0,1].$$

**Example 12.** Let $\alpha = \frac{p}{q}\pi$, where $p$ and $q$ are positive integers and g.c.d. $(p,q) = 1$. It is proved in [20] that the sequence

$$x_n = n\cos(n\cos n\alpha) \bmod 1, \quad n = 1,2,\dots$$

has $G(x_n) = \{g(x)\}$, where

$$g(x) = \begin{cases} x & \text{if } q \text{ is odd },\\ \left(1-\frac{1}{q}\right)x + \frac{1}{q}c_0(x) & \text{if } q \text{ is even }, \end{cases}$$

and $c_0(x)$ is introduced in Example 11 for $\alpha = 0$. Theorem 37 implies

$$\Lambda(x_n) = \begin{cases} \emptyset & \text{if } q \text{ is odd },\\ \{0\} & \text{if } q \text{ is even }. \end{cases}$$

(I) Fridy has shown that the set $\Lambda(x_n)$ need not be closed or open in $\mathbb{R}$. In [90] is proved that the set $\Lambda(x_n)$ is an $F_\sigma$-set in $\mathbb{R}$ for an arbitrary sequence $x_n$ and vice-versa for any given $F_\sigma$-set $X$ there exists a sequence $x_n$ such that $X = \Lambda(x_n)$.

## 3.4 Statistically convergent sequences

Let $x_n$, $n = 1, 2, \dots$, be a sequence of real numbers, it may be unbounded.
• The sequence $x_n$ is said to be *statistically convergent* to the number $\alpha$ provided such that for each $\varepsilon > 0$,

$$\lim_{N\to\infty} \frac{1}{N}\#\{n \le N; |x_n - \alpha| \ge \varepsilon\} = 0. \tag{67}$$

The definition of statistical convergence was given by H. Fast [50] and I.J. Schoenberg [147], independently. Schoenberg gives the concept of statistical convergence from the viewpoint of d.f.s. For $x \in (-\infty, \infty)$ he define
• $F_N(x) = \frac{1}{N}\#\{n \le N; x_n < x\}$. The sequence $x_n$ is said to have the a.d.f. $g(x)$ if the limiting relation $\lim_{N\to\infty} F_N(x) = g(x)$ holds for every point $x$ of continuity of $g(x)$ and define one-jump function in $x \in (-\infty, \infty)$

$$c_\alpha(x) = \begin{cases} 0, & \text{if } x \le \alpha, \\ 1, & \text{if } x > \alpha. \end{cases}$$

Schoenberg [147] proved the following results (I) and (II).
(I) The sequence $x_n$ is statistically convergent to the number $\alpha$ if and only if the sequence $x_n$ admits the a.d.f. $c_\alpha(x)$.
(II) The sequence $x_n$ is statistically convergent to the number $\alpha$ if and only if $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e^{2\pi\imath t x_n} = e^{2\pi\imath t\alpha}$ for every real $t$.

For bounded sequence $x_n$ the claim (II) can be extended to:
(III) The sequence $x_n$ in $[a, b]$ is statistically convergent to $\alpha$ if and only if for every real-valued continuous function $f(x, y, z, \dots)$ defined on the closed multi-dimensional $[a, b]^s$ cube we have

$$\lim_{M,N,K,\dots\to\infty} \frac{1}{MNK\dots}\sum_{m=1}^{M}\sum_{n=1}^{N}\sum_{k=1}^{K}\dots f(x_m, x_n, x_k, \dots) = f(\alpha, \alpha, \alpha\dots).$$

(IV) The sequence $x_n$ in $[a, b]$ possesses a statistical limit if and only if

$$\lim_{M,N\to\infty} \frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N} |x_m - x_n| = 0.$$

(V) The sequence $x_n$ in $[a, b]$ is statistically convergent to $\alpha$ if and only if

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} x_n = \alpha, \lim_{N\to\infty} \frac{1}{N^2}\sum_{m,n=1}^{N} |x_m - x_n| = 0.$$

(VI) The sequence $x_n$ in $[a,b]$ is statistically convergent to zero, or equivalently, $x_n$ has the limiting distribution $c_0(x)$ if and only if

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}|x_n|=0.$$

(VII) the $L^2$ discrepancy related to $c_\alpha(x)$ becomes the form

$$\int_0^1 \left(F_N(x)-c_\alpha(x)\right)^2 \mathrm{d}x=\frac{1}{N}\sum_{n=1}^{N}|x_n-\alpha|-\frac{1}{2N^2}\sum_{m,n=1}^{N}|x_m-x_n|.$$

In addition, for $\alpha=0$ we have the equality

$$\int_0^1 \left(F_N(x)-c_0(x)\right)^2 \mathrm{d}x=\int_0^1 \left(F_N(x)-1\right)^2 \mathrm{d}x=$$

$$=\frac{1}{N}\sum_{n=1}^{N}\left(2x_n-x_n^2-\frac{1}{3}\right)+\int_0^1 \left(F_N(x)-x\right)^2 \mathrm{d}x,$$

with the classical $L^2$ discrepancy $\int_0^1 \left(F_N(x)-x\right)^2 \mathrm{d}x$.
(VIII) Let $f:[0,1]\to\mathbb{R}$ be a function with continuous derivation on $[0,1]$, and let $x_1,x_2,\dots,x_N$ be the finite sequence of points in $[0,1]$. Then for any $\alpha\in[0,1]$ we have

$$\left|\frac{1}{N}\sum_{n=1}^{N}f(x_n)-f(\alpha)\right|\le$$

$$\le\sqrt{\frac{1}{N}\sum_{n=1}^{N}|x_n-\alpha|-\frac{1}{2N^2}\sum_{m,n=1}^{N}|x_m-x_n|}\cdot\sqrt{\int_0^1 f'^2(x)\mathrm{d}x}.$$

Furthermore, the sequence $x_n$ in $[0,1]$ has the statistical limit $\alpha$ if and only if

$$\lim_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N}|x_n-\alpha|-\frac{1}{2N^2}\sum_{m,n=1}^{N}|x_m-x_n|\right)=0.$$

### 3.4.1 Examples

(I) For a positive integer $n$ and a prime number $p$, let $\mathrm{ord}_p(n) = k$ denotes that the prime-power $p^k$ divides $n$ but $p^{k+1}$ does not divide $n$. Then the sequence

$$x_n = \log p \frac{\mathrm{ord}_p(n)}{\log n}, \quad n = 2, 3, \dots$$

is dense in $[0, 1]$ and statistically convergent to the zero-limit.
(II) For every positive integer $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$ let us denote $h(n) = \min(\alpha_1, \dots, \alpha_k)$ and $H(n) = \max(\alpha_1, \dots, \alpha_k)$. A. Schinzel and T. Šalát proved that the following sequences

$$\log 2 \frac{h(n)}{\log n}, \quad \log 2 \frac{H(n)}{\log n}, \quad n = 2, 3, \dots$$

are everywhere dense in $[0, 1]$ and statistically convergent to 0.
(III) In [68, p. 356] it is defined that $x_n \geq 0$ has the *normal order* $y_n$ if $x_n$ is approximately $y_n$ for almost all values of $n$. More precisely, for every positive $\varepsilon$ and almost all values of $n$ we have

$$(1 - \varepsilon) y_n < x_n < (1 + \varepsilon) y_n.$$

Clearly, as in [68] we mentioned that for $y_n > 0$ the sequence $x_n$ has the normal order $y_n$ if and only if $x_n/y_n$ statistically converges to 1. Here we list some known examples:

(i) The normal order of $\omega(n)$ is $\log \log n$. Here $\omega(n)$ denotes the number of different prime factors of $n$ [68, pp. 356–359].

(ii) The normal order of $\Omega(n)$ is $\log \log n$. Here $\Omega(n)$ denotes the total number of prime factors of $n$ [68, pp. 356–359].

(iii) The normal order of $\log d(n)$ is $\log 2 \log \log n$. Here $d(n)$ denotes the number of divisors of $n$ [68, pp. 356–359].

(iv) The normal order of $\omega(\phi(n))$ is $(\log \log n)^2/2$ [106, p. 36].

(v) The normal order of $\omega(\sigma_k(n))$ is $d(k)(\log \log n)^2/2$, where $\sigma_k(n) = \sum_{d|n} d^k$ and $d(n) = \sum_{d|n} 1$ [106, p. 96].

(vi) $\omega(p \pm 1)$ has the normal order $\log \log p$, where $p$ is a prime [106, p. 171].

## 3.5 Diophantine approximation generalized

See [105]. [10] Let $x_n$ be a sequence in $[0, 1]$ and $z_n > 0$ be a monotone sequence, $z_n \to 0$. The method 3.3 of statistical limit points can be used to study the set $x \in [0, 1]$ for which

$$|x - x_n| < z_n$$

holds for infinitely many $n$. The asymptotic density $d(n)$ of such $n$ can be computed.

From Theorem 37 it follows:

**Theorem 40.** *Let $x_n$ be a sequence in $[0, 1)$ such that the set $G(x_n)$ of all d.f.s of $x_n$ contains only continuous d.f.s. Then for every sequence $z_n > 0$, $z_n \to 0$, and every $x \in [0, 1]$ we have: If $|x - x_{n_k}| < z_{n_k}$, $k = 1, 2, \dots$, then the asymptotic density of indices $d(n_k) = 0$.*

For every u.d. sequence $x_n$ we have $G(x_n) = \{g(x)\}$, $g(x) = x$, and thus $d(n_k) = 0$. Another example:

**Example 13.** As we see in Example 1, the sequence

$$x_n = \log n \bmod 1, \quad n = 1, 2, \dots,$$

has the set of d.f.s

$$G(x_n) = \left\{ g_u(x) = \frac{e^{\min(x,u)} - 1}{e^u} + \frac{1}{e^u}\frac{e^x - 1}{e - 1}; u \in [0, 1] \right\},$$

and $\{\log N_k\} \to u$ implies $F_{N_k}(x) \to g_u(x)$. Thus, by Theorem 40, if

$$|x - \{\log n_k\}| < z_{n_k}, \quad k = 1, 2, \dots$$

then $\frac{k}{n_k} \to 0$ for every sequence $z_n > 0$, $z_n \to 0$.

The so called *uniformly maldistributed sequences* (u.m.) $x_n$ are characterized by that the set $G(x_n)$ containing all one-step d.f.s $c_\alpha(x)$. They were introduced by G. Myerson (1993) [109], see the following Section 3.7. For such sequences we have

[10] Other types of results can be found in D. Berend and A. Dubickas (2009) [21].

**Theorem 41.** *Let $x_n$ be a u.m. sequence in $[0,1)$. Then there exists a decreasing sequence $z_n > 0$, $z_n \to 0$, such that for every $x \in [0,1]$: The sequence of all indices $n_k$, $|x - x_{n_k}| < z_{n_k}$, $k = 1, 2, \dots$, has the upper asymptotic density $\overline{d}(n_k) = 1$.*

G. Myerson [109], gave the following example of a maldistributed sequence:

**Example 14.** The sequence

$$x_n = \{\log \log n\}, \quad n = 2, 3, \dots$$

has the set of d.f.s, by [61] (see Example 20 in Section 4.1)

$$G(x_n) = \{c_\alpha(x); \alpha \in [0,1]\} \cup \{h_\beta(x); \beta \in [0,1]\}$$

where $F_{N_k}(x) \to c_\alpha(x)$ if and only if $\{\log \log N_k\} \to \alpha$ and $F_{N_k}(x) \to h_\beta(x)$ if and only if $\{\log \log N_k\} \to 0$ and

$$\frac{e^{e^{[\log \log N_k]}}}{e^{e^{[\log \log N_k] + \{\log \log N_k\}}}} \to 1 - \beta.$$

In the following, for such sequence $x_n$, we find a concrete example of $z_n$ satisfying Theorem 41: Let

(i) $x_n = \{\log \log n\}$,
(ii) $z_n = Z_k$ for $n \in \left(e^{e^k}, e^{e^{k+1}}\right)$, $k = 0, 1, 2, \dots$,
(iii) $Z_k = \frac{1}{k^c}$ where $c > 0$ is an arbitrary large constant, and
(iv) $N_K = \left[e^{e^{K+x+Z_K}}\right]$, for $K = 1, 2, \dots$.

Then, for every $x \in [0,1]$, the sequence of all indices $n_k$, $|x - x_{n_k}| < z_{n_k}$, $k = 1, 2, \dots$, has the upper asymptotic density $\overline{d}(n_k) = 1$. Precise,

$$\lim_{K \to \infty} \frac{\{n \le N_K; |x - x_n| < z_n\}}{N_K} = 1 \tag{68}$$

for every $x \in [0,1]$.

*Proof.* We have

$$|x - x_n| < z_n \Longleftrightarrow x_n \in (x - z_n, x + z_n). \tag{69}$$

Now, define $z_n$ as a constant $z_n = Z_k$ for all integers $n \in \left(e^{e^k}, e^{e^{k+1}}\right)$, $k = 0, 1, 2, \ldots$. Then by (69)

$$\{n \in \left(e^{e^k}, e^{e^{k+1}}\right); |x - x_n| < z_n\} = \{n \in \left(e^{e^k}, e^{e^{k+1}}\right); x_n \in (x - Z_k, x + Z_k)\}. \tag{70}$$

Let $N_K$ be an integer from $\left(e^{e^K}, e^{e^{K+1}}\right)$ and assume that the sequence $Z_k$, $k = 0, 1, 2, \ldots$ is non-increasing. Then by (70)

$$\{n \leq N_K; |x - x_n| < z_n\} \supset \{n \leq N_K; x_n \in (x - Z_K, x + Z_K)\}. \tag{71}$$

Thus

$$\frac{\#\{n \leq N_K; |x - x_n| < z_n\}}{N_K} \geq \frac{\#\{n \leq N_K; x_n \in (x - Z_K, x + Z_K)\}}{N_K}$$

$$= \sum_{k=0}^{K-1} \frac{\#\{n \in \left(e^{e^k}, e^{e^{k+1}}\right); x_n \in (x - Z_K, x + Z_K)\}}{N_K} \tag{72}$$

$$+ \frac{\#\{n \leq N_K; n \in \left(e^{e^K}, e^{e^{K+1}}\right), x_n \in (x - Z_K, x + Z_K)\}}{N_K}. \tag{73}$$

As $K \to \infty$, then $\frac{e^{e^{K-1}}}{e^{e^K}} \to 0$ and thus the sum (72) also tends to zero. Since

$$n \in \left(e^{e^K}, e^{e^{K+1}}\right) \text{ and } x_n \in (x - Z_K, x + Z_K) \Longleftrightarrow n \in \left(e^{e^{K+x-Z_K}}, e^{e^{K+x+Z_K}}\right) \tag{74}$$

and putting $N_K = \left[e^{e^{K+x+Z_K}}\right]$, then the term (73) can be expressed as

$$\frac{\left[e^{e^{K+x+Z_K}}\right] - \left[e^{e^{K+x-Z_K}}\right] + 1}{\left[e^{e^{K+x+Z_K}}\right]}. \tag{75}$$

Omitting the integer parts in (75) leads to the form

$$1 - \frac{1}{e^{e^{K+x+Z_K}(1-e^{-2Z_K})}}. \tag{76}$$

By Lagrange theorem $(1 - e^{-2Z_K}) > 2Z_K.e^{-2Z_K}$ and the lower bound of (76) converge to

$$1 - \frac{1}{e^{e^{K+x-Z_K}(2Z_K)}} \to 1 \tag{77}$$

if $Z_K = \frac{1}{K^c}$, a constant $c > 0$ is an arbitrary large. $\square$

All the above results follows from the following two general theorems of limit points of $x_n$, $n = 1, 2, \dots$.

**Theorem 42.** *Let $x_0 \in [0,1]$ be a discontinuity point of d.f. $g(x) \in G(x_n)$ with a jump of size $h$. Then there exists a subsequence $x_{n_k}$ of $x_n$ such that*
(i) $\lim_{k\to\infty} x_{n_k} = x_0$,
(ii) $\overline{d}(n_k) = h$,
*where $\overline{d}(n_k)$ is the upper asymptotic density of $n_k$, $k = 1, 2, \dots$. In the opposite case,* (i) *and* (ii) *imply that there exists $g(x) \in G(x_n)$ such that $g(x)$ has at $x_0$ a jump of size $\geq h$.*

**Theorem 43.** *For every sequence $x_n \in [0,1)$ there exists a decreasing sequence $z_n \to 0$ such that*

$$\sup\{\overline{d}(n_k);\ x_{n_k} \to x\} = \overline{d}(\{n \in \mathbb{N};\ |x - x_n| < z_n\}$$

*for every $x \in [0,1]$.*

**Example 15.** Let $I \subset [0,1]$ be an interval with length $0 < |I| < 1$. By Theorem 49, part (iv), there exists sequence $x_n \in [0,1)$ such that $G(x_n) = \{c_\alpha(x); \alpha \in I\}$. By Theorem 42 and 43 there exists a positive monotonic sequence $z_n$, $z_n \to 0$ such that if $x \in I$ then for all $n_k$, $|x - x_{n_k}| < z_{n_k}$, we have $\overline{d}(n_k) = 1$. If $x \notin I$, then $d(n_k) = 0$.

## 3.6 The classical Diophantine approximation

For completeness we attaches: The Duffin-Schaeffer conjecture (D.S.C.) is one of the most important unsolved problems in metric number theory until now, cf. Encyclopaedia of Mathematics, M. Hazewinkel ed.:

Define that a class of sequences $q_n$ $n = 1, 2, \dots,$ of distinct positive integers and a class of functions $f(q)$ are said to satisfy D.S.C. if the divergence $\sum_{n=1}^{\infty} \varphi(q_n) f(q_n)$ implies that for almost all $x \in [0,1]$ there exist infinitely many $n$ such that the diophantine inequality

$$\left| x - \frac{p}{q_n} \right| < f(q_n), \qquad \gcd(p, q_n) = 1 \tag{78}$$

has an integer solution $p$. The inequality $|x - x_n| < z_n$ covers (78) in the form that we consider $x_n$ composed by blocks $A_n$

$$A_n = \left( \frac{1}{q_n}, \frac{a_2}{q_n}, \dots, \frac{a_{\varphi(q_n)}}{q_n} \right), \text{g.c.d.}(a_i, q_n) = 1$$

and if we put $z_n = f(q_m)$ for terms $x_i \in A_n$.
There are tree types of results of $q_n, f(q)$ satisfying D.S.C.:

(i) any one-to-one sequence $q_n$ and special $f(q)$, for example $f(q) = c/q^2$ (P. Erdős 1970);

(ii) any $f(q) \geq 0$ and a special $q_n$, for example $\frac{\varphi(q_n)}{q_n} \geq c > 0$ (R. J. Duffin and A. C. Schaeffer 1941);

(iii) special $q_n, f(q)$, for example $f(q_n)q_n > c_1(\varphi(q_n)/q_n)^{c_2}$ for some $c_1, c_2 > 0$ (G. Harman (1998) [41].

Almost complete known results can be found in Unsolved Problems [166, p. 217].

(iv) For $L^2$ discrepancy of $A_n$ cf. Section 6.2.19.

## 3.7 Uniformly maldistributed sequences

Here we mention some representative results, but complete are in section 6.6.

Let $x_n$, $n = 1, 2, \dots$, be a given sequence in $[0, 1)$.

(I) The sequence $x_n$ is said to be *uniformly maldistributed* (u.m.) if for every nonempty proper subinterval $I \subset [0, 1]$ we have both

$$\liminf_{N\to\infty} \frac{1}{N}\#\{n \leq N; x_n \in I\} = 0 \text{ and } \limsup_{N\to\infty} \frac{1}{N}\#\{n \leq N; x_n \in I\} = 1.$$

The definition of uniform maldistribution is due to G. Myerson [109] , who mentioned that the first condition is superfluous, and he showed that:

(II) The sequence $x_n = \{\log\log n\}$ of fractional parts of the iterated logarithm is u.m. The following results are from [160].

(III) By Theorem 183: The sequence $x_n$ is u.m. if and only if

$$\{c_\alpha(x); \alpha \in [0, 1]\} \subset G(x_n).$$

Thus, in the theory of uniform maldistribution we need not consider d.f.s other than one-jump d.f. $c_\alpha(x)$ which has a jump of size 1 at $\alpha$. This suggests the following definition.

(IV) The sequence $x_n$ is said to be *uniformly maldistributed in the strict sense* (u.m.s.) if $G(x_n) = \{c_\alpha(x); \alpha \in [0, 1]\}$.

(V) By Theorem 184: The sequence $x_n \in [0,1)$ is u.m.s. if and only if

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{m,n=1}^{N} |x_m - x_n| = 0 \text{ and } \limsup_{M,N\to\infty} \frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} |x_m - x_n| = 1,$$

or alternatively $\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_n - \liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_n = 1$.

(VI) By Example 103: Let $x_n$, $n = 1, 2, \dots$ be defined as

$$x_n = \left\{ 1 + (-1)^{\left[\sqrt{[\sqrt{\log_2 n}]}\right]} \left\{ \sqrt{\left[\sqrt{\log_2 n}\right]} \right\} \right\},$$

were $[x]$ denotes the integral part and $\{x\}$ the fractional part of $x$. Then $G(x_n) = \{c_\alpha(x); \alpha \in [0,1]\}$.

(VII) R. Winkler [193] extended definition (I) such that $x_n$, $n = 1, 2, \dots$, is called *maldistribution* if $G(x_n)$ is the set of all d.f.s. Such sequences form residual set in the space of all sequences, but in this space almost all sequences are u.d.

## 3.8 The sequence $\xi(3/2)^n$ mod 1

The question about distribution of $(3/2)^n$ mod 1 is a most difficult. Here we listed a selection of known conjectures. Some proofs and other results are given in Section 6.1.

(I) $(3/2)^n$ mod 1 is uniformly distributed in $[0,1]$.

(II) $(3/2)^n$ mod 1 is dense in $[0,1]$.

(III) (T. Vijayaraghavan [187]) $\limsup_{n\to\infty}\{(3/2)^n\} - \liminf_{n\to\infty}\{(3/2)^n\} > 1/2$, where $\{x\}$ is the fractional part of $x$.

(IV) (K. Mahler [102]) There exists no $\xi \in \mathbb{R}^+$ such that $0 \le \{\xi(3/2)^n\} < 1/2$ for $n = 0, 1, 2, \dots$. [11]

(V) (G. Choquet [27]) There exists no $\xi \in \mathbb{R}^+$ such that the closure of $\{\{\xi(3/2)^n\}; n = 0, 1, 2, \dots\}$ is nowhere dense in $[0,1]$.

[11] Such $\xi$, if exists, are called Mahler's $Z$-numbers.

Few positive results known, for instance:

(VI) L. Flatto, J. C. Lagarias and A. D. Pollington [56] stated that

$$\limsup_{n\to\infty}\{\xi(3/2)^n\} - \liminf_{n\to\infty}\{\xi(3/2)^n\} \geq \tfrac{1}{3}$$

for every $\xi > 0$.

(VII) G. Choquet [27] gave infinitely many $\xi \in \mathbb{R}$ for which

$$\tfrac{1}{19} \leq \{\xi(3/2)^n\} \leq 1 - \tfrac{1}{19} \text{ for } n = 0, 1, 2, \dots .$$

(VIII) R. Tijdeman [179] showed that for every pair of integers $k$ and $m$ with $k \geq 2$ and $m \geq 1$ there exists $\xi \in [m, m+1)$ such that

$$0 \leq \{\xi((2k+1)/2)^n\} \leq \tfrac{1}{2k-1} \text{ for } n = 0, 1, 2, \dots .$$

(IX) There are connection between $(3/2)^n$ mod 1 and Waring's problem (cf. M. Bennett [19]), and Mahler's conjecture (IV) and $3x+1$ problem (cf. [56]).

(X) A. Dubickas [39] proved that for any $\xi \neq 0$ the sequence of fractional parts $\{\xi(3/2)^n\}$, $n = 1, 2, \dots$, has at least one limit point in the interval $[0.238117\dots, 0.761882\dots]$ of the length $0.523764\dots$.

(X)* S. Akiyama, C. Frougny and J. Sakarovitch (2006): There is $\xi \neq 0$ such that $||\xi(3/2)^n|| < 1/3$ for $n = 1, 2, \dots$ The bound is equivalent $\{\xi(3/2)^n\} < 1/3$ or $2/3 < \{\xi(3/2)^n\}$

(XI) H. Helson and J.-P. Kahane [70] established the existence of uncountable many $\xi$ such that the sequence $\xi\theta^n$ does not have an a.d.f. mod 1, where $\theta$ is some fixed real numbers $> 1$.

In this Section and Section 6.1 we study the set of all d.f.s of sequences $\xi(3/2)^n$ mod 1, $\xi \in \mathbb{R}$, see [162]. It is motivated by the fact that some conjectures involving a d.f. $g(x)$ of $\xi(3/2)^n$ mod 1 may be formulated strongly as in (I)–(IV). For example, the following conjecture implies Mahler's conjecture (IV):

(XII) If $g(x) =$ constant for all $x \in I$, where $I$ is a subinterval of $[0, 1]$, then the length $|I| < 1/2$. Except $h_{1/2}(x)$, $c_0(x)$, $c_1(x)$.

(XII)* Precisely, does not exist $g(x) \in G(\xi(3/2)^n \bmod 1)$ such that $g(x) = 1$ for $x \in (1/2, 1]$ except $c_0(x)$.

(XIII) Pjateckiĭ-Šapiro [128], by means of the ergodic theory, proved that a necessary and sufficient condition that the sequence $\xi q^n \bmod 1$ with integer $q > 1$ has a distribution function $g(x)$ is that $g_f(x) = g(x)$ for all $x \in [0, 1]$, where $f(x) = qx \bmod 1$ (also see Notes 20).

The following mapping $g \to g_f$ is the main tool for the study of $G(\xi(3/2)^n \bmod 1)$:

Let $f : [0, 1] \to [0, 1]$ be a function such that, for all $x \in [0, 1]$, $f^{-1}([0, x))$ can be expressed as a sum of finitely many pairwise disjoint subintervals $I_i(x)$ of $[0, 1]$ with endpoints $\alpha_i(x) \le \beta_i(x)$. For any distribution function $g(x)$ we put [12]

$$g_f(x) = \sum_i g(\beta_i(x)) - g(\alpha_i(x)).$$

A basic property of $g_f(x)$ is expressed by the following statement, but complete results and proofs are in Section 6.1.

(XIV) Let $x_n \bmod 1$ be a sequence having $g(x)$ as a distribution function associated with the sequence of indices $N_1, N_2, \dots$. Suppose that any term $x_n \bmod 1$ is repeated only finitely many times. Then the sequence $f(\{x_n\})$ has the distribution functions $g_f(x)$ for the same $N_1, N_2, \dots$, and vice-versa any distribution function of $f(\{x_n\})$ has this form (Theorem 106).

In this part we consider $f(x)$ and $h(x)$ as

$$f(x) = 2x \bmod 1, \text{ and } h(x) = 3x \bmod 1.$$

In this case, for every $x \in [0, 1]$, we have

$$g_f(x) = g(f_1^{-1}(x)) + g(f_2^{-1}(x)) - g(1/2),$$

$$g_h(x) = g(h_1^{-1}(x)) + g(h_2^{-1}(x)) + g(h_3^{-1}(x)) - g(1/3) - g(2/3),$$

with inverse functions

$$f_1^{-1}(x) = x/2, \quad f_2^{-1}(x) = (x + 1)/2,$$

[12] Generally $g_f(x) = \int_{f^{-1}([0,x))} 1.\mathrm{d}g(x)$.

and

$$h_1^{-1}(x) = x/3, \quad h_2^{-1}(x) = (x+1)/3, \quad h_3^{-1}(x) = (x+2)/3.$$

For $\xi(3/2)^n \bmod 1$ we have the following necessity similar to (XIII).

(XV) Any distribution function $g(x)$ of $\xi(3/2)^n \bmod 1$ satisfies $g_f(x) = g_h(x)$ for all $x \in [0,1]$ (Theorem 107).

(XVI) Let $g_1, g_2$ be any two distribution functions satisfying $g_{i_f}(x) = g_{i_h}(x)$ for $i = 1,2$ and $x \in [0,1]$. Denote

$$I_1 = [0, 1/3], \quad I_2 = [1/3, 2/3], \quad I_3 = [2/3, 1].$$

If $g_1(x) = g_2(x)$ for $x \in I_i \cup I_j$, $1 \le i \ne j \le 3$, then $g_1(x) = g_2(x)$ for all $x \in [0,1]$ (Theorem 108).

(XVII) Here we give an integral formula for testing $g_f = g_h$. Denote

$$F(x,y) = |\{2x\} - \{3y\}| + |\{2y\} - \{3x\}| - |\{2x\} - \{2y\}| - |\{3x\} - \{3y\}|.$$

The continuous distribution function $g$ satisfies $g_f = g_h$ on $[0,1]$ if and only if (Theorem 111)

$$\int_0^1 \int_0^1 F(x,y) \mathrm{d}g(x) \mathrm{d}g(y) = 0.$$

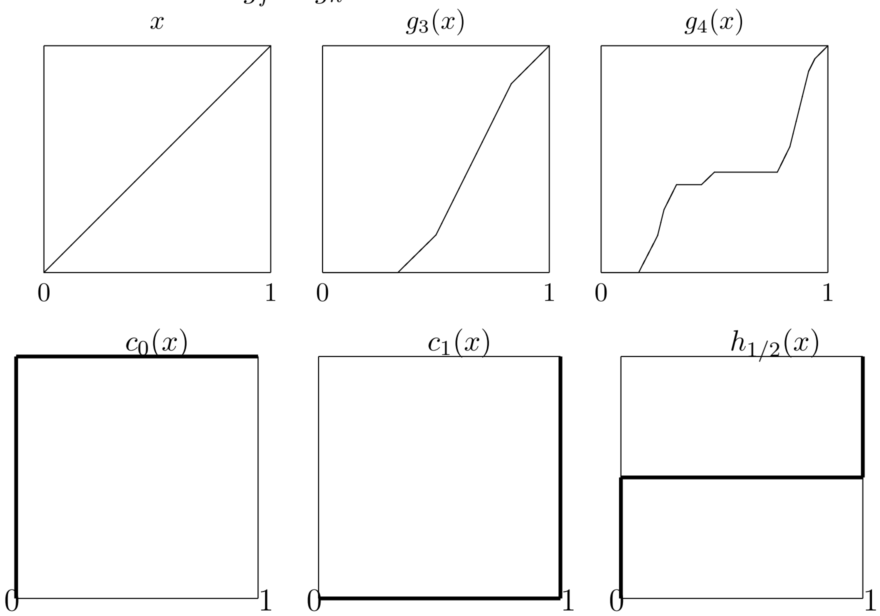
(XVIII) The d.f.s $c_0(x)$, $c_1(x)$, and $x$ solve $g_f(x) = g_h(x)$ for all $x \in [0,1]$. Putting $g_1(x) = x$, a next solution of $g_f = g_h$ we found by Theorem 114 in Example 49:

$$g_3(x) = \begin{cases} 0 & \text{for } x \in [0, 2/6], \\ x - 1/3 & \text{for } x \in [2/6, 3/6], \\ 2x - 5/6 & \text{for } x \in [3/6, 5/6], \\ x & \text{for } x \in [5/6, 1]. \end{cases}$$

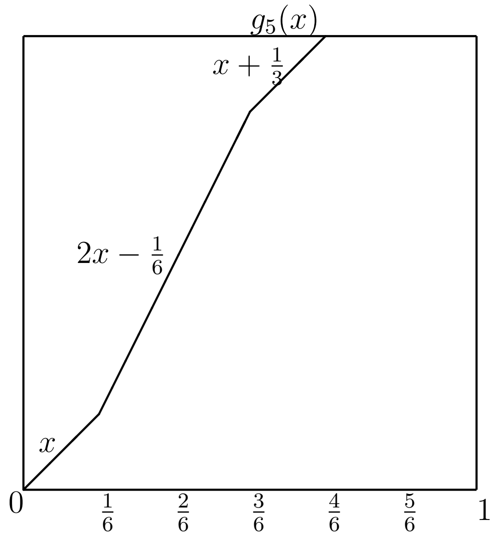
Again applying Theorem 114 to starting $g_3(x)$ we find in Example 50

$$g_4(x) = \begin{cases} 0 & \text{for } x \in [0, 1/6], \\ 2x - 1/3 & \text{for } x \in [1/6, 3/12], \\ 4x - 5/6 & \text{for } x \in [3/12, 5/18], \\ 2x - 5/18 & \text{for } x \in [5/18, 2/6], \\ 7/18 & \text{for } x \in [2/6, 8/18], \\ x - 1/18 & \text{for } x \in [8/18, 3/6], \\ 8/18 & \text{for } x \in [3/6, 7/9], \\ 2x - 20/18 & \text{for } x \in [7/9, 5/6], \\ 4x - 50/18 & \text{for } x \in [5/6, 11/12], \\ 2x - 17/18 & \text{for } x \in [11/12, 17/18], \\ x & \text{for } x \in [17/18, 1]. \end{cases}$$

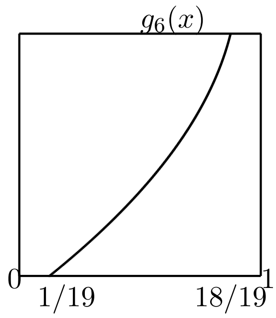
Thus our solution $g_f = g_h$ are



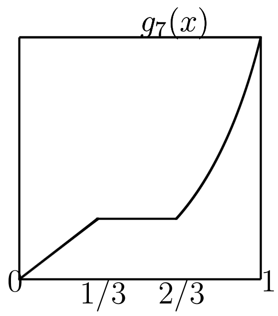

In Example 55 is constructed next solution $g_f = g_h$




G. Choquet's result (VII) gives the existence of infinitely many $g_6(x)$ which solve $g_f = g_h$ with a graph similar to




S. Akiyama, C. Frougny and J. Sakarovitch (2006) result (X)* gives a solution $g_7(x)$ of $g_f = g_h$ of the form



(XIX) Since $x_f = x_h$, Theorem 108 gives the following example of d.f. $g(x)$, which is not d.f. of the sequence $\xi(3/2)^n \bmod 1$, for every $\xi \in \mathbb{R}$

$$g(x) = \begin{cases} x & \text{for } x \in [0, 2/3], \\ x^2 - (2/3)x + 2/3 & \text{for } x \in [2/3, 1]. \end{cases}$$

See Example 56.

(XX) Mahler's conjecture (IV) follows from:

Does not exist d.f. $g(x)$ such that

$g_f(x) = g_h(x)$ for $x \in [0, 1]$ and

$g(x) = 1$ for $x \in [1/2, 1]$.

(XXI) Since $[2\xi(3/2)^n] = 2[\xi(3/2)^n]$ if and only if $\{\xi(3/2)^n\} < 1/2$ (otherwise $[2\xi(3/2)^n] = 2[\xi(3/2)^n]+1$) Mahler's problem (IV), p. 57, can be reformulated as follows: Prove that, for any $\xi > 0$, the sequence $[2\xi(3/2)^n]$, $n = 1, 2, \ldots$, contains infinitely many odd numbers (see A. Dubickas [40].)

(XXII) Any d.f. of the linear combination of solutions $g_f = g_h$ also solve $g_f = g_h$.

By Theorem 112 d.f. $g(x)$ and $1-g(1-x)$ solve $g_f = g_h$ simultaneously.

If $g_1(x)$ solve $g_f = g_h$, then $g_2(x) = g_{1f}(x)$ also solve $g_f = g_h$.

## 3.9 Benford's law

The first digit problem: An infinite sequence $x_n \geq 1$ of real numbers satisfies *Benford's law*, if the frequency (the asymptotic density) of occurrences of a given first digit $a$, when $x_n$ is expressed in the decimal form is given by $\log_{10}\left(1 + \frac{1}{a}\right)$ for every $a = 1, 2, \ldots, 9$ (0 as a possible first digit is not admitted). Since $x_n$ has the first digit $a$ if and only if $\log_{10} x_n \bmod 1 \in [\log_{10} a, \log_{10}(a+1))$ , Benford's law for $x_n$ follows from the u.d. of $\log_{10} x_n \bmod 1$. It was S. Newcomb (1881) who firstly noted "*That the ten digits do not occur with equal frequency must be evident to anyone making use of logarithm tables*". F. Benford (1938) compared the empirical frequency of occurrences of $a$ with $\log_{10}((a+1)/a)$ in twenty different tables having lengths running from 91 entries (atomic weights) to 5000 entries in a mathematical handbook which led him to the conclusion that "the logarithmic law applies particularly to those outlaw numbers that are without known relationships ..." For the asymptotic density of the second-place digit $b$ he found $\sum_{a=1}^{9} \log_{10}\left(1+\frac{1}{10a+b}\right)$.

**Notes 3.** If $x_n$ satisfies B.L., then the asymptotic density of $n$ for which $x_n$ has in the $r$th place the digit $a$ is

$$\sum_{k_1=1}^{b-1}\sum_{k_2=0}^{b-1}\cdots\sum_{k_{r-1}=0}^{b-1} \log_b(k_1.k_2k_3\dots k_{r-1}(a+1)) - \log_b(k_1.k_2k_3\dots k_{r-1}a)$$
$$= \log_b \prod \frac{k_1k_2k_3\dots k_{r-1}(a+1)}{k_1k_2k_3\dots k_{r-1}a}.$$

F. Benford rediscovered Newcomb's observation from (1881).

In the following we apply to the first digit problem theory of d.f.s. A.I. Pavlov [124] was the first who applied this theory however Theorem 170 offers new results.

### 3.9.1 Basic results

Precise: let $b \geq 2$ be an integer considered as a base for the development of positive real number $x > 0$ and $M_b(x)$ be a mantissa of $x$ defined by $x = M_b(x) \times b^{n(x)}$ such that $1 \leq M_b(x) < b$ holds, where $n(x)$ is a uniquely determined integer. Let $K = k_1k_2\dots k_r$ be a positive integer expressed in the base $b$ , that is

$$K = k_1 \times b^{r-1} + k_2 \times b^{r-2} + \cdots + k_{r-1} \times b + k_r,$$

where $k_1 \neq 0$ and at the same time $K = k_1k_2\dots k_r$ is considered as an $r-$consecutive block of integers in the base $b$ . It is clear that the following basic equivalences hold:

$$K \leq M_b(x) \times b^{r-1} < K+1 \iff \tag{79}$$
$$\frac{K}{b^{r-1}} \leq M_b(x) < \frac{K+1}{b^{r-1}} \iff$$
$$\log_b\left(\frac{K}{b^{r-1}}\right) \leq \log_b(M_b(x)) < \log_b\left(\frac{K+1}{b^{r-1}}\right) \iff$$
$$\log_b\left(\frac{K}{b^{r-1}}\right) \leq \log_b x \bmod 1 < \log_b\left(\frac{K+1}{b^{r-1}}\right). \tag{80}$$

Note that for $x$ of the type $x = 0.00\dots0k_1k_2\dots k_r\dots$, the first zero digits we shall omit and $M_b(x) = k_1.k_2\dots k_r\dots$

**Definition 2.** A sequence $x_n$, $n = 1, 2, \dots$, of positive real numbers satisfies Benford law (abbreviated to B.L.) of order $r$ if for every $r$-digits number $K = k_1k_2\dots k_r$ we have

$$\lim_{N\to\infty} \frac{\#\{n \le N; \text{leading block of } r \text{ digits (beginning with} \ne 0) \text{ of } x_n = K}{N}$$
$$= \log_b\left(\frac{K+1}{b^{r-1}}\right) - \log_b\left(\frac{K}{b^{r-1}}\right). \tag{81}$$

**Theorem 44.** *Let $G(\log_b(x_n) \bmod 1)$ be the set of all d.f.s of the sequence $\log_b(x_n) \bmod 1$, $n = 1, 2, \dots$. Then $x_n$ satisfies B.L. of order $r$ if and only if for every $g(x) \in G(\log_b(x_n) \bmod 1)$ we have*

$$g\left(\log_b\left(\frac{K}{b^{r-1}}\right)\right) = \log_b\left(\frac{K}{b^{r-1}}\right), \tag{82}$$

*for every $r$-digits positive integer $K = k_1k_2\dots k_r$.*

*Proof.* Let $x_n > 0$, $n = 1, 2, \dots$, $g(x) \in G(\log_b(x_n) \bmod 1)$ and $g(x) = \lim_{i\to\infty} F_{N_i}(x)$, where

$$F_{N_i}(x) = \frac{\#\{n \le N_i; \log_b(x_n) \bmod 1 \in [0, x)\}}{N_i}. \tag{83}$$

Using equivalent inequalities (79) and (80), then for fixed $N_i$ we have

$$\frac{\#\{n \le N_i;\ \text{first } r \text{ digits of } M_b(x_n) \text{ are equal to } K\}}{N_i}$$
$$= \frac{\#\{n \le N_i; \log_b\left(\frac{K}{b^{r-1}}\right) \le \log_b x_n \bmod 1 < \log_b\left(\frac{K+1}{b^{r-1}}\right)\}}{N_i}$$
$$= F_{N_i}\left(\frac{K}{b^{r-1}}\right) - F_{N_i}\left(\frac{K+1}{b^{r-1}}\right)$$

and (81) can be rewritten as

$$\lim_{i\to\infty}\left(F_{N_i}\left(\frac{K}{b^{r-1}}\right) - F_{N_i}\left(\frac{K+1}{b^{r-1}}\right)\right) = \log_b(K+1) - \log_b K. \tag{84}$$

Thus

$$g\left(\log_b\left(\frac{K+1}{b^{r-1}}\right)\right) - g\left(\log_b\left(\frac{K}{b^{r-1}}\right)\right) = \log_b\left(\frac{K+1}{b^{r-1}}\right) - \log_b\left(\frac{K}{b^{r-1}}\right). \tag{85}$$

Summing up (85) over all $r$-digits integer numbers $K'$, $K' < K$, we find (82). □

Definition 2 and Theorem 44 can be found implicitly in [126].

**Example 16.** For every $r = 1, 2, \dots$ there exists a sequence $x_n$, $n = 1, 2, \dots$ such that $x_n$ satisfies B.L. of order $r$, but does not satisfy B.L. of order $r+1$ in the base $b \geq 2$.

*Proof.* Let

$$A_r = \{\log_b(k_1.k_2k_3\dots k_r); k_i \in \{0, 1, \dots, b-1\}, i = 1, 2, \dots, r, k_1 \neq 0\},$$

where, in this case, $k_1.k_2k_3\dots k_r = \frac{K}{b^{r-1}}$ and $K$ is a positive integer expressed in the base $b$ , i.e. $K = k_1 \times b^{r-1} + k_2 \times b^{r-2} + \cdots + k_{r-1} \times b + k_r$. Define a d.f. $g(x)$ such that $g(x) = x$ for every $x \in A_r$, but $g(x) \neq x$ for some $x \in A_{r+1}$. Now, by the well known theorem stating that for every d.f. $g(x)$ there exists a sequence $y_n \in [0, 1)$, $n = 1, 2, \dots$, such that $g(x)$ is the a.d.f of the sequence $y_n$ , cf. [92, p. 138]. Then let us define a sequence $x_n$ as $x_n = b^{y_n}$ , that is $\log_b x_n = y_n$ , for $n = 1, 2, \dots$. Applying Theorem 44 we see that $x_n$ satisfies B.L. of order $r$ and does not satisfy B.L. of order $r+1$. □

Directly from the proof of Example 16 we obtain the following: If the sequence $x_n$, $n = 1, 2, \dots$ satisfies B.L. of order $r$, then it satisfies of order $r' \leq r$, simultaneously.

**Definition 3** (P. Diaconis (1977)[34])**.** If a sequence $x_n$, $n = 1, 2, \dots$, satisfies B.L. of order $r$, for every $r = 1, 2, \dots$, then it is called that $x_n$ satisfies strong B.L. in base $b$. In the following we shall write strong B.L. again as B.L., i.e. the word *strong* is omitted.

From Theorem 44 it leads directly to: [13]

**Theorem 45.** *A sequence $x_n$, $x_n > 0$, $n = 1, 2, \dots$, satisfies B.L. if and only if the sequence* $\log_b x_n \bmod 1$ *is u.d. in* $[0, 1)$.

[13] Well known Theorem 45 can also be found in [37] and [126]. In the following u.d. of $\log_b x_n \bmod 1$ we characterize by $G(x_n)$, see [13].

*Proof.* Let $g(x) \in G(\log_b(x_n) \bmod 1)$. Theorem 44 implies that $g(x) = x$ for all $x = \log_b\left(\frac{K}{b^{r-1}}\right)$. Since the set of such $x$ are dense for $K = k_1k_2\dots k_r$, $k_1 \neq 0$, $0 \leq k_i < b$, $r = 1, 2, \dots$, then we have $g(x) = x$ for all $x \in [0, 1]$. $\square$

**Example 17.** Fibonacci numbers $F_n$, $n!$ ([93]), $n^n$, $n^{n^2}$, satisfy B.L.

Many authors think that if the sequence $x_n$ does not satisfy B.L., then the relative density of indices $n$ for which the $b$-expansion of $x_n$ start with leading digits $K = k_1k_2\dots k_r$

$$\frac{1}{N}\#\left\{n \leq N; \log_b\left(\frac{K}{b^{r-1}}\right) \leq \{\log_b x_n\} < \log_b\left(\frac{K+1}{b^{r-1}}\right)\right\}.$$

*do not follow any distribution in the sense of natural density*, see S. Eliahou, B. Massé and D. Schneider (2013) [42]. These authors as an alternate result shown that the sequence $\log_{10} n^r \bmod 1$, $n = 1, 2, \dots$, and the sequence $\log_{10} p_n^r \bmod 1$ $n = 1, 2, \dots$, $p_n$ are all prime numbers, have the discrepancy $O(r^{-1})$. Thus, for $r \to \infty$, these sequences tends to u.d. and thus $n^r$ and $p_n^r$ tends to B.L. We propose the following solution:

### 3.9.2 General scheme of solution of the First Digit Problem

**Theorem 46.** *Let* $g(x) \in G(\log_b x_n \bmod 1)$ *and* $\lim_{i\to\infty} F_{N_i}(x) = g(x)$. *Then*

$$\lim_{N_i\to\infty} \frac{\#\{n \leq N_i;\ \textit{first } r \textit{ digits (starting a non-zero digit) of } x_n = K\}}{N_i}$$
$$= g\left(\log_b\left(\frac{K+1}{b^{r-1}}\right)\right) - g\left(\log_b\left(\frac{K}{b^{r-1}}\right)\right). \tag{86}$$

*Proof.* Using step d.f.
$F_N(x) = \frac{1}{N}\#\{n \leq N; \log_b x_n \bmod 1 \in [0, x)\}$ we have
$F_N\left(\log_b\left(\frac{K+1}{b^{r-1}}\right)\right) - F_N\left(\log_b\left(\frac{K}{b^{r-1}}\right)\right)$
$= \frac{1}{N}\#\{n \leq N;$ first $r$ digits (starting a non-zero digit) of $x_n = K\}$. $\square$

This is the general scheme of solution of the First Digit Problem for the sequence $x_n$, $n = 1, 2, \dots$, for which $\log_b x_n \bmod 1$ is not u.d. sequence. This approach is also presented in [13] and [123].

**Example 18.** As it is well known that the increasing sequence of all positive integers $1, 2, 3, \dots$ does not satisfy B.L. (simple B.L. also) in every base $b \geq 2$. It follows from the fact that $\log_b n \bmod 1$ is not u.d. For a density of $n$ for which $r$ initial digits are $K = k_1 k_2 \dots k_r$, A.I. Pavlov [126] proved that

$$\liminf_{N\to\infty} \frac{\#\{n \leq N; n \text{ has the first } r \text{ digits } = K\}}{N} = \frac{1}{K(b-1)}, \tag{87}$$

$$\limsup_{N\to\infty} \frac{\#\{n \leq N; n \text{ has the first } r \text{ digits } = K\}}{N} = \frac{b}{(K+1)(b-1)}. \tag{88}$$

We give the following extension: By G. Pólya and G. Szegö [129] the d.f. of $\log_b n \bmod 1$ is of the form (cf. Theorem 48 and (100))

$$g_u(x) = \frac{1}{b^u}\frac{b^x - 1}{b - 1} + \frac{b^{\min(x,u)} - 1}{b^u}, \tag{89}$$

where the parameter $u$ runs $[0, 1]$. By [61], for increasing sequence $N_i$, $i = 1, 2, \dots$, we have (see def. (83))

$$\log_b N_i \bmod 1 \to u \Longrightarrow F_{N_i}(x) \to g_u(x) \tag{90}$$

and thus

$$\frac{\#\{n \leq N_i; n \text{ has the first } r \text{ digits } = K\}}{N_i} \to g_u(x_2) - g_u(x_1) \tag{91}$$

as $i \to \infty$, where $x_1 = \log_b(k_1.k_2k_3 \dots k_r)$ and $x_2 = \log_b(k_1.k_2k_3 \dots (k_r + 1))$. Now, the Pavlov results (87) and (88) follow from

$$\liminf_{N\to\infty} \frac{\#\{n \leq N; n \text{ has the first } r \text{ digits } = K\}}{N} = \min_{u\in[0,1]} (g_u(x_2) - g_u(x_1)),$$

$$\limsup_{N\to\infty} \frac{\#\{n \leq N; n \text{ has the first } r \text{ digits } = K\}}{N} = \max_{u\in[0,1]} (g_u(x_2) - g_u(x_1)),$$

where the minimum is appeared in $u = x_1$ and maximum in $u = x_2$. Furthermore, if $N_i = [b^{i+u}]$, then $\log_b N_i \bmod 1 \to u$ as $i \to \infty$.

Note that Pavlov formulated his result for more general sequence $an^\alpha$, $n = 1, 2, \dots$, where $a$ and $\alpha$ are fixed positive real numbers. Putting $f(x) = c_1 + c_2 \log x$, where $c_1 = \frac{\log a}{\log b}$ and $c_2 = \frac{\alpha}{\log b}$, we have $\log_b x_n = f(n)$ and applying Theorem 1 in [61] then we find that every d.f. $g_u(x) \in G(\log_b x_n)$ has the form

$$g_u(x) = \frac{1}{b^{\frac{u}{\alpha}}} \frac{b^{\frac{x}{\alpha}} - 1}{b^{\frac{1}{\alpha}} - 1} + \frac{b^{\frac{\min(x,u)}{\alpha}} - 1}{b^{\frac{u}{\alpha}}}, \tag{92}$$

where the parameter $u$ runs $[0,1]$ and for an increasing sequence $N_i$, $i = 1, 2, \dots$, we have

$$\frac{\log a}{\log b} + \alpha \frac{\log N_i}{\log b} \bmod 1 \to u \Longrightarrow F_{N_i}(x) \to g_u(x). \tag{93}$$

Now, computing derivative $\frac{\mathrm{d}}{\mathrm{d}u}(g_u(x_2) - g_u(x_1))$ in (91) we can see that its signum is equal $-$ for $u \in [0, x_1)$, a constant ($+$ or $-$) in $(x_1, x_2)$ and $-$ in $(x_2, 1)$. Thus the $\min_{u\in[0,1]}(g_u(x_2) - g_u(x_1))$ can be appeared in $u = x_1, x_2, 1$ and $\max_{u\in[0,1]}(g_u(x_2) - g_u(x_1))$ in $u = 0, x_2$. Directly by computation we find Pavlov's result

$$\min_{u\in[0,1]}(g_u(x_2) - g_u(x_1)) = g_{x_1}(x_2) - g_{x_1}(x_1) = \frac{1}{b^{\frac{1}{\alpha}} - 1} \frac{(K+1)^{\frac{1}{\alpha}} - K^{\frac{1}{\alpha}}}{K^{\frac{1}{\alpha}}} \tag{94}$$

$$\max_{u\in[0,1]}(g_u(x_2) - g_u(x_1)) = g_{x_2}(x_2) - g_{x_1}(x_2) = \frac{b^{\frac{1}{\alpha}}}{b^{\frac{1}{\alpha}} - 1} \frac{(K+1)^{\frac{1}{\alpha}} - K^{\frac{1}{\alpha}}}{(K+1)^{\frac{1}{\alpha}}}. \tag{95}$$

P. Diaconis [37] and A.I. Pavlov [126] have been the first, who applied u.d. theory to B.L. For instance:

(i) P. Diaconis by using the criterion of P.B Kennedy, see [171, p. 2–13, 2.2.9], proved: *If a sequence $x_n > 0$, $n = 1, 2, \dots$, satisfies B.L. in the base $b$, then*

$$\limsup_{n\to\infty} n \left| \log \frac{x_{n+1}}{x_n} \right| = \infty. \tag{96}$$

(ii) A.I. Pavlov by means of van der Corput difference theorem [92, p. 26, Th.3.1] proved: *Assume $x_n > 0$, $n = 1, 2, \dots$. If for every $k = 1, 2, \dots$ the ratio sequence $\frac{x_{n+k}}{x_n}$, $n = 1, 2, \dots$, satisfies B.L. in the base $b$, then the original sequence $x_n$, $n = 1, 2, \dots$ also satisfies B.L. in the base $b$.*

(iii) In [13] we have: *The positive sequences $x_n$ and $\frac{1}{x_n}$, $n = 1, 2, \dots$ satisfy B.L. in the base $b$ simultaneously.*

*Proof.* Both two sequences $u_n$ and $-u_n$ are u.d. mod 1 simultaneously, since their Weyl's sums are complex conjugate each other. $\square$

(iv) *The positive sequences $x_n$ and $nx_n$, $n = 1, 2, \dots$ satisfy B.L. in the base $b$ simultaneously.*

*Proof.* Both two sequences $u_n$ and $u_n + \log n$ are u.d. mod1 simultaneously, see [171, p. 2–27, 2.3.6.] and Theorem 91. $\square$

(v) *Assume that a sequence $0 < x_1 \leq x_2 \leq \dots$ satisfies B.L. in the integer base $b > 1$. Then*

$$\lim_{n\to\infty} \frac{\log x_n}{\log n} = \infty. \tag{97}$$

*Proof.* It follows from the theorem of H. Niederreiter [171, p. 2–12, 2.2.8] that every monotone u.d. sequence $u_n$ mod 1 must satisfy $\lim_{n\to\infty} \frac{|u_n|}{\log n} = \infty$. Here, it suffices to put $u_n = \log x_n$, instead of $u_n = \log_b x_n$. $\square$

(vi) *For a sequence $x_n > 0$, $n = 1, 2, \dots,$ assume that*

(i) $\lim_{n\to\infty} x_n = \infty$ *monotonically,*

(ii) $\lim_{n\to\infty} \log \frac{x_{n+1}}{x_n} = 0$ *monotonically.*

*Then the sequence $x_n$ satisfies B.L. in every base $b$ if and only if*

$$\lim_{n\to\infty} n \log \frac{x_{n+1}}{x_n} = \infty. \tag{98}$$

*Proof.* It follows from Fejér's difference theorem in the form in [171, p. 2–13, 2.2.11]. $\square$

(vii) [171, p. 2–14, 2.2.12] implies: *Let $x_n > 0$ be a sequence, which satisfies $\lim_{n\to\infty} \log_b \frac{x_{n+1}}{x_n} = \theta$ with $\theta$ is irrational. Then $x_n$ satisfies B.L. in the base $b$.*

**Example 19.** J.L. Brown, Jr. and R.L. Duncan (1970) [24]: Let $x_n$ be a sequence generated by the recurrence relation

$x_{n+k} = a_{k-1}x_{n+k-1} + \dots + a_1 x_{n+1} + a_0 x_n$, $n = 1, 2, \dots,$

where $a_0, a_1, \dots, a_{k-1}$ are non-negative rationals with $a_0 \neq 0$, $k$ is a fixed integer, and $x_1, x_2, \dots, x_k$ are starting points. Assume that the characteristic polynomial

$x^k - a_{k-1}x^{k-1} - \dots - a_1 x - a_0$

has $k$ distinct roots $\beta_1, \beta_2, \ldots, \beta_k$ satisfying
$0 < |\beta_1| < \cdots < |\beta_k|$
and such that none of the roots has magnitude equal to 1.
Then $\log x_n$ mod 1 is u.d.

*Proof.* The general solution of the recurrence $x_n$ is

$$x_n = \sum_{j=1}^{k} \alpha_j \beta_j^n.$$

Let $l$ is the largest value of $j$ for which $\alpha_j \neq 0$. Then $x_n = \sum_{j=1}^{l} \alpha_j \beta_j^n$ and

$$\left|1 - \frac{x_n}{\alpha_l \beta_l^n}\right| = \left|\sum_{j=1}^{l} \frac{\alpha_j \beta_j^n}{\alpha_l \beta_l^n}\right| \to 0.$$

Thus $\lim_{n\to\infty} \frac{x_n}{|\alpha_l \beta_l^n|} = 1$ and thus

$$\lim_{n\to\infty} (\log x_n - \log |\alpha_l \beta_l^n| = 0.$$

Since $\beta_l \neq 1$ is algebraic, then $\log |\beta_l|$ is irrational and then $\log |\alpha_l \beta_l^n|$ and then also $\log x_n$ are u.d. mod1. If $\log_b \beta_l$ is irrational, then also

$$\log_b x_n \bmod 1$$

is u.d., i.e. $x_n$ satisfies B.L. in the base $b$. This implies that Fibonacci and Lucas numbers obey B.L. what rediscovered L.C. Washington (1981)[190]. □

**In Section 6.4 the B.L. continue.**

Selected informations from 6.4:
By Theorem 170: Let $x_n$, $n = 1, 2, \ldots$, be a sequence in $(0, 1)$ and $G(x_n)$ be the set of all d.f.s of $x_n$. Assume that every d.f. $g(x) \in G(x_n)$ is continuous at $x = 0$. Then the sequence $x_n$ satisfies B.L. in the base $b$ if and only if for every $g(x) \in G(x_n)$ we have (458)

$$x = \sum_{i=0}^{\infty} \left( g\left(\frac{1}{b^i}\right) - g\left(\frac{1}{b^{i+x}}\right)\right) \text{ for } x \in [0, 1].$$

For example d.f. (467)

$$g(x) = \begin{cases} x & \text{if } x \in \left[0, \frac{1}{b}\right], \\ 1 + \log_b x + (1 - x)\frac{1}{b-1} & \text{if } x \in \left[\frac{1}{b}, 1\right] \end{cases}$$

is a solution of (458).
By Theorem 175: For a sequence $x_n \in (0,1)$, $n = 1, 2, \dots$, assume that every d.f. $g(x) \in G(x_n)$ is continuous at $x = 0$. Then there exist only finitely many different integer bases $b$ for which the sequence $x_n$ satisfies B.L. simultaneously.
In the contrary by [171, p. 2–117, 2.12.14], the sequence

$$\alpha n \log^\tau n \bmod 1, \quad \alpha \neq 0, 0 < \tau \leq 1,$$

is u.d. From this follows that $x_n = n^n$ satisfies B.L. for an arbitrary integer base $b$, because $\log_b n^n = n \log n \frac{1}{\log b}$.

## 3.10 D.f.s of a two dimensional sequence $(x_n, y_n)$, both $x_n$ and $y_n$ are u.d. (copulas)

See Section 8.5:

Let $F(x, y)$ be a continuous function. The limit points of (565)

$$\frac{1}{N} \sum_{n=1}^{N} F(x_n, y_n),$$

where $x_n$ and $y_n$ are u.d. in $[0, 1)$ can be studied by using d.f.s $g(x, y)$ of $(x_n, y_n)$. These d.f.s satisfies $g(x, 1) = x$, $g(1, y) = y$ and called copulas. As we shall see in the Section 8.5 each copula meets (575)

$$\max(x + y - 1, 0) \leq g(x, y) \leq \min(x, y)$$

then, assuming $\mathrm{d}_x \mathrm{d}_y F(x, y) > 0$, for an arbitrary u.d. $x_n$ and $y_n$, we have in Theorem 214, inequalities (571) and (572)

$$\int_0^1 F(x, 1 - x)\mathrm{d}x \leq \liminf_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} F(x_n, y_n)$$

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} F(x_n, y_n) \leq \int_0^1 F(x, x)\mathrm{d}x.$$

Here the left boundary is attained in the sequence $(x_n, 1 - x_n)$ and the right in $(x_n, x_n)$ where $x_n$ is u.d.

### 3.11 Ratio block sequences

More is described in Section 6.2. Here we present an application of Theorem 143:

Let $x_1 < x_2 < \dots$ be an increasing sequence of positive integers and consider a sequence of blocks $X_n$, $n = 1, 2, \dots$, with blocks

$$X_n = \left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}\right)$$

and denote by $F(X_n, x)$ the step d.f.

$$F(X_n, x) = \frac{\#\{i \le n; \frac{x_i}{x_n} < x\}}{n},$$

for $x \in [0, 1)$ and $F(X_n, 1) = 1$. A d.f. $g$ is a d.f. of the sequence of single blocks $X_n$, if there exists an increasing sequence of positive integers $n_1, n_2, \dots$ such that $\lim_{k\to\infty} F(X_{n_k}, x) = g(x)$ a.e. on $[0, 1]$. Denote by $G(X_n)$ the set of all d.f. of the sequence of single blocks $X_n$. Using boundaries of d.f.s of $G(X_n)$ in [11] we have proved in Theorem 143: For every increasing sequence $x_1 < x_2 < \dots$ of positive integers with lower and upper asymptotic densities $0 < \underline{d} \le \overline{d}$ we have (310)

$$\frac{1}{2}\frac{\underline{d}}{\overline{d}} \le \liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{x_i}{x_n},$$

and (311)

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{x_i}{x_n} \le \frac{1}{2} + \frac{1}{2}\left(\frac{1 - \min(\sqrt{\underline{d}}, \overline{d})}{1 - \underline{d}}\right)\left(1 - \frac{\underline{d}}{\min(\sqrt{\underline{d}}, \overline{d}})\right).$$

Here the equations in (310) and (311) can be attained.

# 4 Calculation of $G(x_n)$

## 4.1 $G(f(n) \bmod 1)$ directly from definition of d.f.s

**Example 20.** Starting with $x_n = \{\log\log n\}$ all the sequences $\{\log\log \dots \log n\}$ have

$$G(x_n) = \{c_\alpha(x); \alpha \in [0, 1]\} \cup \{h_\alpha(x); \alpha \in [0, 1]\}.$$

*Proof.* For the first iterated logarithm we chose an index-sequence $N_k$ as $N_k = [\exp\exp(k+\alpha)]$. Then we have $\lim_{k\to\infty} F_{N_k}(x) = c_\alpha(x)$. For $N_k = [\exp\exp(k+\varepsilon_k)]$, where $\varepsilon_k \to 0$ such that $(\exp\exp(k+\varepsilon_k))/(\exp\exp k) \to \beta$, we have $\lim_{k\to\infty} F_{N_k}(x) = h_\alpha(x)$, where $\alpha = (\beta-1)/\beta$.

On the other hand, let $\lim_{n\to\infty} F_{N_n}(x) = g(x)$. Then $N_n = \exp\exp(k_n+\varepsilon_n)$, where $k_n = [\log\log N_n]$, $\varepsilon_n = \{\log\log N_n\}$, and the sequence $(\varepsilon_n)_{n=1}^\infty$ cannot have different limit points.

The same d.f. are of course obtained if we replace $\log\log t$ by $\log\dots\log t$ and $\exp\exp t$ by $\exp\dots\exp t$ in the above limits. $\square$

The following theorem proved Koksma [87], [88, Kap. 8] (cf. [92, p. 58, Th. 7.7]):

**Theorem 47.** *Let the real-valued continuous function $f(x)$ be strictly increasing and let $f^{-1}(x)$ be its inverse function. Assume that, for $k\to\infty$,*

(i) $f^{-1}(k+1) - f^{-1}(k) \to \infty$,

(ii) $\lim_{k\to\infty} \frac{f^{-1}(k+x)-f^{-1}(k)}{f^{-1}(k+1)-f^{-1}(k)} = \underline{g}(x)$ *exists for* $x \in [0,1]$,

(iii) $\liminf_{k\to\infty} \frac{f^{-1}(k)}{f^{-1}(k+x)} = \chi(x)$.

*Then the sequence*

$$f(n) \bmod 1, \quad n = 1,2,\dots$$

*has the lower d.f. $\underline{g}(x)$ and the upper d.f.*

$$\overline{g}(x) = 1 - \chi(x)(1-\underline{g}(x)), \text{ for } x \in [0,1].$$

Using this we find the lower and the upper d.f. of $\log(n\log n) \bmod 1$ as

$$\underline{g}(x) = \frac{e^x-1}{e-1}, \quad \overline{g}(x) = \frac{e-e^{1-x}}{e-1}$$

and they are the same as for $\log n \bmod 1$, cf. [92, pp. 58-59]. This gives an impulse to the following generalization, cf. [169] and [61]:

In the following Theorems 48, 49 and 50 we assume that

(I) $f(x)$ be a real-valued function defined for $x \geq 1$ such that $f(x)$ is strictly increasing with its inverse function $f^{-1}(x)$.

Assume that the following limits exist:

(II) $\lim_{k\to\infty} \frac{f^{-1}(k+x)-f^{-1}(k)}{f^{-1}(k+1)-f^{-1}(k)} = \tilde{g}(x)$ for each $x \in [0,1]$, of point of continuity of $\tilde{g}(x)$;

(III) $\lim_{k\to\infty} \frac{f^{-1}(k+u)}{f^{-1}(k)} = \psi(u)$ for each $u \in [0,1]$, point of continuity of $\psi(u)$, or $\psi(u) = \infty$ for $u > 0$;

(IV) $\lim_{k\to\infty} f^{-1}(k+1) - f^{-1}(k) = \infty$.

To calculate $G(f(n) \bmod 1)$ we use the following three theorems.

**Theorem 48.** *If* $1 < \psi(1) < \infty$ *and* $f'(x) \to 0$ *as* $x \to \infty$*, then*

$$G(f(n) \bmod 1) = \left\{ g_u(x) = \frac{1}{\psi(u)}\tilde{g}(x) + \frac{\min(\psi(x),\psi(u)) - 1}{\psi(u)}; u \in [0,1] \right\}, \tag{99}$$

*where* $\tilde{g}(x) = \frac{\psi(x)-1}{\psi(1)-1}$ *and* $F_{N_i}(x) \to g_u(x)$ *as* $i \to \infty$ *if and only if* $f(N_i) \bmod 1 \to u$*. The lower d.f.* $\underline{g}(x)$ *and the upper d.f.* $\overline{g}(x)$ *of* $f(n) \bmod 1$ *are*

$$\underline{g}(x) = \tilde{g}(x), \quad \overline{g}(x) = 1 - \frac{1}{\psi(x)}(1 - \tilde{g}(x)).$$

*Furthermore* $\underline{g}(x) = g_0(x) = g_1(x)$ *belongs to* $G(f(n) \bmod 1)$ *but* $\overline{g}(x) = g_x(x)$ *does not.*

In the first, Theorem 48 gives the following generalization of Ex. 1.

**Example 21.** For every base $b > 1$ the sequence $\log_b n \bmod 1$, $n = 1, 2, \dots$, has the set of d.f.s

$$G(x_n) = \left\{ g_u(x) = \frac{1}{b^u}\frac{b^x - 1}{b-1} + \frac{b^{\min(x,u)} - 1}{b^u}; u \in [0,1] \right\}, \tag{100}$$

where $\{\log_b N_k\} \to u$ implies $F_{N_k}(x) \to g_u(x)$. The lower and upper d.f. of $\log_b n \bmod 1$ are

$$\underline{g}(x) = \frac{b^x - 1}{b-1}, \qquad \overline{g}(x) = \frac{1 - b^{-x}}{1 - b^{-1}}, \tag{101}$$

and $\underline{g} \in G(x_n)$ but $\overline{g} \notin G(x_n)$. In this case $f(x) = \log_b x$, $f^{-1}(x) = b^x$, $\tilde{g}(x) = \frac{b^x-1}{b-1}$ and $\psi(u) = b^u$. Note that G. Pólya and G. Szegö (1964) [129] firstly gave all densities (i.e. derivatives) of $g(x) \in G(\log_b n \bmod 1)$ but

$$\{\log N_k\} \to u \Longrightarrow F_{N_k}(x) \to g_u(x) \tag{102}$$

in their result absent. □

**Theorem 49.** *If $\psi(1) = 1$, then the sequence $f(n) \bmod 1$, $n = 1, 2, \dots$ has a.d.f. $\tilde{g}(x)$, i.e.*

$$G(f(n) \bmod 1) = \{\tilde{g}(x)\}. \tag{103}$$

**Theorem 50.** *Let $\psi(u) = \infty$, for every $u > 0$ and for $u = 0$ the limit $\psi(u)$ is not defined in the way that for every $t \in [0, \infty)$ there exists a sequence $u(k) \to 0$ such that* (i) $\lim_{k\to\infty} \frac{f^{-1}(k+u(k))}{f^{-1}(k)} = t$. *Then we have*

$$G(f(n) \bmod 1) = \{c_u(x); u \in [0, 1]\} \cup \{h_\beta(x); \beta \in [0, 1]\}, \tag{104}$$

*where $F_{N_i} \to c_u(x)$ if and only if $f(N_i) \bmod 1 \to u > 0$ and $F_{N_i} \to h_\beta(x)$ if and only if $f(N_i) \bmod 1 \to 0$ and $\frac{f^{-1}([f(N_i)])}{N_i} \to 1 - \beta$.*

**Proofs of Theorems 48, 49 and 50**

For a positive integer $N$ define

- $K = K(N) = [f(N)]$,
- $u(N) = \{f(N)\}$,
- $S_N([x, y)) = \sum_{n=1, f(n)\in[x,y)}^{N} 1$.

Clearly $f^{-1}(K + u(N)) = N$ and for every $x \in [0, 1]$ define d.f. $F_N(x)$ as [14]

$$F_N(x) = \frac{\sum_{k=0}^{K-1} S_N([k, k + x)) + S_N([K, K + x) \cap [K, K + u(N)))}{N} + \frac{O(S_N([0, x))}{N}.$$

From the monotonicity of $f(x)$ (assumption (I)) it follows that $S_N([x, y)) = \sum_{n=1, n\in[f^{-1}(x), f^{-1}(y))}^{N} 1$, and $N = S_N([0, K + u(N))$ and we have

- $S_N([x, y)) = f^{-1}(y) - f^{-1}(x) + \theta(f^{-1}(x), f^{-1}(y))$, where $|\theta| \le 1$,
- $N = f^{-1}(K + u(N)) + \theta(f^{-1}(0), f^{-1}(K + u(N)))$

and thus

$$F_N(x) = \frac{\sum_{k=0}^{K-1} (f^{-1}(k + x) - f^{-1}(k))}{N}$$

[14] Without loss of generality we assume that $f(1) \in [0, 1)$.

$$
\begin{aligned}
&+ \frac{\min(f^{-1}(K+x), f^{-1}(K+u(N))) - f^{-1}(K)}{N} \\
&+ \frac{O(\sum_{k=0}^{K} \theta(f^{-1}(k), f^{-1}(k+x))}{N} + \frac{O(\theta(f^{-1}(K), f^{-1}(K+u(N))))}{N}.
\end{aligned}
$$

Assumption (IV) implies that $1/(f^{-1}(k+1) - f^{-1}(k)) \to 0$; this in turn yields that $K/f^{-1}(K) \to 0$ by the Cauchy-Stolz lemma, which gives

$$\frac{O(\sum_{k=0}^{K} \theta(f^{-1}(k), f^{-1}(k+x)))}{N} = \frac{O(K)}{N} \to 0.$$

Thus

$$F_N(x) = F_N^{(1)}(x) + F_N^{(2)}(x) + o(1),$$

where

$$
\begin{aligned}
F_N^{(1)}(x) &= \frac{\sum_{k=0}^{K-1}(f^{-1}(k+x) - f^{-1}(k))}{f^{-1}(K+u(N))}, \\
F_N^{(2)}(x) &= \frac{\min(f^{-1}(K+x), f^{-1}(K+u(N))) - f^{-1}(K)}{f^{-1}(K+u(N))}.
\end{aligned}
$$

We shall express the first term $F_N^{(1)}(x)$ as

$$F_N^{(1)}(x) = \frac{\sum_{k=0}^{K-1}(f^{-1}(k+x) - f^{-1}(k))}{\sum_{k=0}^{K-1}(f^{-1}(k+1) - f^{-1}(k))} \cdot \frac{f^{-1}(K) - f^{-1}(0)}{f^{-1}(K+u(N))}$$

and by the Cauchy-Stolz lemma [15] and assumption (II),

$$\lim_{K\to\infty} \frac{\sum_{k=0}^{K-1}(f^{-1}(k+x) - f^{-1}(k))}{\sum_{k=0}^{K-1}(f^{-1}(k+1) - f^{-1}(k))} = \tilde{g}(x).$$

Now, let $F_{N_i}(x) \to g(x)$. Then there exists a subsequence $N_i'$ of $N_i$ such that $u(N_i') = u_i' \to u'$ for some $u' \in [0,1]$. Thus, in the following we can assume that

$F_{N_i}(x) \to g(x)$, and

[15]**Cauchy-Stolz lemma:** Let $x_n$ and $y_n$, $n = 1, 2, \ldots$, be the real–valued sequences. If $y_n$ is strictly monotone, $|y_n| \to \infty$, and if the limit (finite or infinite) $\lim_{n\to\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n}$ exists, then the limit of the sequence $\frac{x_n}{y_n}$ also exists and $\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n}$.

$K_i = [f(N_i)]$, and $u(N_i) = u_i \to u$, simultaneously.

We shall prove Theorems 48, 49, and 50 one by one.

*Proof of Theorem 48.* In this case $\tilde{g}(x) = \frac{\psi(x)-1}{\psi(1)-1}$, and the relation $u_i \to u$ implies [16]

$$\frac{f^{-1}(K_i)-f^{-1}(0)}{f^{-1}(K_i+u_i)} \to \frac{1}{\psi(u)};$$

$$F_{N_i}^{(1)}(x) \to \frac{\psi(x)-1}{\psi(1)-1} \cdot \frac{1}{\psi(u)};$$

$$F_{N_i}^{(2)}(x) \to \frac{\min(\psi(x),\psi(u))-1}{\psi(u)}.$$

Thus the d.f. $g(x)$ has the form $g(x) = g_u(x)$. On the other hand, for every $u \in [0,1]$ there exists an increasing sequence of indices $N_i$ such that $u(N_i) = u_i \to u$. It follows from the assumption $f'(x) \to 0$ because for some $\varepsilon_i \to 0$ we can find $N_i \in f^{-1}((K_i + u - \varepsilon_i, K_i + u + \varepsilon_i))$.

Now, we find the lower and upper d.f. by computing $g_u(x)$ for fixed $x \in [0,1]$ and $u \in [0,1]$

$$g_u(x) = \begin{cases} 1 - \frac{1}{\psi(u)}\left(1 - \frac{\psi(x)-1}{\psi(1)-1}\right) & \text{if } u \le x, \\ \frac{\psi(1)}{\psi(u)}\frac{\psi(x)-1}{\psi(1)-1} & \text{if } u \ge x, \end{cases}$$

thus $\inf_{u\le x} g_u(x) = g_0(x)$, $\inf_{u\ge x} g_u(x) = g_1(x)$ and since $g_0(x) = g_1(x)$ we have $\underline{g}(x) = g_0(x) = g_1(x) = \frac{\psi(x)-1}{\psi(1)-1}$. Similarly, $\sup_{u\le x} g_u(x) = g_x(x)$, $\sup_{u\ge x} g_u(x) = g_x(x)$, and thus $\overline{g}(x) = g_x(x) \notin G(f(n) \bmod 1)$. □

*Proof of Theorem 49.* In this case we cannot use the relation $\tilde{g}(x) = \frac{\psi(x)-1}{\psi(1)-1}$ and moreover $\psi(u) = 1$ for all $u \in [0,1]$. Then

$$F_{N_i}^{(1)}(x) \to \tilde{g}(x)\tfrac{1}{1};$$

$$F_{N_i}^{(2)}(x) \to \frac{\min(1,1)-1}{1},$$

and thus $g(x) = \tilde{g}(x)$ for $x \in [0,1)$. □

[16]Note that if $u$ is a point of continuity of $\psi(u)$, the monotonicity of $f^{-1}(x)$ implies the relation $\frac{f^{-1}(K_i+u_i)}{f^{-1}(K_i)} \to \psi(u)$, because $\frac{f^{-1}(K_i+u-\varepsilon)}{f^{-1}(K_i)} \le \frac{f^{-1}(K_i+u_i)}{f^{-1}(K_i)} \le \frac{f^{-1}(K_i+u+\varepsilon)}{f^{-1}(K_i)}$ for $u_i \in (u-\varepsilon, u+\varepsilon)$.

*Proof of Theorem* 50. In this case

$$\tilde{g}(x) = \lim_{k\to\infty} \frac{f^{-1}(k+x) - f^{-1}(k)}{f^{-1}(k+1) - f^{-1}(k)} = 0$$

for every $x \in [0,1)$, thus $F_{N_i}^{(1)}(x) \to 0$. To compute the limit of $F_{N_i}^{(2)}(x)$ we distinguish the two following cases, where $u_i \to u$.
$1^0$. If $u > 0$, then

$$\lim_{i\to\infty} F_{N_i}^{(2)}(x) = \frac{\min(f^{-1}(K_i + x), f^{-1}(K_i + u_i)) - f^{-1}(K_i)}{f^{-1}(K_i + u_i)} = c_u(x).$$

$2^0$. If $u = 0$, then

$$F_{N_i}^{(2)}(x) = 1 - \frac{f^{-1}(K_i)}{f^{-1}(K_i + u_i)}.$$

From $K_i$ we select $K_i'$ (i.e. from $N_i$ we select a subsequence $N_i'$) such that

$$\lim_{i\to\infty} \frac{f^{-1}(K_i')}{f^{-1}(K_i' + u_i')} = t \in [0,1], \tag{105}$$

where again $K_i' = [f(N_i')]$ and $u_i' = \{f(N_i')\}$. Then we have $F_{N_i'}^{(2)}(x) \to h_\beta(x)$, where $\beta = 1 - t$ and $h_\beta(x) = \beta$ for $x \in (0,1)$. On the other hand (by assumptions (i)) for any given $t \in [0,1]$ there exists a sequence of positive integers $K_i$ and real numbers $u_i \to 0$ such that $\lim_{i\to\infty} \frac{f^{-1}(K_i)}{f^{-1}(K_i+u_i)} = t$. Then there exist integers $K_i'$ ($K_i' = K_i$ for almost all $i$) and real numbers $u_i' \in (0,1)$ such that $N_i' = f^{-1}(K_i' + u_i')$ is an integer and

$$|f^{-1}(K_i + u_i) - f^{-1}(K_i' + u_i')| < 1.$$

Thus we have (105) again. $\square$

**Notes 4.** In the case $\psi(1) = 1$ in Theorem 49, the limit (II) it my be some d.f. $\tilde{g}(x)$. To check this, put $H(x) = f^{-1}(x)$ and $H(k+x) = k + \tilde{g}(x)$ for $x \in [0,1]$ and $k = 1,2,\dots$ Then

(III) $\frac{H(k+1)}{H(k)} = \frac{k+1}{k} \to 1$, and

(II) $\frac{H(k+x)-H(k)}{H(k+1)-H(k)} = \tilde{g}(x)$. Similarly, for $H(k+x) = (k + \tilde{g}(x))^2$.

**Notes 5.** The situation is different if we replace the limit

(II) $\lim_{k\to\infty} \frac{f^{-1}(k+x)-f^{-1}(k)}{f^{-1}(k+1)-f^{-1}(k)} = \tilde{g}(x)$ for $x \in [0,1]$ by the

(II') $\lim_{t\to\infty} \frac{f^{-1}(t+x)-f^{-1}(t)}{f^{-1}(t+1)-f^{-1}(t)} = \tilde{g}(x)$ for $x \in [0,1]$, where $t$ is a real variable.

This limit was introduced by J. Cigler [32], generalizing J.F. Koksma's [88, p. 88] result on lower and upper d.f.s of $f(n)$ mod 1. J.H.B. Kemperman [85, Th. 11] proved that in (II') exactly one of the following relations must hold:

(a) $\tilde{g}(x) = x$ for $x \in [0,1]$;

(b) $\tilde{g}(x) = \frac{e^{cx}-1}{e^{c}-1}$ for $x \in [0,1]$;

(c) $\tilde{g}(x) = 0$ for $x \in (0,1)$.

Furthermore he remarked that

(a) In the case $\tilde{g}(x) = x$, the sequence $f(n)$ mod 1 is u.d.

(b) In the case $\tilde{g}(x) = \frac{e^{cx}-1}{e^{c}-1}$ the set $G(f(n) \bmod 1)$ is described by the following Theorem 51, where $t = 1/c$.

(c) The case $\tilde{g}(x) = 0$ is equivalent to $\lim_{t\to\infty} \frac{f^{-1}(t+x)}{f^{-1}(t)} = \infty$ and the set $G(f(n) \bmod 1)$ contains only one-step d.f.s $c_u(x)$, where $F_{N_i}(x) \to c_u(x)$ if and only if $f(N_i)$ mod $1 \to u$.

**Notes 6.** For the sequence $f(n)$ mod 1 with increasing $f(x)$ Kemperman [85] proved [17] the following two theorems which are different in nature from our Theorems 48, 49 and 50.

**Theorem 51** (Kemperman [85, Th.9])**.** *Assume that*

(j) $\lim_{n\to\infty} f(n+1) - f(n) = 0$;

(jj) $\lim_{n\to\infty} n(f(n+1) - f(n)) = t$.

*Then* $G(f(n) \bmod 1)$ *contains only d.f.s of the type*

$$g_u(x) = \begin{cases} \frac{e^{(1+x-u)/t}-e^{(1-u)/t}}{e^{1/t}-1} & \text{if } 0 \le x \le u, \\ 1 - \frac{e^{(1-u)/t}-e^{(x-u)/t}}{e^{1/t}-1} & \text{if } u < x \le 1. \end{cases} \tag{106}$$

*Here the density* $g'_u(x)$ *of* $g_u(x)$ *has the form*

$$g'_u(x) = \frac{e^{\{x-u\}/t}}{t(e^{1/t}-1)}.$$

*Furthermore* $F_{N_i}(x) \to g_u(x)$ *if and only if* $f(N_i) \bmod 1 \to u$.

**Theorem 52** (Kemperman [85, p. 148, Coroll. 1])**.** *Assume that*

(j) $\lim_{n\to\infty} f(n+1) - f(n) = 0$;

(jj) $\lim_{n\to\infty} n(f(n+1) - f(n)) = +\infty$.

[17] In Theorem 51 and 52 we put weights $w(n) = 1$.

(jjj) $f(n+1)-f(n)$ *is monotone in* $n$.

*Then the sequence* $f(n) \bmod 1$*,* $n=1,2,\dots$ *is u.d.*

**Example 22.** We apply Theorem 48 to the function $f(x)=\log(x\log^{(i)}x)$, where $\log^{(i)}x$ is the $i$th iterated logarithm $\log\dots\log x$.

We prove that, for $k\to\infty$,

$$\frac{\log^{(i)} f^{-1}(k+u(k))}{\log^{(i)} f^{-1}(k)}\to 1$$

for every sequence $u(k)\in[0,1]$. Proof: we start with

$$\begin{aligned}0<\log f^{-1}(k+1)-\log f^{-1}(k)&<\log f^{-1}(k+1)-\log f^{-1}(k)+\\&+\log^{(i+1)} f^{-1}(k+1)-\log^{(i+1)} f^{-1}(k)=1\end{aligned}$$

what implies $\frac{\log f^{-1}(k+1)}{\log f^{-1}(k)}\to 1$, and this implies $\log^{(2)} f^{-1}(k+1)-\log^{(2)} f^{-1}(k)\to 0$, what implies $\frac{\log^{(2)} f^{-1}(k+1)}{\log^{(2)} f^{-1}(k)}\to 1$, e.t.c.

Now, every $u(k)$ can be expressed as

$$\begin{aligned}u(k)&=f(f^{-1}(k+u(k)))-f(f^{-1}(k))=\\&=\log\left(\frac{f^{-1}(k+u(k))}{f^{-1}(k)}\cdot\frac{\log^{(i)} f^{-1}(k+u(k))}{\log^{(i)} f^{-1}(k)}\right).\end{aligned}$$

Assuming that $u(k)\to u$ as $k\to\infty$, we have (iii) in the form $\psi(u)=e^u$. To prove (ii) we see that $f(f^{-1}(k+1))-f(f^{-1}(k))=1$ and by mean-value theorem $(f^{-1}(k+1)-f^{-1}(k))f'(x)=1$ for some $x\in(k,k+1)$, where $f'(x)\to 0$ as $k\to\infty$. As summary, $G(\log(n\log^{(i)}n)\bmod 1)=G(\log n\bmod 1)$.

In 2011 Y. Ohkubo [122] proved that in Theorem 48 $f(n)\bmod 1$ can be replaced by $f(p_n)\bmod 1$, where $p_n$ is the increasing sequence of all primes. He proved:

**Theorem 53.** *Let the real-valued function* $f(x)$ *be strictly increasing for* $x\ge 1$ *and let* $f^{-1}(x)$ *be its inverse function. Assume that*

(i) $\lim_{k\to\infty} f^{-1}(k+1)-f^{-1}(k)=\infty$*,*

(ii) $\lim_{k\to\infty}\frac{f^{-1}(k+u_k)}{f^{-1}(k)}=\psi(u)$ *for every sequence* $u_k\in[0,1]$ *for which* $\lim_{k\to\infty}u_k=u$*, where this limit defines the function* $\psi:[0,1]\to[1,\psi(1)]$*,*

(iii) $\psi(1) > 1$. *Then*

$$G(f(p_n) \bmod 1) = \left\{ g_u(x) = \frac{1}{\psi(u)} \frac{\psi(x) - 1}{\psi(1) - 1} + \frac{\min(\psi(x), \psi(u)) - 1}{\psi(u)}; u \in [0, 1] \right\}.$$

*The lower d.f.* $\underline{g}(x)$ *and the upper d.g.* $\overline{g}(x)$ *of* $f(p_n) \bmod 1$ *are*

$$\underline{g}(x) = \frac{\psi(x) - 1}{\psi(1) - 1}, \quad \overline{g}(x) = 1 - \frac{1}{\psi(x)}(1 - \underline{g}(x).)$$

*Here* $\underline{g}(x) = g_0(x) = g_1(x) \in G(f(p_n) \bmod 1)$ *and* $\overline{g}(x) = g_x(x) \notin G(f(p_n) \bmod 1)$.

*Proof.* Let $K$ be an integer and express an $N \in [f^{-1}(K), f^{-1}(K + 1))$ as $N = f^{-1}(K + u)$, $u \in [0, 1)$. Since

$$k < f(p_n) \le k + x \longleftrightarrow f^{-1}(k) < p_n \le f^{-1}(k + x)$$

then we have

$$\frac{\#\{n \le \pi(n); \{f(p_n)\} < x\}}{\pi(N)} = \frac{\sum_{k=0}^{K-1} \pi(f^{-1}(k + x)) - \pi(f^{-1}(k))}{\pi(f^{-1}(K + u))} \tag{107}$$

$$+ \frac{\pi(f^{-1}(K + \min(x, u))) - \pi(f^{-1}(K))}{\pi(f^{-1}(K + u))}. \tag{108}$$

Furthermore,

$$\begin{aligned}\pi(f^{-1}(K + u)) = &\sum_{k=0}^{K-1} \pi(f^{-1}(k + 1)) - \pi(f^{-1}(k)) + \pi(f^{-1}(0)) \\ &+ \pi(f^{-1}(K + \min(x, u))) - \pi(f^{-1}(K)).\end{aligned} \tag{109}$$

To compute (107) via Stoltz lemma

$$\frac{\sum_{k=0}^{K-1} \pi(f^{-1}(k + x)) - \pi(f^{-1}(k))}{\sum_{k=0}^{K-1} \pi(f^{-1}(k + 1)) - \pi(f^{-1}(k))} \cdot \frac{\sum_{k=0}^{K-1} \pi(f^{-1}(k + 1)) - \pi(f^{-1}(k))}{\pi(f^{-1}(K + u))} \tag{110}$$

we need find a limit of

$$\frac{\pi(f^{-1}(k + x)) - \pi(f^{-1}(k))}{\pi(f^{-1}(k + 1)) - \pi(f^{-1}(k))} = \frac{\frac{\pi(f^{-1}(k+x))}{\pi(f^{-1}(k))} - 1}{\frac{\pi(f^{-1}(k+1))}{\pi(f^{-1}(k))} - 1}. \tag{111}$$

Using prime number theorem in the form

$$\frac{\pi(x)}{\frac{x}{\log x}} \to 1$$

if $x \to \infty$ and expressing

$$\frac{\pi(f^{-1}(k+x))}{\pi(f^{-1}(k))} = \frac{\frac{\pi(f^{-1}(k+x))}{\frac{f^{-1}(k+x)}{\log f^{-1}(k+x)}} \frac{f^{-1}(k+x)}{\log f^{-1}(k+x)}}{\frac{\pi(f^{-1}(k))}{\frac{f^{-1}(k)}{\log f^{-1}(k)}} \frac{f^{-1}(k)}{\log f^{-1}(k)}}$$

and using (ii) $\lim_{k\to\infty} \frac{f^{-1}(k+x)}{f^{-1}(k)} = \psi(x)$ as $k \to \infty$ we see that for proof of theorem we need proved

$$\frac{\log f^{-1}(k+1)}{\log f^{-1}(k)} \to 1. \tag{112}$$

Ohkubo proved (112) using the following succession

$$\begin{aligned}\frac{\log f^{-1}(k+1)}{\log f^{-1}(k)} &= \frac{1}{\log f^{-1}(k)}\big(\log f^{-1}(k+1) - \log f^{-1}(k)\big) + 1\\ &= \frac{1}{\log f^{-1}(k)} \log\left(\frac{f^{-1}(k+1)}{f^{-1}(k)}\right) + 1 \to 1\end{aligned}$$

since by assumption $\frac{f^{-1}(k+1)}{f^{-1}(k)} \to \psi(1) > 1$. Thus (107) has the form $\frac{1}{\psi(u)}\frac{\psi(x)-1}{\psi(1)-1}$. Similarly for (108). □

**Example 23.** Theorem 53 implies that the following sequences $\log p_n \bmod 1$, $\log(p_n \log p_n) \bmod 1$ have the same d.f.s as $\log n \bmod 1$. This solve Open problem 1.3 in [166].

## 4.2 Proof of $G(x_n) = H$ using the connectivity of $G(x_n)$

The connectivity of $G(x_n)$ implies the following simple theorem: Define
• For the d.f. $g$ the Graph(g) is continuous curve formed with all the points $(x, g(x))$ for $x \in [0,1]$, and the all line segments connecting points of discontinuity $(x, \liminf_{x'\to x} g(x'))$ and $(x, \limsup_{x'\to x} g(x'))$.
• Denote $\underline{g}_H(x) = \inf_{g\in H} g(x)$ and $\overline{g}_H(x) = \sup_{g\in H} g(x)$.

**Theorem 54** ([161]). *Let $H$ be a non–empty, closed, and connected set of d.f.s. Assume that for every $g \in H$ there exists a point $(x, y) \in \mathrm{Graph(g)}$ such that $(x, y) \notin \mathrm{Graph}(\widetilde{\mathrm{g}})$ for any $\widetilde{g} \in H$ with $\widetilde{g} \neq g$. If*

(i) $G(x_n) \subset H$, and

(ii) $\underline{g} = \underline{g}_H$ and $\overline{g} = \overline{g}_H$ for the lower d.f. $\underline{g}$ and the upper d.f. $\overline{g}$ of the sequence $x_n \in [0, 1)$,

then $G(x_n) = H$.

## 4.3 Proof of $G(x_n) \subset H$ using $L^2$ discrepancy

For proving $G(x_n) \subset H$ it can be used $\lim_{N\to\infty} D_N^{(2)}(x_n, H) = 0$, where

• the $L^2$ discrepancy $D_N^{(2)}(x_n, H)$ of $x_n$ with respect to $H$ is defined as

$$D_N^{(2)}(x_n, H) = \min_{g\in H} \int_0^1 (F_N(x) - g(x))^2 \mathrm{d}x. \tag{113}$$

We prove the following theorem:

**Theorem 55.** *For every sequence $x_n \in [0, 1)$ we have*

$$G(x_n) \subset H \qquad \Longleftrightarrow \qquad \lim_{N\to\infty} D_N^{(2)}(x_n, H) = 0.$$

*Proof.* $\Longrightarrow$. Suppose $\lim_{k\to\infty} F_{N_k}(x) = \tilde{g}(x)$ a.e. and $\tilde{g} \in H$. Then we have $\lim_{k\to\infty} \int_0^1 (F_{N_k}(x) - \tilde{g}(x))^2 \mathrm{d}x = 0$ by the Lebesgue theorem of dominated convergence and because $D_{N_k}^{(2)} \le \int_0^1 (F_{N_k}(x) - \tilde{g}(x))^2 \mathrm{d}x$, we get $\lim_{k\to\infty} D_{N_k}^{(2)} = 0$, for every increasing $N_k$.

$\Longleftarrow$. Suppose that $\lim_{N\to\infty} D_N^{(2)} = 0$. Let $g_N \in H$ be a point of the minimum of $\int_0^1 (F_N(x) - g(x))^2 \mathrm{d}x$ with $g \in H$. The Helly selection principle guarantees a suitable sequence $N_k$, $k = 1, 2, \dots$, such that $\lim_{k\to\infty} F_{N_k}(x) = \tilde{g}_1(x)$ and $\lim_{k\to\infty} g_{N_k}(x) = \tilde{g}_2(x)$ a.e. on $[0, 1]$. Then $\tilde{g}_2 \in H$ and then using the Lebesgue theorem of dominated convergence we get

$$\lim_{k\to\infty} \int_0^1 (F_{N_k}(x) - g_{N_k}(x))^2 \mathrm{d}x = \int_0^1 (\tilde{g}_1(x) - \tilde{g}_2(x))^2 \mathrm{d}x = 0.$$

Thus $\tilde{g}_1 \in H$, and the same is true for every partial limit of $F_N$. To complete the proof we have to shown that the existence of such $g_N$. To do this note that there exists a sequence $g_n \in H$, $n = 1, 2, \dots$, such that $\inf_{g\in H} \int_0^1 (F_N(x) - g(x))^2 \mathrm{d}x = \lim_{n\to\infty} \int_0^1 (F_N(x) - g_n(x))^2 \mathrm{d}x$, and again, by Helly theorem, $\lim_{k\to\infty} g_{n_k}(x) = g_N(x)$ for suitable $n_k$, and thus $g_N \in H$. $\square$

**Notes 7.** It should be noted that that the $L^2$ discrepancy can be expressed as

$$D_N^{(2)}(x_n, H) = \int_0^1 (F_N(x) - g_N(x))^2 \mathrm{d}x = \int_0^1 \int_0^1 F_N(x, y) \mathrm{d}g_N(x) \mathrm{d}g_N(y),$$

and applying (31) we have $F_N(x, y) = F_{\tilde{g}}(x, y)$ putting $\tilde{g}(x) = g_N(x)$ and
$F_{\tilde{g}}(x, y) = \int_0^1 \tilde{g}^2(t)\mathrm{d}t - \int_x^1 \tilde{g}(t)\mathrm{d}t - \int_y^1 \tilde{g}(t)\mathrm{d}t + 1 - \max(x, y)$, and
$\int_0^1 (g(x) - \tilde{g}(x))^2 \mathrm{d}x = \int_0^1 \int_0^1 F_{\tilde{g}}(x, y)\mathrm{d}g(x)\mathrm{d}g(y)$.

• Let $F : [0,1]^2 \to [0,1]$ by symmetric function and $G(F)$ be a set of all d.f. $g(x)$ satisfying

$$\int_0^1 \int_0^1 F(x, y)\mathrm{d}g(x)\mathrm{d}g(y) = 0.$$

Define $\underline{g}_F$ is the lower and $\overline{g}_F$ is the upper d.f of $G(F)$ and

$$\Omega(F) = \cup_{g \in G(F)} \mathrm{Graph(g)}.$$

## 4.4 Proof of $G(x_n) \subset H$ solving $\int_0^1 \int_0^1 F(x, y)\mathrm{d}g(x)\mathrm{d}g(y) = 0$

Theorem 55 we reformulate to

**Theorem 56.** *Let $F(x, y)$ be a continuous function defined on $[0,1]^2$. Then for every sequence $x_n \in [0,1)$ we have*

$$G(x_n) \subset G(F) \iff \lim_{N \to \infty} \frac{1}{N^2} \sum_{m,n=1}^{N} F(x_m, x_n) = 0. \tag{114}$$

*Proof.* Using the definition of the Riemann-Stieltjes integral, we have

$$\int_0^1 \int_0^1 F(x, y)\mathrm{d}F_N(x)\mathrm{d}F_N(y) = \frac{1}{N^2} \sum_{m,n=1}^{N} F(x_m, x_n).$$

Suppose that $\lim_{k\to\infty} F_{N_k}(x) = g(x)$ for all continuity points $x$ of $g(x)$. Then, applying the Helly-Bray lemma, we find

$$\lim_{k\to\infty} \int_0^1 \int_0^1 F(x, y)\mathrm{d}F_{N_k}(x)\mathrm{d}F_{N_k}(y) = \int_0^1 \int_0^1 F(x, y)\mathrm{d}g(x)\mathrm{d}g(y),$$

and the implication $\Longleftarrow$ follows immediately.

In order to show the implication $\Longrightarrow$, assume

$$\lim_{k\to\infty}\frac{1}{{N_k}^2}\sum_{m,n=1}^{N_k}F(x_m,x_n)=\beta>0.$$

By the Helly selection principle (Theorem 1), there exists a subsequence $N'_k$ of $N_k$ such that $\lim_{k\to\infty}F_{N'_k}(x)=g(x)\in G(x_n)$. Again, by Helly-Bray lemma (Theorem 2) we find $\int_0^1\int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y)=\beta$. We conclude $g(x)\notin G(F)$. □

Define
• $G(F=A)$ is the set of all d.f.s $g(x)$ for which $\int_0^1\int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y)=A$,
• $G(A\le F\le B)$ is the set of all d.f.s $g(x)$ for which
$A\le\int_0^1\int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y)\le B$.
Then again:

**Theorem 57.**

$$G(x_n)\subset G(F=A)\Longleftrightarrow\lim_{N\to\infty}\frac{1}{N^2}\sum_{m,n=1}^{N}F(x_m,x_n)=A,$$

$$G(x_n)\subset G(A\le F\le B)\Longleftrightarrow A\le\liminf_{N\to\infty}\frac{1}{N^2}\sum_{m,n=1}^{N}F(x_m,x_n)$$

$$\textit{and}\ \limsup_{N\to\infty}\frac{1}{N^2}\sum_{m,n=1}^{N}F(x_m,x_n)\le B.$$

An application of Theorem 126:

**Theorem 58.** *For any sequence $x_n$ in $[0,1]$ we have*

$$G(x_n)\subset\{c_\alpha(x);\alpha\in[0,1]\}\Longleftrightarrow\lim_{N\to\infty}\frac{1}{N^2}\sum_{m,n=1}^{N}|x_m-x_n|=0.\tag{115}$$

*Moreover, if $G(x_n)\subset\{c_\alpha(x);\alpha\in[0,1]\}$, then $G(x_n)=\{c_\alpha(x);\alpha\in I\}$, where $I$ is a closed subinterval of $[0,1]$ which can be found as*

$$I=\left[\liminf_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}x_n,\ \limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}x_n\right],\tag{116}$$

*and the length $|I|$ of $I$ can also be found as*

$$|I| = \limsup_{M,N\to\infty} \frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} |x_m - x_n|. \tag{117}$$

*Proof.* First observe that $G(|x-y|) = \{c_\alpha(x); \alpha \in [0,1]\}$, and (115) follows from Theorem 126.

Suppose, in order to show the connectivity of $I$, that $\lim_{k\to\infty} F_{M_k}(x) = c_\alpha(x)$, $\lim_{k\to\infty} F_{N_k}(x) = c_\beta(x)$, and $\alpha < \beta$. If $\alpha < \gamma < \beta$, then we can construct a sequence $K_k$ such that $\lim_{k\to\infty} F_{K_k}(\gamma) = 1/2$. By the Helly selection principle there exists a subsequence $K_k'$ such that $\lim_{k\to\infty} F_{K_k'}(x) = c_\gamma(x)$.

In order to find the length of subinterval of such $\alpha$, one observes that

$$\int_0^1 \int_0^1 |x-y| \mathrm{d}c_\alpha(x) \mathrm{d}c_\beta(y) = |\alpha - \beta|$$

implies (117) in the usual way. □

**Example 24.** Let the sequence $x_n$, $n = 1, 2, \dots$ be defined as

$$x_n = \left\{ 1 + (-1)^{\left[\sqrt{[\sqrt{\log_2 n}]}\right]} \left\{ \sqrt{\left[\sqrt{\log_2 n}\right]} \right\} \right\}.$$

Then $G(x_n) = \{c_\alpha(x); \alpha \in [0,1]\}$.

*Proof.* Let $x_n$ be the sequence constructed as follows. Assume that we have a dense sequence $y_n$ in $[0,1]$ with (i) $\lim_{k\to\infty}(y_{k+1} - y_k) = 0$, and an increasing sequence of positive integers $M_k$ with (ii) $\lim_{k\to\infty}(\sum_{i=1}^{k-1} M_i)/M_k = 0$. Now form the sequence $x_n$ by setting $x_n = y_k$ if $\sum_{i=1}^{k-1} M_i \le n < \sum_{i=1}^{k} M_i$. We shall use Theorem 58 which gives that $x_n$ has $c_\alpha(x), \alpha \in [0,1]$, as its d.f.s. For a detailed proof, first take $N = \sum_{i=1}^{k-1} M_i + \theta_k M_k$, where $0 \le \theta_k < 1$. It is not difficult to calculate

$$\frac{1}{N^2} \sum_{m,n=1}^{N} |x_m - x_n| = O\left( \frac{M_{k-1}\theta_k M_k |y_k - y_{k-1}|}{(M_{k-1} + \theta_k M_k)^2} \right).$$

Hence, the limit in Theorem 58 follows immediately. In order to compute the lim sup, one observes, for $N_{i_k} = \sum_{i=1}^{i_k} M_i$ and $N_{j_k} = \sum_{i=1}^{j_k} M_i$, that

$$\limsup_{M,N\to\infty} \frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} |x_m - x_n| =$$

$$= \limsup_{k\to\infty} \frac{1}{M_{i_k}M_{j_k}} \sum_{m=N_{i_k-1}}^{N_{i_k}} \sum_{m=N_{j_k-1}}^{N_{j_k}} |x_m - x_n|$$

$$= \limsup_{k\to\infty} |y_{i_k} - y_{j_k}| = 1 \tag{118}$$

holds if $y_{i_k} \to 0$ and $y_{j_k} \to 1$ as $k \to \infty$. Applying the above construction with

$$y_k = \left\{ 1 + (-1)^{[\sqrt{k}]} \{\sqrt{k}\} \right\}$$

and

$$M_k = 2^{(k+1)^2} - 2^{k^2}$$

for $k = 1, 2, \dots$ we thus have the desired property of $x_n$, $n = 1, 2, \dots$. □

In Section 4.10 ([159]) is studied 3-dimensional body $\mathbf{\Omega}$ formed by points $(X_1, X_2, X_3)$ where
$X_1 = 1 - \int_0^1 x \mathrm{d}g(x) (= \int_0^1 g(x)\mathrm{d}x)$,
$X_2 = \frac{1}{2} - \frac{1}{2}\int_0^1 x^2 \mathrm{d}g(x) (= \int_0^1 xg(x)\mathrm{d}x)$, and
$X_3 = X_1 - \frac{1}{2}\int_0^1\int_0^1 |x-y|\mathrm{d}g(x)\mathrm{d}g(y) (= \int_0^1 g^2(x)\mathrm{d}x)$
and $g(x)$ run all d.f.s. The points $(X_1, X_2, X_3)$ for $g(x) = c_\alpha(x)$ or $g(x) = h_\alpha(x)$, $\alpha \in [0,1]$, lies on the contour of the body $\mathbf{\Omega}$ and for such $(X_1, X_2, X_3)$ the d.f. $g(x)$ is given uniquely. This implies:

**Theorem 59.**

$$\int_0^1 x^2 \mathrm{d}g(x) = \left( \int_0^1 x \mathrm{d}g(x) \right)^2 \iff \exists_{\alpha\in[0,1]} g(x) = c_\alpha(x),$$

$$\int_0^1 x^2 \mathrm{d}g(x) = \int_0^1 x \mathrm{d}g(x) \iff \exists_{\alpha\in[0,1]} g(x) = h_\alpha(x),$$

$$\left(1 - \int_0^1 x \mathrm{d}g(x)\right) \left( \int_0^1 x \mathrm{d}g(x) \right) = \frac{1}{2} \int_0^1 \int_0^1 |x-y| \mathrm{d}g(x)\mathrm{d}g(y)$$
$$\iff \exists_{\alpha\in[0,1]} g(x) = h_\alpha(x)$$

*and also we have*

$$G(x_n) \subset \{c_\alpha(x); \alpha \in [0,1]\} \iff \lim_{N\to\infty} \left( \left( \frac{1}{N} \sum_{n=1}^N x_n \right)^2 - \frac{1}{N} \sum_{n=1}^N x_n^2 \right) = 0,$$

$$G(x_n) \subset \{h_\alpha(x); \alpha \in [0,1]\} \iff \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} (x_n - x_n^2) = 0,$$

$$G(x_n) \subset \{h_\alpha(x); \alpha \in [0,1]\} \iff \lim_{N\to\infty} \left( \frac{1}{N} \sum_{n=1}^{N} x_n - \left( \frac{1}{N} \sum_{n=1}^{N} x_n \right)^2 - \frac{1}{N^2} \sum_{m,n=1}^{N} |x_m - x_n| \right) = 0,$$

*respectively.*

It is well known that the sequence $x_n \in [0,1]$ statistically converges to $\alpha \in [0,1]$ if and only if $G(x_n) = \{c_\alpha(x)\}$. Theorem 58 implies:

**Theorem 60.** *The sequence $x_n$ in $[0,1]$ possesses a statistical limit if and only if*

$$\lim_{M,N\to\infty} \frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} |x_m - x_n| = 0.$$

This is condition of Cauchy type different as in [57]. Also see Section 3.4

**Theorem 61.** *Let $F(x,y)$ be continuous and $G(F)$ be the set of all solutions of $\int_0^1 \int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y) = 0$. Assume that $\underline{g}_F, \overline{g}_F \in G(F)$ and that for every $g \in G(F)$ there exists a point $(x,y) \in \mathrm{Graph(g)}$ such that $(x,y) \notin \mathrm{Graph}(\widetilde{\mathrm{g}})$ for any $\widetilde{g} \in H$ with $\widetilde{g} \neq g$. Then for every sequence $x_n \in [0,1)$ we have $G(x_n) = G(F)$ if and only if*

(i) $\lim_{N\to\infty} \frac{1}{N^2} \sum_{m,n=1}^{N} F(x_m, x_n) = 0,$

(ii) $\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_n - \liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_n = \int_0^1 (\overline{g}_F(x) - \underline{g}_F(x))\mathrm{d}x.$

*Proof.* From (i) Theorem 126 implies $G(x_n) \subset G(F)$. The (ii) similar to Theorem 15 gives $\overline{g}_F(x), \underline{g}_F(x) \in G(x_n)$. Thus $\Omega(x_n) = \Omega(F)$ and if $(x,y) \in \Omega(F)$ then the unique $g \in \Omega(F)$ for which $(x,y) \in \mathrm{Graph(g)}$ gives $g \in G(x_n)$. See also Theorem 54. □

## 4.5 Linear combination $G(x_n) = \{tg_1(x) + (1-t)g_2(x); t \in [0,1]\}$

**Definition 4.** Let $g_1 \neq g_2$ be two d.f.s. Denote

$$F_{g_2}(x,y) := \int_0^x g_2(t)\mathrm{d}t + \int_0^y g_2(t)\mathrm{d}t - \max(x,y) + \int_0^1 (1 - g_2(t))^2 \mathrm{d}t,$$

$$
\begin{aligned}
F_{g_1,g_2}(x) :=& \frac{\int_0^x (g_2(t)-g_1(t))\mathrm{d}t - \int_0^1 (1-g_2(t))(g_2(t)-g_1(t))\mathrm{d}t}{\int_0^1 (g_2(t)-g_1(t))^2\mathrm{d}t},\\
F_{g_1,g_2}(x,y) :=& F_{g_2}(x,y) - F_{g_1,g_2}(x)F_{g_1,g_2}(y).\int_0^1 (g_2(t)-g_1(t))^2\mathrm{d}t.
\end{aligned}
$$

**Theorem 62.** *For given sequence $x_n \in [0,1)$ we have*

$$
G(x_n) = \{tg_1(x) + (1-t)g_2(x); t \in [0,1]\}
$$

*if and only if*

(i) $\lim_{N\to\infty} \frac{1}{N^2}\sum_{m,n=1}^N F_{g_1,g_2}(x_m,x_n) = 0,$

(ii) $\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^N F_{g_1,g_2}(x_n) = 0,$

(iii) $\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^N F_{g_1,g_2}(x_n) = 1.$

*Proof.* For any two d.f.s $g_1, g_2$ the set $H = \{tg_1(x) + (1-t)g_2(x); t \in [0,1]\}$ is nonempty, closed, connected and satisfies the assumption of Theorem 54. In order to compute the $L^2$ discrepancy (113) we shall consider the following quadratic polynomial

$$
P_N(t) := \int_0^1 \big(F_N(x) - (tg_1(x) + (1-t)g_2(x))\big)^2\mathrm{d}x = A_N t^2 + 2B_N t + C_N,
$$

where

$$
\begin{aligned}
A_N = A =& \int_0^1 (g_2(x)-g_1(x))^2\mathrm{d}x,\\
B_N =& \int_0^1 (F_N(x)-g_2(x))(g_2(x)-g_1(x))\mathrm{d}x,\\
C_N =& \int_0^1 (F_N(x)-g_2(x))^2\mathrm{d}x.
\end{aligned}
$$

The minimum $\min_{t\in\mathbb{R}} P_N(t) = (AC_N - B_N^2)/A$ is attained in $t = t_N = -B_N/A$. Therefore

$$
D_N^{(2)}(x_n, H) = \begin{cases} P_N(t_N) & \text{for } 0 \le t_N \le 1,\\ P_N(0) & \text{for } t_N < 0,\\ P_N(1) & \text{for } t_N > 1, \end{cases}
$$

and thus

$$\lim_{N\to\infty} D_N^{(2)}(x_n, H) = 0 \Longleftrightarrow \begin{aligned}&\text{(i)}\ \lim_{N\to\infty} P_N(t_N) = 0,\\ &\text{(ii')}\ \liminf_{N\to\infty} t_N \geq 0,\\ &\text{(iii')}\ \limsup_{N\to\infty} t_N \leq 1\end{aligned}$$

what is equivalent to $G(x_n) \subset H$ by Theorem 55.

Now assume (i),

(ii) $\liminf_{N\to\infty} t_N = 0$ and

(iii) $\limsup_{N\to\infty} t_N = 1$.

Passes to a suitable sequence $N_k$ guarantees that $\lim_{k\to\infty} t_{N_k} = 0$ and $\lim_{k\to\infty} F_{N_k}(x) = \tilde{g}(x)$ a.e. on $[0, 1]$.

The Lebesgue theorem yields

$$\begin{aligned}0 = \lim_{k\to\infty} P_{N_k}(t_{N_k}) &= \lim_{k\to\infty} \int_0^1 \big(F_{N_k}(x) - (t_{N_k} g_1(x) + (1 - t_{N_k})g_2(x))\big)^2 \mathrm{d}x\\ &= \int_0^1 (\tilde{g}(x) - g_2(x))^2 \mathrm{d}x\end{aligned}$$

and thus $g_2 \in G(x_n)$. Similarly, $g_1 \in G(x_n)$. Since $\overline{g}_H(x) = \max(g_1(x), g_2(x))$ and $\underline{g}_H(x) = \min(g_1(x), g_2(x))$ we have [18] $\underline{g} = \underline{g}_H$ and $\overline{g} = \overline{g}_H$. Thus, by Theorem 54, we have $G(x_n) = H$. On the other hand, (i), (ii) and (iii) immediately follows from $G(x_n) = H$. Finally we express

$$P_N(t_N) = \int_0^1 \int_0^1 F_{g_1,g_2}(x, y)\mathrm{d}F_N(x)\mathrm{d}F_N(y), \quad t_N = \int_0^1 F_{g_1,g_2}(x)\mathrm{d}F_N(x)$$

by using Theorem 23 from which for every polynomial $\psi(y) = a(x)y^2 + b(x)y + c(x)$ we have

$$\int_0^1 \psi(F_N(x))\mathrm{d}x = \int_0^1 \int_0^1 F(x, y)\mathrm{d}F_N(x)\mathrm{d}F_N(y) = \frac{1}{N^2} \sum_{m,n=1}^{N} F(x_m, x_n),$$

where

$$F(x, y) = \int_{\max(x,y)}^1 a(t)\mathrm{d}t + \frac{1}{2}\int_x^1 b(t)\mathrm{d}t + \frac{1}{2}\int_y^1 b(t)\mathrm{d}t + \int_0^1 c(t)\mathrm{d}t.$$

□

[18] Here $\underline{g}$ and $\overline{g}$ lower and upper d.f. of $x_n$ $n = 1, 2, \ldots$ respectively.

**Example 25.** For control of Theorem 62 we give:

(I) $\int_0^1 F_{g_1,g_2}(x)\mathrm{d}(tg_1(x)+(1-t)g_2(x)) = t$ because
$\int_0^1 \left( \int_0^x (g_2(u) - g_1(u)\mathrm{d}u) \right) \mathrm{d}g_1(x) = \int_0^1 (1-g_1(x))(g_2(x)-g_1(x))\mathrm{d}x.$
Furthermore

(II) $\int_0^1 \int_0^1 F_{g_1,g_2}(x,y)\mathrm{d}(tg_1(x)+(1-t)g_2(x))\mathrm{d}(tg_1(y)+(1-t)g_2(y))$
$= \int_0^1 \int_0^1 F_{g_2}(x,y)\mathrm{d}(tg_1(x)+(1-t)g_2(x))\mathrm{d}(tg_1(y)+(1-t)g_2(y))$
$- t^2 \int_0^1 (g_2(x)-g_1(x))^2\mathrm{d}x = 0.$

## 4.6 Other characterization of $\{tg_1(x)+(1-t)g_2(x); t \in [0,1]\}$

Some other theorem similar to Theorem 62:

**Definition 5.** Denote

$$\begin{aligned}
F_1(x) &= \int_x^1 (g_2(t)-g_1(t))\mathrm{d}t - \int_0^1 g_2(t)(g_2(t)-g_1(t))\mathrm{d}t,\\
F_1(x,y) &= F_1(x)F_1(y) - F_{g_2}(x,y). \int_0^1 (g_2(t)-g_1(t))^2\mathrm{d}t,\\
F_2(x,y) &= \frac{-F_1(x)-F_1(y)}{2},\\
F_3(x,y) &= \int_0^1 (g_2(t)-g_1(t))^2\mathrm{d}t - F_2(x,y).
\end{aligned}$$

**Theorem 63.** *Let $g_1(x)$, $g_2(x)$ be a given d.f.s such that, $g_1(x) \le g_2(x)$ on the interval $[0,1]$. For a sequence $x_n$, $n = 1, 2, \ldots$ in $[0,1)$ we have*

$$G(x_n) = \{tg_1(x)+(1-t)g_2(x); t \in [0,1]\} \iff \tag{119}$$

$$\text{(i)} \lim_{N\to\infty} \frac{1}{N^2} \sum_{m,n=1}^{N} F_1(x_m, x_n) = 0,$$

$$\text{(ii)} \ \liminf_{N\to\infty} \frac{1}{N^2} \sum_{m,n=1}^{N} F_2(x_m, x_n) \geq 0,$$

$$\text{(iii)} \ \liminf_{N\to\infty} \frac{1}{N^2} \sum_{m,n=1}^{N} F_3(x_m, x_n) \geq 0,$$

$$\text{(iv)} \ \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_n - \liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_n = \int_0^1 (g_2(x) - g_1(x))\mathrm{d}x.$$

*Proof.* Express

$$\int_0^1 \big(g(x) - (tg_1(x) + (1-t)g_2(x))\big)^2 \mathrm{d}x = P(t) = At^2 + 2Bt + C,$$

where

$$\begin{aligned} A &= \int_0^1 (g_2(x) - g_1(x))^2 \mathrm{d}x, \\ B &= \int_0^1 (g(x) - g_2(x))(g_2(x) - g_1(x))\mathrm{d}x, \\ C &= \int_0^1 (g(x) - g_2(x))^2 \mathrm{d}x. \end{aligned}$$

Since $P(t) \geq 0$, the polynomial $P(t)$ has a real root if and only if $B^2 - AC = 0$ and it is equal $t_0 = -B/A$. Simultaneously $t_0 \in [0,1]$ if and only if, $-B \geq 0$ and $-B \leq A$. Together

$$\exists (t \in [0,1]) g(x) = tg_1(x) + (1-t)g_2(x) \text{ a.e. } \Longleftrightarrow \begin{array}{l} \text{(i) } B^2 - AC = 0, \\ \text{(ii) } -B \geq 0, \\ \text{(iii) } A + B \geq 0. \end{array}$$

Then by using Theorem 23 we have

$$B^2 - AC = \int_0^1 \int_0^1 F_1(x,y)\mathrm{d}g(x)\mathrm{d}g(y), -B = \int_0^1 \int_0^1 F_2(x,y)\mathrm{d}g(x)\mathrm{d}g(y),$$

$$A + B = \int_0^1 \int_0^1 F_3(x,y)\mathrm{d}g(x)\mathrm{d}g(y).$$

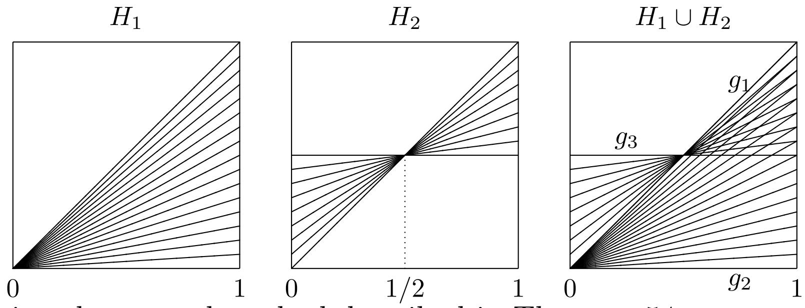
and putting $g(x) = F_N(x)$ we find (119) without (iv). The condition (iv) we add for all possible $g(x) = tg_1(x) + (1-t)g_2(x)$, $t \in [0,1]$. □

**Example 26.** Putt $g_2(x) = c_0(x)$ and $g_1(x) = c_1(x)$. Then $tc_1(x) + (1 - t)c_2(x) = h_{1-t}(x)$ is a constant d.f. Theorem 63 implies

$$
\begin{aligned}
G(x_n) = &\{h_\alpha(x); \alpha \in [0,1]\} \iff \\
&\text{(i)} \lim_{N\to\infty} \frac{1}{N^2} \sum_{m,n=1}^{N} \left( x_m x_n - \frac{x_m + x_n - |x_m - x_n|}{2} \right) = 0, \\
&\text{(ii)} \liminf_{N\to\infty} \frac{1}{N^2} \sum_{m,n=1}^{N} \left( \frac{x_m + x_n}{2} \right) \ge 0, \\
&\text{(iii)} \liminf_{N\to\infty} \frac{1}{N^2} \sum_{m,n=1}^{N} \left( 1 - \frac{x_m + x_n}{2} \right) \ge 0, \\
&\text{(iv)} \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_n - \liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_n = 1.
\end{aligned}
$$

See Example 29. Note that if $x_n = 0$ or $x_n = 1$ then the assumption (i) vanished since $xy - \frac{x+y-|x-y|}{2} = 0$ indentically.

**Example 27.** Denote $g_1(x) = x$, $g_2(x) = c_1(x)$, $g_3(x) = h_{1/2}(x)$, $H_1 = \{tx + (1-t)c_1(x); t \in [0,1]\}$, and $H_2 = \{tx + (1-t)h_{1/2}(x); t \in [0,1]\}$. We can visualize the sets $H_1$, $H_2$, and $H_1 \cup H_2$ as follows




Applying the general method described in Theorem 54 we get

$$
\begin{aligned}
G(x_n) = &H_1 \cup H_2 \iff \\
&\text{(i)} \lim_{N\to\infty} \frac{1}{N^4} \sum_{m,n,k,l=1}^{N} F_{g_1,g_2}(x_m, x_n) F_{g_1,g_3}(x_k, x_l) = 0,
\end{aligned}
$$

$$\text{(ii)}\ \liminf_{N\to\infty} \frac{1}{N^2}\sum_{m,n=1}^{N} F_{g_2}(x_m,x_n)=0,$$

$$\text{(iii)}\ \liminf_{N\to\infty} \frac{1}{N^2}\sum_{m,n=1}^{N} F_{g_3}(x_m,x_n)=0.$$

*Proof.* The maximal interval of reals $t$ for which $tx+(1-t)c_1(x)$ is a d.f. is equal to $[0,1]$. Similarly, for $tx+(1-t)h_{1/2}(x)$. Therefore, by Theorem 126 (cf. Note 1), the inclusion $G(x_n)\subset H_1\cup H_2$ is characterized by the limit (i). If $g_2,g_3\in G(x_n)$ then $G(x_n)=H_1\cup H_2$, since, according to the page 17, it can be seen that the graph $g_2$ can be continuously transformed into the graph $g_3$ over $H_1\cup H_2$ only if the graph $g_2$ can be continuously transformed into the graph $g_1$ over $H_1$ and the graph $g_1$ into the graph $g_3$ over $H_2$. To prove $g_2,g_3\in G(x_n)$ we use $F_{g_i}(x,y)$, $i=2,3$, since $\int_0^1\int_0^1 F_{g_i}(x,y)\mathrm{d}F_N(x)\mathrm{d}F_N(y)=\int_0^1(F_N(x)-g_i(x))^2\mathrm{d}x$. It should be noted that it is possible to replace (ii) by

$$\text{(ii')}\ \liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} F_{g_1,g_2}(x_n)=0$$

because

$$\int_0^1 F_{g_1,g_2}(x)\mathrm{d}(tx+(1-t)c_1(x))=t,$$
$$\int_0^1 F_{g_1,g_2}(x)\mathrm{d}(tx+(1-t)h_{1/2}(x))=\frac{1}{4}t+\frac{3}{4}.$$

These integrals and $G(x_n)\subset H_1\cup H_2$ together with (ii') yield that $c_1(x)\in G(x_n)$.

To finish the example we give explicit formulae of our functions:

$$\begin{aligned}
F_{g_1,g_2}(x,y)&=1-\max(x,y)-\frac{3}{4}(1-x^2)(1-y^2),\\
F_{g_1,g_3}(x,y)&=\frac{x+y}{2}-\max(x,y)+\frac{1}{4}-3(x-x^2)(y-y^2),\\
F_{g_2}(x,y)&=1-\max(x,y),\\
F_{g_3}(x,y)&=\frac{x+y}{2}-\max(x,y)+\frac{1}{4},\\
F_{g_1,g_2}(x)&=\frac{3}{2}(1-x^2).
\end{aligned}$$

□

**Notes 8.** Some extension of Theorem 62 can be found in Example 105.

**Example 28.** Let us again write $H = \{tg_1(x)+(1-t)g_2(x); t \in [0,1]\}$. Let $y_n$ and $z_n$ be two sequences in $[0,1]$ having limit laws $g_1$ and $g_2$, respectively. Let $x_n$ be a sequence mixed from $y_n$ and $z_n$. Given $N$, let $N_1$ and $N_2$ denote the number of terms of $y_n$ and $z_n$, respectively, in the initial segment $x_1, \dots, x_N$. Then

$$D_N^{(2)}(x_n, H) \le \left( \frac{N_1}{N} \sqrt{D_{N_1}^{(2)}(y_n, g_1)} + \frac{N_2}{N} \sqrt{D_{N_2}^{(2)}(z_n, g_2)} \right)^2 .$$

It follows from this that $G(x_n) \subset H$ and

$$G(x_n) = H \Longleftrightarrow \limsup_{N\to\infty} \frac{N_1}{N} = \limsup_{N\to\infty} \frac{N_2}{N} = 1.$$

*Proof.* Let $F_N^{(1)}(x)$ and $F_N^{(2)}(x)$, respectively, denote the $F_N(x)$ (defined previously for $x_n$) applied to $y_n$ and $z_n$. It follows from the definition of the $L^{(2)}$ discrepancy that

$$D_N^{(2)}(x_n, H) \le \int_0^1 \left( F_N(x) - \frac{N_1}{N} g_1(x) - \frac{N_2}{N} g_2(x) \right)^2 \mathrm{d}x.$$

Substituting $F_N(x) = \frac{N_1}{N} F_{N_1}^{(1)}(x) + \frac{N_2}{N} F_{N_2}^{(2)}(x)$ and then using the Cauchy-Schwarz inequality we get the desired upper bound. Now,
$\lim_{N\to\infty} D_N^{(2)}(x_n, H) = 0$
which by Theorem 55 implies $G(x_n) \subset H$ and to prove that $G(x_n) = H$ we can use Theorem 54. □

**Example 29.** The $0 \vee 1$ sequence

$$x_n = \frac{1 + (-1)^{[\log\log n]}}{2}$$

has $G(x_n) = \{h_\alpha(x); \alpha \in [0,1]\}$, where $h_\alpha(x) = \alpha$ for $x \in (0,1)$ (see Example 26) and the sequence

$$y_n = \frac{1}{n} \sum_{i=2}^{n} \frac{1 + (-1)^{[\log\log i]}}{2}$$

is dense in $[0,1]$ with a dispersion

$$d_N(y_n) \le \frac{1}{N^{\frac{1}{e^2} - \frac{1}{e^3}}}.$$

*Proof.* Since for integer part

$$[\log\log n] = k \Longleftrightarrow e^{e^k} < n < e^{e^{k+1}}$$

the sequence $x_n$ is a block sequence constructed with blocks $(0,\dots,0)$ and $(1,\dots,1)$. Then, by Example 28, $G(x_n) = \{tc_0(x) + (1-t)c_1(x)\}$ and the density of $y_n$ follows from Theorem 20. Putting $N_1 = [e^{e^{K-1}}]$ and $N_2 = [e^{e^K}]$ for even $[\log\log N] = K$ (similarly for the odd case) and applying Theorem 5 we can find the desired estimation of $d_N(y_n)$. □

## 4.7 Computing $G(h(x_n, y_n))$ by $\int_{\{h(x,y)\}<t} 1.\mathrm{d}g(x,y)$, where $g(x,y) \in G((x_n, y_n))$

We starting with $h(x,y) = x + y \bmod 1$.

**Theorem 64.** *Let $x_n$ and $y_n$ be two sequences in $[0,1)$ and $G((x_n, y_n))$ denote the set of all d.f.s of the two-dimensional sequence $(x_n, y_n)$. If*

$$z_n = x_n + y_n \bmod 1,$$

*then the set $G(z_n)$ of all d.f.s of $z_n$ has the form*

$$G(z_n) = \left\{ g(t) = \int_{0\le x+y<t} 1.\mathrm{d}g(x,y) + \int_{1\le x+y<1+t} 1.\mathrm{d}g(x,y); \right. \\ \left. g(x,y) \in G((x_n, y_n)) \right\}$$

*assuming that all the used Riemann-Stieltjes integrals exist.*

*Proof.* For bounded $f(x,y)$ and d.f. $g(x,y)$ the Riemann-Stietjes integral

$$\int_{[0,1]^2} f(x,y)\mathrm{d}g(x,y)$$

exists if and only if the set of all discontinuity points of $f(x,y)$ has a zero intersection with discontinuity of $g(x,y)$. Denoting by

$$X(t) = \{(x,y) \in [0,1]^2; 0 \le x+y < t\} \cup \{(x,y) \in [0,1]^2; 1 \le x+y < 1+t\},$$
$$X^{(1)}(t) = \{(x,y); y = t - x, x \in [0,1]\},$$

$$X^{(2)}(t) = \{(x,y); y = t+1-x, x \in [0,1]\},$$
$$X^{(3)} = \{(x,y); y = 1-x, x \in [0,1]\},$$

in the case $f(x,y) = c_{X(t)}(x,y)$ the discontinuity points form the line segments $X^{(1)}(t)$, $X^{(2)}(t)$ and $X^{(3)}$.

Let $(x_n, y_n)$, $n = 1,2,\dots$, be a sequence in $[0,1)^2$ and $z_n = x_n + y_n \bmod 1$. Denote

$$F_N(t) = \frac{1}{N}\sum_{n=1}^{N} c_{[0,t)}(z_n), \quad F_N(x,y) = \frac{1}{N}\sum_{n=1}^{N} c_{[0,x)\times[0,y)}((x_n,y_n)).$$

Since

$$0 \le \{x+y\} < t \Longleftrightarrow (0 \le x+y < t) \text{ or } (1 \le x+y < 1+t),$$

we have $F_N(t) = \frac{1}{N}\sum_{n=1}^{N} c_{X(t)}((x_n,y_n))$ and by means of Riemann-Stieltjes integration

$$F_N(t) = \int_{[0,1]^2} c_{X(t)}(x,y)\mathrm{d}F_N(x,y),$$

assuming that none of the points $(x_n,y_n)$, $1 \le n \le N$, is a point of discontinuity of $c_{X(t)}(x,y)$. It is true that $(x_n,y_n) \notin X^{(1)}(t) \cup X^{(2)}(t)$ for almost all $t \in [0,1]$. By assumption the adjoining diagonal $X^{(3)}$ of $[0,1]^2$ has zero measure with respect to every $g(x,y) \in G((x_n,y_n))$, and thus the set of indices $n$ for which $(x_n,y_n) \in X^{(3)}$ has zero asymptotic density and so it can be omitted.

Now, we assume that $F_{N_k}(x) \to g(x) \in G(z_n)$. Then by the first Helly theorem there exists subsequence $N'_k$ of $N_k$ such that $F_{N'_k}(x,y) \to g(x,y) \in G((x_n,y_n))$ and by the second Helly theorem we have

$$g(t) = \int_{[0,1)^2} c_{X(t)}(x,y)\mathrm{d}g(x,y). \tag{120}$$

Vice-versa, we assume that $F_{N_k}(x,y) \to g(x,y) \in G((x_n,y_n))$. Then by the first Helly theorem we can select subsequence $N'_k$ of $N_k$ such that $F_{N'_k} \to g(x)$. For such $g(x)$, the (120) is true, again. $\square$

In the following we suppose that the sequence $x_n$ and $y_n$ in $[0,1)$ has the asymptotic d.f. $g_1(x)$ or $g_2(x)$, respectively. Let $\Psi : [0,1]^2 \to [0,1]$ be a continuous function and let $z_n$ be the sequence consisting of all the values $\Psi(x_m,y_n)$, $m,n = 1,2,\dots$ and which are linear ordered such that the first $N^2$ term are $\Psi(x_m,y_n)$, $m,n = 1,2,\dots,N$.

**Theorem 65.** *The d.f. $g(x)$ of $z_n$ satisfies*

(i) $\int_0^1\int_0^1 \Psi(x,y)\mathrm{d}g_1(x)\mathrm{d}g_2(y) = \int_0^1 x\mathrm{d}g(x)$,

(ii) $\int_0^1\int_0^1\int_0^1\int_0^1 |\Psi(x,y)-\Psi(u,v)|\mathrm{d}g_1(x)\mathrm{d}g_2(y)\mathrm{d}g_1(u)\mathrm{d}g_2(v)$

$= \int_0^1\int_0^1 |x-y|\mathrm{d}g(x)\mathrm{d}g(y)$,

(iii) $\int_0^1\int_0^1\left(\int_0^{\Psi(x,y)} g(t)\mathrm{d}t\right)\mathrm{d}g_1(x)\mathrm{d}g_2(y) = \int_0^1\left(\int_0^x g(t)\mathrm{d}t\right)\mathrm{d}g(x)$

*which is unique solvable in $g(x)$.*

*Proof.* It follows directly from $L^2$-discrepancy by Theorem 22. □

**Example 30.** [171, p. 2–215, 2.22.2], also see Open problem no. 1.13 in [166]: Assuming u.d. of $x_n$, $n=1,2,\dots$ find a.d.f of the sequences

- $||x_m-x_n|-|x_k-x_l||$, $m,n,k,l=1,2,\dots$,
- $|||x_m-x_n|-|x_k-x_l||-||x_i-x_j|-|x_r-x_s|||$, $m,n,k,l,i,j,r,s=1,2,\dots$, etc.

We can use the following method: Let us denote by $g_j(x)$ an a.d.f of the sequence of $j$th differences (thus by Theorem 104 $g_1(x)=2x-x^2$). For $k$th moment we have

$$\int_0^1 x^k\mathrm{d}g_{j+1}(x) = \int_0^1\int_0^1 |x-y|^k\mathrm{d}g_j(x)\mathrm{d}g_j(y).$$

For $j=1$ we have

$$\int_0^1\int_0^1 |x-y|^k\mathrm{d}g_1(x)\mathrm{d}g_1(y) = \frac{8}{(k+1)(k+2)(k+4)}.$$

which implies

$$g_2(x) = \frac{8}{3}x - 2x^2 + \frac{1}{3}x^4.$$

**Notes 9.** As a Theorem 18 for computing $g_2(x)$ we can used

(i) $\int_0^1\int_0^1(1-|x-y|)\mathrm{d}(2x-x^2)\mathrm{d}(2y-y^2)=\int_0^1 g_2(x)\mathrm{d}x$,

(ii) $\int_0^1\int_0^1\int_0^1\int_0^1\left(1-|x-y|-\frac{||x-y|-|u-v||}{2}\right)$
$\mathrm{d}(2x-x^2)\mathrm{d}(2y-y^2)\mathrm{d}(2u-u^2)\mathrm{d}(2v-v^2)=\int_0^1 g_2^2(x)\mathrm{d}x$

(iii) $\int_0^1\int_0^1\left(\int_0^{|x-y|} g_2(t)\mathrm{d}t\right)\mathrm{d}(2x-x^2)\mathrm{d}(2y-y^2)=\int_0^1 g_2(x)\mathrm{d}x-\int_0^1 g_2^2(x)\mathrm{d}x.$

Also

$\int_0^1\int_0^1\int_0^1\int_0^1\frac{||x-y|-|u-v||}{2}\mathrm{d}(2x-x^2)\mathrm{d}(2y-y^2)$
$\mathrm{d}(2u-u^2)\mathrm{d}(2v-v^2)=\int_0^1 g_2(x)\mathrm{d}x-\int_0^1 g_2^2(x)\mathrm{d}x.$

**Notes 10.** S. Steinerberger (2010) conjectured that the density function $\frac{\mathrm{d}g_j(x)}{\mathrm{d}x}$ of the a.d.f. $g_j(x)$ of $j$th iterated differences is of the form $\frac{\mathrm{d}g_j(x)}{\mathrm{d}x}=\frac{2^{2^j+j-1}}{2^j!}(x-1)^jp(x)$, where $p(x)$ is a polynomial with integer coefficients. For $j=3$ he found
$\frac{\mathrm{d}g_3(x)}{\mathrm{d}x}=\frac{8}{315}(x-1)^3(-132-116x-36x^2+3x^3+x^4)$.

### 4.7.1 Computation of $G((\log n,\log\log n)\bmod 1)$

**Example 31.** The set of all d.f.s of the two-dimensional sequence

$$(\log n,\log\log n)\bmod 1$$

has the form [169]

$$\begin{aligned}G((\log n,\log\log n)\bmod 1)=&\{g_{u,v}(x,y);u\in[0,1],v\in[0,1]\}\\&\cup\{g_{u,0,j,\alpha}(x,y);u\in[0,1],\alpha\in A,j=1,2,\dots\}\\&\cup\{g_{u,0,0,\alpha}(x,y);u\in[\alpha,1],\alpha\in A\},\end{aligned}\tag{121}$$

where $A$ is the set of all limit points of the sequence $e^n$ mod 1, $n=1,2,\dots$ [19] and, for $(x,y)\in[0,1)^2$,

$$g_{u,v}(x,y)=g_u(x)\cdot c_v(y),$$

[19]The exact form of $A$ is an open problem.

$$
\begin{aligned}
g_{u,0,j,\alpha}(x,y) &= g_{u,0,j,\alpha}(x)\cdot c_0(y),\\
g_{u,0,0,\alpha}(x,y) &= g_{u,0,0,\alpha}(x)\cdot c_0(y), \text{ where}\\
g_u(x) &= \frac{e^{\min(x,u)}-1}{e^u}+\frac{1}{e^u}\frac{e^x-1}{e-1},\\
c_v(y) &= \begin{cases} 0 & \text{if } 0\le y\le v,\\ 1 & \text{if } v<y\le 1,\end{cases}\\
\text{and } c_v(0) &= 0, c_v(1)=1,\\
g_{u,0,j,\alpha}(x) &= \frac{e^{\max(\alpha,x)}-e^\alpha}{e^{j+u}}+\frac{e^{\min(x,u)}-1}{e^u}+\frac{1}{e^u}\frac{e^x-1}{e-1}\left(1-\frac{1}{e^{j-1}}\right),\\
g_{u,0,0,\alpha}(x) &= \frac{e^{\max(\min(x,u),\alpha)}-e^\alpha}{e^u},
\end{aligned}
\tag{122}
$$

and, for $x\in[0,1]$ and $y\in[0,1]$, marginal d.f.s are

$$
\begin{aligned}
g_{u,v}(x,1) &= g_u(x) = g_{u,v}(x,1-0),\\
g_{u,v}(1,y) &= c_v(y) = g_{u,v}(1-0,y),\\
g_{u,0,j,\alpha}(x,1) &= g_u(x) > g_{u,0,j,\alpha}(x,1-0), j=0,1,2,\dots,\\
g_{u,0,j,\alpha}(1,y) &= h_\beta(y) = g_{u,0,j,\alpha}(1-0,y), \text{ where } \beta = 1-\frac{1}{e^{j+u-\alpha}}, \text{ and}\\
h_\beta(y) &= \beta \text{ if } y\in(0,1), h_\beta(0)=0, h_\beta(1)=1,
\end{aligned}
\tag{123}
$$

where, the parameters $u,v,j$ and $\alpha$ play the following role: Let $F_N(x,y)$ denote the step d.f. of the sequence $(\log n, \log\log n)$ mod 1 (for definition see Section 7.2) and let $F_{N_k}(x,y)\to g(x,y)$ be a weak convergence as $k\to\infty$. Then

- $F_{N_k}(x,y)\to g_{u,v}(x,y)$ for $\{\log N_k\}\to u$ and $\{\log\log N_k\}\to v>0$,
- $F_{N_k}(x,y)\to g_{u,0,j,\alpha}(x,y)$ for $\{\log N_k\}\to u$, $\{\log\log N_k\}\to 0$, $\{e^J\}\to\alpha$, $K-[e^J]=j>0$, where $K=[\log N_k]$, $J=[\log\log N_k]$, and
- $F_{N_k}(x,y)\to g_{u,0,0,\alpha}(x,y)$ for $\{\log N_k\}\to u$, $\{\log\log N_k\}\to 0$, $\{e^J\}\to\alpha$, $K-[e^J]=0$.

Since

$$\begin{aligned} \mathrm{d}g_{u,v}(x,y) &= \mathrm{d}g_u(x)\mathrm{d}c_v(y) \\ \mathrm{d}g_{u,0,j,\alpha}(x,y) &= \mathrm{d}g_{u,0,j,\alpha}(x)\mathrm{d}c_0(y) \end{aligned}$$

and bearing in mind (122) we have

$\mathrm{d}g_{u,v}(x,y) \neq 0$ only for $(x,y) = (x,v)$,

$\mathrm{d}g_{u,0,j,\alpha}(x,y) \neq 0$ only for $(x,0)$ and $(1,y)$.

Furthermore we see that $X^{(1)}(t) \cup X^{(2)}(t) \cup X^{(3)}$ has zero measure with respect to every $g(x,y) \in G((\log n, \log\log n) \bmod 1)$. We shall find $G(\log(n\log n) \bmod 1)$ by applying Theorem 64 to $G((\log n, \log\log n)) \bmod 1)$. All integrals in Theorem 64 can be computed and every d.f. $g(t) \in G(\log(n\log n) \bmod 1)$ has the form

$$g(t) = \begin{cases} \int_0^{t-v} 1.\mathrm{d}g_u(x) + \int_{1-v}^1 1.\mathrm{d}g_u(x) & \text{if } t \geq v, \\ \int_{1-v}^{1+t-v} 1.\mathrm{d}g_u(x) & \text{if } t < v, \end{cases}$$

for $v > 0$ and for $v = 0$ it has the form

$$g(t) = \int_0^t 1.\mathrm{d}g_{u,0,j,\alpha}(x) + \int_0^t 1.\mathrm{d}(g_u(x) - g_{u,0,j,\alpha}(x)), \quad j = 0,1,2,\dots.$$

Thus we have $G(\log(n\log n) \bmod 1) = \{g_{u,v}(x); u \in [0,1], v \in [0,1]\}$, where

$$g_{u,v}(x) = \begin{cases} g_u(1+x-v) - g_u(1-v) & \text{if } 0 \leq x \leq v, \\ g_u(x-v) + 1 - g_u(1-v) & \text{if } v < x \leq 1. \end{cases} \tag{124}$$

Directly by means of computation we see that $g_{u,v}(x) = g_w(x)$, for $w = u + v \bmod 1$.

*Proof of* (121). For $0 \leq x \leq 1$, $0 \leq y \leq 1$ and a positive integer $N$ denote by $F_N(x,y)$ the step d.f.

$$F_N(x,y) = \frac{\#\{3 \leq n \leq N; (\{\log n\}, \{\log\log n\}) \in [0,x) \times [0,y)\}}{N}$$

and desired d.f.s are all possible weak limits $\lim_{k\to\infty} F_{N_k}(x,y) = g(x,y)$, for suitable sequences $N_1 < N_2 < \dots$. Put

- $K(N) = [\log N]$,

- $J(N) = [\log\log N]$,
- $\mathbf{N} = \{1, 2, \dots, N\}$.

Note that, for given positive integers $J$ and $K$, we have $J(N) = J$ for every $N$ in $(e^{e^J}, e^{e^{J+1}})$ and $K(N) = K$ for every $N \in (e^K, e^{K+1})$. Further, let $k = 1, 2, \dots$ and $j = 0, 1, 2, \dots$. Since

$$0 \le \{\log n\} < x \Longleftrightarrow \exists_k (0 \le \log n - k < x) \Longleftrightarrow \exists_k (e^k \le n < e^{k+x})$$
$$0 \le \{\log\log n\} < y \Longleftrightarrow \exists_j (0 \le \log\log n - j < y) \Longleftrightarrow \exists_j (e^{e^j} \le n < e^{e^{j+y}}),$$

we have

$$F_N(x, y) = \frac{\sum_{j=0}^{J} \#\{[e^{e^j}, e^{e^{j+y}}) \cap (\cup_{k=1}^{K} [e^k, e^{k+x})) \cap \mathbf{N}\}}{N}.$$

Denote by $|X|$ the Lebesgue measure of $X \subset \mathbb{R}$. Using the facts that

- $\#\{[e^k, e^{k+x}) \cap \mathbf{N}\} = e^{k+x} - e^k + O(1)$;
- $1 \le k \le K = [\log N]$;
- $\frac{e^{e^{J-1+y}}}{e^{e^J}} \to 0$ as $J \to \infty$ and for fixed $0 \le y < 1$;
- $N > e^{e^J}$;

we have $F_N(x, y) = \tilde{F}_N(x, y) + o(1)$, where

$$\tilde{F}_N(x, y) = \frac{|[e^{e^J}, e^{e^{J+y}}) \cap (\cup_{k=[e^J]}^{K} [e^k, e^{k+x})) \cap [e^{e^J}, N]|}{N}$$

for every $0 \le x \le 1$ and every $0 \le y < 1$. Put

- $u(N) = \{\log N\}$,
- $v(N) = \{\log\log N\}$.

For any sequence $N_k$ for which $F_{N_k}(x, y) \to g(x, y)$, there exists a subsequence $N'_k$ such that $u(N'_k) \to u$, $v(N'_k) \to v$ and in this case we shall write

$$\lim_{k\to\infty} F_{N'_k}(x, y) = g_{u,v}(x, y).$$

On the other hand, the sequence $(\{\log n\}, \{\log\log n\})$, $n = 2, 3, \dots$, is everywhere dense in $[0,1]^2$. Proof: Let $(u,v) \in [0,1]^2$ and let $J_k$, $k = 1, 2, \dots$, be an increasing sequence of positive integers. Put $N_k = \left[e^{e^{J_k+v}}\right]$. Expressing $N_k = e^{e^{J_k+v_k}}$, then we have $v_k \to v$ as $k \to \infty$. Put $K_k = [\log N_k]$ and $N_k' = \left[e^{K_k+u}\right]$. Express $N_k' = e^{K_k+u_k'} = e^{e^{J_k+v_k'}}$, then $u_k' \to u$ and $v_k' \to v$. The final limit follows from the mean-value theorem

$$N_k' - N_k = e^{e^{J_k+v_k'}} - e^{e^{J_k+v_k}} = (v_k' - v_k)e^{e^{J_k+t}}e^{J_k+t}$$

for $t \in (v_k, v_k')$ and since $N_k, N_k' \in (e^{K_k}, e^{K_k+1})$ we have $(v_k' - v_k) \to 0$.

Thus for every $(u,v) \in [0,1]^2$ there exists a sequence of indices $N_k$ such that $u(N_k) \to u$ and $v(N_k) \to v$ and moreover, by Helly theorem, there exists a subsequence $N_k'$ of $N_k$ such that $F_{N_k'}(x,y)$ weakly converges to $g_{u,v}(x,y)$. We shall see that for $v \neq 0$ the d.f. $g_{u,v}(x,y)$ is defined uniquely.

Thus, in what follows we assume $u(N) \to u$, $v(N) \to v$, $F_N(x,y) \to g_{u,v}(x,y)$ weakly (i.e. $\tilde{F}_N(x,y) \to g_{u,v}(x,y)$ weakly) for a suitable $N = N_k$, $k \to \infty$. We distinguish two main cases: $v > 0$ and $v = 0$.

**1.** $v > 0$.

**1.a.** $0 \le y < v$. In this case

$$\tilde{F}_N(x,y) < \frac{e^{e^{J+y}}}{e^{e^{J+v(N)}}} \to 0$$

for a suitable $N \to \infty$ and thus

$$g_{u,v}(x,y) = 0$$

for every $x \in (0,1)$.

**1.b.** $0 < v < y < 1$. In this case the contributions of the following intervals to the limit $\tilde{F}_N(x,y) \to g_{u,v}(x,y)$ are:

- The first interval $[e^{[e^J]}, e^{[e^J]+1})$ gives
$$\frac{e^{[e^J]+1} - e^{[e^J]}}{e^{e^{J+v(N)}}} \to 0;$$
- the interval $[e^K, e^{K+1})$ gives
$$\frac{e^{\min(K+x,K+u(N))} - e^K}{e^{K+u(N)}} \to \frac{e^{\min(x,u)} - 1}{e^u};$$

- Since $K-[e^J]=([e^{J+v(N)}]-[e^J])\to\infty$, the intervals $[e^k,e^{k+x})$, $k=[e^J]+1,\dots,K-1$ contribute with

$$\frac{1}{e^{K+u(N)}}\sum_{k=[e^J]+1}^{K-1}(e^{k+x}-e^k)\to\frac{1}{e^u}\frac{e^x-1}{e-1}.$$

**1.c.** $v=y$. Taking in consideration the cases 1.a. and 1.b. we see that $g_{u,v}(x,y)$ has for $y=v$ the step d.f. $g_u(x)$.

As summary, in the case $v>0$ we have $g_{u,v}(x,y)=g_u(x)\cdot c_v(y)$, where d.f.

$$g_u(x)=\frac{e^{\min(x,u)}-1}{e^u}+\frac{1}{e^u}\frac{e^x-1}{e-1}$$

for every $x\in[0,1)$ and $g_u(1)=1$. Here $c_v(y)=c_{(v,1]}(y)$, i.e., $c_v(y)=0$ for $0<y\le v$, $c_v(y)=1$ for $v<y\le 1$, and always $c_v(0)=0$, $c_v(1)=1$.

**2.** $v=0$.

We shall see that in this case we need two new parameters $j(N)$ and $\alpha(N)$ defined

- $j(N)=K-[e^J]$,
- $\alpha(N)=\{e^J\}$,

and we again assume that for the sequence of indices $N=N_k$, $(k\to\infty)$ we have $\tilde{F}_N(x,y)\to g(x,y)$, $j(N)\to j$, $\alpha(N)\to\alpha$ and in this case we shall write

$$\tilde{F}_N(x,y)\to g_{u,0,j,\alpha}(x,y).$$

We distinguish three cases: $j=\infty$, $0<j<\infty$ and $j=0$.

**2.a.** $j=\infty$. This case is the same as **1.b**, and thus

$$g_{u,0,\infty,\alpha}(x,y)=g_u(x)\cdot c_0(y).$$

**2.b.** $0<j<\infty$. In this case the contributions of intervals to $\tilde{F}_N(x,y)$ are:

- The first interval $[e^{[e^J]},e^{[e^J]+1})$ gives

$$\frac{\max(e^{e^J},e^{[e^J]+x})-e^{e^J}}{e^{K+u(N)}}\to\frac{e^{\max(\alpha,x)}-e^\alpha}{e^{j+u}};$$

- the interval $[e^K, e^{K+1})$ gives again

$$\frac{e^{\min(K+x,K+u(N))} - e^K}{e^{K+u(N)}} \to \frac{e^{\min(x,u)} - 1}{e^u};$$

- Since $K - [e^J] = j =$ constant , the intervals $[e^k, e^{k+x})$, $k = [e^J] + 1, \dots, K-1$ contribute with

$$\frac{1}{e^{K+u(N)}} \sum_{k=[e^J]+1}^{K-1} (e^{k+x} - e^k) \to \frac{1}{e^u} \frac{e^x - 1}{e - 1} \left(1 - \frac{1}{e^{j-1}}\right).$$

Thus in the case $0 < j < \infty$ we have $g_{u,0,j,\alpha}(x, y) = g_{u,0,j,\alpha}(x) \cdot c_0(y)$, where

$$g_{u,0,j,\alpha}(x) = \frac{e^{\max(\alpha,x)} - e^\alpha}{e^{j+u}} + \frac{e^{\min(x,u)} - 1}{e^u} + \frac{1}{e^u} \frac{e^x - 1}{e - 1} \left(1 - \frac{1}{e^{j-1}}\right)$$

and $g_{u,0,j,\alpha}(1) = 1$.

**2.c.** $j = 0$. In this case we consider only the interval $[e^{[e^J]}, e^{[e^J]+1})$ and its contribution to $\tilde{F}_N(x, y)$ is

$$\frac{\max\left(\min(e^{K+x}, e^{K+u(N)}), e^{e^J}\right) - e^{e^J}}{e^{K+u(N)}} \to \frac{e^{\max(\min(x,u),\alpha)} - e^\alpha}{e^u}$$

and thus we have $g_{u,0,0,\alpha}(x, y) = g_{u,0,0,\alpha}(x) \cdot c_0(y)$, where

$$g_{u,0,0,\alpha}(x) = \frac{e^{\max(\min(x,u),\alpha)} - e^\alpha}{e^u}$$

and $g_{u,0,0,\alpha}(1) = 1$.

We now show that the following triples $(u, j, \alpha)$ can occur: Suppose that for integer sequence $J_1 < J_2 < \dots$ we have $\{e^{J_k}\} \to \alpha$. Then for every fixed $j$, $(j = 0, 1, 2, \dots)$ and $u \in [0, 1]$ (if $j = 0$, then $u$ need to satisfy $\alpha \le u$) there exist two integer sequences $K_k$ and $N_k$ such that

- $K_k - [e^{J_k}] = j$,
- $N_k \in (e^{K_k}, e^{K_k+1})$,
- $\{\log N_k\} \to u$,

- $N_k \in (e^{e^{J_k}}, e^{e^{J_k+1}})$,
- $\{\log\log N_k\} \to 0$.

It suffices to put

- $K_k = [e^{J_k}] + j$, and
- $N_k = [e^{K_k+u_k}]$, where
- $u_k \to u$, $0 < u_k < 1$,
- $u_k > \{e^{J_k}\}$ if $j = 0$,
- $u_k > \log\left(1 + \frac{1}{e^{K_k}}\right)$ if $u = 0$.

Denoting by $A$ the set of all limit points of $\{e^J\}$, $J = 1, 2, \dots$, we have 19.

Finally, by definition, we have $g(0, y) = g(x, 0) = 0$ and for computing $g(x, y) \in G((\{\log n\}, \{\log\log n\}))$ in the case $y = 1$ we can not use $\tilde{F}_N(x, y)$ instead of $F_N(x, y)$. In this case

$$F_N(x, 1) = \frac{(\cup_{k=1}^K [e^k, e^{k+x})) \cap \mathbf{N}}{N}$$

for every $0 \le x \le 1$ and if $u(N) \to u$, then $F_N(x, 1) \to g_u(x)$. Thus $G(\{\log n\}) = \{g_u(x); u \in [0, 1]\}$, which also can be found in [92, p. 59] (see (8) in Example 1 in this book).

All computations of $g(x, y) \in G((\{\log n\}, \{\log\log n\}))$ are also valid for $x = 1$ and we see that $g_{u,v}(1, y) = c_v(y)$ and

$$g_{u,0,j,\alpha}(1, y) = h_\beta(y), \text{ where } \beta = 1 - \frac{e^\alpha}{e^{j+u}}, \quad j = 0, 1, 2, \dots$$

and this $\beta$ covers $[0, 1]$ if $u$ runs $[\alpha, 1]$ for $j = 0$ and $[0, 1]$ for $j = 1, 2, \dots$. Thus

$$g(1, y) \in G(\{\log\log n\}) = \{c_v(y); v \in [0, 1]\} \cup \{h_\beta(y); \beta \in [0, 1]\},$$

where $h_\beta(y) = \beta$ for $0 < y < 1$ and $h_\beta(0) = 0$, $h_\beta(1) = 1$. $\square$

## 4.8 Computation of $G(x_n)$ using mapping $x_n \to f(x_n)$

This leads to solving related functional equations.

### 4.8.1 Mapping $x_n \to f(x_n)$ and mapping $g \to g_f$

We repeat definition $g_f(x)$ in Section 3.8:

Let $f : [0,1] \to [0,1]$ be a function such that, for all $x \in [0,1]$, $f^{-1}([0,x))$ can be expressed as a sum of finitely many pairwise disjoint subintervals $I_i(x)$ of $[0,1]$ with endpoints $\alpha_i(x) \le \beta_i(x)$. For any distribution function $g(x)$ we put

$$g_f(x) = \sum_i g(\beta_i(x)) - g(\alpha_i(x)) \text{ for } x \in [0,1]. \tag{125}$$

The mapping $g \to g_f$ can be used for studying $G(x_n)$ by the following statement:

**Theorem 66.** *Let $x_n \bmod 1$ be a sequence having $g(x)$ as a d.f. associated with the sequence of indices $N_1, N_2, \dots$. Suppose that any term $x_n \bmod 1$ is repeated only finitely many times. Then the sequence $f(\{x_n\})$ has the d.f. $g_f(x)$ for the same $N_1, N_2, \dots$, and vice-versa any distribution function of $f(\{x_n\})$ has this form.*

*Proof.* The form $g_f(x)$ is a consequence of

$$A([0,x); N_k; f(\{x_n\})) = \sum_i A(I_i(x); N_k; x_n) \text{ and}$$

$$A(I_i(x); N_k; x_n) = A([0,\beta_i(x)); N_k; x_n) - A([0,\alpha_i(x)); N_k; x_n) + o(N_k).$$

On the other hand, suppose that $\tilde{g}(x)$ is a distribution function of $f(\{x_n\})$ associated with $N_1, N_2, \dots$. The Helly selection principle guarantee a suitable subsequence $N_{n_1}, N_{n_2}, \dots$ for which some $g(x)$ is a distribution function of $x_n \bmod 1$. Thus $\tilde{g}(x) = g_f(x)$. □

The d.f. $g_f(x)$ is characterized by the following equation: For every continuous $F : [0,1] \to \mathbb{R}$ and every d.f. $g(x)$ we have

$$\int_0^1 F(f(x)) \mathrm{d}g(x) = \int_0^1 F(x) \mathrm{d}g_f(x). \tag{126}$$

If $g(x) = x$, then $g_f(x) = g(x)$ is the d.f. of the random variable $X = f(x)$.

For non-continuous function $f$, some $\alpha(x)$, $\beta(x)$ may be constant $\neq 0,1$ and for exact definition $g_f(x)$ we need used point-wise definition of d.f. (see [92, p. 53, Def. 7.1]):

**Definition 6.** A d.f. $g(x)$ is called the d.f. of a sequence $x_n \bmod 1$, $n = 1, 2, \dots$, if there exists an increasing sequence of positive integers $N_1, N_2, \dots$ such that

$$\lim_{k\to\infty} \frac{A([0,x); N_k; x_n)}{N_k} = g(x) \text{ for every } x \in [0,1].$$

Also $g_1 = g_2$ if and only if $g_1(x) = g_2(x)$ for all $x \in [0,1]$.

**Example 32.** Put $f(x) = 2x \bmod 1$, then $f^{-1}([0,x)) = [0,x/2) \cup [1/2,(1+x)/2)$ and $g_f(x) = g(x/2) + g((1+x)/2) - g(1/2)$. Let $x_n$ and $y_n$ by two sequences in $[0,1)$ such that $x_n \nearrow 1/2$, $y_{2n} \nearrow 1/2$, $y_{2n+1} \searrow 1/2$. Both sequences $x_n$ and $y_n$ have a.d.f. $c_{1/2}(x)$, where for the first sequence $x_n$ we have $c_{1/2}(1/2) = 1$ and for the second one $y_n$ we have $c_{1/2}(1/2) = 1/2$, thus there are two different d.f.s. Furthermore the sequence $f(x_n)$ has a.d.f $c_1(x)$ and $f(y_n)$ has $h_{1/2}(x)$ (i.e. it has constant value $= 1/2$ on $(0,1)$).

Note that if any term $x_n \bmod 1$ is repeated only finitely many times, then the semi-closed interval $[0,x)$ can be instead by the closed interval $[0,x]$. Thus if all of the intervals $I_i(x)$ are semi-closed of the form $[\alpha_i(x), \beta_i(x))$, then $o(N_k) = 0$ and the assumption of finiteness of repetition is superfluous.

Let $f : [0,1] \to [0,1]$ be give Jordan measurable function. We generalize (125) to

$$g_f(x) = \int_{f^{-1}([0,x))} 1.\mathrm{d}g(t) \tag{127}$$

a.e. on $x \in [0,1]$.

In order to find d.f. $g(x)$ of $x_n$ by using $g_f(x)$ of the transforming sequence $f(x_n)$, we can solve e.g.

- the functional equation $g(x) = g_f(x)$ for $x \in [0,1]$, if $x_n$ and $f(x_n)$, $n = 1, 2, \dots$ have the same a.d.f.

- the functional equation $g_f(x) = g_h(x)$ for $x \in [0,1]$ if $f(x_n)$ and $h(x_n)$ coincide.

- or we find one-to-one mapping $g \to g_f$, possibly another map $g \to g_h$ such that $g \to (g_f, g_h)$ is one-to-one.

### 4.8.2 One-to-one map $g \to g_f$

In [156] we have proved $g_f(x) = x_f \iff g(x) = x$ for $f(x,y) = |x-y|$. In a proof we have used

$$\int_0^1 (g_f(x) - x_f)(1-2x)\mathrm{d}x = 2\int_0^1 (g(x)-x)^2\mathrm{d}x - 2\left(\int_0^1 (g(x)-x)\mathrm{d}x\right)^2.$$

Here we actually found $F(x)$ such that

$$\int_0^1 F(x)\mathrm{d}(g_f(x) - \tilde{g}_f(x)) =$$
$$\int_0^1\int_0^1 F(f(x,y))\mathrm{d}g(x)\mathrm{d}g(y) - \int_0^1\int_0^1 F(f(x,y))\mathrm{d}\tilde{g}(x)\mathrm{d}\tilde{g}(y)$$

where

$$\int_0^1\int_0^1 F(f(x,y))\mathrm{d}g(x)\mathrm{d}g(y) = 0$$

has a unique solution $g(x) = \tilde{g}(x)$. In the case $f(x,y) = |x-y|$ and $\tilde{g}(x) = x$ we find

$$F(x) = -\frac{x}{2} + \frac{x^2}{2} + \frac{1}{12}$$

because $F(|x-y|)$ represent diaphony in Example 100, see

$$\begin{aligned}\int_0^1\int_0^1 F(|x-y|)\mathrm{d}g(x)\mathrm{d}g(y) &= \int_0^1 (g(x)-x)^2\mathrm{d}x - \left(\int_0^1 (g(x)-x)\mathrm{d}x\right)^2 \\ &= \iint\limits_{0\le x\le y\le 1} \big(g(y) - g(x) - (y-x)\big)^2\mathrm{d}x\mathrm{d}y = 0,\end{aligned}$$

with the unique solution $g(x) = x$.

The problem of one-to-one $g \to g_f$ can be solve by the following steps:

(i) we find $F(x,y)$ such that the moment problem

$$\int_0^1\int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y) = 0$$

has unique solution $g(x) = \tilde{g}(x)$, and

(ii) $F(x,y)$ can be expressed as

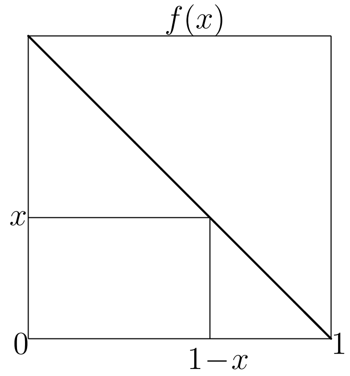
$$F(x,y) = H(f(x,y)),$$

where $f : [0,1]^2 \to [0,1]$ a $H$ are arbitrary continuous functions.

Then $g_f(x) = \tilde{g}_f(x) \Longleftrightarrow g(x) = \tilde{g}(x)$.

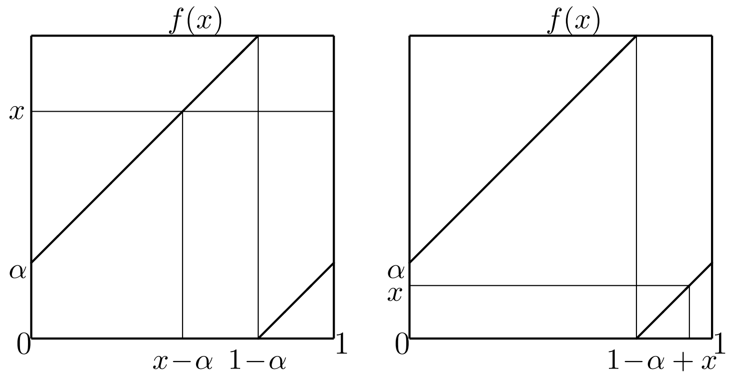
### 4.8.3 Simple examples of $f : [0,1] \to [0,1]$ and $g_f(x)$

**Example 33.** Since $-x = 1 - x \bmod 1$, the mapping $f(x) = -x \bmod 1$ coincide with $f(x) = 1 - x \bmod 1$.




This Fig. implies $g_f(x) = 1 - g(1-x)$ (as an application see Theorem 112).

**Example 34.** For a shift $f(x) = x + \alpha \bmod 1$, $0 < \alpha < 1$, we have




Thus

$$g_f(x) = \begin{cases} g(1-\alpha+x) - g(1-\alpha) & \text{if } 0 \le x \le \alpha \\ g(x-\alpha) + 1 - g(1-\alpha) & \text{if } \alpha < x \le 1. \end{cases} \tag{128}$$

**Example 35.** Applying (128) to the sequence $\log n + \alpha \bmod 1$, then the d.f. $g_u(x)$ of $\log n \bmod 1$ described in Example 1 is transform to $g_{u,f}(x)$ and then using (124) we see that $g_{u,f}(x) = g_w(x)$, where $w = u + \alpha \bmod 1$. Since $u \in [0,1]$ is arbitrary, then we have

$$G((\log n + \alpha) \bmod 1) = G(\log n \bmod 1). \tag{129}$$

Now, let

$$x_{n+1} = x_n - x_n^2, \quad n = 1, 2, \dots \tag{130}$$

with the starting point $x_1 \in (0,1)$. E. Ionescu and P. Stănică (2004) [77] was proved that for every $x_1 \in (0,1)$ there exists the limit

$$v = \lim_{n\to\infty}\left(\frac{1}{x_n} - n - \log n\right). \tag{131}$$

Then they expand

$$x_n = \frac{1}{n} - \frac{\log n}{n^2} - \frac{v}{n^2} + \frac{(\log n)^2}{n^3} + (2v-1)\frac{\log n}{n^3} + o\left(\frac{\log n}{n^3}\right), \tag{132}$$

$$\begin{aligned}\frac{1}{x_n} =& n + \log n + v + \frac{\log n}{n} + \frac{v-(1/2)}{n} - \frac{1}{2}\frac{(\log n)^2}{n^2}\\ &+ (3/2 - v)\frac{\log n}{n^2} + \left(\frac{3}{2}v - \frac{1}{2}v^2 - \frac{5}{6}\right)\frac{1}{n^2} + \frac{1}{3}\frac{(\log n)^3}{n^3}\\ &+ (-2+v)\frac{(\log n)^2}{n^3}\left(\frac{19}{6} - 4v + v^2\right)\frac{\log n}{n^3} + o\left(\frac{\log n}{n^3}\right)\end{aligned} \tag{133}$$

Since the sequence $(\log n + v) \bmod 1$, $n = 1, 2, \dots$, have the same d.f.s as $\log n \bmod 1$, then the sequences

$$n^2 x_n \bmod 1,$$
$$\frac{1}{x_n} \bmod 1,$$

$n = 1, 2, \dots$ have the same d.f.s as $\log n \bmod 1$.

**Example 36.** Using notation of $g_f$ we can write

$$\int_{g^{-1}([0,x))} 1.\mathrm{d}x = x_g$$

for d.f. $g(x)$.

**Example 37.** For fixed d.f. $g(x)$ we can find function $f$ such that $g(x) = g_f(x)$. For example $f(x)$ is called *u.d. preserving* if $x = x_f$.

**Example 38.** For two-dimensional $f(x,y)$ and d.f. $g(x)$ we can define

$$g_f(x) = \int\int_{0\le f(u,v)<x} 1.\mathrm{d}g(u)\mathrm{d}g(v).$$

For $f(x,y) = xy$ and $g(x) = x$ we have $x_f = x(1 - \log x)$.

### 4.8.4 Solution of $g_f = g_1$

Assume that $f$ is continuous with finitely many inverse functions $f_1^{-1}, \dots, f_m^{-1}$, all defined on the whole interval $[0,1]$ and the intervals $f_1^{-1}([0,1]), \dots, f_m^{-1}([0,1])$ are ordered from the left to right, i.e. $f_1^{-1}(x) \le \cdots \le f_m^{-1}(x)$ for every $x \in [0,1]$. Then the d.f. $g_f$ satisfies

$$g_f(x) = \sum_{i=1}^{m} \operatorname{sign}(f_i^{-1})' g(f_i^{-1}(x)) + \frac{1 + \operatorname{sign}(f_1^{-1})'(-1)^m}{2} \tag{134}$$

on $x \in [0,1]$.

A functional equation $g_f = \tilde{g}_f$, where $\tilde{g}$ is known and $g$ is unknown, we can solve by following: Put $[0,\alpha] = f_1^{-1}([0,1])$ and denote

$$x_i(t) = f_i^{-1} \circ f(t), \quad i = 1, \dots, m, \quad t \in [0,\alpha].$$

Into

$$0 = g_f(x) - \tilde{g}_f(x) = \sum_{i=1}^{m} \operatorname{sign}(f_i^{-1})'(g(f_i^{-1}(x)) - \tilde{g}(f_i^{-1}(x)))$$

insert $x = f(t)$, $t \in [0,\alpha]$. We obtain a linear equation with $m$ unknowns $g(x_i)$. Its coefficients depend only on the number $m$ of inverse functions of $f$ and on that $f$ starting with an increasing or decreasing part of a graph. Then selected part of the graph $g$, say $g(x_j(t))$, we compute by remaining parts $g(x_i(t))$ given such that the resulting $g$ is d.f. The $\tilde{g}_f(x)$ can be replaced by arbitrary d.f. $g_1(x)$. As an application we give:

**Theorem 67.** *Let $s(x) = \sin^2(2\pi x)$ and $g_1$ be given absolute continuous d.f. The unknown absolute continuous d.f. $g$ solve the functional equation $g_s = g_1$ on $[0,1]$ if and only if has the form*

$$g(x) = \begin{cases} g_1(s(x)) - \int_0^x \psi(u)\mathrm{d}u & \text{if } x \in [0,1/4], \\ \int_0^{1/2-x} \phi(u)\mathrm{d}u - \int_0^{1/2-x} \psi(u)\mathrm{d}u + 1 - \int_0^{1/4} \phi(u)\mathrm{d}u & \text{if } x \in [1/4,1/2], \\ 1 - \int_0^{x-1/2} \lambda(u)\mathrm{d}u + \int_0^{x-1/2} \phi(u)\mathrm{d}u - \int_0^{1/4} \phi(u)\mathrm{d}u & \text{if } x \in [1/2,3/4], \\ 1 - \int_0^{1-x} \lambda(u)\mathrm{d}u & \text{if } x \in [3/4,1], \end{cases}$$

*where $\psi, \phi, \lambda$ are integrable functions defined on $[0,1]$ and satisfying*

$$0 \le \lambda(x) \le \phi(x) \le \psi(x) \le (g_1(s(x)))' \text{ for } x \in \left[0, \tfrac{1}{4}\right].$$

*Proof.* In our case

$$g_s(x) = g(s_1^{-1}(x)) + g(s_3^{-1}(x)) - g(s_2^{-1}(x)) + 1 - g(s_4^{-1}(x)),$$

where

$$s_1^{-1}(x) = \frac{1}{2\pi} \arcsin \sqrt{x}, \text{ and}$$

$$s_2^{-1}(x) = \frac{1}{2} - s_1^{-1}(x), \quad s_3^{-1}(x) = \frac{1}{2} + s_1^{-1}(x), \quad s_4^{-1}(x) = 1 - s_1^{-1}(x).$$

Substituting $x \to s(x)$ in $g_1 = g_s$ we find

$$\begin{aligned}
&\forall(x \in [0,1]) \quad g_1(x) = g_s(x) \Longleftrightarrow \\
&\forall(x \in [0,1/4]) \quad g_1(s(x)) = g(x) + g(\tfrac{1}{2} + x) - g(\tfrac{1}{2} - x) + 1 - g(1-x).
\end{aligned}$$

Thus we can compute values $g(x)$ on $[0,1/4]$ as $g(x) = g_1(s(x)) - g(\frac{1}{2} + x) + g(\frac{1}{2} - x) - 1 + g(1-x)$. For monotonicity of $g$ we need $0 \le g'(x)$, i.e. $g'(\frac{1}{2}+x) + g'(\frac{1}{2}-x) + g'(1-x) \le (g_1(s(x)))'$. Putting $\psi(x) = g'(\frac{1}{2}+x) + g'(\frac{1}{2}-x) + g'(1-x)$ we find $g(x) = g_1(s(x)) - \int_0^x \psi(u)\mathrm{d}u$ and $g(\frac{1}{2}-x) = g(\frac{1}{2}+x) - g(1-x) + 1 - \int_0^x \psi(u)\mathrm{d}u$. For monotonicity of $g(\frac{1}{2}-x)$ we need $g'(\frac{1}{2}+x) + g'(1-x) \le \psi(x)$ for $x \in [0,1/4]$. Putting $\phi(x) = g'(\frac{1}{2}+x) + g'(1-x)$ we find $g(\frac{1}{2}-x) = \int_0^x \phi(u)\mathrm{d}u - \int_0^x \psi(u)\mathrm{d}u + 1 - \int_0^{1/4} \phi(u)\mathrm{d}u$, etc. □

**Example 39.** For $g_1 = g_0 := (2/\pi)\arcsin\sqrt{x}$ we have $g_1(s(x)) = 4x$ and thus $0 \le \lambda(x) \le \phi(x) \le \psi(x) \le 4$ for $x \in [0,1/4]$.

If $\lambda(x) = 1$, $\phi(x) = 2$ and $\psi(x) = 3$, then $g(x) = x$.

If $\lambda(x) = \phi(x) = \psi(x) = 0$, then $g(x) = \tilde{g}_1(x)$, where

$$\tilde{g}_1(x) = \begin{cases} 4x & \text{for } x \in [0,1/4], \\ 1 & \text{for } x \in [1/4,1]. \end{cases}$$

If $\lambda(x) = \phi(x) = 0$ and $\psi(x) = 2$, then $g(x) = \tilde{g}_2(x)$, where

$$\tilde{g}_2(x) = \begin{cases} 2x & \text{for } x \in [0,1/2], \\ 1 & \text{for } x \in [1/2,1]. \end{cases}$$

Since for $f(x) = 4x(1-x)$ and $h(x) = x(4x-3)^2$ we have $g_{0_f} = g_0$, $g_{0_h} = g_0$, then $x, \tilde{g}_1, \tilde{g}_2$ satisfy $g_{f\circ s} = g_{h\circ s}$ together with they convex combinations.

The general functional equation $g_f = g_h$ is solved in Section 6.1.9.

## 4.9 Hausdorff moment problem (an information)

For completeness, the classical moment problems are:

$$s_n = \int_0^\infty x^n \mathrm{d}g(x),\ n = 0, 1, 2, \dots \text{ T. J. Stieltjes 1892;}$$
$$s_n = \int_{-\infty}^\infty x^n \mathrm{d}g(x),\ n = 0, 1, 2, \dots \text{ H. Hamburger 1920;}$$
$$s_n = \int_0^1 x^n \mathrm{d}g(x),\ n = 0, 1, 2, \dots \text{ F. Hausdorff 1923.}$$

The problem is to recover $g(x)$, given by its moments. Put

$$s_n = \int_0^1 x^n \mathrm{d}g(x), \quad n = 1, 2, \dots, \tag{135}$$

**Theorem 68** (F. Hausdorff)**.** *The moment problem* (135) *has a salution d.f.* $g(x)$ *if and only if*

$$\sum_{i=0}^{m} (-1)^i \binom{m}{i} s_{i+n} \geq 0$$

*for all* $m, n = 0, 1, 2, \dots$.

The set of all points $(s_1, s_2, \dots, s_N)$ in $[0, 1]^N$ for which exist d.f. $g(x)$ satisfying (135) for $n = 1, 2, \dots, N$ is called the $N$-th moment space $\mathbf{\Omega}_N$. It can be shown (see [5]):

(i) the point $(s_1, s_2, \dots, s_N)$ belongs to the moment space $\mathbf{\Omega}_N$ if and only if $\sum_{i=0}^{m} (-1)^i \binom{m}{i} s_{i+n} \geq 0$ for all $m, n = 0, 1, 2, \dots, N$, where $s_0 = 1$, $s_{N+1} = s_{N+2} = \dots = 0$.

(ii) $\mathbf{\Omega}_N$ is a simply connected, convex, and closed subset of $[0, 1]^N$.

(iii) If the point $(s_1, s_2, \dots, s_N)$ belongs to the interior of the moment space $\mathbf{\Omega}_N$ the truncated moment problem (135), $n = 1, 2, \dots, N$, has infinitely many solutions $g(x)$.

(iv) If $(s_1, s_2, \dots, s_N)$ belongs to the boundary of the $\mathbf{\Omega}_N$, the (135) has a unique solution $g(x)$.

(v) If the sequence $x_n \in [0, 1)$, $n = 1, 2, \dots$, satisfies $\lim \frac{1}{N} \sum_{n=1}^{N} x_n = s_1$, $\lim \frac{1}{N} \sum_{n=1}^{N} x_n^2 = s_2$,..., $\lim \frac{1}{N} \sum_{n=1}^{N} x_n^N = s_N$, where $(s_1, s_2, \dots, s_N)$ belongs to the boundary of the $\mathbf{\Omega}_N$, then $x_n$ has an a.d.f. $g(x)$.

Exact characterization of the moment space $\mathbf{\Omega}_N$ can be given in terms of the following matrices, see [83], [5]: Denote the Hankel determinants (different sense for even and odd $n$)

$\underline{H}_{2n} = \det((s_{i+j})_{i,j=0}^n)$,
$\overline{H}_{2n} = \det((s_{i+j+1} - s_{i+j+2})_{i,j=0}^{n-1})$,
$\underline{H}_{2n+1} = \det((s_{i+j+1})_{i,j=0}^n)$,
$\overline{H}_{2n+1} = \det((s_{i+j} - s_{i+j+1})_{i,j=0}^n)$,

**Theorem 69.** (i) *The $N$-dimensional point $\mathbf{s}_N = (s_1, s_2, \dots, s_N)$ belongs to the moment space $\mathbf{\Omega}_N$ if and only if $\underline{H}_n \geq 0$ and $\overline{H}_n \geq 0$ for all $n = 1, 2, \dots, N$.*

(ii) *The point $\mathbf{s}_N$ belongs to the interior of $\mathbf{\Omega}_N$ if and only if $\underline{H}_n > 0$ and $\overline{H}_n > 0$ for all $n = 1, 2, \dots, N$.*

(iii) *The pint $\mathbf{s}_N$ belongs to the boundary of $\mathbf{\Omega}_N$ if and only if there exists a $k$, $1 < k < N$, such that $\underline{H}_l > 0$ and $\overline{H}_l > 0$ for $l = 1, 2, \dots, k-1$ and $\underline{H}_k = 0$ or $\overline{H}_k = 0$. In this case $\underline{H}_n = \overline{H}_n = 0$ for all $n = k, k+1, \dots, N$ and the solution d.f. $g(x)$ in* (135) *is uniquely defined in terms od $\mathbf{s}_N$.*

## 4.10 The moment problem $\mathbf{X} = \mathbf{F}(g)$ for d.f. $g(x)$

See [159]:

In this section we shall solve the moment problem

$$\big(X_1, X_2, X_3\big) = \left(\int_0^1 g(x)\mathrm{d}x, \int_0^1 xg(x)\mathrm{d}x, \int_0^1 g^2(x)\mathrm{d}x\right) \tag{136}$$

and by coordinates

$$X_1 = \int_0^1 g(x)\mathrm{d}x,\ X_2 = \int_0^1 xg(x)\mathrm{d}x \text{ and } X_3 = \int_0^1 g^2(x)\mathrm{d}x$$

for d.f. $g(x)$. It is motivated by $L^2$ discrepancy criterion for $g$-distributed sequences, see Theorem 22, p. 28: A sequence $x_n$ in $[0,1)$ has a.d.f. $g(x)$ if and only if

$$\lim_{N\to\infty}\left(1 + \int_0^1 g^2(x)\mathrm{d}x - 2\int_0^1 g(x)\mathrm{d}x + \frac{2}{N}\sum_{n=1}^N \int_0^{x_n} g(x)\mathrm{d}x \right.$$
$$\left. - \frac{1}{N}\sum_{n=1}^N x_n - \frac{1}{2N^2}\sum_{m,n=1}^N |x_m - x_n|\right) = 0,$$

or equivalently, if and only if

(i) $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} x_n = \int_0^1 x\mathrm{d}g(x)$,

(ii) $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \int_0^{x_n} g(x)\mathrm{d}x = \int_0^1 \left(\int_0^x g(t)\mathrm{d}t\right)\mathrm{d}g(x)$,

(iii) $\lim_{N\to\infty} \frac{1}{N^2}\sum_{m,n=1}^{N} |x_m - x_n| = \int_0^1\int_0^1 |x-y|\mathrm{d}g(x)\mathrm{d}g(y)$.

(cf. [158, p. 176, Th. 1]. Since the left-hand side of the third equation (iii) contains $g(x)$ we shall instead it by the second moment and we solve

$$(s_1, s_2, s_3) = \left(\int_0^1 x\mathrm{d}g(x), \int_0^1 x^2\mathrm{d}g(x), \int_0^1\int_0^1 |x-y|\mathrm{d}g(x)\mathrm{d}g(y)\right).$$

By using

$$(s_1, s_2, s_3) = (1 - X_1, 1 - 2X_2, 2(X_1 - X_3))$$

it can be transform to

$$(X_1, X_2, X_3) = \left(\int_0^1 g(x)\mathrm{d}x, \int_0^1 xg(x)\mathrm{d}x, \int_0^1 g^2(x)\mathrm{d}x\right).$$

### 4.10.1 The body $\mathbf{\Omega}$ of all $\mathbf{F}(g)$ and the boundary $\partial\mathbf{\Omega}$

Define, for every d.f. $g(x)$ the operator

$$\mathbf{F}(g) = \left(\int_0^1 g(x)\mathrm{d}x, \int_0^1 xg(x)\mathrm{d}x, \int_0^1 g^2(x)\mathrm{d}x\right).$$

For $\mathbf{F}$, we introduce its body

$$\mathbf{\Omega} = \{\mathbf{F}(g); g \text{ is d.f. }\},$$

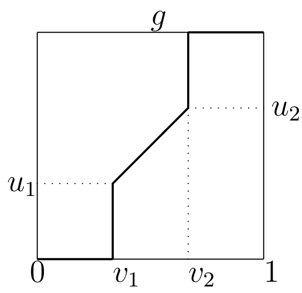
and $\partial\mathbf{\Omega}$ denote the *boundary* of $\mathbf{\Omega}$.

Note that throughout this chapter we shall always denote both the row vector $(X_1, X_2, X_3)$ and the column vector $\left(\begin{smallmatrix}X_1\\X_2\\X_3\end{smallmatrix}\right)$ by the same letter $\mathbf{X}$ and the equation $\mathbf{X} = \mathbf{F}(g)$ is referred to as a moment problem.

Let $g(u_1, v_1, u_2, v_2)$ denote the d.f. $h(x)$ defined by

$$h(x) = \begin{cases} 0 & \text{for } 0 \le x \le v_1\,, \\ \frac{u_2-u_1}{v_2-v_1}x + u_1 - v_1\frac{u_2-u_1}{v_2-v_1} & \text{for } v_1 < x \le v_2\,, \\ 1 & \text{for } v_2 < x \le 1\,. \end{cases}$$

(in any case $h(1) = 1$). Its graph is




Then

$$\mathbf{F}(g(u_1,v_1,u_2,v_2)) = \begin{pmatrix} 1 - v_2 + \frac{1}{2}(v_2 - v_1)(u_1 + u_2) \\ \frac{1}{2} - \frac{1}{6}v_2^2(3 - u_1 - 2u_2) - \frac{1}{6}v_1^2(2u_1 + u_2) - \frac{1}{6}v_1v_2(u_2 - u_1) \\ 1 - v_2 + \frac{1}{3}(v_2 - v_1)(u_1u_2 + u_1^2 + u_2^2) \end{pmatrix}.$$

For varying parameters $u_1$, $v_1$, $u_2$, and $v_2$, the point $\mathbf{F}(g(u_1,v_1,u_2,v_2))$ (= $\mathbf{F}(g)$) describes surfaces $\Pi_1$–$\Pi_6$ or the curve $\Pi_7$ specified by the following list of formulas:

$$\begin{aligned}
\Pi_1 &= \Big\{\mathbf{F}(g);\quad 0 \le v_1 \le v_2 \le 1, u_1 = 0, u_2 = 1\Big\},\\
\Pi_2 &= \Big\{\mathbf{F}(g);\quad v_1 = 0, v_2 = 1, 0 \le u_1 \le u_2 \le 1\Big\},\\
\Pi_3 &= \Big\{\mathbf{F}(g);\quad v_1 = 0, 0 \le v_2 \le 1, u_2 = 1, 0 \le u_1 \le 1\Big\},\\
\Pi_4 &= \Big\{\mathbf{F}(g);\quad 0 \le v_1 \le 1, v_2 = 1, u_1 = 0, 0 \le u_2 \le 1\Big\},\\
\Pi_5 &= \Big\{\mathbf{F}(g);\quad v_1 = 0, 0 \le v_2 \le 1\, 0 \le u_1 = u_2 \le 1, v_2(1 - u_2) > \frac{1}{2}\Big\},\\
\Pi_6 &= \Big\{\mathbf{F}(g);\quad 0 \le v_1 \le 1, v_2 = 1, 0 \le u_1 = u_2 \le 1, u_1(1 - v_1) > \frac{1}{2}\Big\},\\
\Pi_7 &= \Big\{\mathbf{F}(g);\quad v_1 = 0, \frac{1}{2} < v_2 < 1, u_1 = u_2 = 1 - \frac{1}{2v_2}\Big\}\\
&= \Big\{\mathbf{F}(g);\quad 0 < v_1 < \frac{1}{2}, v_2 = 1, u_1 = u_2 = \frac{1}{2(1 - v_1)}\Big\}.
\end{aligned}$$

Eliminating the parameters $u_1$, $v_1$, $u_2$, $v_2$ from the $\mathbf{X} = \mathbf{F}(g(u_1,v_1,u_2,v_2))$ we arrive at the following canonical expressions:

$$\begin{aligned}
\Pi_1 = \Big\{&(X_1, X_2, X_3);\quad X_2 = \frac{1}{2} - \frac{1}{2}(1 - X_1)^2 - \frac{3}{2}(X_1 - X_3)^2,\\
&\max\Big(\frac{4}{3}X_1 - \frac{1}{3}, \frac{2}{3}X_1\Big) \le X_3 \le X_1, 0 \le X_1 \le 1\Big\},
\end{aligned}$$

$$\Pi_2=\Bigg\{(X_1,X_2,X_3);\quad X_2=\frac{1}{2}X_1+\frac{1}{2}\sqrt{\frac{1}{3}(X_3-X_1^2)},$$
$$X_1^2\le X_3\le\min\Big(\frac{4}{3}X_1^2,\frac{4}{3}X_1^2-\frac{2}{3}X_1+\frac{1}{3}\Big),0\le X_1\le 1\Bigg\},$$
$$\Pi_3=\Bigg\{(X_1,X_2,X_3);\quad X_2=\frac{1}{2}-\frac{4}{9}\frac{(1-X_1)^3}{(1+X_3-2X_1)},$$
$$\frac{4}{3}X_1^2-\frac{2}{3}X_1+\frac{1}{3}\le X_3\le\frac{4}{3}X_1-\frac{1}{3},\frac{1}{2}\le X_1\le 1\Bigg\},$$
$$\Pi_4=\Bigg\{(X_1,X_2,X_3);\quad X_2=X_1-\frac{4}{9}\frac{X_1^3}{X_3},\frac{4}{3}X_1^2\le X_3\le\frac{2}{3}X_1,0\le X_1\le\frac{1}{2}\Bigg\},$$
$$\Pi_5=\Bigg\{(X_1,X_2,X_3);\quad X_2=\frac{1}{2}-\frac{1}{2}\frac{(1-X_1)^3}{(1+X_3-2X_1)},$$
$$X_1^2\le X_3\le X_1,0\le X_1<\frac{1}{2}\Bigg\},$$
$$\Pi_6=\Bigg\{(X_1,X_2,X_3);\quad X_2=X_1-\frac{1}{2}\frac{X_1^3}{X_3},X_1^2\le X_3\le X_1,\frac{1}{2}<X_1\le 1\Bigg\},$$
$$\Pi_7=\Bigg\{\Big(\frac{1}{2},\frac{1}{2}-\frac{1}{16X_3},X_3\Big);\quad\frac{1}{4}<X_3<\frac{1}{2}\Bigg\}.\tag{137}$$

Specify the following d.f. $g(u_1,v_1,u_2,v_2)=g^{(i)}$:

$$g^{(1)}=g(0,(1-X_1)-3(X_1-X_3),1,(1-X_1)+3(X_1-X_3)),$$
$$g^{(2)}=g\left(X_1-\sqrt{3(X_3-X_1^2)},0,X_1+\sqrt{3(X_3-X_1^2)},1\right),$$
$$g^{(3)}=g\left(1-\frac{3}{2}\frac{1+X_3-2X_1}{1-X_1},0,1,\frac{4}{3}\frac{(1-X_1)^2}{(1+X_3-2X_1)}\right),$$
$$g^{(4)}=g\left(0,1-\frac{4X_1^2}{3X_3},\frac{3X_3}{2X_1},1\right),$$
$$g^{(5)}=g\left(\frac{X_1-X_3}{1-X_1},0,\frac{X_1-X_3}{1-X_1},\frac{(1-X_1)^2}{1+X_3-2X_1}\right),$$
$$g^{(6)}=g\left(\frac{X_3}{X_1},1-\frac{X_1^2}{X_3},\frac{X_3}{X_1},1\right),$$
$$g^{(7)}=g\left(1-2X_3,0,1-2X_3,\frac{1}{4X_3}\right),$$

$$g^{(7^*)} = g\left(2X_3, 1 - \frac{1}{4X_3}, 2X_3, 1\right). \tag{138}$$

Their graphs are

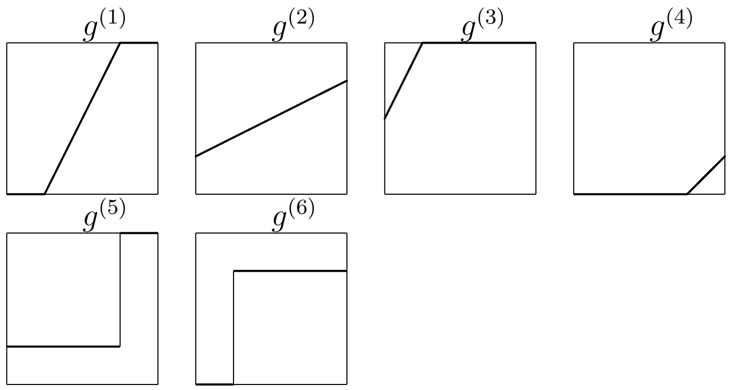



Figure 2

The areas of rectangles bounded by the graphs of $g^5$ and $g^6$ are $\geq 1/2$. We write $g^{(5)} = g^{(7)}$ and $g^{(6)} = g^{(7^*)}$ in the case $= 1/2$.

### 4.10.2 Solution of $\mathbf{X} = \mathbf{F}(g)$ for $\mathbf{X} \in \mathbf{\Omega}$ and $\mathbf{X} \in \partial\mathbf{\Omega}$

We now state main theorems.

**Theorem 70.** *For the moment problem $\mathbf{X} = \mathbf{F}(g)$, to have only a finite number of solutions in d.f.s $g$ it is necessary and sufficient that $\mathbf{X} \in \partial\mathbf{\Omega}$, where $\partial\mathbf{\Omega}$ denotes the boundary of $\mathbf{\Omega}$. We can express the boundary $\partial\mathbf{\Omega}$ as*

$$\partial\mathbf{\Omega} = \bigcup_{1 \leq i \leq 7} \Pi_i.$$

*In addition, for $\mathbf{X} \in \Pi_i$, $i = 1, 2, \dots, 6$, the moment problem $\mathbf{X} = \mathbf{F}(g)$ is uniquely solvable as $g = g^{(i)}$, and for $\mathbf{X} \in \Pi_7$ has precisely two solutions of types $g^{(7)}$ and $g^{(7^*)}$.*

Theorem 70 directly implies

**Theorem 71.** *Let $x_n$, $n = 1, 2, \dots$ be a sequence in $[0, 1]$ with the limits*

$$X_1 = 1 - \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_n,$$

$$X_2 = \frac{1}{2} - \frac{1}{2}\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} x_n^2,$$

$$X_3 = 1 - \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} x_n - \frac{1}{2}\lim_{N\to\infty}\frac{1}{N^2}\sum_{m,n=1}^{N}|x_m - x_n|.$$

*If* $\mathbf{X} = (X_1, X_2, X_3) \in \bigcup_{1\le i\le 7}\Pi_i$, *then the sequence* $x_n$ *has an a.d.f. These a.d.f. are given by formulae* (138). *Exactly, if* $\mathbf{X}\in\Pi_i$, $i = 1,\dots,6$, *then* $x_n$ *has the a.d.f.* $g^{(i)}$, *and if* $\mathbf{X}\in\Pi_7$, *then* $x_n$ *has the a.d.f. either* $g^{(7)}$ *or* $g^{(7^*)}$, *depending on whether*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\int_0^{x_n} g^{(7)}(t)\mathrm{d}t = X_1 - X_3$$

*or*

$$\lim_{n\to\infty}\frac{1}{N}\sum_{0}^{N}\int_0^{x_n} g^{(7^*)}(t)\mathrm{d}t = X_1 - X_3,$$

*respectively. Furthermore,* $\mathbf{X}\in\bigcup_{1\le i\le 7}\Pi_i$ *is testable by* (137).

In the following we also prove that the upper and lower surfaces of $\partial\mathbf{\Omega}$ in the direction $X_2$ are

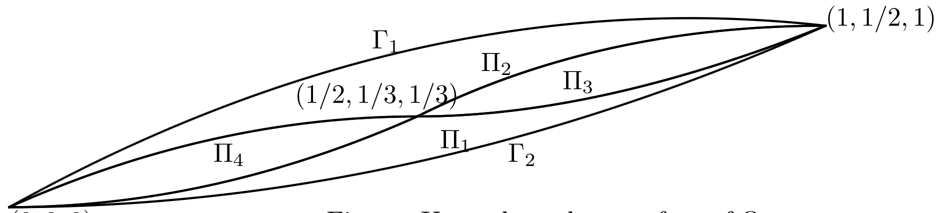



Figure: Upper boundary surface of Ω

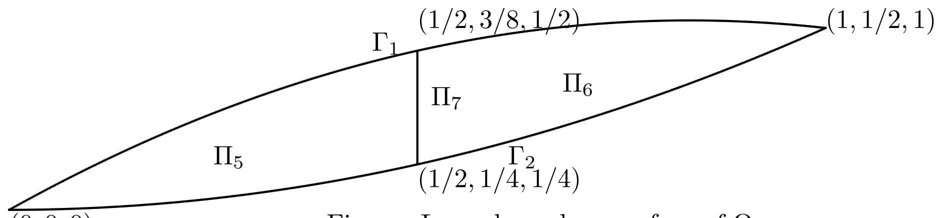



Figure: Lover boundary surface of Ω

For $\Gamma_1$ and $\Gamma_2$ see Theorem 75.

### 4.10.3 Proof of the solutions of $\mathbf{X} = \mathbf{F}(g)$

We shall work with neighbourhoods of points in the body $\mathbf{\Omega}$. For the sake of more clarity, we give an outline of the proof. We shall prove the theorem in six steps.

1. We collected a couple of elementary facts about the body $\mathbf{\Omega}$. We find two affine transformations leaving the body $\mathbf{\Omega}$ fixed. We find projections of the body $\mathbf{\Omega}$ to the planes $X_1 \times X_3$ and $X_1 \times X_2$, and we find two curve-edges of $\mathbf{\Omega}$.

2. The principal technical result establishes the closedness of $\mathbf{\Omega}$ under a linear law of composition as follows : For any finite set of elements $\mathbf{X}^{(0)}, \ldots, \mathbf{X}^{(N)}$ in $\mathbf{\Omega}$, every sum $\sum_{i=0}^{N}(\mathbf{a}_i + \mathbf{B}_i\mathbf{X}^{(i)})$ with vectors $\mathbf{a}_i = a(u_i, v_i, u_{i+1}, v_{i+1})$ and matrices $\mathbf{B}_i = B(u_i, v_i, u_{i+1}, v_{i+1})$, where $u_i, v_i, u_{i+1}, v_{i+1}$ are parameters ($\mathbf{a}$ and $\mathbf{B}$ will be defined later) also belongs to $\mathbf{\Omega}$, and each point $\mathbf{X} \in \mathbf{\Omega}$, for every $N$, can be decomposed as $\mathbf{X} = \sum_{i=0}^{N} \mathbf{a}_i + \mathbf{B}_i\mathbf{X}^{(i)}$ into corresponding terms $\mathbf{X}^{(0)}, \ldots, \mathbf{X}^{(N)} \in \mathbf{\Omega}$.

3. As a consequence, the following is proved : If $\mathbf{X} \in \text{int}\, \mathbf{\Omega}$, then the moment problem $\mathbf{X} = \mathbf{F}(g)$ has infinitely many solutions.

4. Then we make a general observation about neighbourhoods in $\mathbf{\Omega}$. We shall discuss the transformation $\mathbf{O} = \sum_{i=0}^{N} \mathbf{a}_i + \mathbf{B}_i\mathbf{O}_i$ mapping a sequence $\mathbf{O}_0, \ldots, \mathbf{O}_N$ of neighbourhoods $\mathbf{O}_i$ of $\mathbf{X}^{(i)}$ into the neighbourhood $\mathbf{O}$ of $\mathbf{X}$.

5. But every point $\mathbf{X}$ of the boundary $\partial\mathbf{\Omega}$ of $\mathbf{\Omega}$ would not possess a spherical neighbourhood in $\mathbf{\Omega}$, and hence in this way we have a link between the decomposition $\mathbf{X} = \sum_{i=0}^{N}(\mathbf{a}_i + \mathbf{B}_i\mathbf{X}^{(i)})$, and the $\mathbf{X} \in \partial\mathbf{\Omega}$.

6. In accordance with this decomposition and by the theory of Dini derivatives the boundary $\partial\mathbf{\Omega}$ can be expressed as the $\mathbf{F}$-image of d.f.s in Fig. 2.

### 4.10.4 The basic property of $\mathbf{\Omega}$

• The body $\mathbf{\Omega} \subset [0, 1]^3$ starts in $(0, 0, 0)$ and ends in $(1, 1/2, 1)$.
• The boundary $\partial\mathbf{\Omega} = \cup_{1 \leq i \leq 7}\Pi_i$.

**Theorem 72.** *$\mathbf{\Omega}$ and $\partial\mathbf{\Omega}$ are invariant with respect to the transformation group*

$$\{\text{Identity}, \Phi, \Lambda, \Phi \circ \Lambda\},$$

*where*

$$\Phi(X_1, X_2, X_3) = \Big(1 - X_1,\ X_2 - X_1 + \frac{1}{2}, 1 + X_3 - 2X_1\Big),$$

$$\Lambda(X_1, X_2, X_3) = \Big(1 - X_1,\ \frac{1 - X_3}{2},\ 1 - 2X_2\Big).$$

*Proof.* For any d.f. $g$ let d.f. $\tilde{g}$ be defined as [20]

$$\tilde{g}(x) = |\{y \in [0,1]; g(y) < x\}|.$$

Given $\mathbf{F}(g) = \mathbf{X}$, we can compute $\mathbf{F}(\tilde{g}) = \Lambda(\mathbf{X})$ by using

$$\int_0^1 x \mathrm{d}\tilde{g}(x) = \int_0^1 g(x)\mathrm{d}x, \int_0^1 x^2 \mathrm{d}\tilde{g}(x) = \int_0^1 g^2(x)\mathrm{d}x,$$

$$\int_0^1 \int_0^1 |x - y| \mathrm{d}\tilde{g}(x)\mathrm{d}\tilde{g}(y) = \int_0^1 \int_0^1 |g(x) - g(y)|\mathrm{d}x\mathrm{d}y =$$

$$= 4\int_0^1 x g(x)\mathrm{d}x - 2\int_0^1 g(x)\mathrm{d}x.$$

Since the map $\Lambda$ is affine and the matrix of $\Lambda$ is unimodular, it is obvious that $\Lambda(\mathbf{\Omega}) = \mathbf{\Omega}$ and $\Lambda(\partial\mathbf{\Omega}) = \partial\mathbf{\Omega}$. Using $\tilde{g}(x) = 1 - g(1 - x)$, we can obtain analogous results for $\Phi$. [21] $\square$

**Theorem 73.** *Any straight line parallel to the $X_2$ or $X_3$ axis meets $\mathbf{\Omega}$ at a segment. Thus $\mathbf{\Omega}$ is convex in these directions.*

*Proof.* Assuming $X_1 = Y_1$ and $X_2 = Y_2$ for $\mathbf{X} = \mathbf{F}(g)$ and $\mathbf{Y} = \mathbf{F}(f)$, where $f, g$ are d.f.s, we arrive at

$$tX + (1 - t)Y = \left(X_1, X_2, \int_0^1 (tg^2(x) + (1 - t)f^2(x))dx\right).$$

[20] In Example 36 is note $\tilde{g}(x)$ as $\tilde{g}(x) = \int_{g^{-1}([0,x))} 1.\mathrm{d}x = x_g$.

[21] $\Lambda$ can be considered as a transformation $g \to \tilde{g}$. Similarly for $\Phi(g(x)) = 1 - g(1 - x)$.

Since

$$\int_0^1 (tg(x)+(1-t)f(x))^2\,dx \le \int_0^1 \left(tg^2(x)+(1-t)f^2(x)\right)dx \le$$

$$\le \max\left(\int_0^1 g^2(x)dx, \int_0^1 f^2(x)dx\right),$$

from continuity one obtains the existence of $t_0 \in [0,1]$ so that

$$\int_0^1 (t_0 g(x)+(1-t_0)f(x))^2\,\mathrm{d}x = \int_0^1 \left(tg^2(x)+(1-t)f^2(x)\right)\mathrm{d}x.$$

Hence the $\mathbf{F}$-image for nondecreasing $h(x)=t_0g(x)+(1-t_0)f(x)$ takes the form $\mathbf{F}(h)=t\mathbf{X}+(1-t)\mathbf{Y}$.

The rest follows from the properties of the transformation $\Lambda$. □

**Theorem 74.** *The orthogonal projections of the body* $\mathbf{\Omega}$ *onto the* $X_1\times X_3$ *and* $X_1\times X_2$ *planes are equal to*

$$\left\{(X_1,X_3); X_1^2 \le X_3 \le X_1, 0\le X_1\le 1\right\} \tag{139}$$

*and*

$$\left\{(X_1,X_2); \frac{1}{2}X_1 \le X_2 \le X_1-\frac{1}{2}X_1^2,\ \ 0\le X_1\le 1\right\}, \tag{140}$$

*respectively.*

*Proof.* Here we consider $\omega_N$ as an $N$- dimensional vector $\omega_N=(x_1,\dots,x_N)$, where $x_1,\dots,x_N\in[0,1]$ are ordered according to their magnitude, that is $0\le x_1\le x_2\le\cdots\le x_N\le 1$. Defining $F_N(x)$ as step-d.f. of the sequence $\omega_N$. Then

$$\mathbf{F}(F_N)=$$

$$=\left(1-\frac{1}{N}\sum_{n=1}^N x_n, \frac{1}{2}-\frac{1}{2N}\sum_{n=1}^N x_n^2, 1-\frac{1}{N^2}\sum_{n=1}^N(2n-1)x_n\right).$$

Since there is no risk of confusion, we shall write $\mathbf{F}(\omega_N)$ instead of $\mathbf{F}(F_N)$. For fixed $N$, consider the set $\mathbf{\Delta}_N$ of all $\omega_N$ and the set $\mathbf{\Omega}_N$ of all $\mathbf{F}$-images of $\boldsymbol{\omega}_N$. Representing $\mathbf{\Omega}$ and $\mathbf{\Delta}_N$ as

$$\mathbf{\Omega} = \text{closure } \cup_{1\le N<\infty} \mathbf{\Omega}_N,$$

$$\mathbf{\Delta}_N = \text{convex hull}\left\{\omega_N^{(i)} = (0,\dots,0,1,\dots,1); i = 0,1,\dots,N\right\}$$

where $\omega_N^{(i)}$ has $i$-times 0 and $N-i$ times 1. Then we find that

$$\text{projection } \mathbf{\Omega} =$$

$$= \text{closure } \cup_{1\le N<\infty} \text{ projection } \mathbf{F}\left(\text{convex hull } \left\{\omega_N^{(i)}; i = 0,1,\dots,N\right\}\right).$$

Since $\mathbf{X} = \mathbf{F}(\boldsymbol{\omega}_N)$ consists of linear $X_1$ and $X_3$ over $\omega_N$, and

$$(X_1, X_3) = \left(\frac{i}{N}, \left(\frac{i}{N}\right)^2\right)$$

for $\omega_N = \omega_N^{(i)}$, then the orthogonal projection of $\mathbf{\Omega}$ onto the plane $X_1 \times X_3$ is

$$\text{closure } \cup_{1\le N\le\infty} \text{ convex hull }\left\{\left(\frac{i}{N}, \left(\frac{i}{N}\right)^2\right); i = 0,1,\dots,N\right\}.$$

Finally, applying transformation $\Lambda$ to the above projection, we find the orthogonal projection of $\mathbf{\Omega}$ onto $X_1 \times X_2$. $\square$

**Theorem 75.** *The following curves*

$$\Gamma_1 = \left\{(X_1, X_2, X_3); X_2 = \frac{1}{2}X_1(2 - X_1), X_3 = X_1, 0 \le X_1 \le 1\right\}, \tag{141}$$

$$\Gamma_2 = \left\{(X_1, X_2, X_3); X_2 = \frac{1}{2}X_1, X_3 = X_1^2, 0 \le X_1 \le 1\right\}, \tag{142}$$

*intersect in the points* $(0,0,0)$ *and* $(1, \frac{1}{2}, 1)$, *and they belong to* $\mathbf{\Omega}$. *Moreover, their orthogonal projections coincide with the boundary of the projections* (139) *and* (140) *of* $\mathbf{\Omega}$ *and we call* $\Gamma_1$ *and* $\Gamma_2$ *curve-edges of* $\mathbf{\Omega}$.

For d.f.s $g(x)$ we have

**Example 40.**

$$g(x) = x \iff \int_0^1 g(x)(2x - g(x))\mathrm{d}x = \frac{1}{3}.$$

**Example 41.**

$$g(x) = \frac{x}{2} \Longleftrightarrow \int_0^1 g(x)(2x - g(x))\mathrm{d}x = \frac{1}{12}.$$

*Proof.* Projection of $\Omega$ to $X_3 \times X_2$ we have no given explicitly, but we project $\Pi_i$ to $X_3 \times X_2$, $i = 1, 2, \dots, 6$ and compute the maximal $\alpha$ such that straight line $X_2 - X_3 = \alpha$ is touch this projection. We find $\alpha = \frac{1}{12}$ for projection of $\Pi_2$, where $X_2 = \frac{1}{6}$ and $X_3 = \frac{1}{12}$ which have a solution $g(x) = \frac{x}{2}$. Such solution is unique, since the point $\left(\frac{1}{4}, \frac{1}{6}, \frac{1}{12}\right)$ lying in $\Pi_2$ - in the boundary of $\mathbf{\Omega}$. □

**Example 42.** In Section 6.3 we have mentioned that the sequence $\varphi(n)/n$, $n = 1, 2, 3 \dots$, has a singular a.d.f. denoted by $g_0(x)$. Put

$$\mathbf{X}^{(0)} = (X_1^{(0)}, X_2^{(0)}, X_3^{(0)}) = \left(\int_0^1 g_0(x)\mathrm{d}x, \int_0^1 x g_0(x)\mathrm{d}x, \int_0^1 g_0^2(x)\mathrm{d}x\right).$$

Then by (VI) $X^{(0)} \in \operatorname{int} \mathbf{\Omega}$. Since

$$X_1^{(0)} = 1 - \frac{6}{\pi^2}, \quad X_2^{(0)} = \frac{1}{2} - \frac{1}{2}\prod_p \left(1 - \frac{2}{p^2} + \frac{1}{p^3}\right) = 0.285\dots,$$

we have

$$0.250\dots = \min X_3 \le X_3^{(0)} \le \max X_3 = 0.307\dots.$$

*Proof.* Since $\mathbf{\Omega}$ intersect the line $(X_1^{(0)}, X_2^{(0)}, X_3)$ we can find the point $(X_1^{(0)}, X_2^{(0)})$ in the projections $\Pi_4$ and $\Pi_5$ on the plane $X_1 \times X_2$. Then we express $\Pi_4$ and $\Pi_5$ by $X_3$ and we find

$$\min X_3 = \frac{4(X_1^{(0)})^3}{9(X_1^{(0)} - X_2^{(0)})} \text{ and } \max X_3 = 2X_1^{(0)} - 1 + \frac{(1 - X_1^{(0)})^3}{1 - 2X_2^{(0)}}.$$

□

### 4.10.5 The law of composition

Let $((v_i, u_i))_{i=0}^{N+1}$ be a finite sequence of points in $[0, 1]^2$. Suppose that

(i) $(v_0, u_0) = (0, 0)$, $(v_{N+1}, u_{N+1}) = (1, 1)$,

(ii) $v_i \le v_{i+1}$ and $u_i \le u_{i+1}$ for all $i = 0, 1, \ldots, N$.

Defining $\mathbf{a}_i$ and $\mathbf{B}_i$ as

$$\mathbf{a}_i = a(u_i, v_i, u_{i+1}, v_{i+1}) = \begin{pmatrix} (v_{i+1}-v_i)u_i \\ (v_{i+1}-v_i)u_i v_i + \frac{1}{2}(v_{i+1}-v_i)^2 u_i \\ (v_{i+1}-v_i)u_i^2 \end{pmatrix},$$

$$\begin{aligned}\mathbf{B}_i &= B(u_i, v_i, u_{i+1}, v_{i+1}) \\ &= \begin{pmatrix} (u_{i+1}-u_i)(v_{i+1}-v_i) & 0 & 0 \\ v_i(u_{i+1}-u_i)(v_{i+1}-v_i) & (u_{i+1}-u_i)(v_{i+1}-v_i)^2 & 0 \\ 2u_i(u_{i+1}-u_i)(v_{i+1}-v_i) & 0 & (u_{i+1}-u_i)^2(v_{i+1}-v_i) \end{pmatrix},\end{aligned}$$

we form the sum

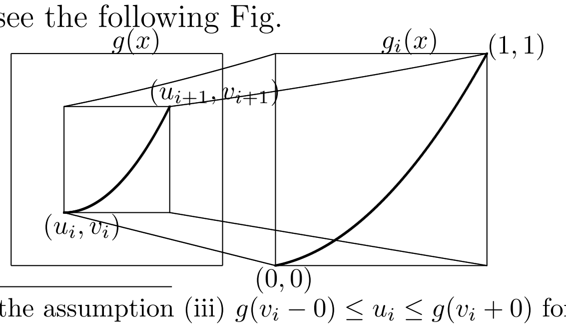
$$\mathbf{X} = \sum_{i=0}^{N} \mathbf{a}_i + \mathbf{B}_i \mathbf{X}^{(i)}\,. \tag{143}$$

**Theorem 76.** *Let* $\mathbf{X}$ *be determined by* (143)*, then we have* $\mathbf{X} \in \mathbf{\Omega}$*, and vice versa, for any* $\mathbf{X} \in \mathbf{\Omega}$ *(i.e.* $\mathbf{X} = \mathbf{F}(g)$*,* $g$ *is d.f.) a sequence* $(\mathbf{X}^{(i)})_{i=0}^{N} \subset \mathbf{\Omega}$*, which satisfies* (143)*, can be found.* [22]

*Proof.* We shall first demonstrate the second statement. We have $\mathbf{X} = \mathbf{F}(g)$, where $g$ is nondecreasing and $g/(v_i, v_{i+1}) : (v_i, v_{i+1}) \to [u_i, u_{i+1}]$ denotes a restriction of $g$. Then by elementary reasoning one shows that the graph of $g/(v_i, v_{i+1})$ has the linear expansion on $[0,1]^2$ given by

$$g_i(x) = \begin{cases} \frac{g(x(v_{i+1}-v_i)+v_i)}{u_{i+1}-u_i} - \frac{u_i}{u_{i+1}-u_i}, & \text{if } u_i < u_{i+1}, \\ 0, & \text{if } u_i = u_{i+1}, \end{cases} \tag{144}$$

for all $x \in (0,1)$, see the following Fig.




[22]We need to add the assumption (iii) $g(v_i - 0) \le u_i \le g(v_i + 0)$ for all $i = 1, \ldots, N$.

Putting $\mathbf{X}^{(i)} = \mathbf{F}(g_i)$ and assuming $v_i < v_{i+1}, u_i < u_{i+1}$, we find that

$$
\begin{aligned}
X_1^{(i)} &= \int_0^1 g_i(x)\mathrm{d}x \\
&= \frac{1}{(u_{i+1}-u_i)(v_{i+1}-v_i)} \int_{v_i}^{v_{i+1}} g(x)\mathrm{d}x - \frac{u_i}{u_{i+1}-u_i}, \\
X_2^{(i)} &= \int_0^1 x g_i(x)\mathrm{d}x \\
&= \frac{1}{(u_{i+1}-u_i)(v_{i+1}-v_i)^2} \left( \int_{v_i}^{v_{i+1}} x g(x)\mathrm{d}x - v_i \int_{v_i}^{v_{i+1}} g(x)\mathrm{d}x \right) \\
&- \frac{1}{2}\frac{u_i}{u_{i+1}-u_i}, \\
X_3^{(i)} &= \int_0^1 g_i^2(x)\mathrm{d}x \\
&= \frac{1}{(u_{i+1}-u_i)^2(v_{i+1}-v_i)} \left( \int_{v_i}^{v_{i+1}} g^2(x)\mathrm{d}x - 2u_i \int_{v_i}^{v_{i+1}} g(x)\mathrm{d}x \right) \\
&+ \left( \frac{u_i}{u_{i+1}-u_i} \right)^2 .
\end{aligned} \tag{145}
$$

As a result of these equalities, we have

$$
\left( \int_{v_i}^{v_{i+1}} g(x)\mathrm{d}x, \int_{v_i}^{v_{i+1}} x g(x)\mathrm{d}x, \int_{v_i}^{v_{i+1}} g^2(x)\mathrm{d}x \right) = \mathbf{a}_i + \mathbf{B}_i \mathbf{X}^{(i)},
$$

which holds also for $v_i = v_{i+1}$ and $u_i = u_{i+1}$. The equality (143) is shown.

In order to prove the first statement of theorem, one observes that, for a given $\left(\mathbf{X}^{(i)}\right)_{i=0}^{N} \subset \mathbf{\Omega}$ and $\left((v_i, u_i)\right)_{i=0}^{N+1} \subset [0,1]^2$, where $\mathbf{X}^{(i)} = \mathbf{F}(g_i)$, and $g_i$ is nondecreasing, one can guarantee the existence of a nondecreasing $g : [0,1] \to [0,1]$ having $g_i$ as in (144). Indeed, putting

$$
g/(v_i, v_{i+1}) = \begin{cases} g_i\left(\frac{x-v_i}{v_{i+1}-v_i}\right)(u_{i+1}-u_i) + u_i, & \text{if } v_i < v_{i+1}, \\ u_i, & \text{if } v_i = v_{i+1}, \end{cases} \tag{146}
$$

we have (144) for all $i = 0, 1, \ldots, N$, and hence the second statement of the theorem can be applied to $g$. □

**Theorem 77.** *If $\mathbf{X} \in \operatorname{int} \mathbf{\Omega}$, then the moment problem $\mathbf{X} = \mathbf{F}(g)$ has infinitely many solutions in d.f.s $g$.*

*Proof.* We start from (143). Assuming $\mathbf{X} \in \mathrm{int}\mathbf{\Omega}$, with the help of Theorems 74 and 75, we can find an index $i$ such that $\mathbf{B}_i \neq \mathbf{0}$. The expression for $\mathbf{X}^{(i)}$ takes the form

$$\mathbf{X}^{(i)} = \mathbf{B}_i^{-1}\left(\mathbf{X} - \mathbf{a}_i - \sum_{j=0, j\neq i}^{N} \mathbf{a}_j + \mathbf{B}_j\mathbf{X}^{(j)}\right) \tag{147}$$

where

$$\mathbf{B}_i^{-1} = \mathbf{B}^{-1}(u_i, v_i, u_{i+1}, v_{i+1}) =$$

$$= \begin{pmatrix} \frac{1}{(u_{i+1}-u_i)(v_{i+1}-v_i)} & 0 & 0 \\ \frac{-v_i}{(u_{i+1}-u_i)(v_{i+1}-v_i)^2} & \frac{1}{(u_{i+1}-u_i)(v_{i+1}-v_i)^2} & 0 \\ \frac{-2u_i}{(u_{i+1}-u_i)^2(v_{i+1}-v_i)} & 0 & \frac{1}{(u_{i+1}-u_i)^2(v_{i+1}-v_i)} \end{pmatrix}.$$

Here we consider the right hand side of (147) as a vector-valued function of the variables $\left(\mathbf{X}^{(j)}\right)_{j=0,j\neq i}^{N} \subset \mathbf{\Omega}$ and $((v_j, u_j))_{j=0}^{N+1} \subset [0,1]^2$. Calculating the limits of $\mathbf{B}_i^{-1}$,$\mathbf{a}_j$,$\mathbf{B}_j$ as $(v_i, u_i) \to (0,0)$ and $(v_{i+1}, u_{i+1}) \to (1,1)$, we obtain [23]

$$\mathbf{B}_i^{-1} \to \mathbf{1}, \mathbf{a}_j \to \mathbf{0} \text{ for all } j, \mathbf{B}_j \to \mathbf{0} \text{ for all } j \neq i.$$

Consequently,

$$\mathbf{B}_i^{-1}\left(\mathbf{X} - \mathbf{a}_i - \sum_{j=0, j\neq i}^{N} \mathbf{a}_j + \mathbf{B}_j\mathbf{\Omega}\right) \subset \mathrm{int}\mathbf{\Omega} \tag{148}$$

for any $(v_i, u_i)$ and $(v_{i+1}, u_{i+1})$ sufficiently near to $(0,0)$ and $(1,1)$, respectively. Thus, we have shown that for $\left(\mathbf{X}^{(j)}\right)_{j=0,j\neq i}^{N} \subset \Omega$ it is possible to compute $\mathbf{X}^{(i)} \in \Omega$ so that (143) is valid. Now, we can represent each $\mathbf{X}^{(j)}, j = 0, 1, \ldots, N$ in the form $\mathbf{X}^{(j)} = \mathbf{F}(g_j)$, and finally, applying construction (146), we infer that $\mathbf{X} = \mathbf{F}(g)$, where for different $\mathbf{X}^{(j)}$ we get different $g$. □

### 4.10.6 Linear neighbourhoods; Definition and construction

The purpose of the following is to study neighbourhoods $O$ of $\mathbf{X}$ in the body $\mathbf{\Omega}$. Here we shall use the notation

$$\mathbf{O} = \mathbf{X} + V = \{\mathbf{X} + \mathbf{Y}; \mathbf{Y} \in V\},$$

[23] Here we use the symbols $\mathbf{1}$ and $\mathbf{0}$ to denote the unit and zero matrices or vectors.

where $V$ denotes a suitable set of three-dimensional vectors. With the help of operation (143) (as $u_i$, $v_i$, $u_{i+1}$, $v_{i+1}$ vary continuously), we can find new surfaces and bodies lying in $\mathbf{\Omega}$, containing $\mathbf{X}$, and forming neighbourhoods of $\mathbf{X}$ in $\mathbf{\Omega}$. Using local coordinates, we can give the following classification:

Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be non-coplanar vectors. Let

$$[\mathbf{a}]\,, \quad [\pm\mathbf{a}]\,, \quad [\mathbf{a},\mathbf{b}]\,, \quad [\pm\mathbf{a},\pm\mathbf{b}]\,, \quad [\pm\mathbf{a},\pm\mathbf{b},\mathbf{c}]$$

denote the following sets of three-dimensional vectors:

$$\begin{aligned}
[\mathbf{a}] &= \big\{\mathbf{a}u + \boldsymbol{\omega}u\,;\ \ u \in [0,\varepsilon]\big\},\\
[\pm\mathbf{a}] &= \big\{\mathbf{a}u + \boldsymbol{\omega}u\,;\ \ u \in [-\varepsilon,\varepsilon]\big\},\\
&\dots\\
[\pm\mathbf{a},\pm\mathbf{b},\mathbf{c}] &= \Big\{\mathbf{a}u + \mathbf{b}v + \mathbf{c}w + \boldsymbol{\omega}\sqrt{u^2+v^2+w^2};\ \ u,v \in [-\varepsilon,\varepsilon]\,,\ \ w \in [0,\varepsilon]\Big\}\,,
\end{aligned}$$

where $u$, $v$, $w$ are variables, $\boldsymbol{\omega}$ is a continuously differentiable vector-valued function such that $\boldsymbol{\omega} \to \mathbf{0}$ as $u \to 0$, $v \to 0$, $w \to 0$, and $\varepsilon$ is a sufficiently small positive number. The sets

$$\mathbf{X} + [\mathbf{a}]\,, \quad \mathbf{X} + [\pm\mathbf{a}]\,, \quad \mathbf{X} + [\mathbf{a},\mathbf{b}]\,, \quad \mathbf{X} + [\pm\mathbf{a},\pm\mathbf{b}]\,, \quad \mathbf{X} + [\pm\mathbf{a},\pm\mathbf{b},\mathbf{c}] \subset \mathbf{\Omega}$$

are called *linear neighbourhoods* of $\mathbf{X}$ in $\mathbf{\Omega}$. For the sake of brevity, we call these neighbourhoods *half-line, line, angular, planar* and *half-spherical* neighbourhoods of $\mathbf{X}$ in $\mathbf{\Omega}$, respectively.

**Theorem 78.** *All the following sets*

$$\begin{aligned}
(0,0,0) &+ [(1,1,1)],\\
(1,1/2,1) &+ [(-1,0,-2)],\\
(1/2,1/3,1/3) &+ [\pm(3,1,2),\pm(3,2,4)],\\
(1-v,(1/2)(1-v^2),1-v) &+ [\pm(1,v,1)], \quad 0 < v < 1,\\
(u,(1/2)u,u^2) &+ [\pm(1,1/2,2u)], \quad 0 < u < 1
\end{aligned}$$

*are linear neighbourhoods in* $\mathbf{\Omega}$. *Here*

$$(0,0,0), (1,1/2,1), (1/2,1/3,1/3), (1-v,(1/2)(1-v^2),1-v), (u,(1/2)u,u^2)$$

*are* $\mathbf{F}$*-images of the functions from* Fig. 5*, respectively.*

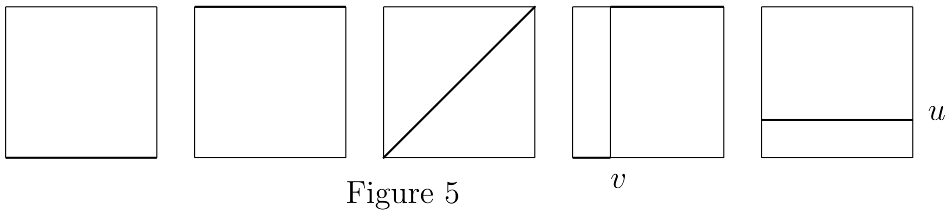


Figure 5

*Proof.* We shall not give the details of the proof of Theorem 78, we show only $(1/2, 1/3, 1/3)+[\pm(3,1,2), \pm(3,2,4)] \subset \mathbf{\Omega}$. Put $\mathbf{X} = \mathbf{F}(g(u_1, v_1, u_2, v_2))$. Fig. 1 indicates that

$$\begin{aligned}\mathbf{X} =& \mathbf{a}(0,0,u_1,v_1) + \mathbf{B}(0,0,u_1,v_1)(0,0,0)\\ &+\mathbf{a}(u_1,v_1,u_2,v_2) + \mathbf{B}(u_1,v_1,u_2,v_2)(1/2,1/3,1/3)\\ &+\mathbf{a}(u_2,v_2,1,1) + \mathbf{B}(u_2,v_2,1,1)(1,1/2,1)\\ &=\mathbf{X}(u_1,v_1,u_2,v_2)\\ &=\begin{pmatrix} 1 - v_2 + \frac{1}{2}(v_2 - v_1)(u_1 + u_2) \\ \frac{1}{2} - \frac{1}{6}v_2^2(3 - u_1 - 2u_2) - \frac{1}{6}v_1^2(2u_1 + u_2) - \frac{1}{6}v_1 v_2(u_2 - u_1) \\ 1 - v_2 + \frac{1}{3}(v_2 - v_1)(u_1 u_2 + u_1^2 + u_2^2) \end{pmatrix}. \end{aligned} \tag{149}$$

Since $\mathbf{X} = (1/2, 1/3, 1/3)$ for $u_1 = v_1 = 0$, $u_2 = v_2 = 1$ and

$$\frac{\partial \mathbf{X}}{\partial v_1} = (-1/2, -1/6, -1/3), \quad \frac{\partial \mathbf{X}}{\partial v_2} = (-1/2, -1/3, -2/3),$$
$$\frac{\partial \mathbf{X}}{\partial u_1} = (1/2, 1/6, 1/3), \quad \frac{\partial \mathbf{X}}{\partial u_2} = (1/2, 1/3, 2/3),$$

we have

$$\begin{aligned}\mathbf{X} =&(1/2,1/3,1/3) + (-1/2,-1/6,-1/3)(v_1 - 0) + (-1/2,-1/3,-2/3)(v_2 - 1)\\ &+(1/2,1/6,1/3)(u_1 - 0) + (1/2,1/3,2/3)(u_2 - 1)\\ &+\omega\sqrt{u_1^2 + v_1^2 + (u_2-1)^2 + (v_2-1)^2},\end{aligned}$$

where $\omega \to 0$ as $v_1, u_1 \to 0$ and $u_2, v_2 \to 1$. This completes the proof. □

**Notes 11.** Almost clear

**Theorem 79.** *If* $\mathbf{X} = \sum_{i=0}^{N} \mathbf{a}_i + \mathbf{B}_i \mathbf{X}^{(i)}$, *and* $\mathbf{X}^{(i)} + V_i \subset \mathbf{\Omega}$ *for all* $i = 0, 1, \ldots, N$, *then* $\mathbf{X} + \sum_{i=0}^{N} \mathbf{B}_i V_i \subset \mathbf{\Omega}$.

**Theorem 80.** *If $\mathbf{X} + V \subset \mathbf{\Omega}$, and $\tilde{V}$ denotes the convex hull of $V$ in the $X_2$ and $X_3$ directions, then $\mathbf{X} + \tilde{V} \subset \mathbf{\Omega}$.*

**Theorem 81.** *A sufficient condition that $\mathbf{X} \in \operatorname{int} \mathbf{\Omega}$ is that the following three conditions be satisfied:*

(i) *There is a half-spherical neighbourhood of $\mathbf{X}$ in $\Omega$ such that $\mathbf{X} + [\pm a, \pm b, c] \subset \mathbf{\Omega}$ has a normal vector $\mathbf{n}$ with second co-ordinate different from zero.*

(ii) *There is a point $\tilde{\mathbf{X}} \in \mathbf{\Omega}$ with a planar neighbourhood $\tilde{\mathbf{X}} + [\pm\tilde{a}, \pm\tilde{b}] \subset \mathbf{\Omega}$ such that $\tilde{X}_1 = X_1$, $\tilde{X}_3 = X_3$, $\tilde{X}_2 \neq X_2$, and the vector product $\tilde{a} \times \tilde{b}$ has a nonzero second co-ordinate.*

(iii) *The second co-ordinates of $\mathbf{n}$ and $X_2 - \tilde{X}_2$ have mutually opposite signs.*

**Theorem 82.** *Assume that*

(i) *$X$ can be decomposed as $X = \sum_{i=0}^{N} a_i + B_i X^{(i)}$, where $\mathbf{X}^{(i)} \in \mathbf{\Omega}$ for $i = 0, 1, \ldots, N$,*

(ii) *there is an index $i$ such that $\mathbf{X}^{(i)} \in \operatorname{int} \mathbf{\Omega}$.*

*Then $\mathbf{X} \in \operatorname{int} \mathbf{\Omega}$.*

**Theorem 83.** *Suppose there are two planar neighbourhoods of $\mathbf{X}$ in $\mathbf{\Omega}$ such that*

(i) *$\mathbf{X} + [\pm\mathbf{a}_1, \pm\mathbf{b}_1]$, $\mathbf{X} + [\pm\mathbf{a}_2, \pm\mathbf{b}_2] \subset \mathbf{\Omega}$, and*

(ii) *the vector product $(\mathbf{a}_1 \times \mathbf{b}_1) \times (\mathbf{a}_2 \times \mathbf{b}_2)$ has a nonzero first co-ordinate.*

*Then $\mathbf{X} \in \operatorname{int} \mathbf{\Omega}$.*

Here we give a definition which we shall need below.

For $\mathbf{X} + [\pm\mathbf{a}, \pm\mathbf{b}, \mathbf{c}]$, consider a vector $\mathbf{n}$ satisfying $\mathbf{n} \times (\mathbf{a} \times \mathbf{b}) = 0$ and $\mathbf{n} \cdot \mathbf{c} < 0$ (scalar product). $\mathbf{n}$ is termed a *normal vector* of the given half-spherical neighbourhood of $\mathbf{X}$.

**Theorem 84.** *Assume that*

(i) *$\mathbf{X}$ can be decomposed as $\mathbf{X} = \sum_{i=0}^{N} \mathbf{a}_i + \mathbf{B}_i \mathbf{X}^{(i)}$, where $\mathbf{X}^{(i)} \in \mathbf{\Omega}$ for $i = 0, 1, \ldots, N$,*

(ii) *there are three indexes $i \neq j \neq k \neq i$ such that $\mathbf{X}^{(i)}$, $\mathbf{X}^{(j)}$, $\mathbf{X}^{(k)}$ have line neighbourhoods $\mathbf{X}^{(i)} + [\pm\mathbf{a}]$, $\mathbf{X}^{(j)} + [\pm\mathbf{b}]$, $\mathbf{X}^{(k)} + [\pm\mathbf{c}] \subset \mathbf{\Omega}$,*

(iii) *the vectors $\mathbf{B}_i\mathbf{a}$, $\mathbf{B}_j\mathbf{b}$, $\mathbf{B}_k\mathbf{c}$ are non-complanar.*

*Then $\mathbf{X} \in \operatorname{int} \mathbf{\Omega}$.*

Theorem 84 admit generalizations which involve also a limiting process.

**Theorem 85.** *Let us suppose that*

(i) *a given point* $\mathbf{X} \in \mathbf{\Omega}$ *can be decomposed as* $\mathbf{X} = \sum_{i=0}^{N} \mathbf{a}_i + \mathbf{B}_i \mathbf{X}^{(i)}$, *where* $\mathbf{X}^{(i)} \in \mathbf{\Omega}$ *for* $i = 0, 1, \ldots, N$, [24]

(ii) *there are three indexes* $0 \leq i, j, k \leq N$, *with* $|i-j|, |i-k|, |j-k| > 1$, *such that the limits*

$$\mathbf{X}^{(i)} \to \mathbf{X}_0^{(i)}, \qquad \mathbf{X}^{(j)} \to \mathbf{X}_0^{(j)}, \qquad \mathbf{X}^{(k)} \to \mathbf{X}_0^{(k)}$$

*exist as*

$$\begin{aligned}
&(u_i, v_i, u_{i+1}, v_{i+1}) \to (u_i^{(0)}, v_i^{(0)}, u_{i+1}^{(0)}, v_{i+1}^{(0)}),\\
&(u_j, v_j, u_{j+1}, v_{j+1}) \to (u_j^{(0)}, v_j^{(0)}, u_{j+1}^{(0)}, v_{j+1}^{(0)}),\\
&(u_k, v_k, u_{k+1}, v_{k+1}) \to (u_k^{(0)}, v_k^{(0)}, u_{k+1}^{(0)}, v_{k+1}^{(0)}),
\end{aligned}$$

*respectively.*

*Further, assume that*

(iii) *the points* $\mathbf{X}_0^{(i)}$, $\mathbf{X}_0^{(j)}$, $\mathbf{X}_0^{(k)}$ *have line neighbourhoods*

$$\mathbf{X}_0^{(i)} + [\pm \mathbf{a}]\,,\ \mathbf{X}_0^{(j)} + [\pm \mathbf{b}]\,,\ \mathbf{X}_0^{(k)} + [\pm \mathbf{c}] \subset \mathbf{\Omega}\,,$$

(iv) *the limits*

$$\frac{\mathbf{B}_i \mathbf{a}}{(u_{i+1}-u_i)(v_{i+1}-v_i)} \to \tilde{\mathbf{a}}, \qquad \frac{\mathbf{B}_j b}{(u_{j+1}-u_j)(v_{j+1}-v_j)} \to \tilde{\mathbf{b}},$$
$$\frac{\mathbf{B}_k c}{(u_{k+1}-u_k)(v_{k+1}-v_k)} \to \tilde{\mathbf{c}},$$

*exist, where*
$(u_i, v_i, u_{i+1}, v_{i+1})$, $(u_j, v_j, u_{j+1}, v_{j+1})$, $(u_k, v_k, u_{k+1}, v_{k+1})$
*converge as in* (ii).

*Moreover, suppose that*

(v) *the vectors* $\tilde{\mathbf{a}}$, $\tilde{\mathbf{b}}$, $\tilde{\mathbf{c}}$ *are non-complanar.*

*Then* $\mathbf{X} \in \operatorname{int} \mathbf{\Omega}$.

*Proof.* See [159, Lemma 14]. □

[24]Here $X^{(i)} = \mathbf{X}^{(i)}(u_i, v_i, u_{i+1}, v_{i+1}) = \mathbf{F}(g_i)$, $g_i$ is defined by (144), and the variables $(u_i, v_i, u_{i+1}, v_{i+1})$, $i = 0, 1, \ldots, N$, must satisfy the assumptions (i), (ii), and (iii) of Theorem 76.

### 4.10.7 Criteria for $\mathbf{F}(g) \in \partial\boldsymbol{\Omega}$

Let $g : [0,1] \to [0,1]$ be a given d.f. Define the following four sets depending on the mapping $g$:

$A =$ the set of points $v^{(1)}$ in which $g$ has a one-side derivative with $0 < g'(v^{(1)}) < +\infty$;

$B =$ the set of points $v^{(2)}$ in which $g$ has a jump discontinuity and $0 < v^{(2)} < 1$;

$C =$ the set of constancy intervals $(v^{(3)}, v^{(4)})$ of $g$ in which $g$ has a value with $0 < g < 1$;

$D =$ the set of continuity points $v^{(5)}$ of $g$ in which $g$ has Dini derivatives such that $D^+g = D^-g = +\infty$, and $D_+g = D_-g = 0$.

**Theorem 86.** *If $\mathbf{F}(g) \in \partial\Omega$, then there exists a straight line passing through the following set of points*

$$\begin{aligned}\left\{\left(v^{(1)}, g(v^{(1)})\right); v^{(1)} \in A\right\} &\cup \left\{\left(v^{(2)}, \frac{g(v^{(2)}+0) + g(v^{(2)})}{2}\right); v^{(2)} \in B\right\}\\ &\cup \left\{\left(\frac{v^{(3)}+v^{(4)}}{2}, g\left(\frac{v^{(3)}+v^{(4)}}{2}\right)\right); (v^{(3)}, v^{(4)}) \in C\right\}\\ &\cup \left\{\left(v^{(5)}, g(v^{(5)})\right); v^{(5)} \in D\right\}.\end{aligned}$$

*Proof.* Consider an arbitrary finite set of elements from $A \dots D$. We shall derive, first of all, that $\mathbf{F}(g) = \mathbf{X}$ can be decomposed as $\mathbf{X} = \sum_{i=0}^{N} \mathbf{a}_i + \mathbf{B}_i \mathbf{X}^{(i)}$ (setting as usual $\mathbf{X}^{(i)} = \mathbf{F}(g_i)$, where $g_i$ is the linear expansion of $g\big|(v_i, v_{i+1})$ given by (9)), where $\mathbf{X}^{(i)} \to \mathbf{X}_0^{(i)}$ and

$$\begin{aligned}X_0^{(i)} \in \Big\{&(0,0,0), (1,1/2,1), (1/2,1/3,1/3), (1-v, (1/2)(1-v^2), 1-v),\\ &(u, (1/2)u, u^2); u, v \in (0,1)\Big\} \qquad (150)\end{aligned}$$

for such $i$ which correspond to the choice of elements from $A \dots D$. Further, with the help of neighbourhoods of these limiting vectors (Theorem 78), the corresponding neighbourhood of $X$ can be immediately (Theorem 85) found. We shall complete the proof using the assumption $\mathbf{X} \in \partial\boldsymbol{\Omega}$.

To do this, let us discuss the following four cases.

a) Let $g$ be a distribution function that has a non-zero finite left derivative at $v^{(1)} \in (0,1]$. Suppose that the independent variable point $(v_i, u_i)$ tends to fixed $(v_{i+1}, u_{i+1}) = (v^{(1)}, g(v^{(1)}))$. Using these assumptions, the following limit can be established

$$\mathbf{X}^{(i)} \to \left(\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{3}\right).$$

Here we shall only show the case $X_3^{(i)} \to \frac{1}{3}$; the cases $X_1^{(i)} \to \frac{1}{2}$ and $X_2^{(i)} \to \frac{1}{3}$ are completely similar.

Indeed, we can write

$$g(x) = g(v_{i+1}) + g'(v_{i+1})(x - v_{i+1}) + \omega(x, v_{i+1})(x - v_{i+1})$$

for all $x \in [v, v_{i+1}]$, where $\omega(x, v_{i+1}) \to 0$ as $x \to v_{i+1}$. Substituting that into the formula (145), after some arrangements, we find

$$\begin{aligned} X_3^{(i)} = {} & \frac{1}{\left(\frac{g(v_{i+1})-g(v_i)}{v_{i+1}-v_i}\right)^2 (v_{i+1} - v_i)^3} \Big\{ (g(v_{i+1}) - g(v_i))^2 (v_{i+1} - v_i) \\ & + g'(v_{i+1})^2 (1/3)(v_{i+1} - v_i)^3 + \omega_1(v_i, v_{i+1})(1/3)(v_{i+1} - v_i)^3 \\ & + 2g'(v_{i+1})\omega_2(v_i, v_{i+1})(1/3)(v_{i+1} - v_i)^3 \\ & - 2(g(v_{i+1}) - g(v_i))g'(v_{i+1})(1/2)(v_{i+1} - v_i)^2 \\ & + 2(g(v_{i+1}) - g(v_i))\omega_3(v_i, v_{i+1})(1/2)(v_{i+1} - v_i)^2 \Big\}, \end{aligned}$$

where

$$\begin{aligned} \omega_1(v_i, v_{i+1})(1/3)(v_{i+1} - v_i)^3 &= \int_{v_i}^{v_{i+1}} \omega(x, v_{i+1})^2 (x - v_{i+1})^2 \mathrm{d}x, \\ \omega_2(v_i, v_{i+1})(1/3)(v_{i+1} - v_i)^3 &= \int_{v_i}^{v_{i+1}} \omega(x, v_{i+1})(x - v_{i+1})^2 \mathrm{d}x, \\ \omega_3(v_i, v_{i+1})(1/3)(v_{i+1} - v_i)^2 &= \int_{v_i}^{v_{i+1}} \omega(x, v_{i+1})(x - v_{i+1}) \mathrm{d}x \end{aligned}$$

and, since $\omega_1, \omega_2, \omega_3 \to 0$ as $v_i \to v_{i+1}$, we finally obtain

$$X_3^{(i)} \to \frac{1}{g'(v_{i+1})^2} \Big\{ g'(v_{i+1})^2 + (1/3)g'(v_{i+1})^2 - g'(v_{i+1})^2 \Big\} = \frac{1}{3}.$$

b) Let us consider the case that $g$ has a jump in $v^{(2)} \in (0,1)$. We choose variable points $(v_j, u_j)$, $(v_{j+1}, u_{j+1})$ such that $v_j < v^{(2)} < v_{j+1}$, $u_j = g(v_j)$,

$u_{j+1} = g(v_{j+1})$, $v_j, v_{j+1} \to v^{(2)}$, and

$$\frac{v^{(2)} - v_j}{v_{j+1} - v_j} = v,$$

where $v \in (0,1)$ is an arbitrary constant. Then, if $(u_j, v_j, u_{j+1}, v_{j+1})$ runs through these variables, we have

$$\mathbf{X}^{(j)} \to (1 - v, (1/2)(1 - v^2), 1 - v).$$

c) Begin with the case when $\big(v^{(3)}, v^{(4)}\big)$ is any interval of a constant value of $g$, where $0 < g < 1$. In the same way as we choose the variables $(u_j, v_j, u_{j+1}, v_{j+1})$, now select $(u_k, v_k, u_{k+1}, v_{k+1})$ such that $v_k \leq v^{(3)} < v^{(4)} \leq v_{k+1}$, $u_k, u_{k+1} \to g$, and

$$\frac{g - u_k}{u_{k+1} - u_k} = u.$$

Then

$$X^{(k)} \to (u, (1/2)u, u^2).$$

For a later application of Theorem 85 in this proof, we note that for neighbouring $v^{(2)}$ and $\big(v^{(3)}, v^{(4)}\big)$ the corresponding variables coincide. In this case we take $(v_{j+1}, u_{j+1}) = (v_k, u_k) \to (v^{(2)}, g)$, preferably. Then the difference $\mathbf{X} - \mathbf{X}_0$ in the proof of Theorem 85 can be expressed as

$$\begin{aligned}\mathbf{X} - \mathbf{X}_0 = B_i\Big(\mathbf{X}^{(i)} - \mathbf{X}_0^{(i)}\Big) + \mathbf{B}_j\left(\mathbf{X}^{(j)} - \mathbf{X}_0^{(j)}\right) + \mathbf{B}_k\left(\mathbf{X}^{(k)} - \mathbf{X}_0^{(k)}\right)\\ + \left(\int_{v^{(2)}}^{v_k} (g - u_k)\mathrm{d}x, \int_{v^{(2)}}^{v_k} x(g - u_k)\mathrm{d}x, \int_{v^{(2)}}^{v_k} (g - u_k)^2\mathrm{d}x\right),\end{aligned}$$

and the final vector on the right-hand side is $O(t^2)$. Thus again the limit in $(\mathbf{X} - \mathbf{X}_0)/t \to \mathbf{0}$ as $t \to 0$.

d) Let us suppose that the distribution function $g$ has the Dini derivatives $D^+g = D^-g = +\infty$ and $D_+g = D_-g = 0$ at a continuity point $v^{(5)} \in (0,1)$. Having in mind the geometrical interpretation of the Dini derivatives, by selecting two suitable sequences of variable vectors $(u_s, v_s, u_{s+1}, v_{s+1})$ one can guarantee the existence of the limits

$$\mathbf{X}^{(s)} \to (0,0,0) \text{ and } \mathbf{X}^{(s)} \to (1, 1/2, 1).$$

These cases are obtained in the limit when all of the sides of the rectangles shown in [159, Fig. 6] suitably tend to zero.

For our further aims, to apply Theorem 85, we need to establish the limits in (iv). To do this, we have already found in Theorem 78 the neighbourhoods of all limiting vectors from a)–d). When

$$(u_i, v_i, u_{i+1}, v_{i+1}), \ldots, (u_s, v_s, u_{s+1}, v_{s+1})$$

run through the variables in a)–d) we verify that

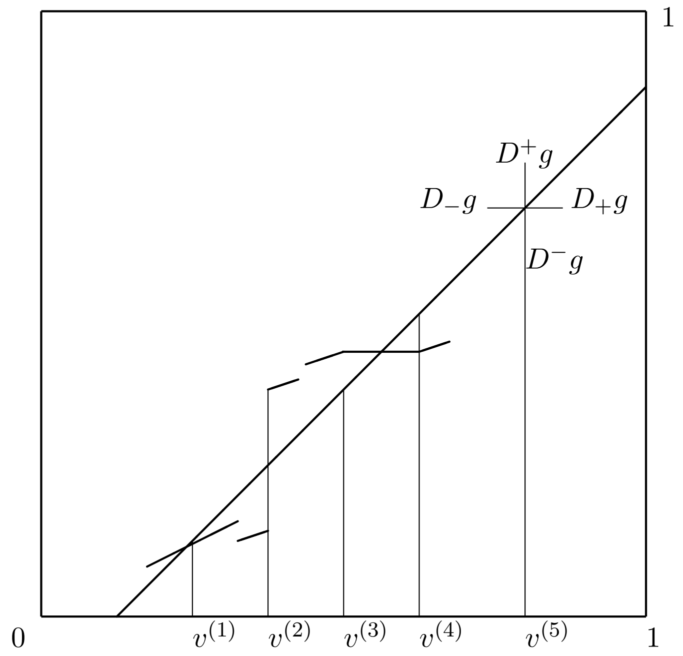
$$\begin{aligned}
\frac{\mathbf{B}_i(3,1,2)}{(u_{i+1}-u_i)(v_{i+1}-v_i)} &\to \left(3,\, 3v^{(1)},\, 6g(v^{(1)})\right), \\
\frac{\mathbf{B}_j(1,v,1)}{(u_{j+1}-u_j)(v_{j+1}-v_j)} &\to \left(1,\, v^{(2)}, g(v^{(2)}+0)+g(v^{(2)})\right), \\
\frac{\mathbf{B}_k(1,(1/2),2u)}{(u_{k+1}-u_k)(v_{k+1}-v_k)} &\to \left(1, \frac{v^{(3)}+v^{(4)}}{2}, 2g\left(\frac{v^{(3)}+v^{(4)}}{2}\right)\right), \\
\frac{\mathbf{B}_s(1,1,1)}{(u_{s+1}-u_s)(v_{s+1}-v_s)} &\to \left(1, v^{(5)}, 2g(v^{(5)})\right), \\
\frac{\mathbf{B}_s(-1,0,-2)}{(u_{s+1}-u_s)(v_{s+1}-v_s)} &\to \left(-1, -v^{(5)}, -2g(v^{(5)})\right).
\end{aligned}$$

Finally, the corresponding vector product of limit vectors has the co-ordinate $X_3$ different from zero for all interesting cases. Thus the additional parallelity assumption in (v) of Theorem 85 is valid.

As a consequence of this theorem, using $\mathbf{F}(g) \in \partial\mathbf{\Omega}$, we easily obtain that all the limiting vectors are co-planar and we have therefore shown Theorem 86. In conclusion, it should only be noted that in cases a)–d) we can choose an arbitrary finite number of elements from $A \ldots D$, respectively, and at these there must exist independently specified variables $(u_n, v_n, u_{n+1}, v_{n+1})$. [25] $\square$

For the sake of more clarity, see a geometrical illustration of this result in [159, Fig. 7.]:

[25] This assumption may be unrealizable if $v^{(2)}$ and $(v^{(3)}, v^{(4)})$ are neighbouring, and therefore we use (20') or the note in c).




Theorem 86 implies the following restriction upon the nature of those distribution functions which can be **F**-mapped on the surface $\partial\mathbf{\Omega}$.

**Theorem 87.** *In order that the d.f.* $g(x)$ *should satisfy* $\mathbf{F}(g) \in \partial\mathbf{\Omega}$*, it is necessary that either*

(i) *$g$ is continuous in* $(0,1)$ *and the part of its graph included in the open square* $(0,1)^2$ *takes the form of a line-segment in* $(0,1)^2$ (see a list of these graphs in Fig. 2). *Express it as $y = ax + b$. If $a \neq 0$, then* $\mathbf{F}(g)$ *possesses a half-spherical neighbourhood having the normal vector* $(b, a, -1/2)$*; or*

(ii) *$g$ is a step-function such that all midpoints of its jumps and intervals of constancy from the open square* $(0,1)^2$ *lie on a common straight line. Write it as $y = ax + b$ and suppose that at least one step*[26] *of the graph*

[26]A step of $g$ consists of a vertical segment associated to a jump of $g$ and the neighbouring interval of constancy of $g$.

*of $g$ lies in the open square $(0,1)^2$. Then a half-spherical neighbourhood of $\mathbf{F}(g)$ can be found, with the normal vector $(-b,-a,1/2)$.*

*Proof.* Let $\mathbf{F}(g) \in \partial\Omega$. We adopt the notations of Theorem 86 and assume first that $g$ has a jump in $v_0^{(2)} \in (0,1)$. Claim: It is not possible to choose a sequence of points of type $v^{(1)}$ so that $v^{(1)} \to v_0^{(2)}$. Suppose, on the contrary, that

$$\begin{aligned} \big(v^{(1)}, g(v^{(1)})\big) &\to \left(v_0^{(2)}, g(v_0^{(2)})\right), \text{ or} \\ \big(v^{(1)}, g(v^{(1)})\big) &\to \left(v_0^{(2)}, g(v_0^{(2)}+0)\right). \end{aligned}$$

Since, by Theorem 86 all of these $\big(v^{(1)}, g(v^{(1)})\big)$ and $\left(v_0^{(2)}, \frac{g(v_0^{(2)}+0)+g(v_0^{(2)})}{2}\right)$ must be lying on a fixed straight-line, the only possibility is that

$$\big(v^{(1)}, g(v^{(1)})\big) \to \left(v_0^{(2)}, \frac{g(v_0^{(2)}+0)+g(v_0^{(2)})}{2}\right).$$

This is impossible, and the claim is proved.

A similar analysis can be done for sequences of $v^{(2)}$, $(v^{(3)}, v^{(4)})$ and $v^{(5)}$ tending to $v_0^{(2)}$. Thus, for a given $v_0^{(2)}$, there exists a suitable $\varepsilon$ so that $g$ has, on $(v_0^{(2)}-\varepsilon, v_0^{(2)}) \cup (v_0^{(2)}, v_0^{(2)}+\varepsilon)$, the following properties: $g$ is continuous; $g$ has derivative zero at all points of differentiability; $g$ cannot have a point of type $v^{(5)}$ and an interval $(v^{(3)}, v^{(4)})$. Since (see K.M. Garg [59]) *for any continuous strictly increasing function $f$ which has a zero-derivative almost everywhere* [27] *there exists a residual set of points with $D^+f = D^-f = +\infty$, $D_+f = D_-f = 0$*, then $g$ is a constant function on $(v_0^{(2)}-\varepsilon, v_0^{(2)})$ and $(v_0^{(2)}, v_0^{(2)}+\varepsilon)$.

Along the same lines, it can be shown that if $g$ has an interval $(v^{(3)}, v^{(4)})$ of constant value of $g$, $0 < g < 1$, the boundary points $v^{(3)}$ and $v^{(4)}$ are the jump-points of $g$. □

**Notes 12.** Here we give a complete proof from [159, pp. 201–204]:

*Proof.* Collecting all these results, we obtain that whenever the function $g$ has at most one point of type $v^{(2)}$ or an interval of type $(v^{(3)}, v^{(4)})$, then $g$ is a step-function.

To complete the possible cases, let us assume that $g$ is continuous in $(0,1)$, but has no element of type $\big(v^{(3)}, v^{(4)}\big)$. Then the unit interval can be split into three sub-intervals, say

[27] such function is said to be *singular*.

$[0,c]$, $(c,d)$, $[d,1]$, so that $g$ takes the value 0 in $[0,c]$, 1 in $[d,1]$, and is strictly increasing in $(c,d)$. If in addition to this, the set of points $v^{(1)}$ is dense in $(c,d)$, then the set of points $\big(v^{(1)}, g(v^{(1)})\big)$ is also dense in the graph of $g\big|_{(c,d)}$, and according to Theorem 86 the graph is a line-segment. If $g$ cannot have $v^{(1)}$-point in a sub-interval $(e,f)\subset(c,d)$, then the restriction $g\big|_{(e,f)}$ of $g$ is singular. Making use of Garg's (above mentioned) theorem, we may argue that the set of points $v^{(5)}$ is dense in $(e,f)$. But then, since $g$ is continuous, the set of points $\big(v^{(5)}, g(v^{(5)})\big)$ is also dense in the graph of $g\big|_{(e,f)}$. Using here Theorem 86, one derives that this graph forms a line-segment. This is impossible because of the existence of $v^{(5)}$.

Let us proceed to find an expression for the half-spherical neighbourhoods of points $\mathbf{F}(g)$ of $g$ specified by (i) and (ii). We start with a decomposition $\mathbf{F}(g)=\sum_{i=0}^{N} a_i + B_i X^{(i)}$ constructed as in the proof of Theorem 76.

In case (i) we may assume that $(v_i,u_i)$ and $(v_{i+1},u_{i+1})$ are chosen from the straight-line $y=ax+b$ with suitable $i$. Then $\mathbf{X}^{(i)}=(\frac{1}{2},\frac{1}{3},\frac{1}{3})$, and we begin with the construction of that half-spherical neighbourhood

$$\left(\frac{1}{2},\frac{1}{3},\frac{1}{3}\right)+\big[\pm(3,1,2),\pm(3,2,4),(0,-1,0)\big]\,. \tag{151}$$

There is a point $(\frac{1}{2},\frac{5}{16},\frac{1}{3})$ lying under $(\frac{1}{2},\frac{1}{3},\frac{1}{3})$. Since it may be put in the form

$$\left(\frac{1}{2},\frac{5}{16},\frac{1}{3}\right)=\mathbf{X}\left(\frac{1}{3},0,\frac{1}{3},\frac{3}{4}\right)$$

by virtue of the expression (149) for $\mathbf{X}(u_1,v_1,u_2,v_2)$ and since

$$\frac{\partial X}{\partial v_2}=\left(-\frac{2}{3},-\frac{1}{2},-\frac{8}{9}\right)\qquad \frac{\partial X}{\partial u_2}=\left(\frac{3}{8},\frac{3}{16},\frac{1}{14}\right),$$

we find that $(\frac{1}{2},\frac{5}{16},\frac{1}{3})$ has a planar neighbourhood in $\mathbf{\Omega}$. Summing up the result and using convexity, provided by Theorem 73, under the lines parallel to the $X_2$-axis, we shall then complete our proof of (151).

Now the decomposition of $\mathbf{F}(g)$ and Theorem 79 imply

$$\mathbf{F}(g)+\big[\pm\mathbf{B}_i(3,1,2),\,\pm\mathbf{B}_i(3,2,4),\,\mathbf{B}_i(0,-1,0)\big]\subset\Omega\,. \tag{152}$$

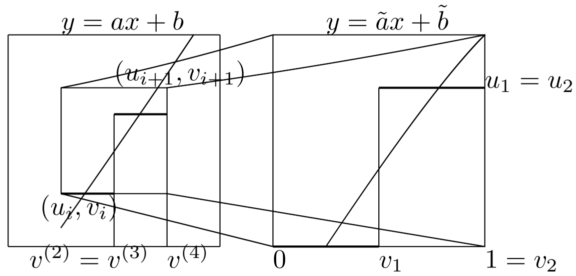
One easily sees that if $(c,d,e)$ is a normal vector to (151) (compare the definition over Theorem 82, then $(c,d,e)\mathbf{B}_i^{-1}$ is a normal vector to (152). Since (151) has $(0,1,-\frac{1}{2})$ as its normal vector, and

$$\left(0,1,-\frac{1}{2}\right)\mathbf{B}_i^{-1}=\frac{1}{(u_{i+1}-u_i)^2(v_{i+1}-v_i)}\left(u_i-v_i\frac{u_{i+1}-u_i}{v_{i+1}-v_i},\,\frac{u_{i+1}-u_i}{v_{i+1}-v_i},\,-\frac{1}{2}\right) \tag{153}$$

$$=\frac{1}{(u_{i+1}-u_i)^2(v_{i+1}-v_i)}(b,a,-\frac{1}{2})\,, \tag{154}$$

then the half-spherical neighbourhood (152) of the image $\mathbf{F}(g)$ has a normal vector $(b,a,-\frac{1}{2})$ as asserted.

Using the same procedure from the above construction, we can now find a half-spherical neighbourhood of $\mathbf{F}(g)$, where $g$ is specified by (ii). Assume that there are neighbouring $v^{(2)}$ and $(v^{(3)}, v^{(4)})$. We distinguish two cases, depending on whether $v^{(2)} = v^{(3)}$ or $v^{(2)} = v^{(4)}$. Observe that the transformation $\Lambda$ transfers immediately the case $v^{(2)} = v^{(4)}$ to $v^{(2)} = v^{(3)}$, so that it remains to consider the case $v^{(2)} = v^{(3)}$. In this case $\mathbf{F}(g)$ can be decomposed into the sum $\mathbf{F}(g) = \sum\limits_{i=0}^{N} a_i + \mathbf{B}_i X^{(i)}$, where some $X^{(i)}$ may be put in the form $X^{(i)} = \mathbf{F}(\tilde{g})$ for a one-step function $\tilde{g}$. We shall use the representation for $\tilde{g}$ given in (1), where $\tilde{g} = g(u_1, v_1, u_2, v_2)$, $0 < u_1 = u_2 < 1$, $0 < v_1 < 1$, $v_2 = 1$, and we again use the expression (149) that gives $\mathbf{X}^{(i)} = \mathbf{X}(u_1, v_1, u_2, v_2)$. Now let $y = ax + b$ be the line joining the points $\left(v^{(2)}, \frac{g(v^{(2)}+0)+g(v^{(2)})}{2}\right)$, $\left(\frac{v^{(3)}+v^{(4)}}{2}, g\left(\frac{v^{(3)}+v^{(4)}}{2}\right)\right)$, and $y = \tilde{a}x + \tilde{b}$ joining $\left(v_1, \frac{u_1}{2}\right)$ and $\left(\frac{v_1+1}{2}, u_1\right)$. Graphically, see following Fig.




Now we can show that the partial derivatives of $\mathbf{X}$ with respect to the parameters $v_1$, $v_2$, $u_1$, and $u_2$ are

$$\frac{\partial \mathbf{X}}{\partial v_1} = -u_1(1, v_1, u_1)\,, \qquad \frac{\partial \mathbf{X}}{\partial v_2} = (-1 + u_1)(1, 1, 1 + u_1)$$
$$\frac{\partial \mathbf{X}}{\partial u_1} = (1 - v_1)\Big(\frac{1}{2}, \frac{1}{6}(1 + 2v_1), u_1\Big) \qquad \frac{\partial \mathbf{X}}{\partial u_2} = (1 - v_1)(\frac{1}{2}, (2 + v_1), u_1).$$

Moreover, the increments $\mathrm{d}v_1$, $\mathrm{d}v_2$, $\mathrm{d}u_1$, and $\mathrm{d}u_2$ must satisfy $\mathrm{d}v_2 \leq 0$ and $\mathrm{d}u_1 \leq \mathrm{d}u_2$, and, in addition, the differential of $X$ becomes

$$\mathrm{d}X = \frac{\partial \mathbf{X}}{\partial v_1}\mathrm{d}v_1 + \frac{\partial \mathbf{X}}{\partial v_2}\mathrm{d}v_2 + \left(\frac{\partial \mathbf{X}}{\partial u_1} + \frac{\partial \mathbf{X}}{\partial u_2}\right)\left(\frac{\mathrm{d}u_2 + \mathrm{d}u_1}{2}\right) + \left(\frac{\partial \mathbf{X}}{\partial u_2} - \frac{\partial \mathbf{X}}{\partial u_1}\right)\left(\frac{\mathrm{d}u_2 - \mathrm{d}u_1}{2}\right).$$

So we arrive at

$$\mathbf{F}(\tilde{g}) + \Big[\pm(1, v_1, u_1),\ \pm\Big(1, \frac{1+v_1}{2}, 2u_1\Big),\ \Big(0, \frac{1-v_1}{6}, 0\Big),\ (1, 1, 1 + u_1)\Big] \subset \Omega\,. \tag{155}$$

Now we make use of the inner products

$$\Big(-\tilde{b}, -\tilde{a}, \frac{1}{2}\Big)(1, v_1, u_1) = 0 \qquad \Big(-\tilde{b}, -\tilde{a}, \frac{1}{2}\Big)\Big(1, \frac{1+v_1}{2}, 2u_1\Big) = 0$$

$$\Big( - \tilde{b}, -\tilde{a}, \frac{1}{2}\Big)\Big(0, \frac{1-v_1}{6}, 0\Big) < 0\,, \qquad \Big( - \tilde{b}, -\tilde{a}, \frac{1}{2}\Big)(1,1,1+u_1) = \frac{1}{2} - u_1\,,$$

and conclude that (155) will be either a half-spherical neighbourhood of $\mathbf{F}(\tilde{g})$ with normal vector $(-\tilde{b}, -\tilde{a}, \frac{1}{2})$ if $\frac{1}{2} \leq u_1$, or a spherical neighbourhood for $u_1 < \frac{1}{2}$.

But again the decomposition of $\mathbf{F}(g)$, Theorem 79, and

$$\begin{aligned}
&(-\tilde{b}, -\tilde{a}, \tfrac{1}{2})\mathbf{B}_i^{-1}\\
=&\Big(-\tilde{b}(u_{i+1}-u_i) + \tilde{a}v_i\frac{u_{i+1}-u_i}{v_{i+1}-v_i} - u_i, -\tilde{a}\frac{u_{i+1}-u_i}{v_{i+1}-v_i}, \frac{1}{2}\Big)\frac{1}{(u_{i+1}-u_i)^2(v_{i+1}-v_i)}\\
=&\Big( - b, -a, \frac{1}{2}\Big)\frac{1}{(u_{i+1}-u_i)^2(v_{i+1}-v_i)}
\end{aligned}$$

yield that

$$\mathbf{F}(g) + \Big[\pm\mathbf{B}_i(1, v_1, u_1), \pm\mathbf{B}_i\Big(1, \frac{1+v_1}{2}, 2u_1\Big), \mathbf{B}_i\Big(0, \frac{1-v_1}{6}, 0\Big)\Big] \subset \mathbf{\Omega} \tag{156}$$

is a half-spherical neighbourhood of $\mathbf{F}(g)$ with the normal vector $(-b, -a, \frac{1}{2})$, and the proof is complete. $\square$

Having done all this, we may now conclude the proof of Theorem 70.

**Theorem 88.** *We have*

$$\overline{\partial}\mathbf{\Omega} = \bigcup_{1\leq i\leq 4} \Pi_i.$$

*Proof.* Suppose that $g$ satisfies condition (i) and $\tilde{g}$ condition (ii) from Theorem 87. For the points $\mathbf{X} = \mathbf{F}(g)$ and $\tilde{\mathbf{X}} = \mathbf{F}(\tilde{g})$ we have, according to this theorem, two half-spherical neighbourhoods in $\mathbf{\Omega}$, with normal vectors $(b, a, -1/2)$ and $(-\tilde{b}, -\tilde{a}, 1/2)$, respectively. Now we state the result:

If $\mathbf{X}, \tilde{\mathbf{X}} \in \partial\mathbf{\Omega}$ and $X_1 = \tilde{X}_1$, $X_3 = \tilde{X}_3$, then $X_2 > \tilde{X}_2$.

Indeed, let us assume, to the contrary, that $X_2 \leq \tilde{X}_2$. Then the points $\mathbf{X}$ and $\tilde{\mathbf{X}}$ (or, in the case $X_2 = \tilde{X}_2$, its small shift) satisfy all the three conditions in Lemma 11, and thus they have spherical neighbourhoods in $\mathbf{\Omega}$; this is a contradiction. We thus obtain a separation of $\mathbf{F}$-images of $g$ which are described in (i) and (ii) of Theorem 87 either to the upper and lower surfaces of $\partial\mathbf{\Omega}$, respectively, or to the interior of $\mathbf{\Omega}$.

Now we use that the functions $g$ which satisfy (i) of Lemma 16 can have only the graphs of types $g^{(1)} \ldots g^{(4)}$ from Fig. 2. Then, using notations (2), let $\Pi_1 \ldots \Pi_4$ be their $\mathbf{F}$-images in $\mathbf{\Omega}$. By virtue of the expression $g(u_1, v_1, u_2, v_2)$ in (1) for $g^{(1)} \ldots g^{(4)}$, we obtain a parametric representation of $\Pi_1 \ldots \Pi_4$. Eliminating parameters from these equations, we obtain the canonical equation of $\Pi_1 \ldots \Pi_4$ as in (4). One sees immediately that any intersection $\Pi_i \cap \Pi_j$

of two different surfaces from $\Pi_1 \dots \Pi_4$ coincide with the intersection of their boundaries. If a point $\mathbf{X}$ runs through this common curve, one can also compute the identity $g^{(i)} = g^{(j)}$ at the equation $X = \mathbf{F}(g^{(i)}) = \mathbf{F}(g^{(j)})$, $(1 \le i, j \le 4)$. In an alternative proof, one uses the normal vectors $(b_i, a_i, -1/2)$ and $(b_j, a_j, -1/2)$ to half-spherical neighbourhoods of $\mathbf{F}(g^{(i)})$ and $\mathbf{F}(g^{(j)})$, respectively, constructed as in the proof of (i) of Theorem 87. Then the direction vector considered in Theorem 83 can be found as the vector product

$$(b_i, a_i, -1/2) \times (b_j, a_j, -1/2).$$

Applying $\mathbf{F}(g_i) = \mathbf{F}(g_j) \in \partial\mathbf{\Omega}$ and Theorem 83, we find that the first coordinate of the product must be zero. The only possibility is that $a_i = a_j$, and consequently $b_i = b_j$ and $g^{(i)} = g^{(j)}$. □

In fact, *the mapping* $\mathbf{F}$ *specifies a one-to-one correspondence between the functions of Fig.* 2 *and the points* $\mathbf{X}$ *of the upper boundary surface of* $\mathbf{\Omega}$.

**Theorem 89.** *We have*

$$\underline{\partial}\mathbf{\Omega} = \bigcup_{5 \le i \le 7} \Pi_i \,.$$

*Proof.* Let us consider one-step functions listed in Fig. 3. Their expression $g(u_1, v_1, u_2, v_2)$ in (137) is determined by two collections of conditions $v_1 = 0$, $0 < u_1 = u_2 < 1$, $0 < v_2 < 1$ or $0 < v_1 < 1$, $0 < u_1 = u_2 < 1$, $v_2 = 1$. Using the $\mathbf{F}$-mapping, we can extend the surfaces $\Pi_5$, $\Pi_6$ (without boundary) to the following enlarged sets

$$\begin{aligned} \Pi_8 &= \big\{\mathbf{F}(g); v_1 = 0, 0 < u_1 = u_2 < 1, 0 < v_2 < 1\big\}, \\ \Pi_9 &= \big\{\mathbf{F}(g); 0 < v_1 < 1, 0 < u_1 = u_2 < 1, v_2 = 1\big\}, \end{aligned}$$

respectively. Taking into account expression (149), we can rewrite parametrically

$$\begin{aligned} \Pi_8 &= \left\{ \begin{pmatrix} 1-v_2+v_2u_2 \\ \frac{1}{2}-\frac{1}{2}v_2^2(1-u_2) \\ 1-v_2+v_2u_2^2 \end{pmatrix} ;\ 0 < u_2 < 1\,,\ \ 0 < v_2 < 1 \right\}, \\ \Pi_9 &= \left\{ \begin{pmatrix} (1-v_1)u_1 \\ \frac{1}{2}-\frac{1}{2}(1-u_1)-\frac{1}{2}v_1^2u_1 \\ (1-v_1)u_1^2 \end{pmatrix} ;\ 0 < u_1 < 1\,,\ \ 0 < v_1 < 1 \right\}. \end{aligned}$$

Since the parameters $u_2$ and $v_2$ in $\Pi_8$ can be eliminated as

$$u_2 = \frac{X_1 - X_3}{1 - X_1}\,, \qquad v_2 = \frac{(1 - X_1)^2}{1 + X_3 - 2X_1}\,, \tag{157}$$

the expression for $\Pi_8$ takes the form

$$\Pi_8 = \Big\{(X_1, X_2, X_3); X_2 = \frac{1}{2} - \frac{1}{2}\frac{(1-X_1)^3}{1+X_3-2X_1},$$
$$X_1^2 < X_3 < X_1, 0 < X_1 < 1\Big\}.$$

Similarly, in the case $\Pi_9$,

$$u_1 = \frac{X_3}{X_1}\,, \qquad 1 - v_1 = \frac{X_1^2}{X_3}\,, \tag{158}$$

and we can write

$$\Pi_9 = \Big\{(X_1, X_2, X_3); X_2 = X_1 - \frac{1}{2}\frac{X_1^3}{X_3};$$
$$X_1^2 < X_3 < X_1, 0 < X_1 < 1\Big\}.$$

Together with Theorem 74, we obtain that the projections of $\Pi_8$ and $\Pi_9$ on the $X_1 \times X_3$-plane are the same as the projection of the domain $\mathbf{\Omega}$.

Moreover, it is easy to verify that $\Pi_8$ and $\Pi_9$ intersect in a curve

$$\Pi_7 = \Big\{\Big(\frac{1}{2}, \frac{1}{2} - \frac{1}{16X_3}, X_3\Big)\,;\ \ \frac{1}{4} < X_3 < \frac{1}{2}\Big\}\,,$$

where $\frac{1}{2} = X_1$, and we simply deduce that every $\mathbf{X} \in \Pi_8$ with $X_1 < \frac{1}{2}$ must lie under $\Pi_9$ and every $\mathbf{X} \in \Pi_9$ with $X_1 > \frac{1}{2}$ must lie under $\Pi_8$, in both cases with respect to the $X_1 \times X_3$-plane.

Recall that every point from $\Pi_8 \cup \Pi_9$ is the $\mathbf{F}$-image of a one-step function and we may apply (ii) of Theorem 87, so that then each one has a half-spherical neighbourhood with a normal vector $(-b, -a, \frac{1}{2})$, where the second co-ordinate is negative. But then we make again use of Theorem 81 and conclude that all the points from $\Pi_8$ lying above $\Pi_9$ and all the points from $\Pi_9$ lying above $\Pi_8$ must lie in the interior of $\mathbf{\Omega}$. We simply write this relation as $\Pi_8^0 \cup \Pi_9^0 \subset \operatorname{int} \mathbf{\Omega}$, where

$$\Pi_8^0 = \Big\{\mathbf{F}(g)\,;\ \ v_1 = 0\,,\ \ 0 < u_1 = u_2 < 1\,,\ \ 0 < v_2 < 1\,,\ \ v_2(1-u_2) < \frac{1}{2}\Big\}\,,$$
$$\Pi_9^0 = \Big\{\mathbf{F}(g)\,;\ \ 0 < v_1 < 1\,,\ \ 0 < u_1 = u_2 < 1\,,\ \ v_2 = 1\,,\ \ (1-v_1)u_1 < \frac{1}{2}\Big\}.$$

□

### 4.10.8 Open problems for dimension $s = 4$

Solve the moment problem

$$(X_1, X_2, X_3, X_4) = \left(\int_0^1 g(x)\mathrm{d}x, \int_0^1 xg(x)\mathrm{d}x, \int_0^1 x^2g(x)\mathrm{d}x, \int_0^1 g^2(x)\mathrm{d}x\right).$$

E.g. for $g(x) = 2x - x^2$ it has the unique solution.

# 5 Operations with d.f.s

Given a sequence $x_n$ in $[0, 1)$, let the sequence $y_n$ be defined by one of the following ways:

$y_n = f(x_n)$, $f : [0, 1] \to [0, 1]$,
$y_n = x_1 + \cdots + x_n \bmod 1$,
$y_n = \frac{x_1+\cdots+x_n}{n}$,
$y_n = nx_n \bmod 1$,
$y_n = \alpha x_n \bmod 1$,
$y_n = x_n u_n$,
$y_n = \frac{x_n}{u_n} \bmod 1$,
$y_n = \max(x_n, x_{n+1}) = \frac{x_n+x_{n+1}+|x_n-x_{n+1}|}{2}$,
$\mathbf{y}_n = (x_{n+1}, \dots, x_{n+s})$,
$\mathbf{y}_n = (x_{2n-1}, x_{2n})$,
$y_n$ is the sequence $F(x_m, x_n) \bmod 1$ for $m, n = 1, 2, \dots$, ordered in such way that the values $F(x_m, x_n) \bmod 1$ with $m, n = 1, 2, \dots, N$, form the first $N^2$ terms of $y_n$, $n = 1, 2, \dots$, where $F : [0, 1]^2 \to \mathbb{R}$.

In every of the above cases the connection between $G(x_n)$ and $G(y_n)$ is an open problem. In what follows some results will be presented:

## 5.1 Statistical independence

The following part continues Section 3.2.

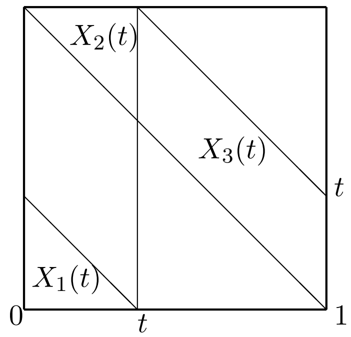
**Theorem 90.** *Let $x_n$ and $y_n$ be two sequences in $(0, 1)^2$. If*
(i) *$x_n$ and $y_n$ are statistically independent;*
(ii) *$x_n$ is u.d.;*
(iii) *$g(x) \in G(y_n)$ are continuous;*
*then the sequence $x_n + y_n \bmod 1$, $n = 1, 2, \dots$ is u.d.*

*Proof.* By (i) and (ii) every $g(x,y) \in G(x_n, y_n)$ has the form $g(x,y) = xg(y)$. Divide unit square $[0,1]^2$ into three parts

$X_1(t) = \{(x,y) \in [0,1]; x+y < t\}$,

$X_2(t) = \{(x,y) \in [0,1]; 1 < x+y < t+1, x \leq t\}$,

$X_3(t) = \{(x,y) \in [0,1]; 1 < x+y < t+1, x > t\}$, see




By integration

$$\int_{X_1(t)} 1.\mathrm{d}g(x,y) = \int_0^t \mathrm{d}x \int_0^{t-x} 1.\mathrm{d}g(y) = \int_0^t g(t-x)\mathrm{d}x$$
$$\int_{X_2(t)} 1.\mathrm{d}g(x,y) = \int_0^t \mathrm{d}x \int_{1-x}^{1} 1.\mathrm{d}g(y) = \int_0^t (1-g(1-x))\mathrm{d}x$$
$$\int_{X_3(t)} 1.\mathrm{d}g(x,y) = \int_t^1 \mathrm{d}x \int_{1-x}^{t+1-x} 1.\mathrm{d}g(y) = \int_t^1 (g(t+1-x)-g(1-x))\mathrm{d}x.$$

Thus

$$\int_{x+y \bmod 1 \in [0,t)} 1\mathrm{d}g(x,y) = \int_0^t 1.\mathrm{d}x - \int_0^1 g(1-x)\mathrm{d}x + \int_0^t g(t-x)\mathrm{d}x + \int_t^1 g(t+1-x)\mathrm{d}x.$$

Now, integration by substitution gives

(j) $-\int_0^1 g(1-x)\mathrm{d}x = -\int_0^1 g(x)\mathrm{d}x$, (jj) $\int_0^t g(t-x)\mathrm{d}x = \int_0^t g(x)\mathrm{d}x$,

(jjj) $\int_t^1 g(t+1-x)\mathrm{d}x = \int_t^1 g(x)\mathrm{d}x$,

then finally we have

$\int_{x+y \bmod 1 \in [0,t)} 1\mathrm{d}g(x,y) = t.$ □

**Example 43.** Let $x_n$ and $y_n$ be two sequences in $[0,1)$. Assume that

(i) $x_n$ and $y_n$ are u.d.

(ii) $x_n$ and $y_n$ are statistically independent.

Then by Theorem 90 the sequence $x_n + y_n \bmod 1$ is again u.d. It can be proved directly by Weyl's criterion if we prove

$$\frac{1}{N}\sum_{n=1}^{N}(\{x_n+y_n\})^k \to \frac{1}{k+1}, \quad k=1,2,\dots$$

From (ii) follows that the sequence $(x_n, y_n)$ has a.d.f. $g(x,y) = xy$ and by Helly theorem

$$\frac{1}{N}\sum_{n=1}^{N}(\{x_n+y_n\})^k \to \int_0^1\int_0^1 (\{x+y\})^k \mathrm{d}x\mathrm{d}y, \quad k=1,2,\dots$$

Now

$$\int_0^1\int_0^1 (\{x+y\})^k \mathrm{d}x\mathrm{d}y = \iint_{0\le x+y\le 1}(x+y)^k\mathrm{d}x\mathrm{d}y + \iint_{1\le x+y\le 2}(x+y-1)^k\mathrm{d}x\mathrm{d}y$$

which is $\frac{1}{k+1}$ and the proof is finished.

**Theorem 91.** *Let $x_n \in [0,1)$, $n=1,2,\dots,$ be u.d. sequence. Then $x_n$ and* $\log n \bmod 1$ *are statistically independent, i.e. every $g(x,y) \in G(x_n, \{\log n\})$ has the form $g(x,y) = x.g(1,y)$.*

*Proof.* Let $N \in [e^K, e^{K+1})$ i.e., $N = e^{K+\theta_N}$, and divide $n \le N$ to the subsets $n \in [e^k, e^{k+1})$, $k \le K$. For such $n$ we have $\{\log n\} \in [0,y) \iff n \in [e^k, e^{k+y})$. For $n \in [e^k, e^{k+y})$ we ask the number of $x_n \in [0,x)$ which is $x(e^{k+y} - e^k) + O(e^k D_{e^k} + e^{k+y} D_{e^{k+y}})$. Omitting integer parts here we use discrepancy $D_M$ of the initial string $x_1, x_2, \dots, x_M$ and the formula $A([0,x); M; x_n) = xM + O(MD_M)$. [28] Thus

$$\frac{A([0,x)\times[0,y); N; (x_n, \{\log n\}))}{N}$$

[28] $O$-constant can be put $=1$ and in the interval $[e^K, e^{K+\min(y,\theta_N)}]$, for simplification, an error term we put $O(e^k D_{e^K} + e^{K+y} D_{e^{K+y}})$.

$$= \frac{\sum_{k=0}^{K-1} x(e^{k+y} - e^k) + x(e^{K+\min(y,\theta_N)} - e^K) + O(\sum_{k=0}^{K} e^k D_{e^k} + e^{k+y} D_{e^{k+y}})}{N}.$$

As $N \to \infty$ and $\theta_N \to u$, we have

$$\frac{\sum_{k=0}^{K-1} x(e^{k+y} - e^k)}{N} = \frac{\sum_{k=0}^{K-1} x(e^{k+y} - e^k)}{\sum_{k=0}^{K-1}(e^{k+y} - e^k)} \frac{\sum_{k=0}^{K-1}(e^{k+y} - e^k)}{N} \to x\frac{e^y - 1}{e - 1}\frac{1}{e^u},$$

$$\frac{x(e^{K+\min(y,\theta_N)} - e^K)}{N} \to x\frac{e^{\min(y,u)} - 1}{e^u},$$

$$\frac{O(\sum_{k=0}^{K} e^k D_{e^k} + e^{k+y} D_{e^{k+y}})}{N} = O\left(\frac{\sum_{k=0}^{K} e^k D_{e^k} + e^{k+y} D_{e^{k+y}}}{\sum_{k=0}^{K}(e^{k+y} - e^k)}\right) \to 0.$$

In the final parenthesis we have used $\frac{e^k D_{e^k} + e^{k+y} D_{e^{k+y}}}{e^{k+y} - e^k} \to 0$ as $k \to \infty$. Collected all above results we have

$$\frac{A([0,x) \times [0,y); N; (x_n, \{\log n\}))}{N} \to x\left(\frac{e^y - 1}{e - 1}\frac{1}{e^u} + \frac{e^{\min(y,u)} - 1}{e^u}\right) = xg_u(y),$$

where $g_u(y)$ is the same as in (8). □

A consequence of Theorem 90 and 91.

**Theorem 92.** *Given a sequence $x_n$, then the sequences*

$$x_n \bmod 1 \quad \textit{and} \quad (x_n + \log n) \bmod 1$$

*are simultaneously u.d.*

**Notes 13.** Theorem 92 was proved by G. Rauzy (1973), see [129, p. 2–27, 2.3.6.]. Other proof can be found in [124, p. 249–250, Exerc. 5.11] and furthermore it is noted that $\log n$ cannot be instead by $(\log n)^\delta$, $\delta > 1$ since $(\log n)^\delta \bmod 1$, $n = 1, 2, \dots$ is u.d.

In 2011 Y. Ohkubo [122] proved that in Theorem 92 the sequence $\log n$ can be replaced by $\log(n \log n)$.

**Theorem 93.** *An arbitrary u.d. sequence $x_n$ mod 1 and $\log(n \log n)$ mod 1 are statistically independent.*

*Proof.* G. Rauzy [135] proved that two sequences $x_n \bmod 1$ and $y_n \bmod 1$ are statistically independent if and only if

$$\lim_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N}e^{2\pi i(hx_n+ky_n)}-\left(\frac{1}{N}\sum_{n=1}^{N}e^{2\pi ihx_n}\right)\left(\frac{1}{N}\sum_{n=1}^{N}e^{2\pi iky_n}\right)\right)=0$$

for every integers $h$ and $k$. [29] Now, by Abel partial summation we obtain

$$\begin{aligned}&\sum_{n=1}^{N}e^{2\pi i(hx_n+k\log(n\log n))}\\&\sum_{n=1}^{N-1}\left(e^{2\pi ik\log(n\log n)}-e^{2\pi ik\log((n+1)\log(n+1))}\right)\sum_{j=1}^{n}e^{2\pi ihx_n}\\&\quad+e^{2\pi ik\log(N\log N)}\sum_{j=1}^{N}e^{2\pi ihx_n}\end{aligned}\tag{159}$$

and

$$\left|e^{2\pi ik\log(n\log n)}-e^{2\pi ik\log((n+1)\log(n+1))}\right|\le 2\pi|k|\frac{(\log n)+1}{n\log n}.\tag{160}$$

Thus

$$\begin{aligned}&\left|\frac{1}{N}\sum_{n=1}^{N}e^{2\pi i(hx_n+k\log(n\log n))}\right|\le\\&\frac{1}{N}\sum_{n=1}^{N-1}2\pi|k|\frac{(\log n)+1}{n\log n}n\left|\frac{1}{n}\sum_{j=1}^{n}e^{2\pi ihx_j}\right|+\left|\frac{1}{N}\sum_{j=1}^{N}e^{2\pi ihx_j}\right|\end{aligned}\tag{161}$$

which tends to 0. □

Using Theorem 93 Ohkubo [122] proved that in Theorem 91 the $\log n$ can be instead by $\log p_n$, $p_n$ is the increasing sequence of all primes.

**Theorem 94.** *Let $x_n\in[0,1)$, $n=1,2,\dots,$ be u.d. sequence. Then $x_n$ and $\log p_n \bmod 1$ are statistically independent.*

[29] P.J. Grabner and R.F. Tichy [65] characterize the statistical independence by $L^2$ discrepancy $\int_0^1\int_0^1\left(F_N(x,y)-F_N(x,1)F_N(1,y)\right)^2\mathrm{d}x\mathrm{d}y$ as
$\frac{1}{16\pi^4}\sum_{\substack{k,l=-\infty\\k,l\neq 0}}^{\infty}\frac{1}{k^2l^2}\left|\frac{1}{N}\sum_{n=1}^{N}e^{2\pi i(kx_n+ly_n)}-\frac{1}{N^2}\sum_{m,n=1}^{N}e^{2\pi i(kx_n+ly_m)}\right|^2$.

*Proof.* Firstly he proved that

$$\log p_n = \log(n \log n) + o\left(\frac{\log \log n}{\log n}\right) + O\left(\frac{1}{\log p_n}\right). \tag{162}$$

Then Ohkubo extend Theorem 10 to the Theorem 197 of two dimensional case. Then the limit

$$\lim_{n\to\infty} (\log p_n - \log(n \log n)) = 0,$$

given by (162), implies

$$G((x_n, \{\log p_n\})) = G(x_n, \{\log(n \log n)\}) = G((x_n, \{\log n\})). \tag{163}$$

Proof of (162). He starting with the prime number theorem of the form

$$\pi(x) = \frac{x}{\log x \ - 1} + O\left(\frac{x}{(\log x)^3}\right). \tag{164}$$

This implies

$$\frac{p_n}{n} = \log p_n \ - 1 + O\left(\frac{1}{\log p_n}\right).$$

Now we use (see [169])

$$\frac{p_n}{n} = \log n + (\log \log n \ - 1) + o\left(\frac{\log \log n}{\log n}\right)$$

which implies (162). □

Similarly, using

$$\lim_{n\to\infty} \left(\frac{p_n}{n} - \log(n \log n)\right) = -1.$$

Ohkubo (2011) [122] proved that

**Theorem 95.** *Let $p_n$, $n = 1, 2, \ldots,$ be the increasing sequence of all primes. An arbitrary u.d. sequence $x_n \bmod 1$ and $\frac{p_n}{n} \bmod 1$ are statistically independent. Thus, for every sequence $x_n$,*

$$x_n \bmod 1 \ \textit{and} \ \left(x_n + \frac{p_n}{n}\right) \bmod 1$$

*are simultaneously u.d.*

**Example 44.** Y. Ohkubo notes that Theorem 94 and 95 implies u.d. mod 1 of the sequences $p_n\theta + \log p_n$ and $p_n\theta + \frac{p_n}{n}$, $n = 1, 2, \dots$

**Example 45.** (a) The sequences $x_n$ and $y_n$ =constant are statisically independent.

(b) The sequences $(x_n, y_n)$, $(1 - x_n, y_n)$, $(x_n, 1 - y_n)$, $(y_n, x_n)$ are statistically independent, simultaneously.

(c) Assume that the sequence $x_n \in [0, 1)$ has a.d.f. $c_\alpha(x)$. Then an arbitrary sequence $y_n \in [0, 1)$ is statistically independent with $x_n$.

(d) If the sequences $x_n$, $x'_n$, $y_n$, $y'_n$ from [0,1) satisfy

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |x_n - x'_n| + |y_n - y'_n| = 0$$

then $(x_n, y_n)$ and $(x'_n, y'_n)$ are statistically independent simultaneously.

## 5.2 Simple operations with sequences

**Theorem 96.** *Let $x_n$, $n = 1, 2, \dots,$ be a sequence in $(0, 1]$. If every $g(x) \in G(x_n)$ is strictly increasing, then the $2^N$-terms blocks sequence with the block*

$$\sum_{n\in X} x_n \bmod 1, \quad X \subset \{1, 2, \dots, N\}$$

*is u.d., if $N \to \infty$.*

*Proof.* For integer $h > 0$ we starting with the Weyl sum

$$\left|\frac{1}{2^N} \sum_{X\subset\{1,2,\dots,N\}} e^{2\pi i h \sum_{n\in X} x_n}\right| = \left|\frac{1}{2^N} \prod_{n=1}^{N} \left(1 + e^{2\pi i x_n}\right)\right| = \prod_{n=1}^{N} |\cos \pi h x_n|,$$

because $\left|1+e^{2\pi i x_n}\right| = 2|\cos \pi h x_n|$. Now, if $\pi h x_n \in \left[\frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta\right]$ for $0 < \delta < \frac{\pi}{2}$, then $|\cos \pi h x_n| \le 1 - \varepsilon$, and if we put $I_h = \frac{1}{h}\left[\frac{1}{2} - \frac{\delta}{\pi}, \frac{1}{2} + \frac{\delta}{\pi}\right]$, then

$$\prod_{n=1}^{N} |\cos \pi h x_n| \le (1 - \varepsilon)^{A(I_h;N;x_n)}.$$

If for every $x \in (0, 1)$ there exists $g(x) \in G(x_n)$ such that $g(x)$ in $x$ increase, then $x_n$, $n = 1, 2, \dots$ is everywhere dense in $[0, 1]$. Then $A(I_h; N; x_n) \to \infty$ as $N \to \infty$, and the Weyl's sum tends to 0. □

Theorem 96 can be applied to $\log n \bmod 1$, since (see Example 1) all $g(x) \in G(\log n \bmod 1)$ strictly increases.

The following problem in AMM was proposed by A.M. Odlyzko [121] and then solved by D.G. Cantor [25].

**Theorem 97.** *Let the sequence $x_n$ in $[0,1)$ have at least one irrational limit point and $A_n$, $n = 1, 2, \ldots$, be the block of $2^n$ numbers*

$$A_n = (\varepsilon_1 x_1 + \varepsilon_2 x_2 + \cdots + \varepsilon_n x_n; \varepsilon_i = \pm 1) \bmod 1.$$

*Then the sequence of individual blocks $A_n$, $n = 1, 2, \ldots$, is u.d.*

*Proof.* A relevant Weyl sum has the form

$$\frac{1}{2^n} \sum_{\varepsilon_i = \pm 1, i=1,2,\ldots,n} e^{2\pi i h(\varepsilon_1 x_1 + \cdots + \varepsilon_n x_n)} = \prod_{i=1}^{n} \cos(2\pi h x_i).$$

Let $x_{k_i} \to \theta$ as $i \to \infty$ and let $\theta$ be irrational. Then $h x_{k_i} \to h\theta$ and $|\cos 2\pi h\theta| < 1$ for integer $h \neq 0$. This implies zero limit of the above Weyl sum. $\square$

**Theorem 98.** *Let the sequence $x_n$ from $(0,1)$ has a continuous a.d.f. $g(x)$. Then the sequence*

$$y_n = \frac{1}{x_n} \bmod 1$$

*has the a.d.f.*

$$\widetilde{g}(x) = \sum_{n=1}^{\infty} g\left(\frac{1}{n}\right) - g\left(\frac{1}{n+x}\right).$$

**Notes 14.** C.f. I.J. Schoenberg (1928) [145], E.K. Haviland (1941) [69], L. Kuipers (1957) [91], a proof can be found in [92, p. 56, Th. 7.6]. E. Hlawka (1961) [72], (1964) [73] considered the multi–dimensional case. G.Pólya (cf. I.J. Schoenberg (1928) [145]) proved that for $g(x) = x$ we have

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+x}\right) = \int_0^1 \frac{1-t^x}{1-t} \mathrm{d}t.$$

For history consult [92, p. 66, Notes].

Theorem 66 has the following reformulation:

**Theorem 99.** *Let $f : [0,1] \to [0,1]$ be a function such that, for all $x \in [0,1]$ the set $f^{-1}([0,x))$ can be expressed as a union of finitely many pairwise disjoint subintervals $I_i(x) \subseteq [0,1]$. Let $x_n \bmod 1$ be such that each its term appears only finitely many times in it. Then*

$$G(f(x_n \bmod 1)) = \{g_f; g \in G(x_n \bmod 1)\}.$$

**Theorem 100.** *If $x_n \bmod 1$ and $f(x_n \bmod 1)$ have the same a.d.f. $g(x)$, then*

$$g(x) = g_f(x) \text{ for } x \in [0,1].$$

**Theorem 101** (S.H. Molnár [107])**.** *Let $f$ be a real function and let $g_1$ and $g_2$ be two d.f.s. For any $g_1$ and $g_2$ there exists a sequence $x_n$, $n = 1,2,\dots$, such that the a.d.f. of $x_n \bmod 1$ is $g_1$ and the a.d.f. of $f(x_n) \bmod 1$ is $g_2$ if and only if the graph of $f \bmod 1$ is everywhere dense in the unit square $[0,1]^2$.*

**Theorem 102** (I.J. Schoenberg [145])**.** See [92, p. 68, Ex. 7.19]: *If the sequence $x_n \bmod 1$ has a continuous a.d.f. $g(x)$ then the sequence*

$$g(\{x_n\})$$

*is u.d.*

*Proof.* Assume $x_n \in [0,1)$. In the simple case if $\frac{\#\{n \le N; x_n < x\}}{N} \to g(x)$ and $g(x)$ increases, using $g(x_n) < x \Leftrightarrow x_n < g^{-1}(x)$ we find $\frac{\#\{n \le N; g(x_n) < x\}}{N} \to g(g^{-1}(x)) = x$.

In the general case, for non-increasing $g(x)$, we can using Weyl's criterion that the sequence $g(x_n)$ is u.d. if and only if

$$\frac{1}{N}\sum_{n=1}^{N}(g(x_n))^k \to \frac{1}{k+1}$$

and by Helly theorem

$$\frac{1}{N}\sum_{n=1}^{N}(g(x_n))^k \to \int_0^1 (g(x))^k \mathrm{d}g(x).$$

Now, integration by parts $\int_0^1 g(x)\mathrm{d}g(x) = [g(x)g(x)]_0^1 - \int_0^1 g(x)\mathrm{d}g(x)$ which implies $\int_0^1 g(x)\mathrm{d}g(x) = \frac{1}{2}$. Also

$\int_0^1 (g(x))^2\mathrm{d}g(x) = [(g(x))^2 g(x)]_0^1 - 2\int_0^1 g(x)g(x)\mathrm{d}g(x)$
which implies $\int_0^1 (g(x))^2\mathrm{d}g(x) = 1/3$, etc. □

**Theorem 103.** *If u.d. of $x_n$ is proved by Fejér's theorem* [171, 2.2.10, 2.6.1] (*see also Theorem 33 in this book*) *then for every $\alpha > 0$ the sequence $\alpha x_n$* mod 1 *is u.d. again.*

**Theorem 104** ([156])**.** *The sequence*

$$x_n \in [0,1), \quad n = 1, 2, \dots,$$

*is u.d. if and only if the double sequence*

$$|x_m - x_n|, \quad m, n = 1, 2, \dots,$$

*has the a.d.f. $g(x) = 2x - x^2$.*

Here the double sequence $|x_m - x_n|$, for $m, n = 1, 2, \dots$, is ordered to an ordinary sequence $y_n$ in such a way that the first $N^2$ terms of $y_n$ are $|x_m - x_n|$ for $m, n = 1, 2, \dots, N$.

*Proof.* For every continuous $f : [0,1] \to \mathbb{R}$ we have

$$\int_0^1 \int_0^1 f(|x-y|)\mathrm{d}x\mathrm{d}y = \int_0^1 f(x)\mathrm{d}g(x) = \int_0^1 f(x)(2-2x)\mathrm{d}x.$$

The argument is that

$$\int_0^1 \int_0^1 |x-y|^k \mathrm{d}x\mathrm{d}y = \frac{2}{(k+1)(k+2)} = \int_0^1 x^k(2-2x)\mathrm{d}x$$

for all $k = 0, 1, 2, \dots$. Then we use Part (I) of Theorem 21 because for u.d. sequence $x_n$, $n = 1, 2, \dots$ we have

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{m,n=1}^{N} f(|x_m - x_n|) = \int_0^1 \int_0^1 f(|x-y|)\mathrm{d}x\mathrm{d}y.$$

□

For symmetric and continuous $F(x,y)$ on $[0,1]^2$ by (723) we have proved for d.f.s $F_N(x)$ and $g(x)$ that

$$\frac{1}{N^2} \sum_{m,n=1}^{N} F(x_m, x_n) - \int_0^1 \int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y) =$$

$$\int_0^1\int_0^1 F(x,y)\mathrm{d}F_N(x)\mathrm{d}F_N(y)-\int_0^1\int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y)=$$

$$=-2\int_0^1(F_N(x)-g(x))\mathrm{d}_xF(x,1)+\int_0^1\int_0^1(F_N(x)-g(x))(F_N(y)+g(y))\mathrm{d}_y\mathrm{d}_xF(x,y).$$

From it

$$\left|\frac{1}{N^2}\sum_{m,n=1}^N F(x_m,x_n)-\int_0^1\int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y)\right|\le\sqrt{\int_0^1(F_N(x)-g(x))^2\mathrm{d}x}$$

$$\times\left(2\sqrt{\int_0^1(F_x'(x,1))^2\mathrm{d}x}+\sqrt{\int_0^1\int_0^1(F_N(y)+g(y))^2(F_{xy}''(x,y))^2\mathrm{d}x\mathrm{d}y}\right).$$

Specially for $F(x,y)=|x-y|$ we find

$$\left|\frac{1}{N^2}\sum_{m,n=1}^N|x_m-x_n|-\int_0^1\int_0^1|x-y|\mathrm{d}g(x)\mathrm{d}g(y)\right|\le$$

$$2\sqrt{\int_0^1(F_N(x)-g(x))^2\mathrm{d}x}\left(1+\sqrt{\int_0^1(1+g(x))^2\mathrm{d}x}\right).$$

## 5.3 Convolution (an information)

Let $g_1:[0,1]\to[0,1]$ and $g_2:[0,1]\to[0,1]$ be two d.f.s and extend they to d.f.s defined in $g_1:(-\infty,\infty)\to[0,1]$ and $g_2:(-\infty,\infty)\to[0,1]$. By statistic d.f. $g(x)$ is a convolution $g(x)=g_1(x)*g_2(x)$ if

$$g(x)=\int_{-\infty}^{\infty}g_1(x-y)\mathrm{d}g_2(y).\tag{165}$$

Since $\mathrm{d}g_2(y)=0$ if $y\notin[0,1]$, then we have

$$g(x)=\int_0^1 g_1(x-y)\mathrm{d}g_2(y).$$

Convolution $g(x)$ is different other than 0 and 1 only for $0<x<2$. Thus

$$g(x)=\begin{cases}\int_0^x g_1(x-y)\mathrm{d}g_2(y) & \text{if } 0\le x\le 1,\\ \int_{x-1}^1 g_1(x-y)\mathrm{d}g_2(y)+g_2(x-1) & \text{if } 1\le x\le 2\end{cases}\tag{166}$$

since in the second part, for $1 \le x \le 2$ we have $\int_0^1 g_1(x-y)\mathrm{d}g_2(y) = \int_0^{x-1} 1.\mathrm{d}g_2(y) + \int_{x-1}^1 g_1(x-y)\mathrm{d}g_2(y)$.

Now, let $x_n^{(1)}$ and $x_n^{(2)}$ be two sequences in $[0,1)$ and $F_N^{(1)}(x)$ and $F_N^{(2)}(x)$ by two relevant step d.f. defined in (1), respectively. Using (166) we have:

If $0 \le x \le 1$, then

$$F_N^{(1)}(x) * F_N^{(2)}(x) = \int_0^x F_N^{(1)}(x-y)\mathrm{d}F_N^{(2)}(y) = \sum_{\substack{n=1,\\ 0 \le x_n^{(2)} < x}}^N F_N^{(1)}(x - x_n^{(2)})\frac{1}{N}$$

$$= \sum_{\substack{n=1,\\ 0 \le x_n^{(2)} < x}}^N \frac{\#\{m \le N; x_m^{(1)} < x - x_n^{(2)}\}}{N^2} = \frac{\#\{(m,n); m,n \le N, x_m^{(1)} + x_n^{(2)} < x\}}{N^2}.$$

If $1 \le x \le 2$, then

$$F_N^{(1)}(x) * F_N^{(2)}(x) = F_N^{(2)}(x-1) + \int_{x-1}^1 F_N^{(1)}(x-y)\mathrm{d}F_N^{(2)}(y)$$

$$= \frac{\#\{n \le N; x_n^{(2)} < x-1\}}{N} + \sum_{\substack{n=1,\\ x-1 \le x_n^{(2)} < 1}}^N F_N^{(1)}(x - x_n^{(2)})\frac{1}{N}$$

$$= \frac{\#\{(m,n); m,n \le N; x_n^{(2)} < x-1\}}{N^2} + \sum_{\substack{n=1,\\ x-1 \le x_n^{(2)} < 1}}^N \frac{\#\{m \le N; x_m^{(1)} + x_n^{(2)} < x\}}{N^2}$$

$$= \frac{\#\{(m,n); m,n \le N, x_m^{(1)} + x_n^{(2)} < x\}}{N^2}.$$

Then we have

$$F_N^{(1)}(x) * F_N^{(2)}(x) = \frac{\#\{(m,n); m,n \le N, x_m^{(1)} + x_n^{(2)} < x\}}{N^2} (= F_{N^2}(x)) \quad (167)$$

for every $0 \le x \le 2$.

Thus the d.f. $g(x)$ of $x_m^{(1)} + x_n^{(2)} \in [0,2)$ ordered by increasing $m+n$ is $g(x) = g_1(x) * g_2(x)$.

## 5.4 Characteristic function (an information)

Let $g(x)$ be a d.f. The characteristic function $\widehat{g(x)}$ of $g(x)$ is defined as

$$\widehat{g(t)} = \int_{-\infty}^{\infty} e^{itx} \mathrm{d}g(x) \tag{168}$$

for $t \in (-\infty, \infty)$. We have

$$g_1(\widehat{x) * g_2}(x) = \widehat{g_1(x)}.\widehat{g_2(x)}. \tag{169}$$

**Example 46.** For an arbitrary sequence $x_n$ in $[0, 1)$ with step d.f. $F_N(x)$ we have

$$\widehat{F_N(t)} = \int_0^1 e^{itx} \mathrm{d}F_N(x) = \frac{1}{N} \sum_{n=1}^{N} e^{itx_n}.$$

According to (167) we have

$$\widehat{F_{N^2}(t)} = \int_0^2 e^{itx} \mathrm{d}F_{N^2}(x) = \frac{1}{N^2} \sum_{m,n=1}^{N} e^{it(x_m^{(1)} + x_n^{(2)})}$$

$$= \widehat{F_N^{(1)}(t)}.\widehat{F_N^{(2)}(t)} = \frac{1}{N} \sum_{n=1}^{N} e^{itx_n^{(1)}}.\frac{1}{N} \sum_{n=1}^{N} e^{itx_n^{(2)}}.$$

**Theorem 105** (Continuity theorem, Lévy 1925)**.** *Let $g_n(x)$ be a sequence of d.f.s and $\phi_n(t)$ the corresponding sequence of their characteristic functions. Then $g_n(x) \to g(x)$ weakly if and only if $\phi_n(t) \to \phi(t)$ pointwise on $\mathbb{R}$, $\phi(t)$ is continuous at* 0*. In addition, in this case, $\phi(t)$ is the characteristic function of $g(x)$ and the convergence $\phi_n(t) \to \phi(t)$ is uniformly on any compact subset.*

C.f. Tenenbaum [177, p. 285, Th. 3].

(I) Let $X$ be Gauss normal, then the characteristic function $\varphi$ of $X$ is $\varphi(t) = e^{-t^2/2}$.

(II) If the random variable $X$ has characteristic function $\varphi(t)$ then $X + a$ has $\psi(t) = e^{iat}\varphi(t)$.

(III) If random variables $X$ and $Y$ are independent with characteristic functions $\varphi_1$ and $\varphi_2$, respectively, the $X + Y$ has $\varphi(t) = \varphi_1(t)\varphi_2(t)$.

## 5.5 Random variables (Some results)

(I) If $X$ and $Y$ are independent random variables and $f, g : \mathbb{R} \to \mathbb{R}$ are arbitrary functions, then $f(X), g(Y)$ are independent also.
(II) Let $X$ and $Y$ be independent random variables with densities $f_X(x)$, $f_Y(y)$. Then the random variable $W = X + Y$ has density $f_W(z)$ of the form of convolution

$$f_W(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) \mathrm{d}x. \tag{170}$$

# 6 Special sequences (extension of Section 3)

## 6.1 D.f.s of $\xi(3/2)^n \bmod 1$ (continuation 3.8)

In this part we study the set of all d.f.s of sequences $\xi(3/2)^n \bmod 1$, $\xi \in \mathbb{R}$. Most results are from [162]. (For open problems see Section 3.8.)

**Definition 7.** Let $f : [0,1] \to [0,1]$ be a function such that, for all $x \in [0,1]$, $f^{-1}([0,x))$ can be expressed as a sum of finitely many pairwise disjoint subintervals $I_i(x)$ of $[0,1]$ with endpoints $\alpha_i(x) \le \beta_i(x)$. For any distribution function $g(x)$ we put

$$g_f(x) = \sum_i g(\beta_i(x)) - g(\alpha_i(x)). \tag{171}$$

The mapping $g \to g_f$ is the main tool of this part. A basic property is expressed by the following statement (previously written in Theorem 66):

**Theorem 106.** *Let $x_n \bmod 1$ be a sequence having $g(x)$ as a d.f. associated with the sequence of indices $N_1, N_2, \dots$. Suppose that any term $x_n \bmod 1$ is repeated only finitely many times. Then the sequence $f(\{x_n\})$ has the d.f. function $g_f(x)$ for the same $N_1, N_2, \dots$, and vice-versa any d.f. of $f(\{x_n\})$ has this form.*

*Proof.* The form $g_f(x)$ is a consequence of

$$A([0,x); N_k; f(\{x_n\})) = \sum_i A(I_i(x); N_k; x_n) \text{ and}$$

$$A(I_i(x); N_k; x_n) = A([0,\beta_i(x)); N_k; x_n) - A([0,\alpha_i(x)); N_k; x_n) + o(N_k).$$

On the other hand, suppose that $\tilde{g}(x)$ is a d.f. of $f(\{x_n\})$ associated with $N_1, N_2, \dots$. The Helly selection principle guarantee a suitable subsequence $N_{n_1}, N_{n_2}, \dots$ for which some $g(x)$ is a distribution function of $x_n$ mod 1. Thus $\tilde{g}(x) = g_f(x)$. □

### 6.1.1 Functional equation $g_f = g_h$

Let $x_n = \xi(3/2)^n \bmod 1$, $n = 1, 2, \dots$, and denote $f(x) = 2x \bmod 1$ and $h(x) = 3x \bmod 1$. In this case, for every $x \in [0,1]$, we have

$$g_f(x) = g(f_1^{-1}(x)) + g(f_2^{-1}(x)) - g(1/2),$$

$$g_h(x) = g(h_1^{-1}(x)) + g(h_2^{-1}(x)) + g(h_3^{-1}(x)) - g(1/3) - g(2/3),$$

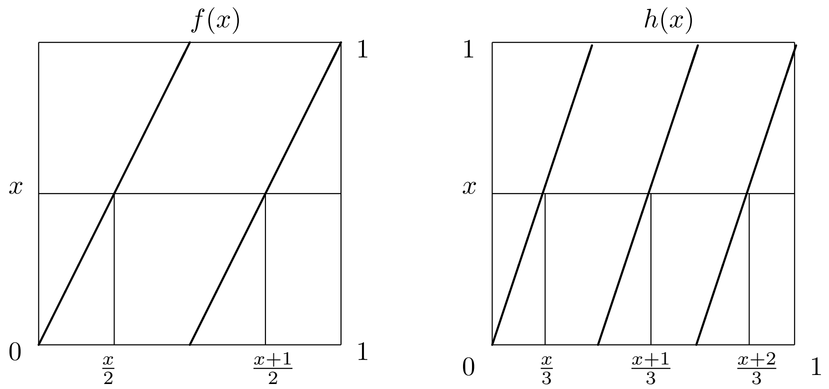
with inverse functions

$$f_1^{-1}(x) = x/2, \quad f_2^{-1}(x) = (x+1)/2,$$

$$h_1^{-1}(x) = x/3, \quad h_2^{-1}(x) = (x+1)/3, \quad h_3^{-1}(x) = (x+2)/3.$$

**Theorem 107.** *Any d.f. $g(x)$ of $\xi(3/2)^n$ mod 1 satisfies $g_f(x) = g_h(x)$ for all $x \in [0,1]$.*

*Proof.* Using $\{q\{x\}\} = \{qx\}$ for any integer $q$, then we have $\{2\{\xi(3/2)^n\}\} = \{3\{\xi(3/2)^{n-1}\}\}$. Therefore $f(\{\xi(3/2)^n\})$ and $h(\{\xi(3/2)^{n-1}\})$ formed the same sequence and the rest follows from Theorem 106. See Fig.




□

### 6.1.2 Intervals of uniqueness for $f(x) = 2x \bmod 1$ and $h(x) = 3x \bmod 1$

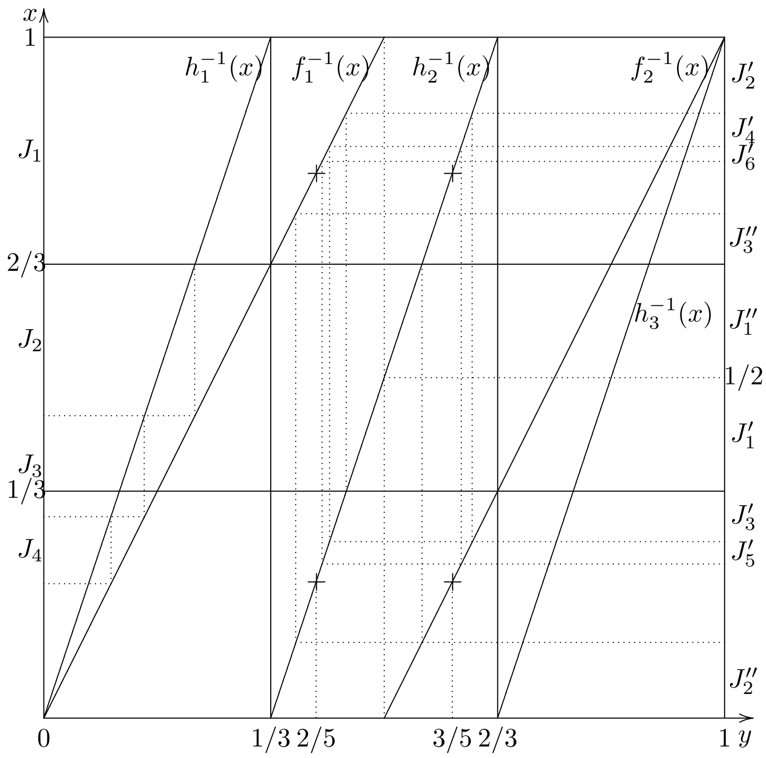
The above theorem yields to the following sets of uniqueness for d.f.s of $\xi(3/2)^n \bmod 1$. Here
• $X \subset [0,1]$ is said to be set of uniqueness for $g_f = g_h$ if for any two $g_1, g_2$ solutions of $g_f = g_h$ the equality $g_1(x) = g_2(x)$ for $x \in X$ implies $g_1(x) = g_2(x)$ for every $x \in [0,1]$.

**Theorem 108.** *Denote $f(x) = 2x \bmod 1$ and $h(x) = 3x \bmod 1$ an let $g_1, g_2$ be any two d.f.s satisfying $g_{i_f}(x) = g_{i_h}(x)$ for $i = 1,2$ and $x \in [0,1]$. Denote*

$$I_1 = [0, 1/3], \quad I_2 = [1/3, 2/3], \quad I_3 = [2/3, 1].$$

*If $g_1(x) = g_2(x)$ for $x \in I_i \cup I_j$, $1 \leq i \neq j \leq 3$, then $g_1(x) = g_2(x)$ for all $x \in [0,1]$.*

*Proof.* Assume that a distribution function $g$ satisfies $g_f = g_h$ on $[0,1]$ and let $J_i, J'_j, J''_k$ be the intervals from $[0,1]$ described in the following Fig.




There are three cases of $I_i \cup I_j$.

$1^\circ$. Consider first the case $I_2 \cup I_3$. Using the values of $g$ on $I_2 \cup I_3$, and the equation $g_f = g_h$ on $J_1$, we can compute $g(h_1^{-1}(x))$ for $x \in J_1$. Mapped $x \in J_1$ to $x' \in J_2$ by using $h_1^{-1}(x) = f_1^{-1}(x')$, we find $g(f_1^{-1}(x))$ for $x \in J_2$. Then, by the equation $g_f = g_h$ on $J_2$ we can compute $g(h_1^{-1}(x))$ for $x \in J_2$, from it we have $g(f_1^{-1}(x))$ for $x \in J_3$, etc. Thus we have $g(x)$ for $x \in I_1$.

$2^\circ$. Similarly for the case $I_1 \cup I_2$.

$3^\circ$. In the case $I_1 \cup I_3$, firstly we compute $g(1/2)$ by using $g_f(1/2) = g_h(1/2)$ and then, an infinite process of computation of $g(x)$ for $x \in I_2$, we shall divide into two parts:

In the first part, using $g(y)$, for $y \in I_1 \cup I_3$, and $g_f = g_h$ on $[0,1]$, we compute $g(h_2^{-1}(x))$ for $x \in J_1'$. Mapped $x \in J_1' \to x' \in J_2'$ by $h_2^{-1}(x) = f_1^{-1}(x')$ and employed $g_f = g_h$ we have find $g(h_2^{-1}(x)$ for $x \in J_2'$. In the same way this leads to $g(f_2^{-1})$ on $J_3'$, $g(h_2^{-1})$ on $J_3'$, $g(f_1^{-1})$ on $J_4'$, $g(h_2^{-1})$ on $J_4'$, and so on.

Similarly, in the second part, from $g$ on $I_1 \cup I_3$ and $g_f = g_h$ on $[0,1]$ we find $g(h^{-1})$ on $J_1''$, $g(f_2^{-1})$ on $J_2''$, $g(h_2^{-1})$ on $J_2''$, $g(f_1^{-1})$ on $J_3''$, $g(h_2^{-1})$ on $J_3''$, etc.

In both parts these infinite processes not covered the values $g(2/5)$ and $g(3/5)$. The rest follows from the equations $g_f(1/5) = g_h(1/5)$ and $g_f(4/5) = g_h(4/5)$. $\square$

Directly from the above Fig. we have

**Theorem 109.** *The set* $X = [2/9, 1/3] \cup [1/2, 1]$ *and the set* $X^* = [0, 1/2] \cup [2/3, 7/9]$ *are also sets of uniqueness.*

*Proof.* Using the values of $g(x)$ on $x \in X$, and the equation $g_f = g_h$ on $J_1$, we can compute $g(f_1^{-1}(x))$ for $x \in J_1$. This gives $g(x)$ on $[1/3, 1]$ which is the set of uniqueness. Symmetrizing $X$ we find the set $X^*$ of uniqueness. $\square$

### 6.1.3 Intervals of uniqueness for $f(x) = 4x(1-x) \bmod 1$ and $h(x) = x(4x-3)^2 \bmod 1$

Now, let

$$f(x) = 4x(1-x), \quad h(x) = x(4x-3)^2 \text{ for } x \in [0,1].$$

In this case

$$g_f(x) = g(f_1^{-1}(x)) + 1 - g(f_2^{-1}(x)),$$

$$g_h(x) = g(h_1^{-1}(x)) + g(h_3^{-1}(x)) - g(h_2^{-1}(x)),$$

where

$$f_1^{-1}(x) = \frac{1-\sqrt{1-x}}{2}, \quad f_2^{-1}(x) = 1 - f_1^{-1}(x),$$

and for $h_j^{-1}$ we known only

$$h_{i,j}^{-1}(x) = \frac{1}{2}\left(\left(\frac{3}{2} - h_k^{-1}(x)\right) \pm \sqrt{\left(\frac{3}{2} - h_k^{-1}(x)\right)^2 - \frac{x}{4h_k^{-1}(x)}}\right), \tag{172}$$

for every $x \in [0,1]$ and $1 \le i \ne j \ne k \le 3$. In this case the equation $g_f = g_h$ has solutions

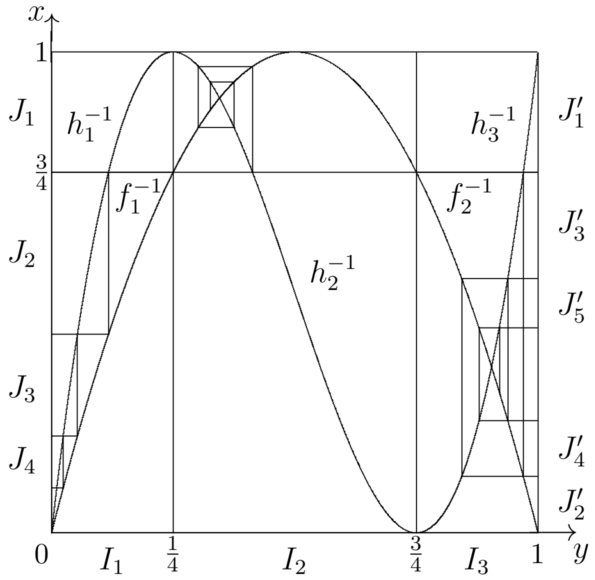
$$c_0(x), c_{\frac{5-\sqrt{5}}{8}}(x), c_{\frac{5+\sqrt{5}}{8}}(x), g_0(x) := \frac{2}{\pi}\arcsin\sqrt{x}.$$

**Theorem 110.** *Let $f(x) = 4x(1-x)$, $h(x) = x(4x-3)^2$ and*

$$I_1 = [0, 1/4], \quad I_2 = [1/4, 3/4], \quad I_3 = [3/4, 1].$$

*If $g_1, g_2$ solve $g_f = g_h$ on whole interval $[0,1]$ and $g_1(x) = g_2(x)$ for $x \in I_i \cup I_j$, $1 \le i \ne j \le 3$, then $g_1(x) = g_2(x)$ for every $x \in [0,1]$.*

*Proof.* It follows directly from the following Fig.




□

### 6.1.4 Equation $\int_0^1\int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y)=0$ characterizes $g_f=g_h$

Next we return to Theorem 107. We derive an integral formula for testing $g_f=g_h$. Denote [30]

$$F(x,y)=|\{2x\}-\{3y\}|+|\{2y\}-\{3x\}|-|\{2x\}-\{2y\}|-|\{3x\}-\{3y\}|. \quad (173)$$

**Theorem 111.** *The continuous d.f. $g(x)$ satisfies $g_f=g_h$ on $[0,1]$ if and only if*

$$\int_0^1\int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y)=0. \quad (174)$$

*Proof.* Let $x_n$, $n=1,2,\dots$ be an auxiliary sequence in $[0,1]$ such that all $(x_m,x_n)$ are points of continuity of $F(x,y)$, and let $c_X(x)$ be the characteristic function of the set $X$. Applying $c_{[0,x)}(x_n)=c_{(x_n,1]}(x)$, we can compute

$$\int_0^1\left(\frac{1}{N}\sum_{n=1}^N c_{f^{-1}([0,x))}(x_n)-\frac{1}{N}\sum_{n=1}^N c_{h^{-1}([0,x))}(x_n)\right)^2\mathrm{d}x=\frac{1}{N^2}\sum_{m,n=1}^N F_{f,h}(x_m,x_n),$$

where

$$F_{f,h}(x,y)=\max(f(x),h(y))+\max(f(y),h(x))-\max(f(x),f(y))$$

[30] Here again $f(x)=2x \bmod 1$ and $h(x)=3x \bmod 1$.

$$- \max(h(x), h(y)) =$$
$$= \frac{1}{2}\big(|f(x) - h(y)| + |f(y) - h(x)| - |f(x) - f(y)| - |h(x) - h(y)|\big). \tag{175}$$

Applying the Helly lemma we have (cf. Example 102)

$$\int_0^1 (g_f(x) - g_h(x))^2 \mathrm{d}x = \int_0^1 \int_0^1 F_{f,h}(x, y) \mathrm{d}g(x) \mathrm{d}g(y) \tag{176}$$

for any continuous d.f. $g$. Here $2F_{f,h}(x, y) = F(x, y)$ in (173). $\square$

*Second proof.* Helly theorem gives

$$\frac{1}{N^2} \sum_{m,n=1}^{N} F(x_m, x_n) \to \int_0^1 \int_0^1 F(x, y) \mathrm{d}g(x) \mathrm{d}g(y)$$

$$\frac{1}{N^2} \sum_{m,n=1}^{N} F(f(x_m), h(x_n)) \to \begin{cases} \int_0^1 \int_0^1 F(f(x), h(y)) \mathrm{d}g(x) \mathrm{d}g(y) \\ \int_0^1 \int_0^1 F(x, y) \mathrm{d}g_f(x) \mathrm{d}g_h(y). \end{cases} \tag{177}$$

Then

$$\int_0^1 \int_0^1 |\{2y\} - \{3x\}| \mathrm{d}g(x) \mathrm{d}g(y) = \int_0^1 \int_0^1 |x - y| \mathrm{d}g_f(x) \mathrm{d}g_h(y),$$
$$\int_0^1 \int_0^1 |\{2x\} - \{3y\}| \mathrm{d}g(x) \mathrm{d}g(y) = \int_0^1 \int_0^1 |x - y| \mathrm{d}g_h(x) \mathrm{d}g_f(y),$$
$$- \int_0^1 \int_0^1 |\{2x\} - \{2y\}| \mathrm{d}g(x) \mathrm{d}g(y) = - \int_0^1 \int_0^1 |x - y| \mathrm{d}g_f(x) \mathrm{d}g_f(y),$$
$$- \int_0^1 \int_0^1 |\{3x\} - \{3y\}| \mathrm{d}g(x) \mathrm{d}g(y) = - \int_0^1 \int_0^1 |x - y| \mathrm{d}g_h(x) \mathrm{d}g_h(y)$$

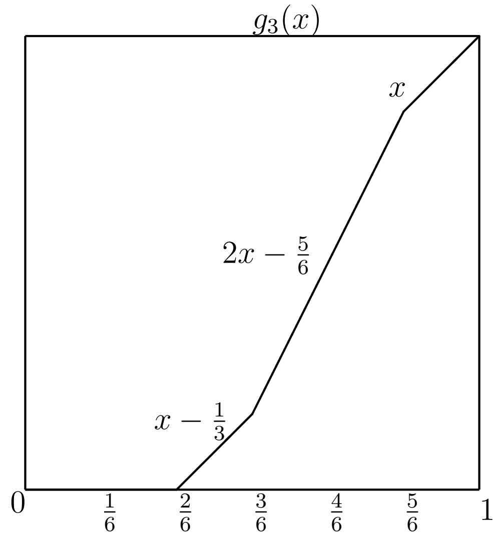
and by integrals in Section 13

$$\int_0^1 \int_0^1 |x - y| \mathrm{d}g_f(x) \mathrm{d}g_h(y) = \int_0^1 g_f(x) \mathrm{d}x + \int_0^1 g_h(x) \mathrm{d}x - 2 \int_0^1 g_f(x) g_h(x) \mathrm{d}x$$
$$\int_0^1 \int_0^1 |x - y| \mathrm{d}g_h(x) \mathrm{d}g_f(y) = \int_0^1 g_h(x) \mathrm{d}x + \int_0^1 g_f(x) \mathrm{d}x - 2 \int_0^1 g_h(x) g_f(x) \mathrm{d}x$$
$$- \int_0^1 \int_0^1 |x - y| \mathrm{d}g_f(x) \mathrm{d}g_f(y) = -2\left( \int_0^1 g_f(x) \mathrm{d}x - \int_0^1 g_f^2(x) \mathrm{d}x \right)$$

$$-\int_0^1\int_0^1 |x-y|\mathrm{d}g_h(x)\mathrm{d}g_h(y) = -2\left(\int_0^1 g_h(x)\mathrm{d}x - \int_0^1 g_h^2(x)\mathrm{d}x\right)$$

which gives (176). $\square$

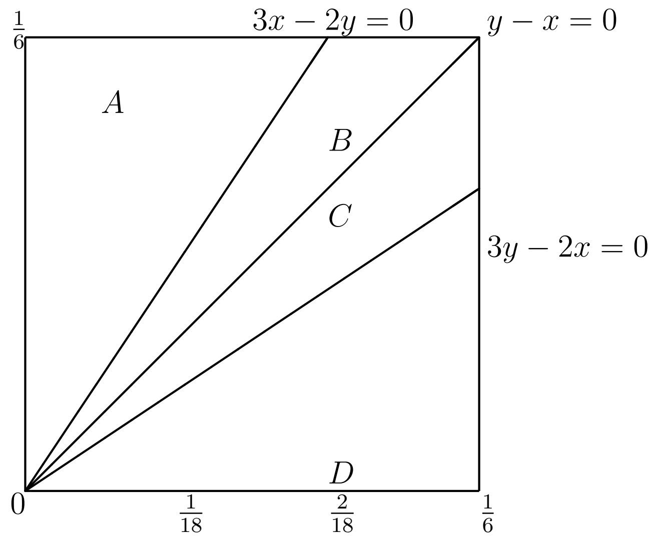
**Example 47.** We verify Theorem 174 for $g_3(x)$ which solve $g_f = g_h$ by Example 49.




Firstly we compute

$$\int_{\frac{j}{6}}^{\frac{j+1}{6}}\left(\int_{\frac{i}{6}}^{\frac{i+1}{6}} F(x,y)\mathrm{d}x\right)\mathrm{d}y, \quad i,j = 0,1,\dots,5 \tag{178}$$

for $F(x,y)$ defined in (173). For example we compute
$\int_0^{\frac{1}{6}}\int_0^{\frac{1}{6}} F(x,y)\mathrm{d}x\mathrm{d}y$, where in this case
$F(x,y) = |2x-3y| + |3x-2y| - |2x-2y| - |3x-3y|$.
By the following Fig. we have $F(x,y) = A, B, C, D$




where

$A = 3y - 2x + 2y - 3x - 5(y - x) = 0,$
$B = 3y - 2x + 3x - 2y - 5(y - x) = 6x - 4y,$
$C = 3y - 2x + 3x - 2y - 5(x - y) = -4x + 6y,$
$D = 2x - 3y + 3x - 2y - 5(x - y) = 0.$

From it

$$\int_0^{\frac{1}{6}} \int_0^{\frac{1}{6}} F(x,y)\mathrm{d}x\mathrm{d}y = \frac{1}{972}.$$

By program "DERIVE" all integrals in (178) are (indexed from the bottom)

$$\begin{pmatrix} 0 & -\frac{1}{432} & \frac{7}{1944} & -\frac{5}{1944} & \frac{1}{3888} & \frac{1}{972} \\ -\frac{1}{432} & -\frac{5}{1944} & \frac{37}{3888} & -\frac{5}{432} & \frac{13}{1944} & \frac{1}{3888} \\ \frac{7}{1944} & \frac{37}{3888} & -\frac{23}{972} & \frac{2}{81} & -\frac{5}{432} & -\frac{5}{1944} \\ -\frac{5}{1944} & -\frac{5}{432} & \frac{2}{81} & -\frac{23}{972} & \frac{37}{3888} & \frac{7}{1944} \\ \frac{1}{3888} & \frac{13}{1944} & -\frac{5}{432} & \frac{37}{3888} & -\frac{5}{1944} & -\frac{1}{432} \\ -\frac{1}{972} & \frac{1}{3888} & -\frac{5}{1944} & \frac{7}{1944} & -\frac{1}{432} & 0 \end{pmatrix} \tag{179}$$

Then we see that sums over lines of matrix (179) are 0, i.e.

$$\int_0^1 \int_0^1 F(x,y)\mathrm{d}x\mathrm{d}y = 0$$

which is equivalent that $x_f = x_g$. Furthermore for every $j = 0,1,2,3,4,5$ we have (cf. (744))

$$\int_{\frac{j}{6}}^{\frac{j+1}{6}} \left( \int_0^1 F(x,y)\mathrm{d}x \right) \mathrm{d}y = 0. \tag{180}$$

New, we shall compute

$$a_{i,j} = \frac{\mathrm{d}}{\mathrm{d}x} g_3(x) \frac{\mathrm{d}}{\mathrm{d}y} g_3(y) \text{ for } x \in \left(\frac{i}{6}, \frac{i+1}{6}\right), y \in \left(\frac{j}{6}, \frac{j+1}{6}\right),$$

where $i,j = 0,1,2,3,4,5$. This $a_{i,j}$ form the matrix (indexed from the bottom)

$$\begin{pmatrix} 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 2 & 4 & 4 & 2 \\ 0 & 0 & 2 & 4 & 4 & 2 \\ 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{181}$$

Combining (179) and (181) we find

$$\begin{aligned} \int_0^1 \int_0^1 F(x,y)\mathrm{d}g_3(x)\mathrm{d}g_3(y) = & \sum_{i,j=0}^{5} a_{i,j} \int_{\frac{j}{6}}^{\frac{j+1}{6}} \left( \int_{\frac{i}{6}}^{\frac{i+1}{6}} F(x,y)\mathrm{d}x \right) \mathrm{d}y \\ = & 1.\frac{2}{81} + 2.\frac{-23}{972} + 2.\frac{37}{3888} + 1.\frac{7}{1944} \\ & + 2.\frac{-23}{972} + 4.\frac{2}{81} + 4.\frac{-5}{432} + 2.\frac{-5}{1944} \\ & + 2.\frac{37}{3888} + 4.\frac{-5}{432} + 4.\frac{13}{1944} + 2.\frac{1}{3888} \\ & + 1.\frac{7}{1944} + 2.\frac{-5}{1944} + 2.\frac{1}{3888} + 1.\frac{1}{972} \\ = & 0. \end{aligned}$$

Again as in (180)

$$\int_{\frac{j}{6}}^{\frac{j+1}{6}} \left( \int_0^1 F(x,y)\mathrm{d}g_3(x) \right) \mathrm{d}g_3(y) = 0 \tag{182}$$

for $j = 0,1,2,3,4,5$.

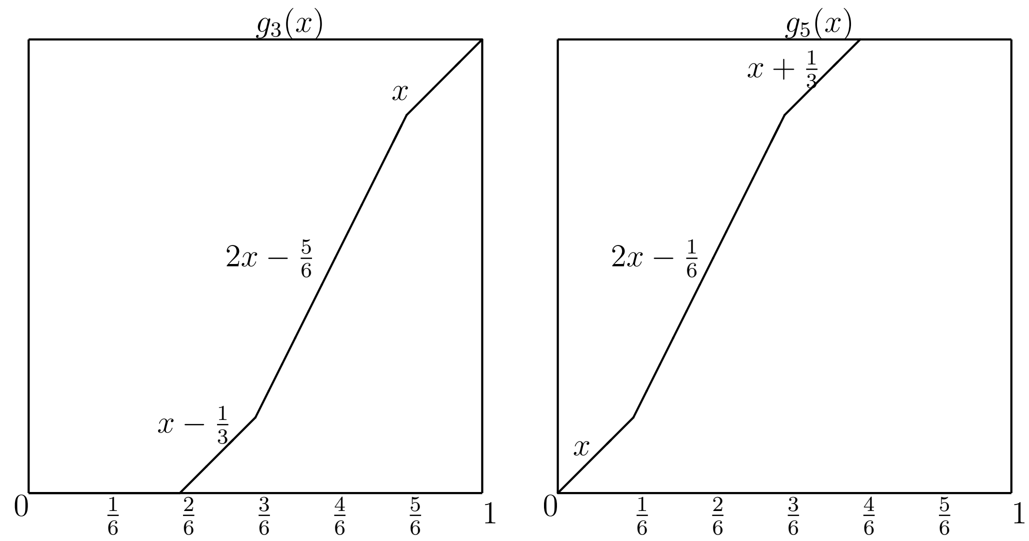
**Notes 15.** [164, Prop. 15]: If d.f. $g(x)$ solve $\int_0^1 \int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y) = 0$, then we have $0 = \int_0^1 F(x,y)\mathrm{d}g(y)$ for every $x \in [0,1]$. Here we assume $F(x,y)$ is a copositive and $g(x)$ is strictly increasing. See also Theorem 247 in this book.

### 6.1.5 $g(x)$ and $1-g(1-x)$ satisfies $g_f = g_h$ simultaneously

We present two proof, second one by using the integral (174).

**Theorem 112.** *The d.f. $g(x)$ and $1-g(1-x)$ satisfies $g_f = g_h$ simultaneously for all $x \in [0,1]$.*

By the following Fig. we see that $g_5(x) = 1 - g_3(1-x)$. Their solution of $g_f = g_h$ are given in Examples 49 and 55.




*First proof.* Using transformation $x = 1 - y$ then we have

1. $1 - \frac{x}{2} = \frac{y+1}{2}$,
2. $1 - \frac{x+1}{2} = \frac{y}{2}$,
3. $1 - \frac{x}{3} = \frac{y+2}{3}$,
4. $1 - \frac{x+1}{3} = \frac{y+1}{3}$,
5. $1 - \frac{x+2}{3} = \frac{y}{3}$.

Put $g_1(x) = 1 - g(1-x)$. Then

$$g_f(y) - g_h(y) = g_{1f}(x) - g_{1h}(x) \tag{183}$$

for every $x = 1 - y \in [0,1]$. $\square$

*Second proof.* We shall prove that for all $(x,y) \in [0,1]^2$ except $(x,y)$ in lines

$$\begin{aligned} &x = 0, x = \frac{1}{3}, x = \frac{1}{2}, x = \frac{2}{3}, x = 1 \text{ and} \\ &y = 0, y = \frac{1}{3}, y = \frac{1}{2}, y = \frac{2}{3}, y = 1 \end{aligned} \tag{184}$$

we have

$$F(x,y) = F(1-x, 1-y). \tag{185}$$

We start with

$$\begin{aligned} |\{2x\} - \{3y\}| &= |2x - [2x] - (3y - [3y])| = |(2x - 3y + [3y] - [2x])| \\ &= \begin{cases} 2x - 3y + [3y] - [2x] > 0 \\ -(2x - 3y + [3y] - [2x]) > 0. \end{cases} \end{aligned} \tag{186}$$

Thus from integer parts $[3y] - [2x]$ in Fig. 1 we have the fractional part $|\{2x\} - \{3y\}|$ in Fig. 2 without straight lines (184).

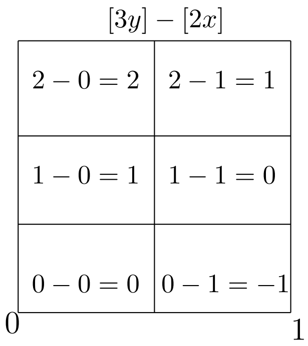


Figure 1

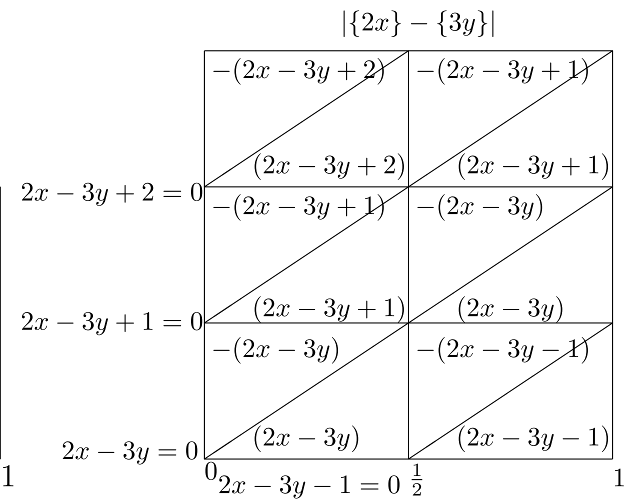


Figure 2

Similarly to (186) we have

$$\begin{aligned}|\{2(1-x)\}-\{3(1-y)\}| &= |\{-2x\}-\{-3y\}| \\ &= |-2x-[-2x]-(-3y-[-3y])| \\ &= |(3y-2x+[-3y]-[-2x])| \\ &= \begin{cases}3y-2x+[-3y]-[-2x]>0 \\ -(3y-2x+[-3y]-[-2x])>0\end{cases}\end{aligned} \tag{187}$$

and similarly to Fig.1 and Fig. 2 we have

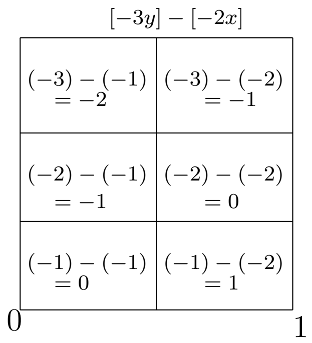



Figure 3

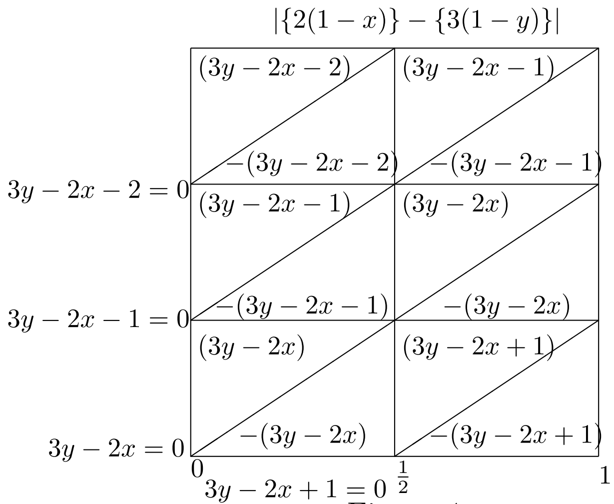



Figure 4

Thus from Fig. 2 and Fig. 4 we see that

$$|\{2x\}-\{3y\}| = |\{2(1-x)\}-\{3(1-y)\}| \tag{188}$$

for every $(x, y) \in [0, 1]^2$ without straight lines (184). Similarly it holds also for the parts $|\{2y\}-\{3x\}|$, $|\{2x\}-\{2y\}|$ and $|\{3x\}-\{3y\}|$ of $F(x, y)$ and thus

$$F(x, y) = F(1-x, 1-y) \tag{189}$$

for all $(x, y) \in [0, 1]^2$ except straight lines (184). If d.f. $g(x)$ is continuous at $x = 0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1$, then

$$\int\int_{\text{over (184)}} F(x, y)\mathrm{d}g(x)\mathrm{d}g(y) = 0$$

and thus

$$\int_0^1 \int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y) = \int_0^1 \int_0^1 F(1-x,1-y)\mathrm{d}g(x)\mathrm{d}g(y). \qquad (190)$$

Using transformation $x \to 1-x$, $y \to 1-y$ we have

$$\begin{aligned}&\int_0^1 \int_0^1 F(1-x,1-y)\mathrm{d}g(x)\mathrm{d}g(y)\\ &= \int_0^1 \int_0^1 F(x,y)\mathrm{d}(g(1-x))\mathrm{d}(g(1-y)) \end{aligned} \qquad (191)$$

and thus

$$\begin{aligned}&\int_0^1 \int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y) = 0 \Longleftrightarrow\\ &\int_0^1 \int_0^1 F(x,y)\mathrm{d}(1-g(1-x))\mathrm{d}(1-g(1-y)) = 0. \end{aligned} \qquad (192)$$

Finally we use Theorem 111. $\square$

In the following Fig. 5 and 6 we give different values of $|\{2x\} - \{3y\}|$ and $|\{2(1-x)\} - \{3(1-y)\}|$ on the straight lines (184).

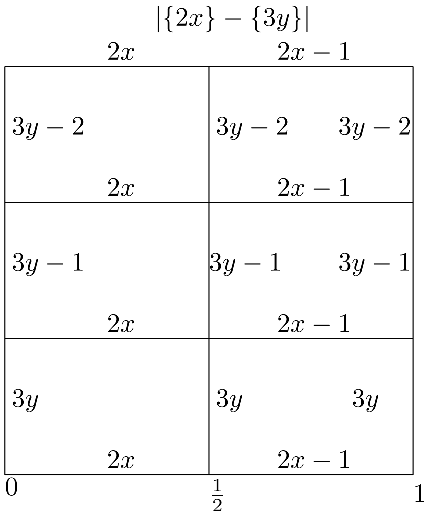


Figure 5

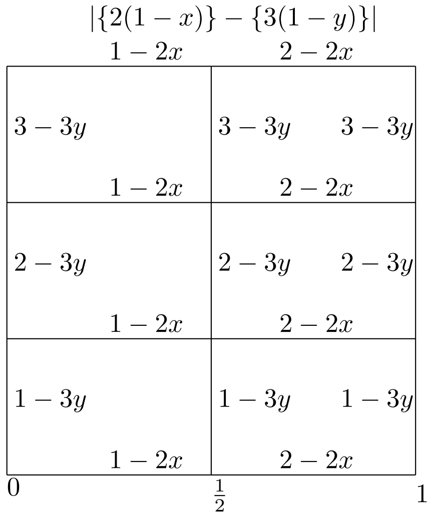


Figure 6

### 6.1.6 Explicit formula for $F(x,y) = |\{2x\} - \{3y\}| + |\{2y\} - \{3x\}| - |\{2x\} - \{2y\}| - |\{3x\} - \{3y\}|$

We give explicit formulas of $|\{2x\} - \{3y\}|$, $|\{2y\} - \{3x\}|$, $-|\{2x\} - \{2y\}|$, $-|\{3x\} - \{3y\}|$ respectively, in the following Figures 7–14:

$|\{2x\} - \{3y\}|$ on (184)

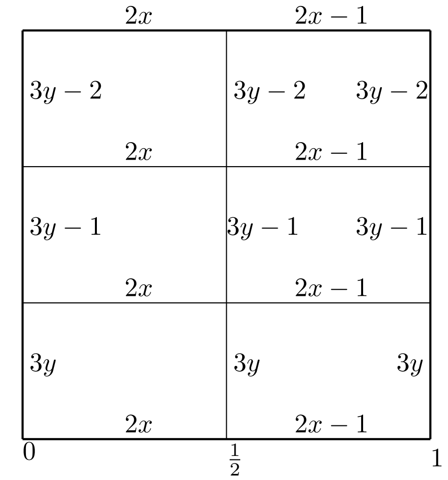



Figure 7

$|\{2x\} - \{3y\}|$

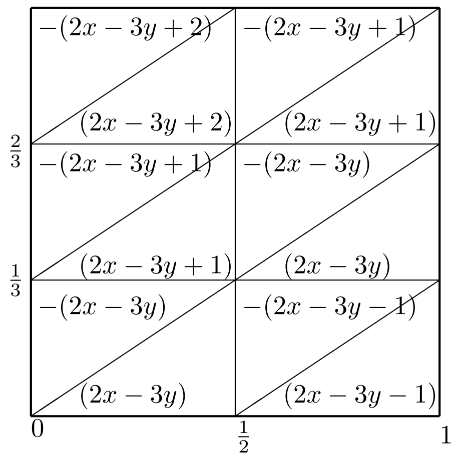



Figure 8

$|\{2y\} - \{3x\}|$ on (184)

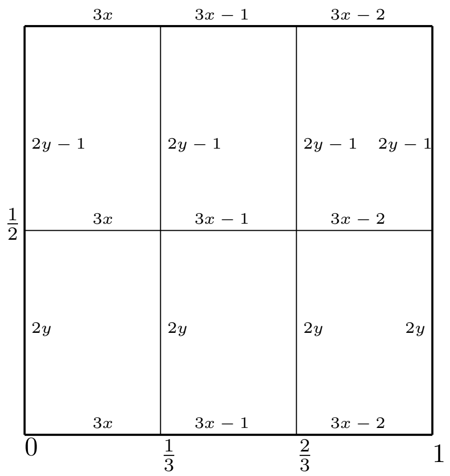



Figure 9

$|\{2y\} - \{3x\}|$

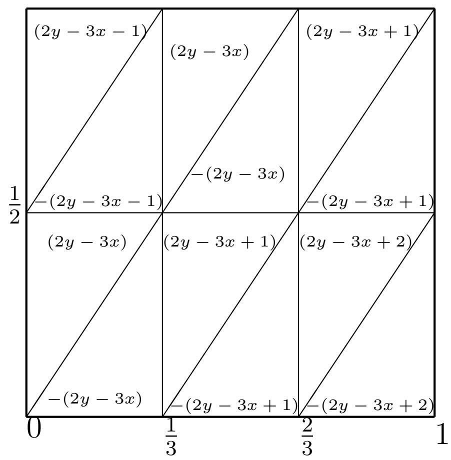



Figure 10

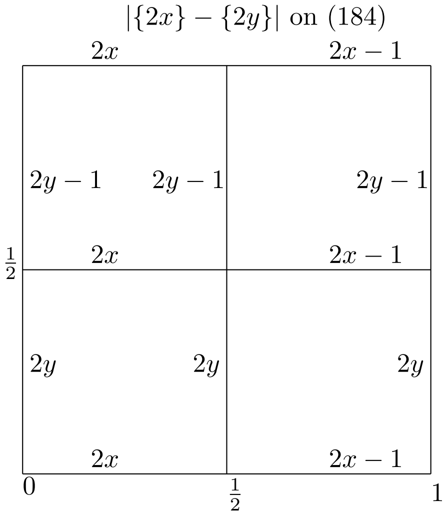


Figure 11

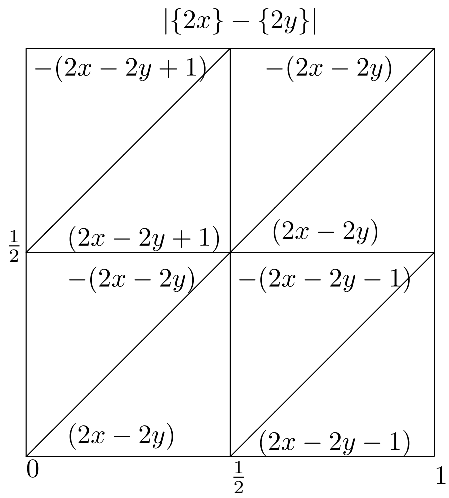


Figure 12

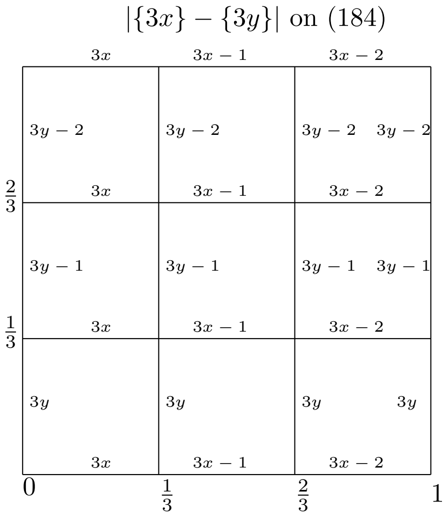


Figure 13

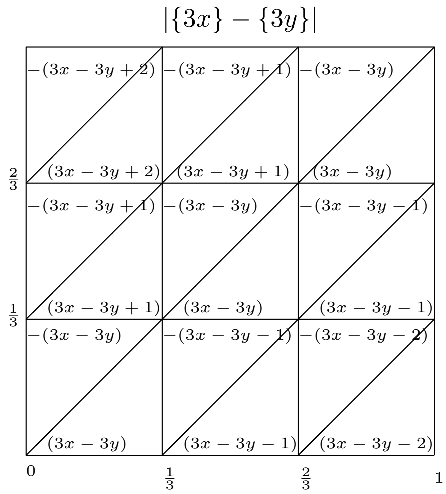


Figure 14

### 6.1.7 Integration $\int_0^1 \int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y)$ by parts

Since $F(x,y)$ in (173) is discontinuous we cannot apply (541) in the form

$$\int_0^1 \int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y) = F(1,1) - \int_0^1 g(y)\mathrm{d}_y F(1,y)$$

$$- \int_0^1 g(x)\mathrm{d}_x F(x,1) + \int_0^1 \int_0^1 g(x)g(y)\mathrm{d}_x\mathrm{d}_y F(x,y). \tag{193}$$

Here for $F(x,y)$ in (173) we have

$F(1,1) = 0,$

$F(1,y) = \{3y\} + \{2y\} - \{2y\} - \{3y\} = 0,$

$F(x,1) = 0.$

In the following we apply W.H. Young's [196] repeated integration by parts (see (206), (207) and (208)) on every continuous part of $F(x,y)$ defined on subs-squares of $[0,1]^2$ by Fig. 8,10,12 and 14.

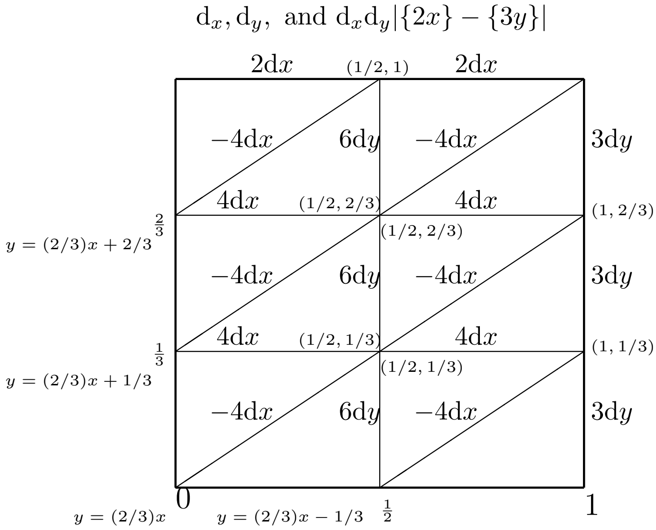



Figure 15

By Fig. 15 for arbitrary d.f. $g(x)$ we have

$$\int_0^1 \int_0^1 |\{2x\} - \{3y\}|\mathrm{d}g(x)\mathrm{d}g(y)$$

$$= -g(1/2)g(1) - g(1)g(1/3) - g(1)g(2/3) - 2g(1/2)g(1/3) - 2g(1/2)g(2/3) \tag{194}$$

$$+ \int_0^{1/2} g(x)g((2/3)x + 2/3)(-4)\mathrm{d}x + \int_0^1 g(x)g((2/3)x + 1/3)(-4)\mathrm{d}x$$

$$+ \int_0^1 g(x)g((2/3)x)(-4)\mathrm{d}x + \int_{1/2}^1 g(x)g((2/3)x - 1/3)(-4)\mathrm{d}x \tag{195}$$

$$+ \int_0^1 g(1/2)g(y)6\mathrm{d}y + \int_0^1 g(1)g(y)3\mathrm{d}y + \int_0^1 g(x)g(1)2\mathrm{d}x$$

$$+\int_0^1 g(x)g(1/3)4\mathrm{d}x+\int_0^1 g(x)g(2/3)4\mathrm{d}x. \tag{196}$$

Here (194) are jumps, (195) are integrals over diagonals and (196) integrals over orthogonal lines.

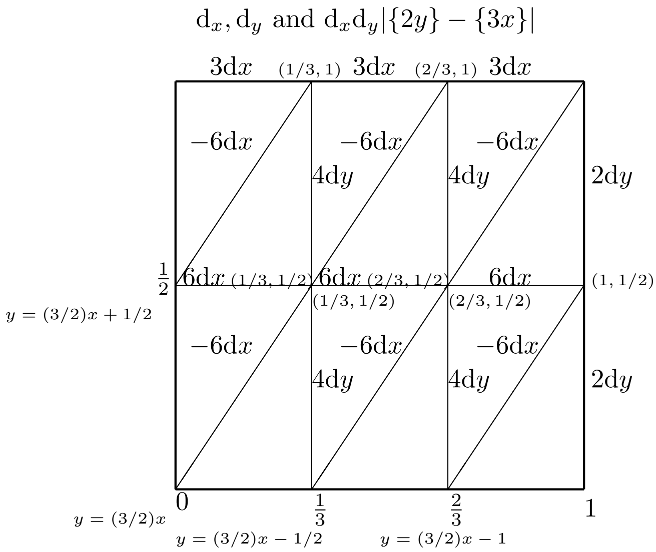


Figure 16

By Fig. 16 for arbitrary d.f. $g(x)$ we have

$$\int_0^1\int_0^1 |\{2y\}-\{3x\}|\mathrm{d}g(x)\mathrm{d}g(y)$$
$$=-g(1/3)g(1)-g(2/3)g(1)-2g(1/3)g(1/2)-2g(2/3)g(1/2)-g(1)g(1/2) \tag{197}$$
$$+\int_0^{1/3} g(x)g((3/2)x+1/2)(-6)\mathrm{d}x+\int_0^{2/3} g(x)g((3/2)x)(-6)\mathrm{d}x$$
$$+\int_{1/3}^1 g(x)g((3/2)x-1/2)(-6)\mathrm{d}x+\int_{2/3}^1 g(x)g((3/2)x-1)(-6)\mathrm{d}x \tag{198}$$
$$+\int_0^1 g(1/3)g(y)4\mathrm{d}y+\int_0^1 g(2/3)g(y)4\mathrm{d}y+\int_0^1 g(1)g(y)2\mathrm{d}y$$
$$+\int_0^1 g(x)g(1)3\mathrm{d}x+\int_0^1 g(x)g(1/2)6\mathrm{d}x. \tag{199}$$

Here (197) are jumps, (198) are integrals over diagonals and (199) integrals over orthogonal lines.

**Notes 16.** Since

$$\int_0^1\int_0^1 |\{2y\}-\{3x\}|\mathrm{d}g(x)\mathrm{d}g(y) = \int_0^1\int_0^1 |x-y|\mathrm{d}g_f(x)\mathrm{d}g_h(y),$$
$$\int_0^1\int_0^1 |\{2x\}-\{3y\}|\mathrm{d}g(x)\mathrm{d}g(y) = \int_0^1\int_0^1 |x-y|\mathrm{d}g_h(x)\mathrm{d}g_f(y),$$

then we have (194), (195) and (196) equal to (197), (198) and (199), respectively. For (195)=(198) we give

$$\int_0^{1/2} g(x)g((2/3)x+2/3)(-4)\mathrm{d}x = \int_{2/3}^1 g(x)g((3/2)x-1)(-6)\mathrm{d}x,$$
$$\int_0^1 g(x)g((2/3)x+1/3)(-4)\mathrm{d}x = \int_{1/3}^1 g(x)g((3/2)x-1/2)(-6)\mathrm{d}x,$$
$$\int_0^1 g(x)g((2/3)x)(-4)\mathrm{d}x = \int_0^{2/3} g(x)g((3/2)x)(-6)\mathrm{d}x,$$
$$\int_{1/2}^1 g(x)g((2/3)x-1/3)(-4)\mathrm{d}x = \int_0^{1/3} g(x)g((3/2)x+1/2)(-6)\mathrm{d}x.$$

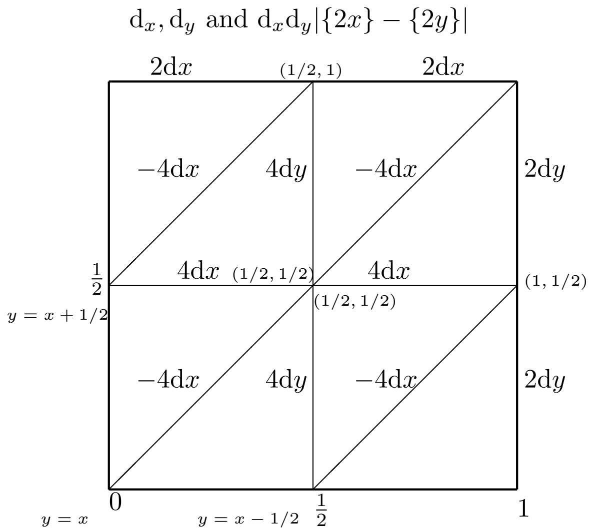


Figure 17

By Fig. 17, for an arbitrary d.f. $g(x)$ we have

$$\int_0^1\int_0^1 |\{2x\}-\{2y\}|\mathrm{d}g(x)\mathrm{d}g(y)$$
$$= -g(1/2)g(1) - g(1)g(1/2) - g(1/2)g(1/2) - g(1/2)g(1/2) \tag{200}$$

$$+\int_0^{1/2} g(x)g(x+1/2)(-4)\mathrm{d}x+\int_0^1 g(x)g(x)(-4)\mathrm{d}x+\int_{1/2}^1 g(x)g(x-1/2)(-4)\mathrm{d}x \tag{201}$$

$$+\int_0^1 g(1/2)g(y)4\mathrm{d}y+\int_0^1 g(1)g(y)2\mathrm{d}y+\int_0^1 g(x)g(1/2)4\mathrm{d}x+\int_0^1 g(x)g(1)2\mathrm{d}x. \tag{202}$$

Here (200) are jumps, (201) are integrals over diagonals and (202) integrals over orthogonal lines.

$\mathrm{d}_x, \mathrm{d}_y$ and $\mathrm{d}_x\mathrm{d}_y|\{3x\}-\{3y\}|$

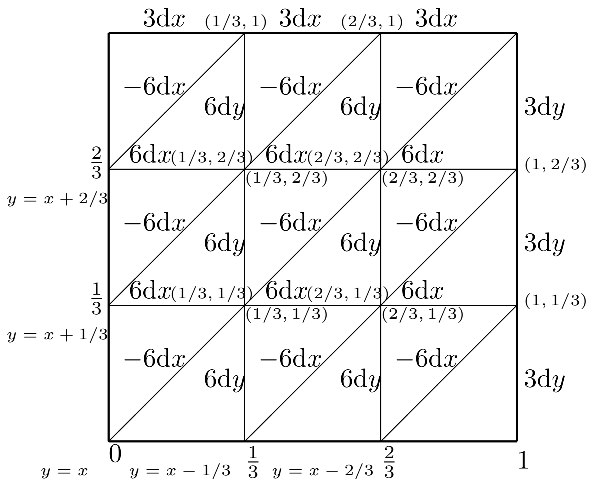



Figure 18

By Fig. 18 for an arbitrary d.f. $g(x)$ we have

$$\int_0^1\int_0^1 |\{3x\}-\{3y\}|\mathrm{d}g(x)\mathrm{d}g(y)$$
$$= -g(1/3)g(1)-g(2/3)g(1)-g(1)g(1/3)-g(1)g(2/3)-2.g(1/3)g(2/3)$$
$$-2g(2/3)g(2/3)-2.g(1/3)g(1/3)-2g(2/3)g(1/3) \tag{203}$$
$$+\int_0^{1/3} g(x)g(x+2/3)(-6)\mathrm{d}x+\int_0^{2/3} g(x)g(x+1/3)(-6)\mathrm{d}x+\int_0^1 g(x)g(x)(-6)\mathrm{d}x$$
$$+\int_{1/3}^1 g(x)g(x-1/3)(-6)\mathrm{d}x+\int_{2/3}^1 g(x)g(x-2/3)(-6)\mathrm{d}x \tag{204}$$
$$+\int_0^1 g(1/3)g(y)6\mathrm{d}y+\int_0^1 g(2/3)g(y)6\mathrm{d}y+\int_0^1 g(1)g(y)3\mathrm{d}y$$

$$+\int_0^1 g(x)g(1)3\mathrm{d}x + \int_0^1 g(x)g(2/3)6\mathrm{d}x + \int_0^1 g(x)g(1/3)6\mathrm{d}x. \tag{205}$$

Here (203) are jumps, (204) are integrals over diagonals and (205) integrals over orthogonal lines.

*Proof.* Let $F(x,y)$ be continuous on the interval $[x_1,x_2]\times[y_1,y_2]\subset[0,1]^2$ and $g(x)$ be a d.f. Two-times integration by parts given by W.G. Young [196]

$$\begin{aligned}
&\int_{x_1}^{x_2}\int_{y_1}^{y_2} F(x,y)\mathrm{d}g(x)\mathrm{d}g(y)\\
=&g(x_2)g(y_2)F(x_2,y_2) - g(x_1)g(y_2)F(x_1,y_2) - g(x_2)g(y_1)F(x_2,y_1)\\
&+ g(x_1)g(y_1)F(x_1,y_1) &(206)\\
&-\int_{x_1}^{x_2} g(x)g(y_2)\mathrm{d}F(x,y_2) + \int_{x_1}^{x_2} g(x)g(y_1)\mathrm{d}F(x,y_1)\\
&-\int_{y_1}^{y_2} g(x_2)g(y)\mathrm{d}F(x_2,y) + \int_{y_1}^{y_2} g(x_1)g(y)\mathrm{d}F(x_1,y) &(207)\\
&+\int_{x_1}^{x_2}\int_{y_1}^{y_2} g(x)g(y)\mathrm{d}_x\mathrm{d}_yF(x,y). &(208)
\end{aligned}$$

For example, we shall apply this to subs-quare (II) in the following Fig. 19.

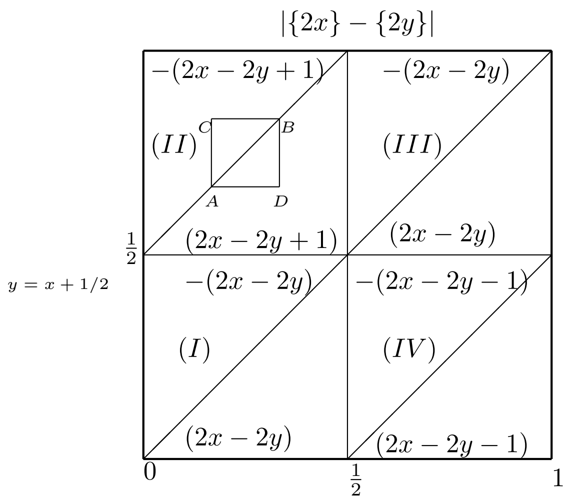



Figure 19

In this case

$$F(x,y)=\begin{cases}-(2x-2y+1) & \text{if } -(2x-2y+1)\geq 0,\\ (2x-2y+1) & \text{if } (2x-2y+1)\geq 0\end{cases} \tag{209}$$

$(x_1,y_1)=(0,1/2)$, $(x_2,y_2)=(1/2,1)$,
$F(x_1,y_1)=0, F(x_2,y_2)=0$ since $(x_1,y_1)$ an $(x_2,y_2)$ lies on diagonals.
Put
$A=(x,y)$, $B=(x+\mathrm{d}x,y+\mathrm{d}y)$, $C=(x,y+\mathrm{d}y)$, $D=(x+\mathrm{d}x,y)$, where $y=x+1/2$. By definition of differential $\mathrm{d}_x\mathrm{d}_yF(x,y)$ we have

$$\begin{aligned}\mathrm{d}_x\mathrm{d}_yF(x,y)&=F(x,y)+F(x+\mathrm{d}x,y+\mathrm{d}y)-F(x,y+\mathrm{d}y)-F(x+\mathrm{d}x,y)\\ &=(2x-2y-1)(=0)+(2(x+\mathrm{d}x)-2(y+\mathrm{d}y)-1)(=0)\\ &-(-(2x-2(y+\mathrm{d}y)-1))-(2(x+\mathrm{d}x)-2y-1)\\ &=2\mathrm{d}y-2\mathrm{d}x=-4\mathrm{d}x. \end{aligned} \tag{210}$$

Now applying (206), (207), (208) to $F(x,y)$ in (209) we find

$$\begin{aligned}&\int_0^{1/2}\int_{1/2}^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y)\\ &=g(1/2)g(1)F(1/2,1)(=0)-g(0)g(1)F(0,1)(=0)\\ &-g(1/2)g(1/2)F(1/2,1/2)(=1)+g(0)g(1/2)F(0,1/2)(=0) \qquad (211)\\ &-\int_0^{1/2}g(x)g(1)\mathrm{d}F(x,1)\left(=\int_0^{1/2}g(x)g(1)2\mathrm{d}x\right)\\ &+\int_0^{1/2}g(x)g(1/2)\mathrm{d}F(x,1/2)\left(=\int_0^{1/2}g(x)g(1/2)2\mathrm{d}x\right)\\ &-\int_{1/2}^1 g(1/2)g(y)\mathrm{d}F(1/2,y)\left(=\int_{1/2}^1 g(1/2)g(y)2\mathrm{d}y\right)\\ &+\int_{1/2}^1 g(0)g(y)\mathrm{d}F(0,y)(=0) \qquad (212)\\ &+\int_0^{1/2}\int_{1/2}^1 g(x)g(y)\mathrm{d}_x\mathrm{d}_yF(x,y)\left(=\int_0^{1/2}g(x)g(x+1/2)(-4)\mathrm{d}x\right). \qquad (213)\end{aligned}$$

Here (206), (207), (208) are equivalent (211), (212), (213). Similarly, using all sub-rectangles of Fig. 15, 16, 17 and 18 for $F(x,y)$ in (173) we find the integral $\int_0^1\int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y)$ as a sums of (194), (195), (196), (197), (198), (199), (200), (201), (202), (203), (204), (205), see Theorem 113. $\square$

**Theorem 113.** *For every continuous d.f. $g(x)$ we have*

$$
\begin{aligned}
&2\int_0^1 (g_f(x)-g_h(x))^2\mathrm{d}x = \int_0^1\int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y)\\
&= -4g(1/2)g(1/3) - 4g(1/2)g(2/3) + 2g(1/2)g(1/2) + 4g(1/3)g(2/3)\\
&+ 2g(2/3)g(2/3) + 2g(1/3)g(1/3) \qquad (214)\\
&- 8\Big(\int_0^{1/2} g(x)g\big((2/3)x+2/3\big)\mathrm{d}x + \int_0^1 g(x)g\big((2/3)x+1/3\big)\mathrm{d}x\\
&+ \int_0^1 g(x)g\big((2/3)x\big)\mathrm{d}x + \int_{1/2}^1 g(x)g\big((2/3)x-1/3\big)\mathrm{d}x\Big)\\
&+ \int_0^{1/2} g(x)g(x+1/2)\mathrm{d}x + 10\int_0^1 g(x)g(x)\mathrm{d}x\\
&+ 12\int_0^{1/3} g(x)g(x+2/3)\mathrm{d}x + 12\int_0^{2/3} g(x)g(x+1/3)\mathrm{d}x \qquad (215)\\
&+ \int_0^1 g(x)\mathrm{d}x\Big(4g(1/2) - 4g(1/3) - 4g(2/3)\Big). \qquad (216)
\end{aligned}
$$

*Proof.* Using (194), (781), -(200), and -(203) in (214); using (195), (198), -(201), and -(204) in (215); and using (196), (199), -(202), and -(205) in (216) we find Theorem 113. □

Theorem 113 can be using for testing $g_f = g_h$. But if $g(x)$ is defined by finite intervals then we can compute $g_f$ and $g_h$ directly by definition, see:

**Example 48.** Transform the graph of $g_3(x)$ to the interval $[0,1/2]$ then we find

$$
g(x) = \begin{cases} 0 & \text{if } x \in [0, 2/12],\\ 2x-(4/12) & \text{if } x \in [2/12, 3/12],\\ 4x-(10/12) & \text{if } x \in [3/12, 5/12],\\ 2x & \text{if } x \in [5/12, 1/2],\\ 1 & \text{if } x \in [1/2, 1]. \end{cases} \qquad (217)
$$

and directly by definition $g_f(x) = g(x/2) + g((x+1)/2) - g(1/2)$ we have

$$
g_f(x) = \begin{cases} 0 & \text{if } x \in [0, 4/12],\\ x-(4/12) & \text{if } x \in [4/12, 6/12],\\ 2x-(10/12) & \text{if } x \in [6/12, 10/12],\\ x & \text{if } x \in [10/12, 1], \end{cases} \qquad (218)
$$

and directly by $g_h(x) = g(x/3) + g((x+1)/3) + g((x+2)/3) - g(1/3) - g(2/3)$ we have

$$g_h(x) = \begin{cases} (4/3)x & \text{if } x \in [0, 3/12], \\ (2/3)x + 2/12 & \text{if } x \in [3/12, 9/12], \\ (4/3)x - 4/12 & \text{if } x \in [9/12, 1], \end{cases} \tag{219}$$

which gives $g_f \neq g_h$.

### 6.1.8 Copositivity of $F(x, y)$

By Section 13 (IV)

$$2\int_0^1 (g_f(x) - g_h(x))^2 \mathrm{d}x = \int_0^1 \int_0^1 F(x, y)\mathrm{d}g(x)\mathrm{d}g(y) \geq 0. \tag{220}$$

Thus $F(x, y)$ is copositive if we not taking into account the continuity of $F(x, y)$. Assuming that Theorem 242 (I) holds also in this case and thus

$$F(x, x).F(y, y) \geq (F(x, y))^2 \tag{221}$$

for $(x, y) \in [0, 1]^2$ or $F(x, y) \geq 0$.

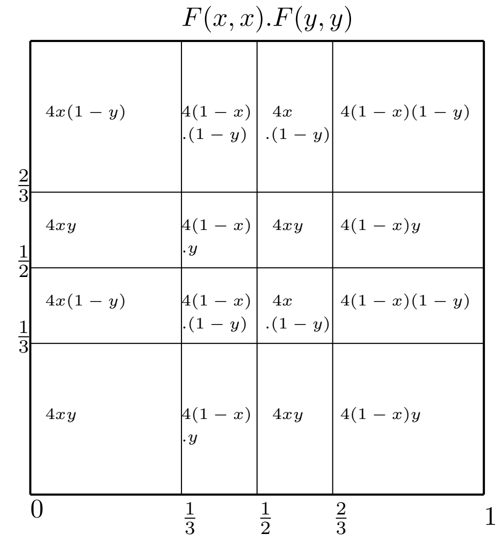
Now, using $F(x, x) = 2|\{2x\} - \{3x\}|$ and

$$|\{2x\} - \{3x\}| = |2x - 3x + [3x] - [2x]| = \begin{cases} (-x + [3x] - [2x]) \geq 0, \\ -(-x + [3x] - [2x]) \end{cases}$$

then we have

$$|\{2x\} - \{3x\}| = \begin{cases} x & \text{if } x \in [0, 1/3), \\ 1 - x & \text{if } x \in [1/3, 1/2), \\ x & \text{if } x \in [1/2, 2/3), \\ 1 - x & \text{if } x \in [2/3, 1). \end{cases}$$

and then we find upper bounds for $F^2(x, y)$ by




### 6.1.9 Solution of $g_f = g_h$

We continue Section 4.8.4, see also Example 113.

Let $f; [0,1] \to [0,1]$ and $h : [0,1] \to [0,1]$ satisfy:

(i) $f$ and $h$ commute,

(ii) $f$ has $m$ inverse functions $f_1^{-1}, \dots, f_m^{-1}$, defined on whole interval $[0,1]$,

(iii) $h$ has $k$ inverse functions $h_1^{-1}, \dots, h_k^{-1}$, defined on the whole interval $[0,1]$,

(iv) These inverse functions are ordered such that

$$f_1^{-1}(x) \le \cdots \le f_m^{-1}(x), \quad h_1^{-1}(x) \le \cdots \le h_k^{-1}(x) \text{ for all } x \in [0,1].$$

The functional equations $g_f = \tilde{g}_f$ and $g_h = \tilde{g}_h$ can be adding by following method: Put

$$[0,\alpha] = f_1^{-1} \circ h_1^{-1}([0,1]),$$

$$x_{ij}(t) = f_i^{-1} \circ h_j^{-1} \circ h \circ f(t), \quad i = 1,\dots,m, j = 1,\dots,k, \quad t \in [0,\alpha].$$

Now, to the equation

$$0 = g_f(x) - \tilde{g}_f(x) = \sum_{i=1}^{m} \operatorname{sign}(f_i^{-1})'(g(f_i^{-1}(x)) - \tilde{g}(f_i^{-1}(x))),$$

input $x = h_j^{-1} \circ h \circ f(t)$, for $j = 1,\dots,k$, where $t \in [0,\alpha]$. We find $k$ linear equations on $mk$ unknowns $g(x_{ij}(t))$, $i = 1,\dots,m$,$j = 1,\dots,k$. Similarly, into

$$0 = g_h(x) - \tilde{g}_h(x) = \sum_{j=1}^{k} \operatorname{sign}(h_j^{-1})'(g(h_j^{-1}(x)) - \tilde{g}(h_j^{-1}(x))),$$

input $x = f_i^{-1} \circ f \circ h(t)$, for $i = 1, \dots, m$. We find $m$ linear equations with $mk$ unknowns $g(x_{ls}(t))$, $l = 1, \dots, m$,$s = 1, \dots, k$, where $t \in [0, \alpha]$. All coefficients of such system of equations are dependent on the number $m$ and $k$ of inverse functions of $f$ and $h$ (respect.), on that $f_1^{-1}$ and $h_1^{-1}$ increase or decrease and are independent on concrete form of $f$ and $h$.

Using this system of linear equations some $g(x_{ij}(t))$ cam be computed by others $g(x_{ls}(t))$, where we can select such that the resulting $g$ will be d.f. in $[0, 1]$.

The above method we apply on the following two pairs of $f, h$:

$f(x) = 2x \bmod 1$ and $h(x) = 3x \bmod 1$;

$f(x) = 4x(1 - x)$ and $h(x) = x(4x - 3)^2$

where $x \in [0, 1]$. A motivation for searching $f(x) = 2x \bmod 1$ and $h(x) = 3x \bmod 1$ is Theorem 107. A motivation for searching $f(x) = 4x(1 - x)$ and $h(x) = x(4x - 3)^2$ is the fact that for $s(x) = \sin^2 2\pi x$ we have $f(s(x)) = s(2x)$ and $h(s(x)) = s(3x)$. Thus by Theorem 107 arbitrary d.f $g(x) \in G(\xi(3/2)^n \bmod 1)$ satisfies $g_{f\circ s} = g_{h\circ s}$. It is holds also for [31] $g(x) = x$ as we see from the following: $x_{2t \bmod 1} = x$ and $x_{3t \bmod 1} = x$ implies $x_{f\circ s} = x_s$ and $x_{h\circ s} = x_s$. Furthermore $x_s = \frac{2}{\pi} \arcsin \sqrt{x} (:= g_0(x))$. Thus $g_0$ solve $g_f = g_h$ as we mentioned after (172).

**Theorem 114.** *Let $g_1, g_2$ be two absolutely continuous d.f.s satisfying $g_{1_h}(x) = g_{2_f}(x)$ for $x \in [0, 1]$, where $f(x) = 2x \bmod 1$ and $h(x) = 3x \bmod 1$. Then the absolutely continuous d.f. $g(x)$ satisfies $g_f(x) = g_1(x)$ and $g_h(x) = g_2(x)$ for $x \in [0, 1]$ if and only if $g(x)$ has the form*

$$g(x) = \begin{cases} \Psi(x), & \text{for } x \in [0, 1/6], \\ \Psi(1/6) + \Phi(x - 1/6), & \text{for } x \in [1/6, 2/6], \\ \Psi(1/6) + \Phi(1/6) + g_1(1/3) - \Psi(x - 2/6) & \\ +\Phi(x - 2/6) - g_1(2x - 1/3) + g_2(3x - 1), & \text{for } x \in [2/6, 3/6], \\ 2\Phi(1/6) + g_1(1/3) - g_1(2/3) + g_2(1/2) & \\ -\Psi(x - 3/6) + g_1(2x - 1), & \text{for } x \in [3/6, 4/6], \\ -\Psi(1/6) + 2\Phi(1/6) + g_1(1/3) - g_1(2/3) + g_2(1/2) & \\ -\Phi(x - 4/6) + g_1(2x - 1), & \text{for } x \in [4/6, 5/6], \\ -\Psi(1/6) + \Phi(1/6) + g_1(1/3) + \Psi(x - 5/6) & \\ -\Phi(x - 5/6) - g_1(2x - 5/3) + g_2(3x - 2), & \text{for } x \in [5/6, 1], \end{cases}$$

*where*

$$\Psi(x) = \int_0^x \psi(t) \mathrm{d}t, \quad \Phi(x) = \int_0^x \phi(t) \mathrm{d}t, \text{ for } x \in [0, 1/6],$$

[31] In opposite case we will have simple proved that the sequence $(3/2)^n \bmod 1$ is not u.d.

*and $\psi(t)$, $\phi(t)$ are Lebesgue integrable functions on* $[0, 1/6]$ *satisfying*

$0 \le \psi(t) \le 2g_1'(2t)$,

$0 \le \phi(t) \le 2g_1'(2t + 1/3)$,

$2g_1'(2t) - 3g_2'(3t + 1/2) \le \psi(t) - \phi(t) \le -2g_1'(2t + 1/3) + 3g_2'(3t)$,

*for almost all* $t \in [0, 1/6]$.

*Proof.* We shall use a method which is applicable for any two commutable $f, h$ having finitely many inverse functions.

The starting point is the set of new variables $x_i(t)$

$$\begin{aligned}
x_1(t) :=& f_1^{-1} \circ h_1^{-1} \circ h \circ f(t) = h_1^{-1} \circ f_1^{-1} \circ f \circ h(t),\\
x_2(t) :=& f_1^{-1} \circ h_2^{-1} \circ h \circ f(t) = h_1^{-1} \circ f_2^{-1} \circ f \circ h(t),\\
x_3(t) :=& f_1^{-1} \circ h_3^{-1} \circ h \circ f(t) = h_2^{-1} \circ f_1^{-1} \circ f \circ h(t),\\
x_4(t) :=& f_2^{-1} \circ h_1^{-1} \circ h \circ f(t) = h_2^{-1} \circ f_2^{-1} \circ f \circ h(t),\\
x_5(t) :=& f_2^{-1} \circ h_2^{-1} \circ h \circ f(t) = h_3^{-1} \circ f_1^{-1} \circ f \circ h(t),\\
x_6(t) :=& f_2^{-1} \circ h_3^{-1} \circ h \circ f(t) = h_3^{-1} \circ f_2^{-1} \circ f \circ h(t).
\end{aligned}$$

Here the different expressions of $x_i(t)$ follows from the fact that $f(h(x)) = h(f(x))$, $x \in [0, 1]$. For $t \in [0, 1/6]$ we have

$$x_i(t) = t + (i - 1)/6, \quad i = 1, \ldots, 6.$$

Substituting $x = h_j^{-1} \circ h \circ f(t)$, $j = 1, 2, 3$, into $g_f(x) = g_1(x)$ and $x = f_i^{-1} \circ f \circ h(t)$, $i = 1, 2$, into $g_h(x) = g_2(x)$ we have five linear equations over $g(x_k(t))$, $k = 1, \ldots, 6$. Abbreviating the composition $f_i^{-1} \circ h_j^{-1} \circ h \circ f(t)$ as $f_1^{-1}h_2^{-1}hf(t)$, and $x_i(t)$ as $x_i$, we can write

$$\begin{aligned}
&g(x_1) + g(x_4) - g(1/2) = g_1(h_1^{-1}hf(t)),\\
&g(x_2) + g(x_5) - g(1/2) = g_1(h_2^{-1}hf(t)),\\
&g(x_3) + g(x_6) - g(1/2) = g_1(h_3^{-1}hf(t)),\\
&g(x_1) + g(x_3) + g(x_5) - g(1/3) - g(2/3) = g_2(f_1^{-1}fh(t)),\\
&g(x_2) + g(x_4) + g(x_6) - g(1/3) - g(2/3) = g_2(f_2^{-1}fh(t)).
\end{aligned}$$

Summing up the first three equations and, respectively, the next two equations, we have find the necessity condition

$$g_1(1/3)+g_1(2/3)+3g(1/2)+g_{1_h}(hf(t)) = g_{2_f}(fh(t))+g_2(1/2)+2(g(1/3)+g(2/3)),$$

for $t \in [0, 1/6]$, which is equivalent to

$$g_1(1/3)+g_1(2/3)-g_2(1/2) = 2(g(1/3)+g(2/3))-3g(1/2) \text{ and } g_{1_h}(x) = g_{2_f}(x)$$

for $x \in [0,1]$. Eliminating the fourth equation which is depending on the others we can compute $g(x_3), \ldots, g(x_6)$ by using $g(x_1)$, $g(x_2)$, $g(1/3)$, $g(1/2)$, and $g(2/3)$ as follows

$$\begin{aligned} g(x_3) &= g(1/3) + g(2/3) - g(1/2) - g(x_1) + g(x_2) \\ &\quad - g_1(h_2^{-1}hf(t)) + g_2(f_1^{-1}fh(t)), \\ g(x_4) &= g(1/2) - g(x_1) + g_1(h_1^{-1}hf(t)), \\ g(x_5) &= g(1/2) - g(x_2) + g_1(h_2^{-1}hf(t)), \\ g(x_6) &= g(1/3) + g(2/3) - g(1/2) + g(x_1) - g(x_2) \\ &\quad - g_1(h_1^{-1}hf(t)) + g_2(f_2^{-1}fh(t)), \end{aligned} \tag{222}$$

for all $t \in [0, 1/6]$. Putting $t = 0$ and $t = 1/6$, respectively, we have find

$$\begin{aligned} g(1/2) &= 2g(1/3) - 2g(1/6) + g_1(1/3) - g_1(2/3) + g_2(1/2), \\ g(2/3) &= 2g(1/3) - 3g(1/6) + 2g_1(1/3) - g_1(2/3) + g_2(1/2). \end{aligned}$$

These values satisfy the necessity $g_1(1/3) + g_1(2/3) - g_2(1/2) = 2(g(1/3) + g(2/3)) - 3g(1/2)$. Moreover, $g(1/3) = g(x_2(1/6))$, $g(1/6) = g(x_2(0))$, and thus $g(x_3), \ldots, g(x_6)$ can be expressed by only using $g(x_1)$, $g(x_2)$. Next, we simplify (222) by using

$h_1^{-1}hf(t) = ff_2^{-1}h_1^{-1}hf(t) = f(x_4)$ for $g(x_4)$,
$h_2^{-1}hf(t) = ff_2^{-1}h_2^{-1}hf(t) = f(x_5)$ for $g(x_5)$,
$f_1^{-1}fh(t) = hh_2^{-1}f_1^{-1}fh(t) = h(x_3)$ and $h_2^{-1}hf(t) = ff_1^{-1}h_2^{-1}hf(t) = f(x_2)$ for $g(x_3)$,
$f_2^{-1}fh(t) = hh_3^{-1}f_2^{-1}fh(t) = h(x_6)$ and $h_1^{-1}hf(t) = ff_1^{-1}h_1^{-1}hf(t) = f(x_1)$ for $g(x_6)$.

Now, any $g(x_i)$ can be expressed as $g(x)$, $x \in [(i-1)/6, i/6]$. Using $g(x)$, $x \in [0, 2/6]$ and given $g_1(x)$ and $g_2(x)$, then (222) has the form

$$\begin{aligned} g(x_3) = g(x) &= g(1/3) - g(1/6) + g_1(1/3) - g(x - 2/6) + g(x - 1/6) \\ &\quad - g_1(2x - 1/3) + g_2(3x - 1) \text{ for } x \in [2/6, 3/6], \end{aligned}$$

$$\begin{aligned}
g(x_4) = g(x) &= 2g(1/3) - 2g(1/6) + g_1(1/3) - g_1(2/3) + g_2(1/2) \\
&\quad - g(x-3/6) + g_1(2x-1) \text{ for } x \in [3/6, 4/6], \\
g(x_5) = g(x) &= 2g(1/3) - 2g(1/6) + g_1(1/3) - g_1(2/3) + g_2(1/2) \\
&\quad - g(x-3/6) + g_1(2x-1) \text{ for } x \in [4/6, 5/6], \\
g(x_6) = g(x) &= g(1/3) - g(1/6) + g_1(1/3) + g(x-5/6) - g(x-4/6) \\
&\quad - g_1(2x-5/3) + g_2(3x-2) \text{ for } x \in [5/6, 1]. \qquad (223)
\end{aligned}$$

Now, assuming the absolute continuity of $g(x_1) = g(x)$ for $x \in [0, 1/6]$ and $g(x_2) = g(x)$ for $x \in [1/6, 2/6$ we can write

$$g(x_1) = g(x) = \int_0^x \psi(u)\mathrm{d}u, \text{ and } g(x_2) = g(x) = \int_0^{1/6} \psi(u)\mathrm{d}u + \int_0^x \phi(u)\mathrm{d}u \qquad (224)$$

for $x \in [0, 1/6]$.

Summing up (223) and (224) we have find the expression $g(x)$ in the theorem. The graph of $g(x)$ defined by (223) is continuous. For the condition of monotonicity of $g(x)$ we investigated $g'(x_i(t)) \geq 0$ for $t \in [0, 1/6]$ and $i = 1, \ldots, 6$ which leads to the inequalities, for $\psi$ and $\phi$, described in our theorem, immediately. $\square$

If $g_1 \neq g_2$ then Theorem 114 not give a new solution $g_f = g_h$. If $g_1 = g_2$, $g_{1f} = g_{1h}$ then we find a new solution $g_f = g_h$. Thus Theorem 114 can be using to construct a chain of solutions $g_f = g_h$ in the following form:

**Theorem 115.** *Let $g_1$ be given absolutely continuous d.f. satisfying $g_{1_h}(x) = g_{1_f}(x)$ for $x \in [0, 1]$, where $f(x) = 2x \bmod 1$ and $h(x) = 3x \bmod 1$. Then the absolutely continuous d.f. $g(x)$ satisfies $g_f(x) = g_h(x) = g_1(x)$ for $x \in [0, 1]$ if and only if $g(x)$ has the form*

$$g(x) = \begin{cases}
\Psi(x), & \text{for } x \in [0, 1/6], \\
\Psi(1/6) + \Phi(x-1/6), & \text{for } x \in [1/6, 2/6], \\
\Psi(1/6) + \Phi(1/6) + g_1(1/3) - \Psi(x-2/6) & \\
+\Phi(x-2/6) - g_1(2x-1/3) + g_1(3x-1), & \text{for } x \in [2/6, 3/6], \\
2\Phi(1/6) + g_1(1/3) - g_1(2/3) + g_1(1/2) & \\
-\Psi(x-3/6) + g_1(2x-1), & \text{for } x \in [3/6, 4/6], \\
-\Psi(1/6) + 2\Phi(1/6) + g_1(1/3) - g_1(2/3) + g_1(1/2) & \\
-\Phi(x-4/6) + g_1(2x-1), & \text{for } x \in [4/6, 5/6], \\
-\Psi(1/6) + \Phi(1/6) + g_1(1/3) + \Psi(x-5/6) & \\
-\Phi(x-5/6) - g_1(2x-5/3) + g_1(3x-2), & \text{for } x \in [5/6, 1],
\end{cases}$$

*where*

$$\Psi(x) = \int_0^x \psi(t)\mathrm{d}t, \quad \Phi(x) = \int_0^x \phi(t)\mathrm{d}t, \text{ for } x \in [0, 1/6],$$

*and* $\psi(t)$, $\phi(t)$ *are Lebesgue integrable functions on* $[0, 1/6]$ *satisfying*
$0 \le \psi(t) \le 2g_1'(2t)$,
$0 \le \phi(t) \le 2g_1'(2t + 1/3)$,
$2g_1'(2t) - 3g_1'(3t + 1/2) \le \psi(t) - \phi(t) \le -2g_1'(2t + 1/3) + 3g_1'(3t)$,
*for almost all* $t \in [0, 1/6]$.

### 6.1.10 Examples of solutions $g_f = g_h$

**Example 49.** The functions $c_0(x)$, $c_1(x)$, and $x$ solve $g_f(x) = g_h(x)$ for all $x \in [0, 1]$. Putting $g_1(x) = x$, the next solution of $g_f = g_h$ can be found by Theorem 115. In this case we have
$0 \le \psi(t) \le 2$,
$0 \le \phi(t) \le 2$,
$-1 \le \psi(t) - \phi(t) \le 1$,
for every $t \in [0, 1/6]$. Putting $\psi(t) = \phi(t) = 0$, the resulting d.f. is

$$g_3(x) = \begin{cases} 0 & \text{for } x \in [0, 2/6], \\ x - 1/3 & \text{for } x \in [2/6, 3/6], \\ 2x - 5/6 & \text{for } x \in [3/6, 5/6], \\ x & \text{for } x \in [5/6, 1]. \end{cases}$$

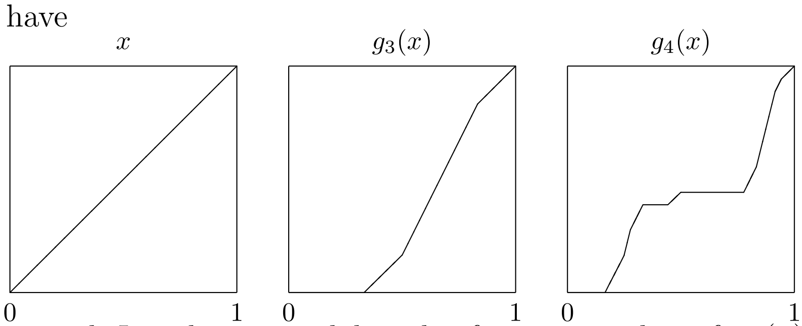
**Example 50.** Taking $g_1(x) = g_3(x)$, this $g_3(x)$ can be used as a starting point for a further application of Theorem 115. We need
$0 \le \psi(t) \le 0$ for $t \in [0, 1/6]$,
$0 \le \phi(t) \le 2$ for $t \in [0, 1/12]$,
$0 \le \phi(t) \le 4$ for $t \in [1/12, 1/6]$,
$-6 \le -\phi(t) \le -2$ for $t \in [0, 1/12]$,
$-6 \le -\phi(t) \le -4$ for $t \in [1/12, 1/9]$,
$-3 \le -\phi(t) \le -1$ for $t \in [1/9, 1/6]$.
Putting $\Psi(x) = \int_0^x 0\mathrm{d}t = 0$ and

$$\Phi(x) = \int_0^x \phi(t)\mathrm{d}t = \begin{cases} 2x & \text{for } x \in [0, 1/12], \\ 4x - 1/6 & \text{for } x \in [1/12, 1/9], \\ 2x + 1/18 & \text{for } x \in [1/9, 1/6]. \end{cases}$$

and applying Theorem 115 we find

$$g_4(x) = \begin{cases} 0 & \text{for } x \in [0, 1/6], \\ 2x - 1/3 & \text{for } x \in [1/6, 3/12], \\ 4x - 5/6 & \text{for } x \in [3/12, 5/18], \\ 2x - 5/18 & \text{for } x \in [5/18, 2/6], \\ 7/18 & \text{for } x \in [2/6, 8/18], \\ x - 1/18 & \text{for } x \in [8/18, 3/6], \\ 8/18 & \text{for } x \in [3/6, 7/9], \\ 2x - 20/18 & \text{for } x \in [7/9, 5/6], \\ 4x - 50/18 & \text{for } x \in [5/6, 11/12], \\ 2x - 17/18 & \text{for } x \in [11/12, 17/18], \\ x & \text{for } x \in [17/18, 1]. \end{cases}$$

Then we have




The interval $I$ with maximal length of constat value of $g_4(x)$ is $I = [1/2, 7/9]$ (see open problem XII in Section 3.8).

Other chain of solutions of $g_f = g_h$ by Theorem 115 can be construct by following theorem:

**Theorem 116.** *Assume that $g_1(x)$ satisfies $g_f = g_h$. Then $g_2(x)$,*

$g_2(x) = g_{1f}(x) = g_1(x/2) + g_1((x+1)/2) - g(1/2)$, *or*
$g_2(x) = g_{1h}(x) = g_1(x/3) + g_1((x+1)/3) + g_1((x+2)/3) - g_1(1/3) - g_1(2/3)$

*also satisfies $g_f = g_h$.*

*Proof.* Assume $g_{1f}(x) = g_{1h}(x) = g_2(x)$. Then $g_{2f} = g_{1hf}$ and $g_{2h} = g_{1fh}$. But the functions $f$ and $h$ commute $fh = hf$. $\square$

**Example 51.** If we start in Theorem 116 with $g_1(x) = g_4(x)$ then we find resulting $g_2(x) = g_3(x)$. If we start with $g_1(x) = g_4(x)$ then we find $g_2(x) = g_3(x)$. Putting $g_3(x)$ in our scheme as $g_1(x)$, then the resulting $g_2(x) = x$.

### 6.1.11 Problem $g_f = g_h$ and $g(x) = 1$ for $x \in [1/2, 1]$

**Example 52.** Assume
d.f. $g(x) = 1$ for $x \in [1/2, 1]$ and assume
$g_f = g_h$. Denote
$g_1(x) = 1 - g(1 - x)$;
$g_\alpha(x) = (1 - \alpha)g(x) + \alpha g_1(x)$ for $\alpha \in [0, 1]$. Then
$g_\alpha(x) = \alpha + (1 - \alpha)g(x) - \alpha g(1 - x)$ and
$g_\alpha(x)$ satisfies $g_f = g_h$ again. Now put $\alpha = 1/2$. Then
$g_{1/2}(1/2) = 1/2$ and
$g_{1/2}(1/2 - x) = (1/2)g(1/2 - x)$. Then
$g_{1/2}(1/2 + x)) = 1 - g_{1/2}(1/2 - x)$ for $x \in [0, 1/2]$.
This leads to the functional equation

$$\tilde{g}(x) + \tilde{g}(1 - x) = 1 \tag{225}$$

but for all $x \in [0, 1]$, where $\tilde{g}(x)$ satisfies $g_f = g_h$. For example such d.f. $\tilde{g}(x)$ exists, e.g. $\tilde{g}(x) = x$ and $\tilde{g}(x) = h_{1/2}(x)$

**Example 53.** Assume that
d.f. $g(x) = 1$ for $x \in [1/2, 1]$. Then $g_f = g_h$ has the form

$$g(x/2) = g(x/3) + g((x + 1)/3) - g(1/3) \tag{226}$$

for $x \in [0, 1]$.

Now assume that $(x + 1)/3 \geq 1/2$, i.e. $x \geq 1/2$. Then $g((x + 1)/3) = 1$ and the equation (226) has the form

$$g(x/2) = g(x/3) + 1 - g(1/3). \tag{227}$$

for $x \in [1/2, 1]$. Putting $x/3 = z$ (then $x/2 = (3/2)z$) we simplify (227) to

$$g((3/2)z) = g(z) + 1 - g(1/3). \tag{228}$$

for $z \in [1/6, 1/3]$, or putting $(3/2)z := z$

$$g(z) = g((2/3)z) + 1 - g(1/3) \tag{229}$$

for $z \in [1/4, 1/2]$.

In the other hand for $x \in [0, 1/2]$, putting $x/2 = z$ (then $x/3 = (2/3)z$ and $(x+1)/3 = (2/3)z + 1/3$) the equation (226) gives

$$g((2/3)z) = g(z) - g((2/3)z + 1/3) + g(1/3) \tag{230}$$

for $z \in [0, 1/4]$. Repeating (230) we find

$$g((2/3)^k z) = g(z) - \sum_{j=1}^{k} \Big( g\big((2/3)^j z + 1/3\big) - g(1/3) \Big) \tag{231}$$

for $z \in [0, 1/4]$. Using $k \to \infty$ we have

$$g(z) = \sum_{k=1}^{\infty} \Big( g\big((2/3)^k z + (1/3)\big) - g(1/3) \Big) \tag{232}$$

for $z \in [0, 1/4]$. Thus

$$g(z) = \begin{cases} \sum_{k=1}^{\infty} \Big( g\big((2/3)^k z + (1/3)\big) - g(1/3) \Big), & \text{if } z \in [0, 1/4], \\ g((2/3)z) + 1 - g(1/3), & \text{if } z \in [1/4, 1/2]. \end{cases} \tag{233}$$

Using (230) we have a finite form of (233)

$$g(z) = \begin{cases} (\mathrm{I}) g(2/3)z + g\big((2/3)z + 1/3\big) - g(1/3), & \text{if } z \in [0, 1/4], \\ (\mathrm{II}) g((2/3)z) + 1 - g(1/3), & \text{if } z \in [1/4, 1/2]. \end{cases} \tag{234}$$

Since for $z \geq 1/4$ we have $g((2/3)z + (1/3)) = 1$, then (234) has the form

$$g(z) = g(2/3)z + g\big((2/3)z + 1/3\big) - g(1/3), \text{ if } z \in [0, 1/2]. \tag{235}$$

Similarly, (233) has the form

$$\sum_{k=1}^{\infty} \Big( g\big((2/3)^k z + (1/3)\big) - g(1/3) \Big), \text{ if } z \in [0, 1/2]. \tag{236}$$

Since $2/9 < 1/4$ then by (I) we have

$$g(2/9) = g((2/3)(2/9)) + g\big((2/3)(2/9) + (1/3)\big) - g(1/3)$$

$$= g(4/27) + g(13/27) - g(1/3). \tag{237}$$

Since $13/27 > 1/4$ then by (II)

$$g(13/27) = g((2/3)(13/27)) + 1 - g(1/3) = g(26/81) + 1 - g(1/3). \tag{238}$$

Then

$$g(2/9) = g(4/27) + g(26/81) + 1 - 2g(1/3). \tag{239}$$

**Notes 17.** If we extend $[1/2, 1]$ to $g(x) = 1$ for $x \in [1/3, 1]$, then (226) gives
$g(x/2) = g(x/3)$
which implies
$g(x) = c_0(x)$
the same result as in Theorem 108.

**Example 54.** Assuming that d.f. $g(x)$ satisfies $g_f = g_h$, $g(x) = 1$ for $x \in [1/2, 1]$ and $g(x)$ is constructed by Theorem 115 using starting d.f. $g_1(x)$ also satisfactory $g_f = g_h$. Then we have

$$\begin{aligned}
&1 = 2\Phi(1/6) + g_1(1/3) - g_1(2/3) + g_1(1/2) - \Psi(x - 3/6)\\
&+ g_1(2x - 1), \text{ for } x \in [3/6, 4/6],\\
&1 = -\Psi(1/6) + 2\Phi(1/6) + g_1(1/3) - g_1(2/3) + g_1(1/2) - \Phi(x - 4/6)\\
&+ g_1(2x - 1), \text{ for } x \in [4/6, 5/6],\\
&1 = -\Psi(1/6) + \Phi(1/6) + g_1(1/3) + \Psi(x - 5/6) - \Phi(x - 5/6) - g_1(2x - 5/3)\\
&+ g_1(3x - 2), \text{ for } x \in [5/6, 1].
\end{aligned} \tag{240}$$

From (240) follows

$$\begin{aligned}
&\Psi(y) - g_1(2y) = 2\Phi(1/6) + g_1(1/3) - g_1(2/3) + g_1(1/2) - 1, y \in [0, 1/6],\\
&\Phi(y) - g_1(2y + 1/3) = -\Psi(1/6) + \Phi(1/6) + g_1(1/3) - g_1(2/3)\\
&+ g_1(1/2) - 1, y \in [0, 1/6],\\
&\Phi(y) - \Psi(y) + g_1(2y) - g_1(3y + 1/2) = -\Psi(1/6) + \Phi(1/6)\\
&+ g_1(1/3) - 1, y \in [0, 1/6].
\end{aligned} \tag{241}$$

Then

$$\Phi(y) - \Psi(y) + g_1(2y) = g_1(2y + 1/3) - \Psi(1/6), y \in [0, 1/6],$$

$$\Phi(y) - \Psi(y) + g_1(2y) = g_1(3y + 1/2) - \Psi(1/6) + \Phi(1/6)$$
$$+ g_1(1/3) - 1, y \in [0, 1/6]. \tag{242}$$

From it
$$g_1(3y + 1/2) = g_1(2y + 1/3) + 1 - \Phi(1/6) - g_1(1/3) \tag{243}$$
for $y \in [0, 1/6]$. From $y = 0$ we find $1 - \Phi(1/6) = g_1(1/2)$ and the equation (243) has the form

$$g_1(3y + 1/2) = g_1(2y + 1/3) + g_1(1/2) - g_1(1/3) \tag{244}$$

for $y \in [0, 1/6]$.
Using $3y + 1/2 = z$, then
$y = (1/3)(z - 1/2))$,
$2y + 1/3 = (2/3)z$
$z \in [1/2, 1]$
and we can transform (244) to the final form

$$g_1((2/3)z) = g_1(z) - g_1(1/2) + g_1(1/3) \tag{245}$$

for every $z \in [1/2, 1)$. [32]

**Notes 18.** But such d.f. $g_1(x)$ satisfying (245) exist, e.g.

$$g_1(x) = \begin{cases} x & \text{for } x \in [0, 4/9], \\ 3x - (8/9) & \text{for } x \in [4/9, 1/2], \\ (2/3)x + (5/18) & \text{for } x \in [1/2, 2/3], \\ 2x - (11/18) & \text{for } x \in [2/3, 3/4], \\ (8/18)x + (10/18) & \text{for } x \in [3/4, 1]. \end{cases} \tag{246}$$

Denote $\Delta = 5/18$. Then
$g_1(1/2 = g_1(1/3) + \Delta = 11/18$,
$g_1(2/3) = g_1(4/9) + \Delta = 13/18$,
$g_1(3/4) = g_1(1/2) + \Delta = 16/18$, and
$g_1(1) = g_1(2/3) + \Delta = 1$.
Insert (246) to (240) we find

(i) $\Psi(y) = g_1(y)$ for $y \in [0, 1/6]$,

(ii) $\Psi(1/6) = g_1(1/3) = 6/18$,

(iii) $\Phi(y) = g_1(2y + 1/3) - \Psi(1/6)$,

[32] Similar functional equation is in (230).

and then we find

$$g(x) = \begin{cases} 2x & \text{for } x \in [0, 2/9], \\ 6x - (8/9) & \text{for } x \in [2/9, 1/4], \\ (4/3)x + (5/18) & \text{for } x \in [1/4, 1/3], \\ x + (7/18) & \text{for } x \in [1/3, 13/27], \\ 7x - (45/18) & \text{for } x \in [13/27, 1/2], \\ 1 & \text{for } x \in [1/2, 1]. \end{cases} \tag{247}$$

But the d.f. $g_1(x)$ (246) does not satisfies the equation $g_f = g_h$ because $g_{1f}(1/3) \neq g_{1h}(1/3)$. Then also d.f. $g(x)$ from (247) does not satisfies $g_f = g_h$.

**Example 55.** The Mahler conjecture follows from that does not exist d.f. $g(x)$ such that $g_f(x) = g_h(x)$ for $x \in [0, 1]$ and $g(x) = 1$ for $x \in [1/2, 1]$. Yet is not proven, therefore we are looking for the biggest possible interval $[\beta, 1]$ such that $g(x) = 1$ for $x \in [\beta, 1]$ without $c_0(x)$. Here we found d.f. $g_5(x)$ such that $g_5(x) = 1$ for $x \in [4/6, 1]$. Starting with Theorem 114, we put

$$\begin{aligned} &1 = -\Psi(1/6) + 2\Phi(1/6) + g_1(1/3) - g_1(2/3) + g_1(1/2) - \Phi(x - 4/6) \\ &+ g_1(2x - 1), \text{ for } x \in [4/6, 5/6], \\ &1 = -\Psi(1/6) + \Phi(1/6) + g_1(1/3) + \Psi(x - 5/6) - \Phi(x - 5/6) - g_1(2x - 5/3) \\ &+ g_1(3x - 2), \text{ for } x \in [5/6, 1], \end{aligned}$$

From it

$$\begin{aligned} &\Phi(x - 4/6) = g_1(2x - 1) - \Psi(1/6) + 2\Phi(1/6) + g_1(1/3) - g_1(2/3) \\ &+ g_1(1/2) - 1, \text{ for } x \in [4/6, 5/6], \qquad (248) \\ &\Phi(x - 5/6) - \Psi(x - 5/6) = -g_1(2x - 5/3) + g_1(3x - 2) - \Psi(1/6) + \Phi(1/6) \\ &+ g_1(1/3) - 1, \text{ for } x \in [5/6, 1]. \qquad (249) \end{aligned}$$

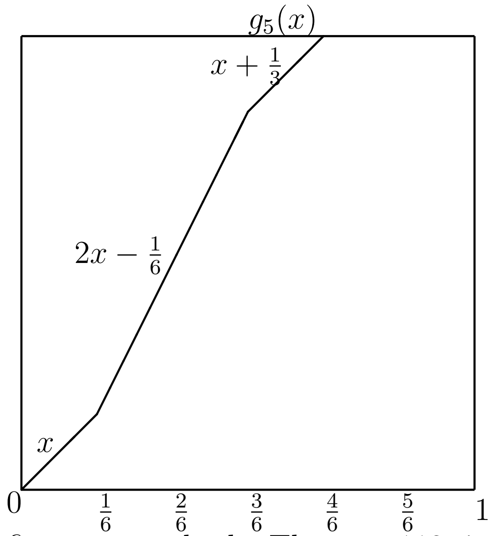
Putting $g_1(x) = x$ and $y = x - 4/6$ from (248) we find

$$\Phi(y) = 2y + 2\Phi(1/6) - \Psi(1/6) - 1/2 \text{ for } y \in [0, 1/6]. \tag{250}$$

Putting $y = x - 5/6$, (250) and (248) we find

$$\Phi(y) = y + 2\Phi(1/6) - 1/3 \text{ for } y \in [0, 1/6]. \tag{251}$$

Using continuity of $g_5(x)$ in $x = 0$ we put $\Phi(1/6) = 1/3$ and Theorem 114 gives




Note that $g_5(x)$ satisfies $g_f = g_h$ also by Theorem 112 since $g_5(x) = 1-g_3(1-x)$ and by Example 47 d.f. $g_3(x)$ satisfies them. Also note that Theorem 108 of uniqueness implies that if $g_f = g_h$ and $g(x) = 1$ for $x \in [1/3, 1]$ then $g(x) = c_0(x)$.

**Example 56.** Since $x_f = x_h$, Theorem 108 gives the following example of d.f. $g(x)$, which is not d.f. of the sequence $\xi(3/2)^n$ mod 1, for every $\xi \in \mathbb{R}$

$$g(x) = \begin{cases} x & \text{for } x \in [0, 2/3], \\ x^2 - (2/3)x + 2/3 & \text{for } x \in [2/3, 1]. \end{cases}$$

**Theorem 117.** *Let $f(x) = 4x(1-x)$ a $h(x) = x(4x-3)^2$ and let $g_1, g_2$ be two absolutely continuous d.f.s solvable the functional equation $g_{1_h} = g_{2_f}$ on $[0,1]$. Then an absolutely continuous d.f. $g$ satisfying $g_f = g_1$ and $g_h = g_2$ if and only if $g$ has the form*

$$g(x) = \begin{cases} \int_0^{x_1(x)} \psi(u)\mathrm{d}u \quad (:= \Psi(x)) & \textit{for } x \in \left[0, \frac{2-\sqrt{3}}{4}\right], \\ \int_{x_1(x)}^{(2-\sqrt{3})/4} \phi(u)\mathrm{d}u + \int_0^{(2-\sqrt{3})/4} \psi(u)\mathrm{d}u \quad (:= \Phi(x)) & \textit{for } x \in \left[\frac{2-\sqrt{3}}{4}, \frac{1}{4}\right], \\ \Psi(x) + \Phi(x) + 1 - g_1(f(x_1(x))) - g_2(h(x)) & \textit{for } x \in \left[\frac{1}{4}, \frac{1}{2}\right], \\ \Psi(x) + \Phi(x) + 1 - g_1(f(x_2(x))) - g_2(h(x)) & \textit{for } x \in \left[\frac{1}{2}, \frac{3}{4}\right], \\ \Phi(x) + 1 - g_1(f(x)) & \textit{pre } x \in \left[\frac{3}{4}, \frac{2+\sqrt{3}}{4}\right], \\ \Psi(x) + 1 - g_1(f(x)) & \textit{for } x \in \left[\frac{2+\sqrt{3}}{4}, 1\right], \end{cases}$$

*where* $\psi$, $\phi$ *are integrable on* $\left[0, \frac{2-\sqrt{3}}{4}\right]$ *and a.e.*

$0 \le \psi(t) \le (g_1(f(t)))'$,
$0 \le \phi(t) \le -(g_1(f(x_2(t))))'$,
$(g_1(f(t)) + g_2(1-h(t)))' \le \psi(t) - \phi(t) \le (g_1(f(x_2(t))) + g_2(h(t)))'$.

*The* $x_1(x)$ *and* $x_2(x)$ *are given in* (254) *and* (255).

*Proof.* In this case the new variables has the form

$$\begin{aligned}
x_1(t) =& f_1^{-1} \circ h_1^{-1} \circ h \circ f(t) = h_1^{-1} \circ f_1^{-1} \circ f \circ h(t),\\
x_2(t) =& f_1^{-1} \circ h_2^{-1} \circ h \circ f(t) = h_1^{-1} \circ f_2^{-1} \circ f \circ h(t),\\
x_3(t) =& f_1^{-1} \circ h_3^{-1} \circ h \circ f(t) = h_2^{-1} \circ f_2^{-1} \circ f \circ h(t),\\
x_4(t) =& f_2^{-1} \circ h_3^{-1} \circ h \circ f(t) = h_2^{-1} \circ f_1^{-1} \circ f \circ h(t),\\
x_5(t) =& f_2^{-1} \circ h_2^{-1} \circ h \circ f(t) = h_3^{-1} \circ f_1^{-1} \circ f \circ h(t),\\
x_6(t) =& f_2^{-1} \circ h_1^{-1} \circ h \circ f(t) = h_3^{-1} \circ f_2^{-1} \circ f \circ h(t),
\end{aligned}$$

Here we double indexing $x_{ij}(t)$ changed on simple $x_i(t)$ such that the intervals $\{x_i(t); t \in [0, \alpha]\}$ are ordered by growth. The values $x_1(t), \dots, x_6(t)$, for $t \in \left[0, \frac{2-\sqrt{3}}{4}\right] = f_1^{-1} \circ h_1^{-1}([0,1])$, divide the unit interval $[0,1]$ on the six intervals

$$\left[0, \tfrac{2-\sqrt{3}}{4}\right], \quad \left[\tfrac{2-\sqrt{3}}{4}, \tfrac{1}{4}\right], \quad \left[\tfrac{1}{4}, \tfrac{1}{2}\right], \quad \left[\tfrac{1}{2}, \tfrac{3}{4}\right], \quad \left[\tfrac{3}{4}, \tfrac{2+\sqrt{3}}{4}\right], \quad \left[\tfrac{2+\sqrt{3}}{4}, 1\right].$$

Now, assume that the system of functional equations $g_f = g_1 \wedge g_h = g_2$ has a solution $\tilde{g}$. Input $x = h_j^{-1} \circ h \circ f(t)$, $j = 1,2,3$, into $g_f(x) = \tilde{g}_f(x)$ and input $x = f_i^{-1} \circ f \circ h(t)$, $i = 1,2$, into $g_h(x) = \tilde{g}_h(x)$ we find five linear equation between $g(x_k(t))$, $k = 1, \dots, 6$. Leave out depending equations we find

$$g_f = \tilde{g}_f \wedge g_h = \tilde{g}_h \Longleftrightarrow$$

$$\begin{pmatrix} 0\\0\\0\\0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1\\ 0 & 1 & 0 & 0 & -1 & 0\\ 0 & 0 & 1 & -1 & 0 & 0\\ 0 & 0 & 0 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} g(x_1) - \tilde{g}(x_1)\\ g(x_2) - \tilde{g}(x_2)\\ g(x_3) - \tilde{g}(x_3)\\ g(x_4) - \tilde{g}(x_4)\\ g(x_5) - \tilde{g}(x_5)\\ g(x_6) - \tilde{g}(x_6) \end{pmatrix} \text{ for } t \in \left[0, \tfrac{2-\sqrt{3}}{4}\right].$$

It is equivalent to [33]

$$
\begin{aligned}
g(x_6) =& g(x_1)+\tilde{g}(x_6)-\tilde{g}(x_1)\\
g(x_5) =& g(x_2)+\tilde{g}(x_5)-\tilde{g}(x_2)\\
g(x_4) =& g(x_1)+g(x_2)-(\tilde{g}(x_1)+\tilde{g}(x_2))+\tilde{g}(x_4)\\
g(x_3) =& g(x_1)+g(x_2)-(\tilde{g}(x_1)+\tilde{g}(x_2))+\tilde{g}(x_3).
\end{aligned}
\tag{252}
$$

Thus we can compute $g(x_3),\dots,g(x_6)$ by using $g(x_1)$ a $g(x_2)$. Here $g(x_1)$ a $g(x_2)$ we must choice such that resulting $g(x)$ is non-decreasing on $[0,1]$. For absolutely continuous $g(x)$ and $\tilde{g}(x)$ the monotonicity of $g(x)$ is equivalent to $g'(x_i(t))\geq 0$ pre $i=1,\dots,6$ a.e. on $t\in\left[0,\frac{2-\sqrt{3}}{4}\right]$. Its has the form

$$
\begin{aligned}
g'(x_1)x_1'\geq 0,\quad g'(x_2)x_2' \leq& 0,\\
g'(x_1)x_1'+\tilde{g}'(x_6)x_6'-\tilde{g}'(x_1)x_1' \leq& 0,\\
g'(x_2)x_2'+\tilde{g}'(x_5)x_5'-\tilde{g}'(x_2)x_2' \geq& 0,\\
g'(x_1)x_1'+g'(x_2)x_2'-\tilde{g}'(x_1)x_1'-\tilde{g}'(x_2)x_2'+\tilde{g}'(x_3)x_3' \geq& 0,\\
g'(x_1)x_1'+g'(x_2)x_2'-\tilde{g}'(x_1)x_1'-\tilde{g}'(x_2)x_2'+\tilde{g}'(x_4)x_4' \leq& 0.
\end{aligned}
\tag{253}
$$

In (278) and (279) we shall add all terms $\tilde{g}(x_i)$ occur in $\tilde{g}_f$ and $\tilde{g}_h$, e.g.

$$
\begin{aligned}
g(x_4) &= g(x_1)+g(x_2)+1-\tilde{g}_f(h_2^{-1}\circ h\circ f(t))-\tilde{g}_h(f_1^{-1}\circ f\circ h(t))\\
&= g(x_1)+g(x_2)+1-\tilde{g}_f(f(x_2))-\tilde{g}_h(h(x_4)).
\end{aligned}
$$

In the next step modified (278) and (279) (in above way) we shall prepare by using $x_1(t)=t$ for $t\in\left[0,\frac{2-\sqrt{3}}{4}\right]$ and the identity

$$x_i(x_j(t))=x_i(t) \text{ pre } t\in[0,1] \text{ a } 1\leq i,j\leq 6,$$

which follows immediately from

$$f_i^{-1}\circ h_j^{-1}\circ h\circ f\circ f_k^{-1}\circ h_l^{-1}\circ h\circ f(t)=f_i^{-1}\circ h_j^{-1}\circ h\circ f(t).$$

[33] The same system of linear equations we find for the following functions

$$
f(x)=\begin{cases}2x, & \text{if } x\in[0,1/2]\\ 2-2x, & \text{if } x\in[1/2,1]\end{cases}
\qquad
h(x)=\begin{cases}3x, & \text{if } x\in[0,1/3],\\ 2-3x, & \text{if } x\in[1/3,2/3]\\ 3x-2, & \text{if } x\in[2/3,1]\end{cases}
$$

with the variables $x_1(t)=t$, $x_2(t)=2/6-t$, $x_3(t)=2/6+t$, $x_4(t)=4/6-t$, $x_5(t)=4/6+t$, $x_6(t)=1-t$, where $t\in[0,1/6]$.

E.g.

$$g(x_4) = g(x_1(x_4)) + g(x_2(x_4)) + 1 - \tilde{g}_f(f(x_2(x_4))) - \tilde{g}_h(h(x_4))$$

is the same as

$$g(x) = g(x_1(x)) + g(x_2(x)) + 1 - \tilde{g}_f(f(x_2(x))) - \tilde{g}_h(h(x)) \text{ pre } x \in [\tfrac{1}{2}, \tfrac{3}{4}].$$

Finally, we put

$$\psi(t) = g'(x_1(t))x_1'(t), \quad \phi(t) = -g'(x_2(t))x_2'(t) \text{ pre } t \in \left[0, \tfrac{2-\sqrt{3}}{4}\right],$$

and we find

$$\begin{aligned} g(x_1(t)) &= \int_0^t \psi(u)\mathrm{d}u, \\ g(x_2(t)) &= \int_t^{(2-\sqrt{3})/4} \phi(u)\mathrm{d}u + \int_0^{(2-\sqrt{3})/4} \psi(u)\mathrm{d}u. \end{aligned}$$

and we solve modified (279) with respec to $\psi$ a $\phi$. This gives theorem. Note that the continuity of $g$ of boundary points of using intervals follows directly. □

For possible application of Theorem 117 we add explicit formulas for $x_i(x), i = 1, 2, 3$ and the others are given by

$$x_4(x) = 1 - x_3(x), \quad x_5(x) = 1 - x_2(x), \quad x_6(x) = 1 - x_1(x) \text{ pre } x \in [0, 1].$$

Starting with (172) for $h_i^{-1}$, $h_j^{-1}$, $h_k^{-1}$, and putting $k = 1$ for $x \in [0, 1/4]$, $k = 2$ for $x \in [1/4, 3/4]$, and $k = 3$ for $x \in [3/4, 1]$, we find

$$h_{i,j}^{-1}(h(x)) = \frac{3-2x}{4} \pm \frac{\sqrt{3x(1-x)}}{2}.$$

Denoting

$$\begin{aligned} a_1(x) :=& f_1^{-1}\left(\frac{3-2f(x)}{4} - \frac{\sqrt{3f(x)(1-f(x))}}{2}\right) \\ =& \frac{1}{2} - \frac{1}{4}\sqrt{4|2x-1|\sqrt{3x(1-x)} - 8x^2 + 8x + 1} \end{aligned}$$

$$
\begin{aligned}
a_2(x) :=& f_1^{-1}\left(\frac{3-2f(x)}{4}+\frac{\sqrt{3f(x)(1-f(x))}}{2}\right)\\
=&\frac{1}{2}-\frac{1}{4}\sqrt{-4|2x-1|\sqrt{3x(1-x)}-8x^2+8x+1}
\end{aligned}
$$

we find

$$
x_1(x)=\begin{cases} x & \text{for } x\in\left[0,\frac{2-\sqrt{3}}{4}\right],\\ a_1(x) & \text{for } x\in\left[\frac{2-\sqrt{3}}{4},\frac{2+\sqrt{3}}{4}\right],\\ 1-x & \text{for } x\in\left[\frac{2+\sqrt{3}}{4},1\right],\end{cases} \tag{254}
$$

$$
x_2(x)=\begin{cases} a_1(x) & \text{for } x\in\left[0,\frac{2-\sqrt{3}}{4}\right],\\ x & \text{for } x\in\left[\frac{2-\sqrt{3}}{4},\frac{1}{4}\right],\\ a_2(x) & \text{for } x\in\left[\frac{1}{4},\frac{3}{4}\right],\\ 1-x & \text{for } x\in\left[\frac{3}{4},\frac{2+\sqrt{3}}{4}\right],\\ a_1(x) & \text{for } x\in\left[\frac{2+\sqrt{3}}{4},1\right],\end{cases} \tag{255}
$$

$$
x_3(x)=\begin{cases} a_2(x) & \text{for } x\in\left[0,\frac{1}{4}\right],\\ x & \text{for } x\in\left[\frac{1}{4},\frac{1}{2}\right],\\ 1-x & \text{for } x\in\left[\frac{1}{2},\frac{3}{4}\right],\\ a_2(x) & \text{for } x\in\left[\frac{3}{4},1\right].\end{cases} \tag{256}
$$

**Notes 19.** For $g_f = g_h$ see also Examples 115, 113, 112 in Section 10.6. For $g_f = g_1$ see also Theorem 67 and Example 39.

**Notes 20.** Let $a$ be nonzero integer. Pjateckiĭ-Šapiro in [128] proved:

1. The d.f. $g(x)$ is an a.d.f. of the sequence $\xi a^n \bmod 1$, $n = 1, 2, \ldots$ if and only if $g_f(x) = g(x)$, $x \in [0, 1]$ for $f(x) = xa \bmod 1$.
2. If the a.d.f. $g(x)$ of $\xi a^n \bmod 1$, $n = 1, 2, \ldots$ is absolute continuous, then $g(x) = x$, $x \in [0, 1]$.
3. If every d.f. $g(x)$ of $\xi a^n \bmod 1$, $n = 1, 2, \ldots$ satisfies $g(y) - g(x) \le c(y - x)$ for some $c > 0$ and for all $0 \le x \le y \le 1$, then the sequence $\xi a^n \bmod 1$ is u.d.

### 6.1.12 Equations for $g(x) \in G((3/2)^n \bmod 1)$ other than $g_f = g_h$ (open questions)

In the previous part for the sequence $\left\{\left(\frac{3}{2}\right)^n\right\}$ we study the sequence $\left\{\left\{\frac{3}{2}\right\}+\left\{\left(\frac{3}{2}\right)^2\right\}+\cdots+\left\{\left(\frac{3}{2}\right)^n\right\}\right\}$. By the following

$$
\begin{aligned}
&\left\{\left\{\frac{3}{2}\right\}+\left\{\left(\frac{3}{2}\right)^2\right\}+\cdots+\left\{\left(\frac{3}{2}\right)^n\right\}\right\}=\left\{\frac{3}{2}+\left(\frac{3}{2}\right)^2+\cdots+\left(\frac{3}{2}\right)^n\right\}\\
&=\left\{\frac{3}{2}\frac{\left(\frac{3}{2}\right)^n-1}{\frac{3}{2}-1}\right\}=\left\{3\left(\frac{3}{2}\right)^n\right\}=\left\{2\left(\frac{3}{2}\right)^{n+1}\right\}\\
&=\left\{3\left\{\left(\frac{3}{2}\right)^n\right\}\right\}=\left\{2\left\{\left(\frac{3}{2}\right)^{n+1}\right\}\right\}
\end{aligned}
\tag{257}
$$

we have that d.f.s of $\left\{\left\{\frac{3}{2}\right\}+\left\{\left(\frac{3}{2}\right)^2\right\}+\cdots+\left\{\left(\frac{3}{2}\right)^n\right\}\right\}$ are the same as $\left\{3\left\{\left(\frac{3}{2}\right)^n\right\}\right\}$ or d.f.s of $\left\{2\left\{\left(\frac{3}{2}\right)^{n+1}\right\}\right\}$. Other expression of such d.f.s are

$$
\begin{aligned}
&\left\{\left\{\frac{3}{2}\right\}+\left\{\left(\frac{3}{2}\right)^2\right\}+\cdots+\left\{\left(\frac{3}{2}\right)^n\right\}\right\}=\left\{3\left(\frac{3}{2}\right)^n\right\}=\left\{\frac{9}{2}\left(\frac{3}{2}\right)^{n-1}\right\}\\
&=\left\{\frac{9}{2}\left\{\left(\frac{3}{2}\right)^{n-1}\right\}+\frac{9}{2}\left[\left(\frac{3}{2}\right)^{n-1}\right]\right\},
\end{aligned}
\tag{258}
$$

where

$$
\left\{\frac{9}{2}\left(\frac{3}{2}\right)^{n-1}\right\}=\begin{cases}\left\{\frac{9}{2}\left\{\left(\frac{3}{2}\right)^{n-1}\right\}\right\} & \text{if } \left[\left(\frac{3}{2}\right)^{n-1}\right] \text{ is even,}\\ \left\{\frac{9}{2}\left\{\left(\frac{3}{2}\right)^{n-1}\right\}+\frac{1}{2}\right\} & \text{if } \left[\left(\frac{3}{2}\right)^{n-1}\right] \text{ is odd.}\end{cases}
\tag{259}
$$

Put

$$
\begin{aligned}
f(x)&=2x \bmod 1,\\
h(x)&=3x \bmod 1,\\
v(x)&=(9/2)x \bmod 1,\\
u(x)&=(9/2)x+(1/2) \bmod 1.
\end{aligned}
\tag{260}
$$

For some sequence $N=N_k$, $k\to\infty$, we have

$$
\begin{aligned}
&\frac{\#\{n\le N;\{(3/2)^n\}\in[0,x)\}}{N}\to g(x),\\
&\frac{\#\{n\le N;\{(3/2)^n\}\in[0,x),[(3/2)^n]\text{ is odd}\}}{N}\to g^{(1)}(x)
\end{aligned}
$$

$$\frac{\#\{n \le N; \{(3/2)^n\} \in [0,x), [(3/2)^n] \text{ is even}\}}{N} \to g^{(2)}(x) \qquad (261)$$

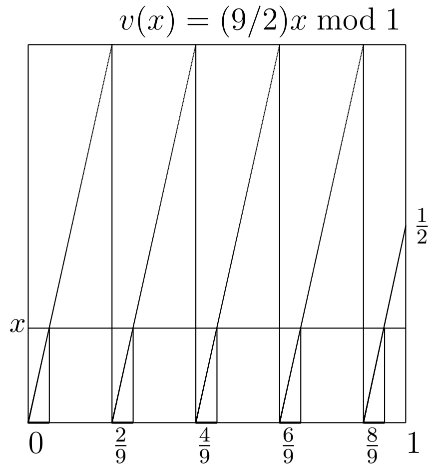
[34] As in (171): Let $f : [0,1] \to [0,1]$ be a function such that, for all $x \in [0,1]$, $f^{-1}([0,x))$ can be expressed as a sum of finitely many pairwise disjoint subintervals $I_i(x)$ of $[0,1]$ with endpoints $\alpha_i(x) \le \beta_i(x)$. For any distribution function $g(x)$ we put $g_f(x) = \sum_i g(\beta_i(x)) - g(\alpha_i(x))$. By Theorem 106 the sequence $f(\{x_n\})$ has the distribution functions $g_f(x)$ if $x_n$ has the d.f. $g(x)$.

**Theorem 118.** *Every d.f. $g(x)$ of the sequence $(3/2)^n \bmod 1$ can be expressed as $g(x) = g^{(1)}(x) + g^{(2)}(x)$ for every $x \in [0,1]$ such that*

$$g_u^{(1)}(x)+g_v^{(2)}(x) = g_f(x) = g_h(x) = g_f^{(1)}(x)+g_f^{(2)}(x) = g_h^{(1)}(x)+g_h^{(2)}(x). \quad (262)$$

Note that a solution of (262) is an open problem.

*Proof.* Dividing the sequence $(3/2)^n \bmod 1$ into two parts which respect to $[(3/2)^n]$ is odd or $[(3/2)^n]$ is even and then transform they applying $v(x)$ and $u(x)$, respectively, then we find the sequence (258) which is the same as (257). $\square$




For $x \le 1/2$ we have

$$g_v^{(2)}(x) = g^{(2)}\big((2/9)x\big) + g^{(2)}\big((2/9)x + (2/9)\big) - g^{(2)}\big((2/9)\big)$$

[34]By Section 3.8 if $g^{(1)}(x) > 0$ then Mahler's conjecture valid for $(3/2)^n \bmod 1$.

$$
\begin{aligned}
&+g^{(2)}\big((4/9)+(2/9)x\big)-g^{(2)}\big((4/9)\big)\\
&+g^{(2)}\big((6/9)+(2/9)x\big)-g^{(2)}\big((6/9)\big)\\
&+g^{(2)}\big((8/9)+(2/9)x\big)-g^{(2)}\big((8/9)\big). \qquad (263)
\end{aligned}
$$

The procedure (258) can be extend as

$$
\begin{aligned}
&\left\{\left\{\frac{3}{2}\right\}+\left\{\left(\frac{3}{2}\right)^2\right\}+\dots+\left\{\left(\frac{3}{2}\right)^n\right\}\right\}=\left\{3\left(\frac{3}{2}\right)^n\right\}\\
&=\left\{3\left(\frac{3}{2}\right)^k\left(\frac{3}{2}\right)^{n-k}\right\}=\left\{2\left(\frac{3}{2}\right)^{k+1}\left(\frac{3}{2}\right)^{n-k}\right\}. \qquad (264)
\end{aligned}
$$

Then the sequence $\left\{\left\{\frac{3}{2}\right\}+\left\{\left(\frac{3}{2}\right)^2\right\}+\dots+\left\{\left(\frac{3}{2}\right)^n\right\}\right\}$ has the same d.f.s as $\left\{\left(\frac{3}{2}\right)^n\right\}$ transformed by

$$
\left\{3\left(\frac{3}{2}\right)^k\left(\frac{3}{2}\right)^{n-k}\right\}=\left\{3\left(\frac{3}{2}\right)^k\left\{\left(\frac{3}{2}\right)^{n-k}\right\}+3\left(\frac{3}{2}\right)^k\left[\left(\frac{3}{2}\right)^{n-k}\right]\right\}
$$

i.e. by

$$
3\left(\frac{3}{2}\right)^k x+\frac{i(n)}{2^k} \bmod 1, \text{ where } \frac{i(n)}{2^k}=\left\{3\left(\frac{3}{2}\right)^k\left[\left(\frac{3}{2}\right)^{n-k}\right]\right\}. \qquad (265)
$$

Let $N_1 < N_2 < \dots$ be a sequence of indices such that

$$
\begin{aligned}
&\frac{\#\{n\le N_j;\{(3/2)^n\}\in[0,x)\}}{N_j}\to g(x),\\
&\frac{\#\{n\le N_j;\{(3/2)^n\}\in[0,x), i(n)=i\}}{N}\to g_i(x). \qquad (266)
\end{aligned}
$$

and denote

$$
\begin{aligned}
&u_i(x)=3\left(\frac{3}{2}\right)^k x+\frac{i}{2^k} \bmod 1,\\
&u_i^{-1}([0,x))=\cup_{j=1}^{s}[\alpha_j(x),\beta_j(x)),\\
&g_{i,u_i}(x)=\sum_{j=1}^{S}(g_i(\beta_j(x))-g_i(\alpha_j(x))).
\end{aligned}
$$

Theorem 118 can be extend to

**Theorem 119.** *Every d.f. $g(x)$ of the sequence $(3/2)^n \bmod 1$ can be expressed as*

$$g(x) = \sum_{i=0}^{2^k-1} g_i(x), \tag{267}$$

$$\sum_{i=0}^{2^k-1} g_{i,u_i}(x) = g_f(x) = g_h(x). \tag{268}$$

## 6.2 Ratio block sequences (continuation of 3.11)

Let $x_n$, $n = 1, 2, \dots$ be an increasing sequence of positive integers. The double sequence $x_m/x_n$, $m, n = 1, 2, \dots$ is called *the ratio sequence* of $x_n$; it was introduced by T. Šalát [140]. He studied its everywhere density. For further study of the ratio sequences, O. Strauch and J.T. Tóth [173] introduced a sequence $X_n$ of blocks

$$X_n = \left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}\right), \quad n = 1, 2, \dots$$

and they studied the set $G(X_n)$ of its d.f.s. The motivation is that the existence of strictly increasing $g(x) \in G(X_n)$ implies everywhere density of $x_m/x_n$, the basic problem by Šalát [140]. Further motivation is that the block sequences are a tool for study of d.f.s of sequences. Also for computing boundaries of limit points e.g. of the sequence $\frac{1}{n}\sum_{i=1}^{n} \frac{x_i}{x_n}$, $n = 1, 2, \dots$ we apply the Riemann-Stieltjes integration of $g(x) \in G(X_n)$.

### 6.2.1 Basic notations of $X_n = (x_1/x_n, \dots, x_n/x_n)$

Denote by $F(X_n, x)$ the step distribution function

$$F(X_n, x) = \frac{\#\{i \le n; \frac{x_i}{x_n} < x\}}{n},$$

for $x \in [0, 1)$ and $F(X_n, 1) = 1$. Directly from definition

$$F(X_m, x) = \frac{n}{m} F\left(X_n, x\frac{x_m}{x_n}\right) \tag{269}$$

for every $x \in [0, 1)$.

For any increasing integer sequence $x_n$, $n = 1, 2, \dots$, we define a counting function $A(t)$ as

$$A(t) = \#\{n \in \mathbb{N}; x_n < t\}.$$

Then for every $x \in (0, 1]$ we have

$$\frac{nF(X_n, x)}{xx_n} = \frac{A(xx_n)}{xx_n}. \tag{270}$$

A d.f. $g$ is a d.f. of the sequence of single blocks $X_n$, if there exists an increasing sequence of positive integers $n_1, n_2, \dots$ such that

$$\lim_{k\to\infty} F(X_{n_k}, x) = g(x)$$

a.e. on $[0, 1]$.

Denote by $G(X_n)$ the set of all d.f. of the sequence of single blocks $X_n$. $G(X_n)$ has the following properties (some of them will be proven later):

### 6.2.2 Overview of basic results of d.f.s $g(x) \in G(X_n)$

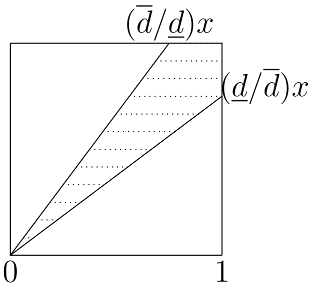
(i) If $g(x) \in G(X_n)$ increases and is continuous at $x = \beta$ and $g(\beta) > 0$, then there exists $1 \le \alpha < \infty$ such that $\alpha g(x\beta) \in G(X_n)$. If every d.f. of $G(X_n)$ is continuous at 1, then $\alpha = 1/g(\beta)$, [173, Prop. 3.1, Th. 3.2].

(ii) Assume that all d.f.s in $G(X_n)$ are continuous at 0 and $c_1(x) \notin G(X_n)$. Then for every $\tilde{g}(x) \in G(X_n)$ and every $1 \le \alpha < \infty$ there exists $g(x) \in G(X_n)$ and $0 < \beta \le 1$ such that $\tilde{g}(x) = \alpha g(x\beta)$ a.e., [173, Prop. 3.1, Th. 3.3].

(iii) Assume that all d.f.s in $G(X_n)$ are continuous at 1. Then all d.f.s in $G(X_n)$ are continuous on $(0, 1]$, i.e. only possible discontinuity is in 0, [173, Th. 4.1].

(iv) If $\underline{d}(x_n) > 0$, then for every $g(x) \in G(X_n)$ we have [173, Th. 6.2(iii)]

$$\frac{\underline{d}(x_n)}{\overline{d}(x_n)} x \le g(x) \le \frac{\overline{d}(x_n)}{\underline{d}(x_n)} x \text{ for every } x \in [0, 1]$$




Thus $\underline{d}(x_n) = \overline{d}(x_n) > 0$ implies u.d. of the block sequence $X_n$, $n = 1, 2, \dots$.

(v) If $\underline{d}(x_n) > 0$, then every $g(x) \in G(X_n)$ is continuous on $[0, 1]$, [173, Th. 6.2 (iv)].

(vi) If $\underline{d}(x_n) > 0$, then there exists $g(x) \in G(X_n)$ such that $g(x) \geq x$ for every $x \in [0, 1]$, [173, Th. 6.2(ii)]. Generally [11, Th. 6)] every $G(X_n)$ contains $g(x) \geq x$.

(vii) If $\overline{d}(x_n) > 0$, then there exists $g(x) \in G(X_n)$ such that $g(x) \leq x$ for every $x \in [0, 1]$, [173, Th. 6.2].

(viii) Assume that $G(X_n)$ is singleton, i.e. $G(X_n) = \{g(x)\}$. Then either $g(x) = c_0(x)$ for $x \in [0, 1]$; or $g(x) = x^\lambda$ for some $0 < \lambda \leq 1$ and $x \in [0, 1]$. Moreover, if $\overline{d}(x_n) > 0$, then $g(x) = x$, [173, Th. 8.2].

(ix) $\max_{g \in G(X_n)} \int_0^1 g(x)\mathrm{d}x \geq \frac{1}{2}$, c.f. (vi), [173, Th. 7.1.].

(x) Assume that every d.f. $g(x) \in G(X_n)$ has a constant value on the fixed interval $(u, v) \subset [0, 1]$ (maybe different). If $\underline{d}(x_n) > 0$ then all d.f.s in $G(X_n)$ has infinitely many intervals with constant values, [174].

(xi) There exists an increasing sequence $x_n$, $n = 1, 2, \dots$, of positive integers such that $G(X_n) = \{h_\alpha(x); \alpha \in [0, 1]\}$, where $h_\alpha(x) = \alpha$, $x \in (0, 1)$ is the constant d.f., [66, Ex. 1].

(xii) There exists an increasing sequence $x_n$, $n = 1, 2, \dots$, of positive integers such that $c_1(x) \in G(X_n)$ but $c_0(x) \notin G(X_n)$, where $c_0(x)$ and $c_1(x)$ are one–jump d.f.s with the jump of height 1 at $x = 0$ and $x = 1$, respectively.

(xiii) There exists an increasing sequence $x_n$, $n = 1, 2, \dots$, of positive integers such that $G(X_n)$ is non-connected, [66, Ex. 2].

(xiv) $G(X_n) = \{x^\lambda\}$ if and only if $\lim_{n\to\infty}(x_{k.n}/x_n) = k^{1/\lambda}$ for every $k = 1, 2, \dots$. Here as in (viii) we have $0 < \lambda \le 1$, [54].

(xv) L. Mišík: If $\underline{d}(x_n) > 0$, then all d.f.s $g(x) \in G(X_n)$ are continuous, nonsingular and bounded by $h_1(x) \le g(x) \le h_2(x)$, where [35]

$$h_1(x) = \begin{cases} x\frac{\underline{d}}{\overline{d}} & \text{if } x \in \left[0, \frac{1-\overline{d}}{1-\underline{d}}\right], \\ \frac{\underline{d}}{\frac{1}{x}-(1-\underline{d})} & \text{otherwise,} \end{cases} \qquad h_2(x) = \min\left(x\frac{\overline{d}}{\underline{d}}, 1\right).$$

Furthermore, there exists $x_n$, $n = 1, 2, \dots$, such that $h_2(x) \in G(X_n)$ and for every $x_n$ we have $h_1(x) \notin G(X_n)$, [11, Th. 7] and moreover

(xvi) for a given fixed $g(x) \in G(X_n)$ we have $h_{1,g}(x) \le g(x) \le h_{2,g}(x)$, where

$$h_{1,g}(x) = \begin{cases} x\frac{\underline{d}}{d_g} & \text{if } x < y_0 = \frac{1-d_g}{1-\underline{d}}, \\ x\frac{1}{d_g} + 1 - \frac{1}{d_g} & \text{if } y_0 \le x \le 1, \end{cases}$$
$$h_{2,g}(x) = \min\left(x\frac{\overline{d}}{d_g}, 1\right),$$

see [11, Th.6].

(xvii) These boundaries are established by observing that for every $g(x) \in G(X_n)$

$$0 \le \frac{g(y) - g(x)}{y - x} \le \frac{1}{d_g}$$

for $x < y$, $x, y \in [0, 1]$.

In the following we give proofs of some (i)–(xvii).

### 6.2.3 Basic theorems of $G(X_n)$

Using

$$x_i < x x_m \Longleftrightarrow x_i < \left(x\frac{x_m}{x_n}\right) x_n$$

[35] $\underline{d} = \underline{d}(x_n)$, $\overline{d} = \overline{d}(x_n)$.

and that these inequalities imply $i < m$, it directly follows from definition $F(X_n, x)$ it follows (269)

$$F(X_m, x) = \frac{n}{m} F\left(X_n, x\frac{x_m}{x_n}\right),$$

for every $m \le n$ and $x \in [0, 1)$. Also for any increasing sequence of positive integers $x_n$, $n = 1, 2, \ldots$, we define a counting function $A(t)$ as

$$A(t) = \#\{n \in \mathbb{N};\ x_n < t\}.$$

Then for every $x \in (0, 1]$ we have the equality (270)

$$\frac{nF(X_n, x)}{xx_n} = \frac{A(xx_n)}{xx_n},$$

which we shall use to compute the asymptotic density of $x_n$. We have the lower asymptotic density $\underline{d}$, and the upper asymptotic density $\overline{d}$ of $x_n$, $n = 1, 2, \ldots$ as

$$\underline{d} = \liminf_{t\to\infty} \frac{A(t)}{t} = \liminf_{n\to\infty} \frac{n}{x_n}, \quad \overline{d} = \limsup_{t\to\infty} \frac{A(t)}{t} = \limsup_{n\to\infty} \frac{n}{x_n}.$$

Using Helly's selection principle from the sequence $(m, n)$ we can select a subsequence $(m_k, n_k)$ such that $F(X_{n_k}, x) \to g(x)$, $F(X_{m_k}, x) \to \tilde{g}(x)$ as $k \to \infty$, furthermore $x_{m_k}/x_{n_k} \to \beta$ and $m_k/n_k \to \alpha$, but $\alpha$ may be infinity. These limits have the following connection.

**Theorem 120** ([173, Prop. 3.1])**.** *Let $m_k$ and $n_k$ be two increasing integer sequences satisfying $m_k \le n_k$, for $k = 1, 2, \ldots$ and assume that*

(i) $\lim_{k\to\infty} F(X_{n_k}, x) = g(x)$ *a.e.,*

(ii) $\lim_{k\to\infty} F(X_{m_k}, x) = \tilde{g}(x)$ *a.e.,*

(iii) $\lim_{k\to\infty} \frac{x_{m_k}}{x_{n_k}} = \beta > 0$*,*

(iv) $g(\beta - 0) > 0$.

*Then there exists* $\lim_{k\to\infty} \frac{n_k}{m_k} = \alpha < \infty$ *such that*

$$\tilde{g}(x) = \alpha g(x\beta) \text{ a.e. on } [0, 1], \text{ and } \alpha = \frac{\tilde{g}(1 - 0)}{g(\beta - 0)}. \tag{271}$$

*Proof.* Firstly we prove

$$\lim_{k\to\infty} F\left(X_{n_k}, x\frac{x_{m_k}}{x_{n_k}}\right) = g(x\beta). \tag{272}$$

Denoting $\beta_k = x_{m_k}/x_{n_k}$ and substituting $u = x\beta_k$, we find

$$\begin{aligned} 0 \le \int_0^1 (F(X_{n_k}, x\beta_k) - g(x\beta_k))^2 \mathrm{d}x &= \frac{1}{\beta_k}\int_0^{\beta_k} (F(X_{n_k}, u) - g(u))^2 \mathrm{d}u \\ &\le \frac{1}{\beta_k}\int_0^{1} (F(X_{n_k}, u) - g(u))^2 \mathrm{d}u \to 0, \end{aligned}$$

which leads to $(F(X_{n_k}, x\beta_k) - g(x\beta_k)) \to 0$ a.e. as $k\to\infty$ (here necessarily $\beta > 0$). Furthermore,

$$\int_0^1 (F(X_{n_k}, x\beta_k) - g(x\beta))^2 \mathrm{d}x = \int_0^1 (F(X_{n_k}, x\beta_k) - g(x\beta_k) + g(x\beta_k) - g(x\beta))^2 \mathrm{d}x$$

$$\le 2\left(\int_0^1 (F(X_{n_k}, x\beta_k) - g(x\beta_k))^2 \mathrm{d}x + \int_0^1 (g(x\beta_k) - g(x\beta))^2 \mathrm{d}x\right).$$

Since $g(x)$ is continuous a.e. on $[0,1]$ then $(g(x\beta_k) - g(x\beta)) \to 0$ a.e. and applying the Lebesgue theorem of dominant convergence we find $\int_0^1 (g(x\beta_k) - g(x\beta))^2 \mathrm{d}x \to 0$. This gives (272). The existence of the limit $\lim_{k\to\infty}\frac{n_k}{m_k} = \alpha < \infty$ follows from (1) and (iv). Now, let $t_n \in [0,1)$ increases to 1 and $\tilde{g}(x)$ be continuous in $t_n$. Then $g(x\beta)$ is also continuous in $t_n$ and $\tilde{g}(t_n) = \alpha g(t_n\beta)$ for $n = 1, 2, \dots$. The limit of this equation gives the desired form of $\alpha$. $\square$

In the simple form we have

Let $g(x) \in G(X_n)$, $\beta \in (0,1)$, and assuming that

(i) $g(x)$ is continuous at $\beta$,

(ii) $g(x)$ increases at $\beta$, [36]

(iii) $g(\beta) > 0$,

(iv) all d.f. in $G(X_n)$ are continuous at 1.

Then

$$\frac{g(x\beta)}{g(\beta)} \in G(X_n). \tag{273}$$

[36] The assumption (ii) can be replaced by a requirement that $\beta$ is a limit point of $\frac{x_i}{x_{n_k}}$, $i = 1, 2, \dots, n_k$, $k = 1, 2, \dots$, where weakly $F(X_{n_k}, x) \to g(x)$.

**Theorem 121** ([173, Prop.6.1]). *Assume for a sequence $n_k$, $k = 1, 2, \dots$ that*

(i) $\lim_{k\to\infty} F(X_{n_k}, x) = g(x)$,

(ii) $\lim_{k\to\infty} \frac{n_k}{x_{n_k}} = d_g$.

*Then there exists*

(iii) $\lim_{k\to\infty} \frac{A(xx_{n_k})}{xx_{n_k}} = d_g(x)$ *and*

$$g(x) = \frac{x}{d_g} d_g(x). \tag{274}$$

*Here the limits* (i) *and* (iii) *can be considered for all $x \in (0, 1]$ or all continuity points $x \in (0, 1]$ of $g(x)$.*

*Proof.* Directly by (270). □

**Definition 8.** If $\lim_{k\to\infty} F(X_{n_k}, x) = g(x)$ and $\lim_{k\to\infty} \frac{n_k}{x_{n_k}} = d_g$. we shall call $d_g$ as a *local asymptotic density* for d.f. $g(x)$.

### 6.2.4 Continuity of $g \in G(X_n)$

If all $g \in G(X_n)$ are everywhere continuous on $[0, 1]$, then relation (271) is of the form

$$\frac{g(x\beta)}{g(\beta)} \in G(X_n). \tag{275}$$

As a criterion for continuity of all $g \in G(X_n)$ we can adapt the Wiener-Schoenberg theorem (cf. [92, 6, p. 55]), but here we give the following simple sufficient condition.

**Theorem 122** ([173, Th. 4.1]). *Assume that all d.f.s in $G(X_n)$ are continuous at 1. Then all d.f. in $G(X_n)$ are continuous on $(0, 1]$, i.e. the only discontinuity point may be 0.*

*Proof.* Assume that $x_{m_k}/x_{n_k} \to \beta$ and $F(X_{n_k}, x) \to g(x)$ as $k \to \infty$. If from $(m_k, n_k)$ we can select two sequences $(m'_k, n'_k)$ and $(m''_k, n''_k)$ such that $n'_k/m'_k \to \alpha_1$ and $n''_k/m''_k \to \alpha_2$ with a finite $\alpha_1 \neq \alpha_2$, then $\alpha_1 g(x\beta), \alpha_2 g(x\beta) \in G(X_n)$ and thus one of such d.f. $\tilde{g}(x)$ must be discontinuous at 1 (it holds also for $g$ continuous at $\beta$). Thus, assuming that $G(X_n)$ has only continuous d.f.s at 1, the limits $x_{m_k}/x_{n_k} \to \beta > 0$ and $F(X_{n_k}, x) \to g(x)$ imply the convergence of $n_k/m_k$. Now by [173, Th. 3.2]: If $\beta$ is a point of discontinuity of $g(x)$ with $g(\beta+0) - g(\beta-0) = h > 0$, then there exists a closed interval $I \subset [0,1]$, with length $|I| \geq h$ such that for every $\frac{1}{\alpha} \in I$ we have $\alpha g(x\beta) \in G(X_n)$. Thus $g(x)$ cannot have a discontinuity point in $(0,1]$. $\square$

**Theorem 123** ([173, Th. 6.2])**.** (i) *If $\overline{d} > 0$, then there exits $g \in G(X_n)$ such that $g(x) \leq x$ for every $x \in [0,1]$.*

(ii) *If $\underline{d} > 0$, then there exits $g \in G(X_n)$ such that $g(x) \geq x$ for every $x \in [0,1]$.*

(iii) *If $\underline{d} > 0$, then for every $g \in G(X_n)$ we have*

$$(\underline{d}/\overline{d})x \leq g(x) \leq (\overline{d}/\underline{d})x \tag{276}$$

*for every $x \in [0,1]$.*

(iv) *If $\underline{d} > 0$, then every $g \in G(X_n)$ is everywhere continuous in $[0,1]$.*

(v) *If $\underline{d} > 0$, then for every limit point $\beta > 0$ of $x_m/x_n$ there exist $g \in G(X_n)$ and $0 \leq \alpha < \infty$ such that $\alpha g(x\beta) \in G(X_n)$.*

*Proof.* (i). Assume that $n_k/x_{n_k} \to \overline{d}$ as $k \to \infty$. Select a subsequence $n'_k$ of $n_k$ such that $F(X_{n'_k}, x) \to g(x)$ a.e. on $[0,1]$. Since $d_g(x) \leq \overline{d}$ a.e. in (274) gives $(g(x)/x)\overline{d} \leq \overline{d}$ a.e., which leads to $g(x) \leq x$ a.e. and implies $g(x) \leq x$ for every $x \in [0,1]$.

(ii). Similarly to (i), let $n_k/x_{n_k} \to \underline{d}$ as $k \to \infty$. Select a subsequence $n'_k$ of $n_k$ such that $F(X_{n'_k}, x) \to g(x)$ a.e. on $[0,1]$. Since $d_2(x) \geq \underline{d}$ a.e., (274) implies $(g(x)/x)\underline{d} \geq \underline{d}$ a.e. again, which gives $g(x) \geq x$ a.e., whence, $g(x) \geq x$ everywhere on $x \in [0,1]$.

(iii). For any $g \in G(X_n)$ there exists $n_k$ such that $F(X_{n_k}, x) \to g(x)$ a.e. From $n_k$ we can choose a subsequence $n'_k$ such that $n'_k/x_{n'_k} \to d_1$. Using (274) and the fact that $\underline{d} \leq d_1 \leq \overline{d}$ and $\underline{d} \leq d_2 \leq \overline{d}$ we have $(g(x)/x)\underline{d} \leq \overline{d}$ and $(g(x)/x)\overline{d} \geq \underline{d}$ a.e. If $\underline{d} > 0$, these inequalities are valid for every $x \in (0,1]$.

(iv). Continuity of $g \in G(X_n)$ at 1 follows from [173, Prop 4.2]: Denote

$$\overline{d}(\varepsilon) = \limsup_{n\to\infty} \frac{\#\{i \leq n; (1-\varepsilon)x_n < x_i < x_n\}}{n}.$$

Every $g\in G(X_n)$ is continuous at 1 if and only if $\lim_{\varepsilon\to 0}\overline{d}(\varepsilon)=0$. Since

$$\overline{d}(\varepsilon)\le\limsup_{n\to\infty}\varepsilon\frac{x_n}{n}=\frac{\varepsilon}{\underline{d}},$$

applying [173, Th. 4.1]= Theorem 18, we have continuity of $g$ in $(0,1]$. Continuity at 0 follows from (276).

(v). It follows from the fact that if $\underline{d}>0$ and $\lim_{k\to\infty}x_{m_k}/x_{n_k}=\beta>0$ for $m_k<n_k$, then $\limsup_{k\to\infty}n_k/m_k<\infty$. More precisely, if we pick $(m_k',n_k')$ from $(m_k,n_k)$ such that $n_k'/m_k'\to\alpha$, then

$$\frac{\underline{d}}{\overline{d}\beta}\le\alpha\le\frac{\overline{d}}{\underline{d}\beta}. \tag{277}$$

This is so because if we select $(m_k'',n_k'')$ from $(m_k',n_k')$ such that $n_k''/x_{n_k''}\to d_1$ and $m_k''/x_{m_k''}\to d_2$, then, by

$$\frac{n_k''}{m_k''}=\frac{\frac{n_k''}{x_{n_k''}}x_{n_k''}}{\frac{m_k''}{x_{m_k''}}x_{m_k''}},$$

we see $\alpha=d_1/(d_2\beta)$. □

### 6.2.5 Singleton $G(X_n)=\{g\}$

For general $G(X_n)$, the connection between $G(X_n)$ and $G(x_m/x_n)$ is open, but for singleton $G(X_n)$ we have

**Theorem 124** ([173, Th. 8.1]). *If $G(X_n)=\{g\}$, then $G(x_m/x_n)=\{g\}$.*

*Proof.* A proof of the theorem is the same as the proof of [130, Prop. 1, (ii)], since

$$\lim_{n\to\infty}\frac{|X_n|}{|X_1|+\cdots+|X_n|}=\lim_{n\to\infty}\frac{n}{n(n+1)/2}=0.$$

□

**Theorem 125** ([173, Th. 8.2]). *Assume that $G(X_n)=\{g\}$. Then either*

(i) *$g(x)=c_0(x)$ for $x\in[0,1]$ or*

(ii) *$g(x)=x^\lambda$ for some $0<\lambda\le 1$ and $x\in[0,1]$. Moreover,*

(iii) *if* $\overline{d} > 0$ *then* $g(x) = x$.

*Proof.* Let $G(X_n) = \{g\}$. We divide the proof into the following six steps.

(I). By [173, Th. 7.1], we have $\int_0^1 g(x)\mathrm{d}x \geq \frac{1}{2}$ which implies $g(x) \neq c_1(x)$.

(II). $g$ must be continuous on $(0, 1)$, since otherwise [173, Th. 3.2], for a discontinuity point $\beta \in (0, 1)$, guarantees the existence of $\alpha_1 \neq \alpha_2$ such that $\alpha_1 g(x\beta) = \alpha_2 g(x\beta) = g(x)$ a.e. which is a contradiction.

(III). Assume that $g(x)$ increases in every point $\beta \in (0, 1)$. In this case relation (5) gives the well-known Cauchy equation $g(x)g(\beta) = g(x\beta)$ for a.e. $x, \beta \in [0, 1]$ For a monotonic $g(x)$ the Cauchy equation has solutions only of the type $g(x) = x^\lambda$.

(IV). Assume that $g(x)$ has a constant value on the interval $(\gamma, \delta) \subset [0, 1]$. For $\beta \in (0, 1]$ satisfying (j) $g(x)$ increases in $\beta$ and (jj) $g(\beta) > 0$ the basic relation (271) gives $g(x) = \alpha g(x\beta)$ which implies that $g(x)$ has a constant value also on $\beta(\gamma, \delta)$ and if $\delta \leq \beta$ then also on $\beta^{-1}(\gamma, \delta)$. Thus, if $(\gamma_i, \delta_i)$, $i \in \mathcal{I}$ is a system of all intervals (maximal under inclusion) in which $g(x)$ possesses constant values, then for every $i \in \mathcal{I}$ there exists $j \in \mathcal{I}$ such that $\beta(\gamma_i, \delta_i) = (\gamma_j, \delta_j)$ and vice-versa for every $j \in \mathcal{I}$, $\delta_j \leq \beta$, there exists $i \in \mathcal{I}$ such that $\beta^{-1}(\gamma_j, \delta_j) = (\gamma_i, \delta_i)$. This is true also for $\beta = \beta_1^{n_1}\beta_2^{n_2}\dots$, where $\beta_1, \beta_2, \dots$ satisfy (j) and (jj) and $n_1, n_2, \dots \in \mathbb{Z}$. Thus, there exists $0 < \theta < 1$ such that every such $\beta$ has the form $\theta^n$, $n \in \mathbb{N}$. The end points $\gamma_i$, $\delta_i$ (without $\gamma_i = 0$) satisfy (j) and (jj) and thus the intervals $(\gamma_i, \delta_i)$ is of the form $(\theta^n, \theta^{n-1})$, $n = 1, 2, \dots$ and all discontinuity points of $g(x)$ are $\theta^n$, $n = 1, 2, \dots$, a contradiction with (II). For $g(x) = c_0(x)$ there exists no $\beta \in (0, 1]$ satisfying (j) and (jj).

(V). We have the possibilities $g(x) = c_0(x)$ and $g(x) = x^\lambda$ for some $\lambda > 0$. Applying [173, Th. 7.1] we have $\int_0^1 g(x)\mathrm{d}x \geq 1/2$ which reduces $\lambda$ to $\lambda \leq 1$.

(VI). If $\overline{d} > 0$, then by [173, Th. 6.2, (i)]= Theorem 125 must be $g(x) \leq x$ which is contrary to $x^\lambda > x$ for $\lambda < 1$. □

**Notes 21.** The possibilities (i), (ii) are achievable. Trivially, for $x_n = [n^\lambda]$, $G(X_n) = \{x^{1/\lambda}\}$ and for $x_n$ satisfying $\lim_{n\to\infty} x_n/x_{n+1} = 0$ we have $G(X_n) = \{c_0(x)\}$. Less trivially, every lacunary $x_n$, i.e. $x_n/x_{n+1} \leq \lambda < 1$, gives $G(X_n) = \{c_0(x)\}$.

The following limit covers all of $G(X_n) = \{g\}$.

**Theorem 126** ([173, Th. 8.3])**.** *The set* $G(X_n)$ *is a singleton if and only if*

$$\lim_{m,n\to\infty} \left( \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \left| \frac{x_i}{x_m} - \frac{x_j}{x_n} \right| \right.$$

$$-\frac{1}{2m^2}\sum_{i,j=1}^{m}\left|\frac{x_i}{x_m}-\frac{x_j}{x_m}\right|-\frac{1}{2n^2}\sum_{i,j=1}^{n}\left|\frac{x_i}{x_n}-\frac{x_j}{x_n}\right|\Bigg)=0. \tag{278}$$

*Proof.* It follows directly from the limit (278) in the form

$$\lim_{m,n\to\infty}\int_0^1(F(X_m,x)-F(X_n,x))^2\mathrm{d}x=0,$$

after applying

$$\int_0^1(g(x)-\tilde{g}(x))^2\mathrm{d}x=\int_0^1\int_0^1|x-y|\mathrm{d}g(x)\mathrm{d}\tilde{g}(y)$$
$$-\frac{1}{2}\int_0^1\int_0^1|x-y|\mathrm{d}g(x)\mathrm{d}g(y)-\frac{1}{2}\int_0^1\int_0^1|x-y|\mathrm{d}\tilde{g}(x)\mathrm{d}\tilde{g}(y) \tag{279}$$

for $g(x)=F(X_m,x)$ and $\tilde{g}(x)=F(X_n,x)$. □

### 6.2.6 One-step d.f. $c_\alpha(x)$ of $G(X_n)$

In [173] there is proved that singleton $G(X_n)=\{c_1(x)\}$ does not exist, since (by [173, Th. 7.1]) for every increasing sequence $x_n$ of positive integers we have

$$\max_{g(x)\in G(X_n)}\int_0^1 g(x)\mathrm{d}x\geq\frac{1}{2}. \tag{280}$$

In [173] is also proved (see Th. 8.4, 8.5) that

**Theorem 127.**

$$G(X_n)=\{c_0(x)\}\Longleftrightarrow\lim_{n\to\infty}\frac{1}{nx_n}\sum_{i=1}^{n}x_i=0, \tag{281}$$

$$G(X_n)=\{c_0(x)\}\Longleftrightarrow\lim_{n\to\infty}\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left|\frac{x_i}{x_m}-\frac{x_j}{x_n}\right|=0, \tag{282}$$

$$G(X_n)\subset\{c_\alpha(x);\alpha\in[0,1]\}\Longleftrightarrow\lim_{n\to\infty}\frac{1}{n^2x_n}\sum_{i,j=1}^{n}|x_i-x_j|=0. \tag{283}$$

*Proof.* (281). $\int_0^1 x\mathrm{d}g(x)=1-\int_0^1 g(x)\mathrm{d}x=0$ only if $g(x)=c_0(x)$.

(282). Assume that $F(X_{m_k},x)\to\tilde{g}(x)$ and $F(X_{n_k},x)\to g(x)$ a.e. as $k\to\infty$. Riemann-Stieltjes integration yields

$$\frac{1}{m_kn_k}\sum_{i=1}^{m_k}\sum_{j=1}^{n_k}\left|\frac{x_i}{x_{m_k}}-\frac{x_j}{x_{n_k}}\right|=\int_0^1\int_0^1|x-y|\mathrm{d}F(X_{m_k},x)\mathrm{d}F(X_{n_k},y)\quad(284)$$

which, after using Helly's theorem, tends to

$$\int_0^1\int_0^1|x-y|\mathrm{d}\tilde{g}(x)\mathrm{d}g(y)\quad(285)$$

as $k\to\infty$. Then (285) is equal to 0 if and only if $\tilde{g}(x)=g(x)=c_\alpha(x)$ for some fixed $\alpha\in[0,1]$. By Theorem 125, $\alpha$ must be 0. ($\overline{d}=0$ follows from Theorem 123, part (i)).

(283). Again $\int_0^1\int_0^1|x-y|\mathrm{d}g(x)\mathrm{d}g(y)=0$ if and only if $g(x)=c_\alpha(x)$ for $\alpha\in[0,1]$ and thus

$$\lim_{k\to\infty}\frac{1}{n_kn_k}\sum_{i=1}^{n_k}\sum_{j=1}^{n_k}\left|\frac{x_i}{x_{n_k}}-\frac{x_j}{x_{n_k}}\right|=0$$

for every $n_k\to\infty$. □

Furthermore, if $G(X_n)\subset\{c_\alpha(x);\alpha\in[0,1]\}$, then $\underline{d}(x_n)=0$. Here we prove that

**Theorem 128** ([66, Th. 6])**.** *Let $x_n$, $n=1,2,\dots$, be an increasing sequence of positive integers. Assume that $G(X_n)\subset\{c_\alpha(x);\alpha\in[0,1]\}$. Then $c_0(x)\in G(X_n)$ and if $G(X_n)$ contains two different d.f.s, then also $c_1(x)\in G(X_n)$.*

*Proof.* We start from the equation (2) (see [173, p. 756, (1)])

$$F(X_m,x)=\frac{n}{m}F\left(X_n,x\frac{x_m}{x_n}\right),$$

which is valid for every $m\le n$ and $x\in[0,1]$. Assuming, for two increasing sequences of indices $m_k\le n_k$, that, as $k\to\infty$

(i) $F(X_{m_k},x)\to c_{\alpha_1}(x)$ a.e.,

(ii) $F(X_{n_k},x)\to c_{\alpha_2}(x)$ a.e.,

(iii) $\frac{n_k}{m_k} \to \gamma$,

(iv) $\frac{x_{m_k}}{x_{n_k}} \to \beta$,

(such sequences $m_k \leq n_k$ exist by Helly theorem) then we have:

a) If $\beta > 0$ and $\gamma < \infty$ (see (3) in [173]), then

$$c_{\alpha_1}(x) = \gamma c_{\alpha_2}(x\beta) \tag{13}$$

for almost all $x \in [0, 1]$.

b) If $\beta = 0$ and $\gamma < \infty$, then by Helly theorem there exists subsequence $(m'_k, n'_k)$ of $(m_k, n_k)$ such that $F\left(X_{n'_k}, x\frac{x_{m'_k}}{x_{n'_k}}\right) \to h(x)$ a.e. and since

$$F\left(X_{n_k}, x\frac{x_{m'_k}}{x_{n'_k}}\right) \leq F(X_{n_k}, x\beta')$$

for every $\beta' > 0$ and sufficiently large $k$, we get $h(x) \leq c_{\alpha_2}(x\beta')$. Summarizing, we have

$$c_{\alpha_1}(x) \leq \gamma c_{\alpha_2}(x\beta') \tag{14}$$

for every $\beta' > 0$ a.e. on $[0, 1]$.

We distinguish the following steps (notions (i)-(iv), a) and b) are preserve):

$1^0$. Let $c_{\alpha_1}(x) \in G(X_n)$, $0 \leq \alpha_1 < 1$, and let $m_k$, $k = 1, 2, \dots$, be an increasing sequence of positive integers for which

(i)$F(X_{m_k}, x) \to c_{\alpha_1}(x)$.

Relatively to the $m_k$, we choose an arbitrary sequence $n_k$, $m_k \leq n_k$, such that

(iii)$\frac{n_k}{m_k} \to \gamma$, $1 < \gamma < \infty$.

From $(m_k, n_k)$ we select a subsequence $(m'_k, n'_k)$ such that

(ii)$F(X_{n'_k}, x) \to c_{\alpha_2}(x)$ a.e. on $[0, 1]$,

(iv)$\frac{x_{m'_k}}{x_{n'_k}} \to \beta$ for some $\beta \in [0, 1]$.

a) If $\beta > 0$, then (13) $c_{\alpha_1}(x) = \gamma c_{\alpha_2}(x\beta)$ a.e. is impossible, because $\gamma > 1$ and for $x > \alpha_1$ we have $c_{\alpha_1}(x) = 1$. Thus $\beta = 0$.

b) The condition $\beta = 0$ implies (14) $c_{\alpha_1}(x) \leq \gamma c_{\alpha_2}(x\beta')$ for every $\beta' > 0$ and a.e. on $x \in [0, 1]$. If $\alpha_2 > 0$, then $c_{\alpha_2}(x\beta') = 0$ for all $x < \frac{\alpha_2}{\beta'}$, which implies, using $\beta' \leq \alpha_2$, that $c_{\alpha_1}(x) = 0$ for $x \in (0, 1)$, and this is contrary to the assumption $\alpha_1 < 1$.

Thus $\alpha_2 = 0$ and we have: If $0 \le \alpha_1 < 1$ and $c_{\alpha_1}(x) \in G(X_n)$ then $c_0(x) \in G(X_n)$. Now, applying [173, Th. 7.1] we have $\max_{c_\alpha(x)\in G(X_n)} \int_0^1 c_\alpha(x)\mathrm{d}x = 1-\alpha \ge \frac{1}{2}$. Then the assumption $c_{\alpha_1}(x) \in G(X_n)$, $0 \le \alpha_1 < 1$ is true, thus $c_0(x) \in G(X_n)$ holds.
$2^0$. In this case we start with the sequence $n_k$ and we assume that $c_{\alpha_2}(x) \in G(X_n)$, $0 < \alpha_2 \le 1$, and

(ii)$F(X_{n_k}, x) \to c_{\alpha_2}(x)$ a.e. on $[0,1]$.

Then we choose arbitrary $m_k$ such that $m_k \le n_k$ and

(iii)$\frac{n_k}{m_k} \to \gamma$, $1 < \gamma < \infty$.

From $(m_k, n_k)$ we select a subsequence $(m'_k, n'_k)$ such that

(ii)$F(X_{m'_k}, x) \to c_{\alpha_1}(x)$ a.e. on $[0,1]$,

(iv)$\frac{x_{m'_k}}{x_{n'_k}} \to \beta$ for some $\beta \in [0,1]$.

a) If $\beta > 0$, then by (13) $c_{\alpha_1}(x) = \gamma c_{\alpha_2}(x\beta)$ a.e. If $\alpha_1 < 1$, then $\gamma > 1$ implies $c_{\alpha_1}(x) > 1$ for some $x \in (0,1)$, a contradiction. Thus $\alpha_1 = 1$ (in this case $\beta \le \alpha_2$).
b) Now, $\beta = 0$ implies (14) $c_{\alpha_1}(x) \le \gamma c_{\alpha_2}(x\beta')$ for every $\beta' > 0$ and a.e. on $x \in [0,1]$ and the assumption $\alpha_2 > 0$ implies $c_{\alpha_2}(x\beta') = 0$ for all $x < \frac{\alpha_2}{\beta'}$, which gives $\alpha_1 = 1$. Summarizing, if $G(X_n)$ contains two different d.f.s, then it contains $c_0(x)$ and $c_1(x)$ simultaneously. □

### 6.2.7 Connectivity of $G(X_n)$

As we have proved in Theorem 6, for a usual sequence $y_n$ the set $G(y_n)$ of all d.f. of $y_n$ is nonempty, closed and connected in the weak topology, and consists either of one or infinitely many functions. The closedness of $G(X_n)$ is clear, but connectivity of $G(X_n)$ is does not have to pay. A general block sequence $Y_n$ with non-connected $G(Y_n)$ can be found trivially. For our special $X_n$ we have only the following sufficient condition.

**Theorem 129** ([173, Th. 5.1])**.** *If*

$$\lim_{n\to\infty}\left(\frac{1}{n(n+1)}\sum_{i=1}^{n+1}\sum_{j=1}^{n}\left|\frac{x_i}{x_{n+1}}-\frac{x_j}{x_n}\right| - \frac{1}{2(n+1)^2}\sum_{i,j=1}^{n+1}\left|\frac{x_i}{x_{n+1}}-\frac{x_j}{x_{n+1}}\right| - \frac{1}{2n^2}\sum_{i,j=1}^{n}\left|\frac{x_i}{x_n}-\frac{x_j}{x_n}\right|\right) = 0, \tag{286}$$

*then $G(X_n)$ is connected in the weak topology.*

*Proof.* The connection follows from the limit

$$\lim_{n\to\infty}\int_0^1 (F(X_{n+1},x)-F(X_n,x))^2\,\mathrm{d}x = 0,$$

since by a theorem of H. G. Barone [16] (Theorem 8 in this book) if $t_n$ is a sequence in a metric space $(X,\rho)$ satisfying

(i) any subsequence of $t_n$ contains a convergent subsequence and

(ii) $\lim_{n\to\infty}\rho(t_n,t_{n+1})=0$,

then the set of all limit points of $t_n$ is connected. Next we use the expression

$$\int_0^1 (g(x)-\tilde{g}(x))^2\mathrm{d}x = \int_0^1\int_0^1 |x-y|\mathrm{d}g(x)\mathrm{d}\tilde{g}(y) - \frac{1}{2}\int_0^1\int_0^1 |x-y|\mathrm{d}g(x)\mathrm{d}g(y) - \frac{1}{2}\int_0^1\int_0^1 |x-y|\mathrm{d}\tilde{g}(x)\mathrm{d}\tilde{g}(y).$$

Putting $g(x)=F(X_{n+1},x)$ and $\tilde{g}(x)=F(X_n,x)$ we get the desired limit. [37] □

As a consequence we have:

**Theorem 130.** *If $\lim_{n\to\infty}\frac{x_n}{x_{n+1}}=1$, then $G(X_n)$ is connected.*

*Proof.* After some manipulation (437) it follows from

$$\lim_{n\to\infty}\left(\frac{1}{nx_n}\sum_{i=1}^{n}x_i\right)\left(1-\frac{x_n}{x_{n+1}}\right)=0.$$

□

Note that by [173, Th. 4.1] all d.f.'s in $G(X_n)$ are continuous everywhere on $[0,1]$ if they are continuous at 0 and 1.

In [173, Th. 3.2] is proved that if $g(x)\in G(X_n)$, $g(x)$ increases at $\beta\in[0,1)$, $g(\beta)>0$, then there exists $\alpha\in[1,\infty)$ such that $\alpha g(x\beta)\in G(X_n)$.

[37] $\rho^2(g,\tilde{g})=\int_0^1(g(x)-\tilde{g}(x))^2\mathrm{d}x$.
In Example 99 is given $X_n$ such that $G(X_n)$ is connected but $\limsup_{n\to\infty}\rho(t_{n+1},t_n)=1$.

Using this fact, we can define on $G(X_n)$ the relation $\tilde{g}(x) \prec g(x)$ if there exist $\alpha, \beta$ such that $\tilde{g}(x) = \alpha g(x\beta)$. For every element $g(x) \in G(X_n)$ we define $[g(x)]$ as the set of all $\tilde{g}(x) \in G(X_n)$ for which $\tilde{g}(x) \prec g(x)$. Assuming that all d.f.s in $G(X_n)$ are continuous and strictly increasing, then we have

$$[g(x)] = \{g(x\beta)/g(\beta); \beta \in (0, 1]\}.$$

Denote as $G(g(x))$ the set of all possible limits $\lim_{k\to\infty} g(x\beta_k)/g(\beta_k)$, where $\beta_k \to 0$ and put

$$[g(x)]^* = [g(x)] \cup G(g(x)).$$

**Theorem 131.** *Assume that all d.f.s in $G(X_n)$ are continuous and strictly increasing. If $G(X_n) = \cup_{i=1}^k [g_i(x)]^*$, then $G(X_n)$ is connected if and only if $g_i(x)$, $i = 1, 2, \dots, k$ can be reordered into $g_{i_n}(x)$, $n = 1, 2, \dots, k$ such that*

*(i)* $[g_{i_n}(x)]^* \cap [g_{i_{n+1}}(x)]^* \neq \emptyset$, $n = 1, 2, \dots, k-1$.

*Proof.* $1^0$. Firstly we prove that $[g(x)]^*$ is nonempty, closed and connected, for every $g(x) \in G(X_n)$. Note that, in the following we say that we can go connectively $g_1(x) \to g_2(x)$ through the set $H$ if for every $\varepsilon > 0$ there exists a chain $g_{i_n}(x) \in H$, $n = 1, 2, \dots, m$ such that $\rho(g_1, g_{i_1}) < \varepsilon$, $\rho(g_{i_2}, g_{i_3}) < \varepsilon, \dots,$ $\rho(g_{i_m}, g_2) < \varepsilon$.

*Connectivity:* If $g_1(x) = g(x\beta_1)/g(\beta_1)$ and $g_2(x) = g(x\beta_2)/g(\beta_2)$ then we can go connectively $g_1(x) \to g_2(x)$ through $g(x\beta)/g(\beta)$, where $\beta$ is between $\beta_1$ and $\beta_2$, since

$$\frac{g(x\beta)}{g(\beta)} - \frac{g(x\beta')}{g(\beta')} = \left(\frac{g(x\beta) - g(x\beta')}{g(\beta)} + g(x\beta')\frac{g(\beta') - g(\beta)}{g(\beta)g(\beta')}\right) \to 0$$

as $(\beta' - \beta) \to 0$, where $\beta, \beta' \geq \varepsilon > 0$.

If $g_1(x) = \lim_{k\to\infty} g(x\beta_k)/g(\beta_k)$ and $g_2(x) = \lim_{k\to\infty} g(x\beta'_k)/g(\beta'_k)$, then we can go connectively

$$g_1(x) \to g(x\beta_k)/g(\beta_k) \to g(x\beta'_k)/g(\beta'_k) \to g_2(x)$$

through $[g(x)]$. Similarly for the rest

$$g_1(x) = g(x\beta_1)/g(\beta_1) \text{ and } g_2(x) = \lim_{k\to\infty} g(x\beta_k)/g(\beta_k).$$

*Closedness:* If $\lim_{k\to\infty} g(x\beta_k)/g(\beta_k) = g_1(x)$, we can select $\beta_k$ such that $\beta_k \to \beta$. If $\beta > 0$, then from continuity $g(x)$ we have $g_1(x) = g(x\beta)/g(\beta)$. The closedness of $G(g(x))$ follows from definition of $G(g(x))$.

$2^0$. Assume that (i) holds and select $g_n^*(x) \in [g_{i_n}(x)]^* \cap [g_{i_{n+1}}(x)]^*$, $i = 1, 2, \dots, k-1$. Let $g_1(x) \in [g_{i_1}(x)]^*$ and $g_2(x) \in [g_{i_3}(x)]^*$. Then we can go connectively

$$g_1(x) \to \frac{g_{i_1}(x\beta_1)}{g_{i_1}(\beta_1)} \to g_1^*(x) \to \frac{g_{i_2}(x\beta_2)}{g_{i_2}(\beta_2)} \to g_2^*(x) \to \frac{g_{i_3}(x\beta_3)}{g_{i_3}(\beta_3)} \to g_2(x),$$

similarly in a general case.
$3^0$. Assume that (i) does not hold. Then $[g_i(x)]^*$, $i = 1, 2, \dots, k$, can be divided into two parts such that

$$(\cup_{i\in A}[g_i(x)]^*) \cap (\cup_{i\in B}[g_i(x)]^*) = \emptyset,$$

where $A \cup B = \{1, 2, \dots, k\}$. From closedness of such sets follows $\rho(g, \tilde{g}) \geq \delta > 0$ for some $\delta$ and every $g(x) \in \cup_{i\in A}[g_i(x)]^*$ and $\tilde{g}(x) \in \cup_{i\in B}[g_i(x)]^*$, which contradicts the connectivity of $G(X_n)$. $\square$

### 6.2.8 Everywhere density of $x_m/x_n$, $m, n = 1, 2, \dots$

**Theorem 132** ([172]). *Let $x_n$, $n = 1, 2, \dots$ be an increasing sequence of positive integers. If $\underline{d}(x_n) \geq (1/2)$ then the ratio sequence $x_m/x_n$, $m, n = 1, 2, \dots$, is everywhere dense in $[0, \infty)$. Conversely, if $0 \leq \gamma < 1/2$ then there exists an $x_n \in \mathbb{N}$ such that $\underline{d}(x_n) = \gamma$ and $x_m/x_n$, $m, n = 1, 2, \dots$, is not everywhere dense in $[0, \infty)$.*

The proof immediately follows from the following theorem.

**Theorem 133.** *Let $x_n \in \mathbb{N}$ and the interval $(\alpha, \beta)$, $0 \leq \alpha < \beta \leq 1$ be such that $(\alpha, \beta) \cap \{x_m/x_n, m, n = 1, 2, \dots\} = \emptyset$. Then*

$$\underline{d}(x_n) \leq \frac{\alpha}{\beta} \min(1 - \overline{d}(x_n), \overline{d}(x_n)) \text{ and} \tag{287}$$

$$\overline{d}(x_n) \leq 1 - (\beta - \alpha). \tag{288}$$

*Proof of* (287). Let $A \subset \mathbb{N}$ be listed in a strictly increasing order as $x_1 < x_2 < \cdots < x_n < \dots$. If $(\alpha, \beta) \cap \{x_m/x_n, m, n = 1, 2, \dots\} = \emptyset$, then the intervals

$$(\alpha x_n, \beta x_n), n = 1, 2, \dots$$

cannot intersect $A$ but they may have mutually nonempty intersections. We can select pairwise disjoint subintervals

$$(\alpha x_{[\theta n]}, \alpha x_{[\theta n]} + \alpha), (\alpha x_{[\theta n]+1}, \alpha x_{[\theta n]+1} + \alpha), \dots,$$

$$(\alpha x_{n-1}, \alpha x_{n-1} + \alpha), \tag{289}$$
$$(\alpha x_n, \beta x_n) \tag{290}$$

for some $0 \le \theta \le 1$ (here we put $x_{[\theta n]} = 0$ if $[\theta n] = 0$). Denote $B = \mathbb{N} - A$ and $B(x) = \#\{b \le x; b \in B\}$. Counting the number of integer points belonging to (289) we obtain

$$B(\beta x_n) \ge (n - [\theta n])(\alpha - 1) + ((\beta - \alpha)x_n - 1) + B(\alpha x_{[\theta n]})$$

for all sufficiently large $n$. To eliminate 1 in $(\alpha - 1)$ we replace $n$ with $nk$ and $\alpha$ with $k\alpha$. Then (289) transforms into pairwise disjoint subintervals of the form

$$\begin{aligned}&(\alpha x_{[\theta n]k}, \alpha x_{[\theta n]k} + k\alpha), (\alpha x_{([\theta n]+1)k}, \alpha x_{([\theta n]+1)k} + k\alpha), \dots, \\ &(\alpha x_{(n-1)k}, \alpha x_{(n-1)k} + k\alpha), (\alpha x_{nk}, \beta x_{nk}).\end{aligned} \tag{291}$$

Thus, we have

$$\frac{B(\beta a_{nk})}{\beta a_{nk}} \ge \frac{(n - [\theta n])(k\alpha - 1)}{\beta a_{nk}} + \frac{((\beta - \alpha)a_{nk} - 1)}{\beta a_{nk}} + \frac{B(\alpha a_{[\theta n]k})}{\alpha a_{[\theta n]k}} \frac{\alpha}{\beta} \frac{a_{[\theta n]k}}{a_{nk}}.$$

To compute the lim sup of the left and right hand side, respectively, use that

(i) $\limsup_{n\to\infty} B(\beta a_{nk})/\beta a_{nk} \le \overline{d}(B) = 1 - \underline{d}(A)$,

(ii) $\limsup_{n\to\infty} nk/a_{nk} = \overline{d}(A)$,

(iii) $\liminf_{n\to\infty} B(\alpha a_{[\theta n]k})/\alpha a_{[\theta n]k} \ge \underline{d}(B) = 1 - \overline{d}(A)$, and

(iv) by selecting indices $n$ for which $\lim_{n\to\infty} nk/a_{nk} = \overline{d}(A)$ we have (assuming $\overline{d}(A) > 0$)

$$\liminf_{n\to\infty} \frac{a_{[\theta n]k}}{a_{nk}} = \liminf_{n\to\infty} \frac{a_{[\theta n]k}}{[\theta n]k} \lim_{n\to\infty} \frac{[\theta n]k}{a_{nk}} \ge \frac{1}{\overline{d}(A)} \overline{d}(A)\theta.$$

Thus, letting $k \to \infty$ we get

$$1 - \underline{d}(A) \ge (1 - \theta)\frac{\alpha}{\beta}\overline{d}(A) + \frac{\beta - \alpha}{\beta} + (1 - \overline{d}(A))\frac{\alpha}{\beta}\theta.$$

Computing the maximum of the right hand side for $0 \le \theta \le 1$ yields

$$1 - \underline{d}(A) \ge \frac{\beta - \alpha}{\beta} + \frac{\alpha}{\beta}\max(\overline{d}(A), 1 - \overline{d}(A)),$$

which justifies (287). □

*Proof of* (288). A proof of (288) is given in [172], p. 69–71: Every infinite set $A \subset \mathbb{N}$ with infinite complement $\mathbb{N} - A$ can be expressed as the set of the integer points lying in the intervals

$$[b_1, c_1], [b_2, c_2], \dots, [b_n, c_n], \dots, \tag{292}$$

whose endpoints form two integer sequences ordered as

$$b_1 \le c_1 < b_2 \le c_2 < \dots < b_n \le c_n < \dots.$$

Clearly

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{1}{b_n} \sum_{i=1}^{n-1} (c_i - b_i + 1), \tag{293}$$

$$\overline{d}(A) = \limsup_{n\to\infty} \frac{1}{c_n} \sum_{i=1}^{n} (c_i - b_i + 1). \tag{294}$$

The points of $A \cap [1, c_n]$ divided by $i$, $i \in [b_n, c_n]$, form a subset $R_n \subset R(A)$; [38] we obtain the intervals

$$\left[\frac{b_1}{i}, \frac{c_1}{i}\right], \left[\frac{b_2}{i}, \frac{c_2}{i}\right], \dots, \left[\frac{b_{n-1}}{i}, \frac{c_{n-1}}{i}\right], \left[\frac{b_n}{i}, \frac{c_n}{i}\right]$$

which have the following property: the distance of any two neighbouring points of $R_n$ lying in $[b_{n-k}/i, c_{n-k}/i]$ is less than $1/b_n$ and the same holds for the union

$$\bigcup_{i=b_n}^{c_n} \left[\frac{b_{n-k}}{i}, \frac{c_{n-k}}{i}\right] = \left[\frac{b_{n-k}}{c_n}, \frac{c_{n-k}}{b_n}\right].$$

Thus, for sufficiently large $n$, every interval $(\alpha, \beta) \subset [0, 1]$ satisfying $(\alpha, \beta) \cap R(A) = \emptyset$ must lie in the complement of $[b_{n-k}/c_n, c_{n-k}/b_n]$, $k = 0, 1, \dots, n-1$, which is formed by the pairwise disjoint intervals

$$\left(\frac{c_{n-k}}{b_n}, \frac{b_{n-k+1}}{c_n}\right), \quad k = 1, 2, \dots, n-1, \tag{295}$$

some of which may be empty. Hence, a necessary condition for $(\alpha, \beta) \cap R(A) = \emptyset$ is the existence of an integer sequence $k_n$, $k_n < n$, such that

$$(\alpha, \beta) \subset \left(\frac{c_{n-k_n}}{b_n}, \frac{b_{n-k_n+1}}{c_n}\right) \tag{296}$$

[38] $R(A) = \{a/b; a, b \in A\}$.

for all sufficiently large $n$. This also gives

$$\frac{b_{n-k_n+1}}{c_n} - \frac{c_{n-k_n}}{c_n} \geq \beta - \alpha.$$

Now we can express the upper asymptotic density as

$$\overline{d}(A) = \limsup_{n\to\infty} \left( \frac{c_n - b_1}{c_n} + \frac{n}{c_n} - \left( \frac{b_2 - c_1}{c_n} + \frac{b_3 - c_2}{c_n} + \cdots + \frac{b_n - c_{n-1}}{c_n} \right) \right) \tag{297}$$

whence

$$\overline{d}(A) - \overline{d}(C) \leq 1 - (\beta - \alpha), \tag{298}$$

where $C$ is the range of $c_n$. For sufficiency of (296) we need the set $R(A)^l$ of all limit points of $R(A)$. [39]

By the above reasoning we see that $(\alpha, \beta) \cap R(A)^l = \emptyset$ if and only if there exists $k_n < n$ satisfying (296) for all sufficiently large $n$. Thus, inequality (298) holds for $(\alpha, \beta)$ satisfying $(\alpha, \beta) \cap R(A)^l = \emptyset$ as well.

Now, for a positive integer $k$, transform

$$[b_n, c_n] \to [kb_n, kc_n + k - 1]$$

and denote by $A_k$ the set of all integer points lying in $[kb_n, kc_n + k - 1]$, $n = 1, 2, \ldots$. Similarly, $C_k$ is the set of all $kc_n + k - 1$. Evidently

$$\overline{d}(A_k) = \overline{d}(A), \quad \overline{d}(C_k) = \overline{d}(C)/k, \quad \text{and} \quad R(A_k)^l = R(A)^l$$

which gives

$$\overline{d}(A) - \overline{d}(C)/k \leq 1 - (\beta - \alpha)$$

and (288) follows. □

**Notes 22.** Everywhere density of $x_m/x_n$ was first investigated by T. Šalát [140]. He proved

**Theorem 134.** *If $d(x_n) > 0$ or $\overline{d}(x_n) = 1$, then $x_m/x_n$, $m, n = 1, 2, \ldots$ is everywhere dense in $[0, \infty)$. Also $x_m/x_n$ is everywhere dense if $A(x)/\frac{cx}{\log^\alpha x} \to 1$ for $x \to \infty$, where $c > 0$, $\alpha > 0$ and $A(x) = \#\{a \leq x; a \in A\}$.*

---

[39] $R(A)^l$ is the set of all limit points $x = \lim_{i\to\infty} \frac{a_{m_i}}{a_{n_i}}$ of $R(A)$.

$R(A)^d$ is the set of all accumulation points of $R(A)$ i.e. the points $x$ which can be expressed as a limit $x = \lim_{i\to\infty} \frac{a_{m_i}}{a_{n_i}}$ of a one-to-one sequence $\frac{a_{m_i}}{a_{n_i}}$.

Using (287) and (288) Theorem 132 has also the form

**Theorem 135.** *For every increasing integers $x_n$, if $\underline{d}(x_n) + \overline{d}(x_n) \geq 1$ then $x_m/x_n$ is everywhere dense in $[0, \infty)$.*

**Theorem 136.** *The ratio sequence $x_m/x_n$, $m, n = 1, 2, \ldots$ is everywhere dense in $[0, 1]$ if*

(i) $\lim_{n\to\infty} \frac{1}{n^2 x_n} \sum_{i,j=1}^{n} |x_i - x_j| = 0$,

(ii) $\liminf_{n\to\infty} \frac{1}{n x_n} \sum_{i=1}^{n} x_i = 0$,

(iii) $\limsup_{n\to\infty} \frac{1}{n x_n} \sum_{i=1}^{n} x_i = 1$.

*Proof.* It follows from Theorem 19. □

### 6.2.9 U.d. of $X_n$

By Theorem 124, u.d. of the single block sequence $X_n$ implies the u.d. of the ratio sequence $x_m/x_n$. Applying [173, Th 6.3, (i)] $(\underline{d}/\overline{d})x \leq g(x) \leq (\overline{d}/\underline{d})x$ for every $x \in [0, 1]$, we have

**Theorem 137.** *If the increasing sequence $x_n$ of positive integers has a positive asymptotic density, i.e. $\underline{d} = \overline{d} > 0$, then the associated ratio sequence $x_m/x_n$, $m = 1, 2, \ldots, n$, $n = 1, 2, \ldots$ is u.d. in $[0, 1]$.*

Positive asymptotic density is not necessary. According to T. Šalát [140] we can use also a sequence $x_n$ with $\underline{d} = 0$.

**Theorem 138** ([173, Th. 9.2])**.** *Let $x_n$ be an increasing sequence of positive integers and $h : [0, \infty) \to [0, \infty)$ be a function satisfying*

(i) $A(x) \sim h(x)$ *as* $x \to \infty$, *where*

(ii) $h(xy) \sim xh(y)$ *as* $y \to \infty$ *and for every* $x \in [0, 1]$, *and*

(iii) $\lim_{n\to\infty} \frac{n}{h(x_n)} = 1$.

*Then $X_n$ (and consequently $x_m/x_n$) is u.d. in $[0, 1]$.*

*Proof.* Starting with (33) $F(X_n, x)n = A(xx_n)$ it follows from (i) that

$$\frac{F(X_n, x)n}{h(xx_n)} \to 1$$

as $n \to \infty$, then by (ii)

$$\frac{F(X_n, x)n}{xh(x_n)} \to 1$$

which gives by (iii) the limit

$$F(X_n, x)\frac{n}{h(x_n)} \to x$$

as $n \to \infty$. $\square$

**Notes 23.** Assuming only (i) and (ii), we have $\liminf_{n\to\infty} n/h(x_n) \geq 1$, since otherwise $n_k/h(x_{n_k}) \to \alpha < 1$ implies $F(X_{n_k}, x) \to x/\alpha$ for every $x \in [0, 1]$ which is a contradiction. Also, $G(X_n) \subset \{x\lambda; \lambda \in [0, 1]\}$.

### 6.2.10 $L^2$ discrepancy of $X_n$

It has the form

$$D^{(2)}(X_n) = \frac{1}{3} + \frac{1}{nx_n^2}\sum_{i=1}^{n} x_i^2 - \frac{1}{nx_n}\sum_{i=1}^{n} x_i - \frac{1}{2n^2x_n}\sum_{i,j=1}^{n} |x_i - x_j|. \qquad (299)$$

*Proof.* Applying (720) we have

$$D^{(2)}(X_n) = \int_0^1 (F(X_n, x) - x)^2 \mathrm{d}x = \frac{1}{n^2}\sum_{i,j=1}^{n} F_0\left(\frac{x_i}{x_n}, \frac{x_j}{x_n}\right),$$

where

$$F_0(x, y) = \frac{1}{3} + \frac{x^2 + y^2}{2} - \frac{x + y}{2} - \frac{|x - y|}{2}.$$

$\square$

Then

**Theorem 139.** *For every increasing sequence $x_n$ of positive integers we have*

$$\lim_{n\to\infty} D^{(2)}(X_n) = 0 \iff \lim_{n\to\infty} F(X_n, x) = x.$$

**Notes 24.** The left hand-side can be divided into three limits (cf. [160, Th. 1])

$$\lim_{n\to\infty} D^{(2)}(X_n) = 0 \Longleftrightarrow \begin{cases} (i) \lim_{n\to\infty} \frac{1}{nx_n} \sum_{i=1}^n x_i = \frac{1}{2}, \\ (ii) \lim_{n\to\infty} \frac{1}{nx_n^2} \sum_{i=1}^n x_i^2 = \frac{1}{3}, \\ (iii) \lim_{n\to\infty} \frac{1}{n^2 x_n} \sum_{i,j=1}^n |x_i - x_j| = \frac{1}{3}. \end{cases}$$

Also note that if $D^{(2)}(X_n) \to 0$, then $x_m/x_n$ is everywhere dense.

**Notes 25.** Weyl's criterion for u.d. of $X_n$ is not well applicable in our case. It says (cf. [145, (7)]).

**Theorem 140.** *$X_n$ is u.d. if and only if*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n e^{2\pi i h \frac{x_k}{x_n}} = 0$$

*for all positive integers $h$.*

### 6.2.11 Boundaries of $g(x) \in G(X_n)$

**Theorem 141** ([11, Th. 5])**.** *For every increasing sequence of positive integers $x_n$, $n = 1, 2, \dots$, there exists $g(x) \in G(X_n)$ such that $g(x) \geq x$ for all $x \in [0, 1]$.*

*Proof.* If $\underline{d} > 0$, select $n_k$ so that $\frac{n_k}{x_{n_k}} \to \underline{d} > 0$, and $F(X_{n_k}, x) \to g(x)$. For such $g(x)$, (274) implies

$$\frac{g(x)}{x}\underline{d} \geq \underline{d}.$$

Now, let $\underline{d} = 0$. Select $n_k$ such that

$$\frac{n_k}{x_{n_k}} = \min_{i \leq n_k} \frac{i}{x_i},$$

and $F(X_{n_k}, x) \to g(x)$. Then for every $x \in (0, 1]$,

$$\frac{A(xx_{n_k})}{xx_{n_k}} \geq \frac{n_k - 1}{x_{n_k}}.$$

Applying (33) yields

$$\frac{F(X_{n_k}, x)}{x} \frac{n_k}{x_{n_k}} \geq \frac{n_k - 1}{x_{n_k}},$$

and taking the limit, as $k \to \infty$, we obtain $g(x) \geq x$ for all $x \in [0, 1]$. [40] □

[40] L. Mišík.

See [11, Th. 6]:

**Theorem 142** (L. Mišík)**.** *Let $x_1 < x_2 < \dots$ be a sequence of positive integers with positive lower asymptotic density $\underline{d} > 0$, and upper asymptotic density $\overline{d}$. Then all d.f.s $g(x) \in G(X_n)$ are continuous, non-singular, and bounded by $h_1(x) \le g(x) \le h_2(x)$, where*

$$h_1(x) = \begin{cases} x\frac{\underline{d}}{\overline{d}}, & \text{if } x \in \left[0, \frac{1-\overline{d}}{1-\underline{d}}\right]; \\ \frac{\underline{d}}{\frac{1}{x}-(1-\underline{d})}, & \text{otherwise}, \end{cases} \tag{300}$$

$$h_2(x) = \min\left(x\frac{\overline{d}}{\underline{d}}, 1\right). \tag{301}$$

*Moreover, $h_1(x)$ and $h_2(x)$ are the best possible in the following sense: for given $0 < \underline{d} \le \overline{d}$, there exists $x_1 < x_2 < \dots$ with lower and upper asymptotic densities $\underline{d}$, $\overline{d}$, such that $\underline{g}(x) = h_1(x)$ for $x \in \left[\frac{1-\overline{d}}{1-\underline{d}}, 1\right]$; also, there exists $x_1 < x_2 < \dots$ with given $0 < \underline{d} \le \overline{d}$ such that $\overline{g}(x) = h_2(x) \in G(X_n)$.*

*Proof.* For $g(x) \in G(X_n)$, let $n_k$, $k = 1, 2, \dots$, be an increasing sequence of indices such that $F(X_{n_k}, x) \to g(x)$. From $n_k$ we can select a subsequence (for simplicity written as the original $n_k$) [41] such that

$$\frac{n_k}{x_{n_k}} \to d_g > 0. \tag{302}$$

Then, by (274), we have

$$g(x) = x\frac{d_g(x)}{d_g}, \text{ where } \frac{A(xx_{n_k})}{xx_{n_k}} \to d_g(x) \tag{303}$$

for arbitrary $x \in (0, 1]$.

We will continue in six steps $1^0$–$6^0$.

$1^0$. We prove the continuity of $g(x)$ at $x = 1$ (improving (iv) in [173, Th. 6.2]) for each $g(x) \in G(X_n)$.

In view of the definition of the counting function $A(t)$

$$0 \le A(x_{n_k}) - A(xx_{n_k}) \le x_{n_k} - xx_{n_k};$$

[41] We call $d_g$ a *local asymptotic density* related to $g(x)$.

thus,

$$0 \le \frac{A(x_{n_k})}{x_{n_k}} - \frac{A(xx_{n_k})}{x_{n_k}} = \frac{n_k - 1}{x_{n_k}} - \frac{A(xx_{n_k})}{xx_{n_k}} x \le 1 - x,$$

and, as $k \to \infty$, we have $0 \le d_g - d_g(x)x \le 1 - x$, which implies

$$0 \le d_g - d_g(x) + d_g(x)(1 - x) \le 1 - x.$$

Consequently, $\lim_{x\to 1} d_g(x) = d_g$, and so $\lim_{x\to 1} g(x) = \lim_{x\to 1} x\frac{d_g(x)}{d_g} = 1$. Since $g(x) \in G(X_n)$ is arbitrary, [173, Th. 4.1,Th. 6.2] gives continuity of $g(x)$ in the whole unit interval $[0, 1]$.

$2^0$. We prove that $g(x)$ has a bounded right derivative for every $x \in (0, 1)$, and for each $g(x) \in G(X_n)$.

For $0 < x < y < 1$ again

$$0 \le A(yx_{n_k}) - A(xx_{n_k}) \le (y - x)x_{n_k} \ ,$$

which implies

$$0 \le \frac{A(yx_{n_k})}{yx_{n_k}} y - \frac{A(xx_{n_k})}{xx_{n_k}} x \le y - x.$$

Letting $k \to \infty$, we get $0 \le d_g(y)y - d_g(x)x \le y - x$, hence

$$0 \le g(y) - g(x) = \frac{d_g(y)y - d_g(x)x}{d_g} \le \frac{y - x}{d_g}.$$

Consequently,

$$0 \le \frac{g(y) - g(x)}{y - x} \le \frac{1}{d_g} \tag{304}$$

for all $x, y \in (0, 1)$, $x < y$, which gives the upper bound of the right derivatives of $g(x)$ for every $x \in (0, 1)$. Note that a singular d.f. (continuous, strictly increasing, having zero derivative a.e.) has infinite right Dini derivatives in a dense subset of $(0, 1)$.

$3^0$. We prove a local form of Theorem 47.

As $\underline{d} \le d_g \le \overline{d}$, (303) implies

$$x\frac{\underline{d}}{d_g} \le g(x) \le x\frac{\overline{d}}{d_g} \tag{305}$$

for every $x \in [0,1]$. It follows from (304), that there exists an extreme point $A_g = (x_g, y_g)$ on the line $y = x\frac{\underline{d}}{d_g}$ such that $g(x)$ has no common point with this line for $x > x_g$. This point $A_g$ is the intersection of the lines

$$y = x\frac{\underline{d}}{d_g} \text{ and, } y = x\frac{1}{d_g} + 1 - \frac{1}{d_g} \tag{306}$$

therefore,

$$A_g = (x_g, y_g) = \left(\frac{1-d_g}{1-\underline{d}}, \frac{\underline{d}}{d_g}\frac{1-d_g}{1-\underline{d}}\right). \tag{307}$$

It means that for a given $g(x) \in G(X_n)$, $h_{1,g}(x) \le g(x) \le h_{2,g}(x)$, where

$$h_{1,g}(x) = \begin{cases} x\frac{\underline{d}}{d_g}\ , & \text{if } x < y_0 = \frac{1-d_g}{1-\underline{d}}; \\ x\frac{1}{d_g} + 1 - \frac{1}{d_g}\ , & \text{if } y_0 \le x \le 1, \end{cases} \tag{308}$$

$$h_{2,g}(x) = \min\left(x\frac{\overline{d}}{d_g}, 1\right). \tag{309}$$

$4^0$. Now we find $h_1(x)$, and $h_2(x)$ such that

$$h_1(x) \le h_{1,g}(x) \le h_{2,g}(x) \le h_2(x)$$

for every $g \in G(X_n)$.

In the parametric expression (307) of $A_g$, the local asymptotic density $d_g$ defined by (302) belongs to the interval $[\underline{d}, \overline{d}]$. The well-known Darboux property of the asymptotic density implies that for an arbitrary $d \in [\underline{d}, \overline{d}]$ there exists an increasing $n_k$, $k = 1, 2, \dots$, such that $\frac{n_k}{x_{n_k}} \to d$ [42], and then the Helly selection principle implies the existence of a subsequence of $n_k$ such that $F(X_{n_k}, x) \to g(x)$ for some $g(x) \in G(X_n)$. Thus, if $g(x)$ runs over $G(X_n)$, then $d_g$ runs over the entire interval $[\underline{d}, \overline{d}]$. Substituting $d_g = 1 - x_g(1 - \underline{d})$ in $A_g = (x_g, y_g)$ we get

$$y_g = y_g(x_g) = \frac{\underline{d}}{\frac{1}{x_g} - (1 - \underline{d})},$$

where $x_g = \frac{1-d_g}{1-\underline{d}}$ runs through the interval $I = \left[\frac{1-\overline{d}}{1-\underline{d}}, 1\right]$ for $d_g \in [\underline{d}, \overline{d}]$. By putting $x_g = x$, and $y_g = h_1$ we find a part of $h_1(x)$ for $x \in I$ in (300).

[42] A simple proof follows from the fact that for every $d \in (\underline{d}, \overline{d})$ there exist infinitely many $n \in \mathbb{N}$ such that $A(n)/n \le d \le A(n+1)/(n+1)$. These $n$ we denote as $n_k$.

The remaining part of $h_1(x)$, and also the whole $h_2(x)$, follow from the basic inequality (305), see [11, Fig. 1.]. The optimality of $h_1(x)$ and $h_2(x)$ are proved in $5^0$ and $6^0$ pages 518–522 of [11]. [43] $\square$

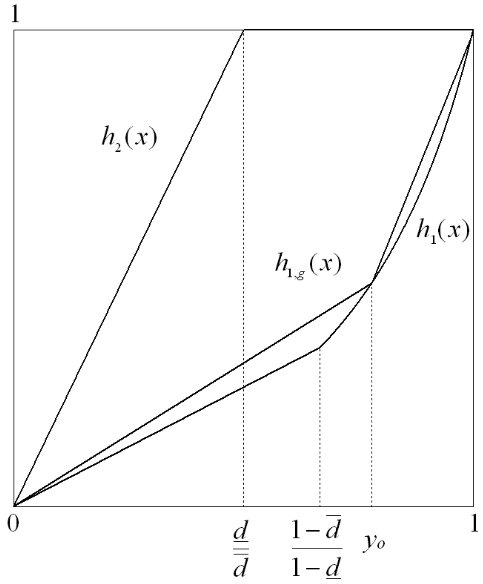



Figure 1: Boundaries of $g(x) \in G(X_n)$

[43] L. Mišík.

### 6.2.12 Applications of boundaries of $g(x)$

An application of d.f.s in Theorem 142 to elementary number theory:

**Theorem 143** ([11, Th. 7])**.** *For every increasing sequence $x_1 < x_2 < \dots$ of positive integers with lower and upper asymptotic densities $0 < \underline{d} \le \overline{d}$ we have*

$$\frac{1}{2}\frac{\underline{d}}{\overline{d}} \le \liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{x_i}{x_n}, \tag{310}$$

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{x_i}{x_n} \le \frac{1}{2} + \frac{1}{2}\left(\frac{1-\min(\sqrt{\underline{d}},\overline{d})}{1-\underline{d}}\right)\left(1-\frac{\underline{d}}{\min(\sqrt{\underline{d}},\overline{d})})\right). \tag{311}$$

*Here the equations in* (310) *and* (311) *can be attained.*

*Proof.* By Helly theorem, if $F(X_{n_k},x)\to g(x)$, then

$$\int_0^1 x\mathrm{d}F(X_{n_k},x) = \frac{1}{n_k}\sum_{i=1}^{n_k}\frac{x_i}{x_{n_k}} \to \int_0^1 x\mathrm{d}g(x) = 1-\int_0^1 g(x)\mathrm{d}x.$$

If $\underline{d} > 0$, then $h_1(x) \le g(x) \le h_2(x)$ which implies

$$1-\int_0^1 h_2(x)\mathrm{d}x \le 1-\int_0^1 g(x)\mathrm{d}x \le 1-\int_0^1 h_1(x)\mathrm{d}x. \tag{312}$$

For $x_1 < x_2 < \dots$ for which $h_2(x) \in G(X_n)$ in the left of (312) we have equation, but because by (xi) in every case $h_1(x) \notin G(X_n)$ for $0 < \underline{d} < \overline{d}$, which implies strong inequality in the right, i.e.

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{x_i}{x_n} < 1 - \frac{1}{2}\frac{\underline{d}}{\overline{d}}\left(\frac{1-\overline{d}}{1-\underline{d}}\right)^2 - \frac{\underline{d}}{(1-\underline{d})^2}\left(\log\frac{\underline{d}}{\overline{d}} - (\overline{d}-\underline{d})\right). \tag{313}$$

Since for every $g(x)\in G(X_n)$ in $3^0$ we have $h_{1,g}(x) \le g(x) \le h_{2,g}(x)$, then

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{x_i}{x_n} \le \max_{g(x)\in G(X_n)}\left(1-\int_0^1 h_{1,g}(x)\mathrm{d}x\right). \tag{314}$$

If the maximum in (314) is attained in $g_0(x)\in G(X_n)$ and $h_{1,g_0}(x)\in G(X_n)$, then $g_0(x) = h_{1,g_0}(x)$ and we have

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{x_i}{x_n} = 1-\int_0^1 h_{1,g_0}(x)\mathrm{d}x. \tag{315}$$

Using (105) we find

$$\int_0^1 h_{1,g}(x)\mathrm{d}x = \frac{1}{2}\left(1 + \frac{1-d_g}{1-\underline{d}}\left(\frac{\underline{d}}{d_g} - 1\right)\right)$$

for $d_g \in [\underline{d}, \overline{d}]$ with derivative $\left(\int_0^1 h_{1,g}(x)\mathrm{d}x\right)' = \frac{1}{2(1-\underline{d})}\left(1 - \frac{\underline{d}}{(d_g)^2}\right)$ and which gives that $\min \int_0^1 h_{1,g}(x)\mathrm{d}x$ is attained in $d_{g_0} = \min(\sqrt{\underline{d}}, \overline{d})$.

Now, to prove (315) we can construct integer $x_1 < x_2 < \dots$ with $0 < \underline{d} \le \overline{d}$ such that $h_{1,g_0}(x) \in G(X_n)$.

We starting with the sequence of indices $n_k$, and then by (105) we must find indices $m'_k < m_k < n_k$ and integers $x_{m'_k} < x_{m_k} < x_{n_k}$ such that

(i) $\frac{n_k}{x_{n_k}} \to d_{g_0}$,

(ii) $\frac{m_k}{n_k} \to \frac{\underline{d}}{d_{g_0}} \frac{1-d_{g_0}}{1-\underline{d}}$,

(iii) $\frac{x_{m_k}}{x_{n_k}} \to \frac{1-d_{g_0}}{1-\underline{d}}$,

(iv) $\frac{x_{m'_k}}{x_{n_k}} \to 0$,

(v) $\frac{m'_k}{n'_k} \to 0$,

(vi) $\frac{m'_k}{x_{m'_k}} \to \overline{d}$.

Then from (i), (ii) and (iii) follows $\frac{m_k}{x_{m_k}} \to \underline{d}$. Furthermore we must again assumed

(v) $x_{m_k} - x_{m'_k} \ge m_k - m'_k$,

(vi) $x_{n_k} - x_{m_k} \ge n_k - m_k$,

(vii) $x_{m'_{k+1}} - x_{n_k} \ge m'_{k+1} - n_k$,

(viii) $n_k < m'_{k+1}$,

(ix) $m'_1 \le x_{m'_1}$.

It can be solved naturally and complement values $x_n$ we define linearly. $\square$

### 6.2.13 Lower and upper d.f.s of $G(X_n)$

In the Introduction 6.2, (xv), we notify the result [11, Th. 7] that for every integer sequence $1 \le x_1 < x_2 < \dots$ with $\underline{d} > 0$ and every d.f. $g(x) \in G(X_n)$ we have $h_1(x) \le g(x) \le h_2(x)$, where $h_1(x)$ and $h_2(x)$ are defined in (14) and (23), respectively. Furthermore, by [11, $6^0$ of Proof], there exists an integer sequence $1 \le x_1 < x_2 < \dots$ with $\underline{d} > 0$ such that $h_2(x) \in G(X_n)$. In this case $h_2(x) = \overline{g}(x)$ and $G(X_n)$ has the following additional properties.

**Theorem 144.** *Let $1 \le x_1 < x_2 < \dots$ be an integer sequence with $\underline{d} > 0$ such that $h_2(x) \in G(X_n)$. Then the set $G(X_n)$ contains uncountable many different d.f.s $g_\alpha(x)$, $\alpha \in [1,\infty)$, of the form*

$$g_\alpha(x) = \begin{cases} x\frac{1}{\alpha\beta}\frac{\overline{d}}{\underline{d}} & \text{if } x \in \left[0, \frac{\underline{d}}{\overline{d}}\beta\right], \\ \frac{1}{\alpha} & \text{if } x \in \left[\frac{\underline{d}}{\overline{d}}\beta, \beta\right], \\ \text{nondecreasing} & \text{if } x \in \left[\beta, 1\right], \end{cases} \tag{316}$$

*where for $\beta = \beta(\alpha)$ we have $1 \le \alpha\beta \le \frac{\overline{d}}{\underline{d}}$. Furthermore, $g(x) = x$ is also in $G(X_n)$.*

*Proof.* We use two steps.

$1^0$. Assume that $F(X_{n_k}, x) \to h_2(x)$ as $k \to \infty$ for $x \in [0,1]$. For every $\alpha \in [1,\infty)$ we can choose $n'_k > n_k$ so that

(i) $\frac{n'_k}{n_k} \to \alpha$.

From the sequence $(n'_k, n_k)$, $k = 1, 2, \dots$, we can select subsequence (with the same notation) such that

(ii) $\frac{x_{n_k}}{x_{n'_k}} \to \beta$,

where $\beta = \beta(\alpha)$ but it is not given uniquely. We have only $\frac{1}{\alpha}\frac{\underline{d}}{\overline{d}} \le \beta \le \frac{1}{\alpha}\frac{\overline{d}}{\underline{d}}$ because

$$\frac{n'_k}{n_k}\frac{x_{n_k}}{x_{n'_k}} = \frac{\frac{n'_k}{x_{n'_k}}}{\frac{n_k}{x_{n_k}}} \to \alpha\beta$$

and which gives $\alpha < \infty \Leftrightarrow \beta > 0$. Now, from $(n'_k, n_k)$ we again select a subsequence such that

(iii) $F(X_{n'_k}, x) \to g(x)$

for all $x \in [0,1]$. Applying the identity (750)

$$F(X_{n_k}, x) = \frac{n'_k}{n_k} F\big(X_{n'_k}, x\frac{x_{n_k}}{x_{n'_k}}\big) \tag{317}$$

and assuming that $\underline{d} > 0$, which implies everywhere continuity of $g(x)$ (see [173, Th. 6.2]) and $g(x) > 0$ for $0 < x \le 1$, then we can take limit in (317) to obtain

$$h_2(x) = \alpha g_\alpha(x\beta) \tag{318}$$

for $x \in [0,1]$. Now, using $h_2(x) = 1$ for $x \in \left[\frac{\underline{d}}{\overline{d}}, 1\right]$, (318) implies $g_\alpha(x) = \frac{1}{\alpha}$ for $x \in \left[\frac{\underline{d}}{\overline{d}}\beta, \beta\right]$ and $h'_2(x) = \frac{\overline{d}}{\underline{d}}$ for $x \in \left[0, \frac{\underline{d}}{\overline{d}}\right]$ implies $g'_\alpha(x) = \frac{\overline{d}}{\underline{d}}\frac{1}{\alpha\beta}$ for $x \in \left[0, \frac{\underline{d}}{\overline{d}}\beta\right]$. Then we obtain (316) and since $g_\alpha(x) \le h_2(x)$, then $1 \le \alpha\beta$.

$2^0$. Again, let $F(X_{n_k},x)\to h_2(x)$ for $x\in[0,1]$. For every limit point [44] $\beta>0$ of $\frac{x_i}{x_{n_k}}$, $i=1,2,\dots,n_k$, $k=1,2,\dots$, we can select $m_k<n_k$ such that

(i) $\frac{x_{m_k}}{x_{n_k}}\to\beta$,
(ii) $\frac{n_k}{m_k}\to\alpha$
(iii) $F(X_{m_k},x)\to g(x)$.

The identity (750) in the form $F(X_{m_k},x)=\frac{n_k}{m_k}F\big(X_{n_k},x\frac{x_{m_k}}{x_{n_k}}\big)$ implies

$$g(x)=\alpha h_2(x\beta)=\frac{h_2(x\beta)}{h_2(\beta)} \tag{319}$$

for $x\in[0,1]$. From the form of $h_2(x)$ we have guaranteed that $\beta\in\big[0,\frac{\underline{d}}{\overline{d}}\big]$ is a limit point of $\frac{x_i}{x_{n_k}}$ and in this case (319) gives

$$g(x)=\frac{x\beta^{\frac{\overline{d}}{\underline{d}}}}{\beta^{\frac{\overline{d}}{\underline{d}}}}=x.$$

For $\beta>\frac{\underline{d}}{\overline{d}}$, if exists, we have $g(x)=h_2(x\beta)$ for $x\in[0,1]$, i.e.,

$$g(x)=\begin{cases}x\beta^{\frac{\overline{d}}{\underline{d}}} & \text{if } x\in\big[0,\frac{\underline{d}}{\overline{d}}\frac{1}{\beta}\big],\\ 1 & \text{if } x\in\big[\frac{\underline{d}}{\overline{d}}\frac{1}{\beta},1\big].\end{cases} \tag{320}$$

□

Finally, for $h_2(x)$ defined in (23) for which $h_2(x)=\overline{g}(x)$ for special $1\le x_1<x_2<\dots$, we see directly that

$$h_2(xy)\le h_2(x)h_2(y) \tag{321}$$

for every $x,y\in[0,1]$. Also for $h_1(x)$ defined in (14), in the case $x\ge\sqrt{\frac{1-\overline{d}}{1-\underline{d}}}$, for which there exists a special sequence $x_n$ (see [173, pp. 774–777, Ex. 11.2]) such that the lower d.f. $\underline{g}(x)=h_1(x)$ we have [45]

$$\left(\frac{\underline{d}}{\frac{1}{x}-(1-\underline{d})}\right)\left(\frac{\underline{d}}{\frac{1}{y}-(1-\underline{d})}\right)\le\frac{\underline{d}}{\frac{1}{xy}-(1-\underline{d})} \tag{322}$$

for $xy\ge\sqrt{\frac{1-\overline{d}}{1-\underline{d}}}$. In the following theorem we extend (321) and (322) for arbitrary lower $\underline{g}(x)$ and upper $\overline{g}(x)$ d.f.s.

[44]In the following $\alpha$ and $\beta$ have another meaning as in $1^0$.
[45]This holds also for arbitrary $x,y\in(0,1)$, since it is equivalent to $x(1-y)\le 1-y$.

**Theorem 145.** *For every increasing sequence of positive integers* $1 \le x_1 < x_2 < \dots$*, with* $\underline{d} > 0$*, the lower d.f.* $\underline{g}(x)$ *and the upper d.f.* $\overline{g}(x)$ *satisfy*

$$\underline{g}(x).\underline{g}(y) \le \underline{g}(x.y) \le \overline{g}(x.y) \le \overline{g}(x).\overline{g}(y) \tag{323}$$

*for every* $x, y \in (0,1)$.

*Proof.* $\underline{d} > 0$ implies that arbitrary $g(x) \in G(X_n)$ is everywhere continuous and $g(x) > 0$ for $x > 0$. Let $y \in (0,1)$.

$1^0$. Firstly we prove the left-hand side of (323).

a) If $y$ is an increasing point [46] of $g(x)$, $n = 1, 2, \dots$ then by (123) we have $\frac{g(xy)}{g(y)} \in G(X_n)$ and thus $\underline{g}(x) \le \frac{g(xy)}{g(y)}$ which implies

$$\underline{g}(x)\underline{g}(y) \le \underline{g}(x)g(y) \le g(xy) \tag{324}$$

for every $x \in (0,1)$.

b) Let $g(x)$ does not increase at $y$. Since every $g(x) \in G(X_n)$ is continuous and $\underline{\underline{d}}x \le g(x) \le \frac{\overline{d}}{\underline{d}}x$ for $x \in [0,1]$, there exists the nearest neighboring point $y_1 < y$, $y_1 > 0$ at which $g(x)$ increases. Thus $\frac{g(xy_1)}{g(y_1)} \in G(X_n)$ which implies $\underline{g}(x) \le \frac{g(xy_1)}{g(y_1)}$. Because $g(y_1) = g(y)$, $g(xy_1) \le g(xy)$, then again

$$\underline{g}(x)\underline{g}(y) \le \underline{g}(x)g(y) = \underline{g}(x)g(y_1) \le g(xy_1) \le g(xy) \tag{325}$$

for every $x \in (0,1)$.

Since $g \in G(X_n)$ is arbitrary, and for $x, y \in (0,1)$ by (323) and (324) we have $\underline{g}(x)\underline{g}(y) \le g(xy)$, then the definition of lower d.f. of $G(X_n)$ as $\underline{g}(xy) = \inf_{g \in G(X_n)} g(xy)$ implies $\underline{g}(x)\underline{g}(y) \le \underline{g}(xy)$.

$2^0$. Now, we prove the right-hand side of (323).

a) Again, if $y$ is an increasing point of $g(x)$, then $\frac{g(xy)}{g(y)} \in G(X_n)$, thus $\frac{g(xy)}{g(y)} \le \overline{g}(x)$ which implies

$$g(xy) \le g(y)\overline{g}(x) \le \overline{g}(y)\overline{g}(x) \tag{326}$$

for $x \in (0,1)$.

b) Let $g(x)$ be non increasing at $y$ and let $y_2$ be the nearest point to the right at which $g(x)$ is increasing. Again, by $\underline{\underline{d}}x \le g(x) \le \frac{\overline{d}}{\underline{d}}x$, this point exists

[46] Either $g(y - \varepsilon) < g(y)$ or $g(y) < g(y + \varepsilon)$, for arbitrary $\varepsilon > 0$.

and thus for given $g(x) \in G(X_n)$ we have $\frac{g(xy_2)}{g(y_2)} \in G(X_n)$, $\frac{g(xy_2)}{g(y_2)} \leq \overline{g}(x)$ which implies

$$g(xy) \leq g(xy_2) \leq g(y_2)\overline{g}(x) \leq g(y)\overline{g}(x) \leq \overline{g}(y)\overline{g}(x) \tag{327}$$

for $x \in (0,1)$. Then $\overline{g}(xy) = \sup_{g \in G(X_n)} g(xy)$ implies $\overline{g}(x.y) \leq \overline{g}(x).\overline{g}(y)$ for $x, y \in (0,1)$. $\square$

**Notes 26.** By J. Aczél [1, p. 144–145, Th. 4] every continuous d.f. $g(xy) = g(x)g(y)$ has the form $g(x) = x^c$ for a constant $c$ and $x \in [0,1]$.

### 6.2.14 Algorithm for constructing $\tilde{g}(x) \leq g(x)$

[12, p. 5]: Let $1 \leq x_1 < x_2 < \dots$ be an increasing sequence of positive integers. Put $x_0 = 0$ and

$$t_n = x_n - x_{n-1}, \quad n = 1, 2, \dots$$

For every $n = 1, 2, \dots$ we compute the finite integer sequence

$$t_1^{(n)}, t_2^{(n)}, \dots, t_n^{(n)}$$

from $t_1, t_2, \dots$ by the following procedure:

$1^0$. For $n = 1$, $t_1^{(1)} = t_1 = x_1$;

$2^0$. For $n = 2$, $t_1^{(2)} = t_1 + t_2 - 1 = x_2 - 1$ and $t_2^{(2)} = 1$;

$3^0$. Assume that for $n - 1 \geq 2$ we have $t_i^{(n-1)}$, $i = 1, 2, \dots, n-1$. For $n$ we first define the initial auxiliary sequence $t'_1, t'_2, \dots, t'_n$ such that $t'_i = t_i^{(n-1)}$, $i = 1, 2, \dots, n-1$, and $t'_n = t_n$. Then we repeatedly modify this sequence using following steps (a) and (b).

(a) If there exists $k$, $1 < k < n$, such that $t'_1 = t'_2 = \dots = t'_{k-1} > t'_k$ and $t'_n > 1$, then we put $t'_k := t'_k + 1$, $t'_n := t'_n - 1$ and $t'_i := t'_i$ in all other cases.

(b) If such $k$ does not exist and $t'_n > 1$, then we put $t'_1 := t'_1 + 1$, $t'_n := t'_n - 1$ and $t'_i := t'_i$ in all other cases.

Repeated application of (a) and (b) shows that the step $3^0$ terminates if $t'_n = 1$ and outputs the sequence $t_1^{(n)} := t'_1, \dots, t_n^{(n)} := t'_n$.

$4^0$. Put $n - 1 := n$ and use the output $t_1^{(n)}, \dots, t_n^{(n)}$ as the new input in $3^0$.

Thus the final output of Algorithm is the infinite sequence of finite integers block $t_1^{(n)}, t_2^{(n)}, \dots, t_n^{(n)}$ for $n = 1, 2, \dots$.

**Theorem 146** ([12, Lemma 1])**.** *Assuming that $t_n \neq 1$ for infinitely many $n$, then the output $t_1^{(n)}, t_2^{(n)}, \dots, t_n^{(n)}$ of the Algorithm can be of the following two possible forms:*

$$\begin{aligned}
&\text{(A)} \quad t_1^{(n)} = \cdots = t_m^{(n)} = D_n > t_{m+1}^{(n)} \geq t_{m+2}^{(n)} = t_{m+3}^{(n)} = \dots t_n^{(n)} = 1,\\
&\text{(B)} \quad t_1^{(n)} = \cdots = t_m^{(n)} = D_n > t_{m+1}^{(n)} = \cdots = t_{m+s}^{(n)} = D_n - 1\\
&\qquad\qquad\qquad\qquad\qquad\qquad \geq t_{m+s+1}^{(n)} = \cdots = t_n^{(n)} = 1,
\end{aligned}$$

*for some $m = m(n)$, $s = s(n)$ and for $D_n := t_1^{(n)}$.*

**Theorem 147** ([12, Lemma 2])**.** *For $D_n$ defined in Theorem 146 there are two possibilities:*
(I) *$D_n$ is bounded;*
(II) *$D_n \to \infty$.*
*In the case* (I) *we have only the form* (A) *and $D_n = const. = c \geq 2$ for all sufficiently large $n$.*
*In the case* (II) *both cases* (A) *and* (B) *are possible.*

**Construction of $\tilde{g}(x) \leq g(x)$**

[12, p. 8]: Assume that, for every $n = 1, 2, \dots$, we have given $n$-terms sequence

$$t_1^{(n)}, t_2^{(n)}, \dots, t_n^{(n)}$$

such that for every $n = 1, 2, \dots$

$$t_1^{(n)} \leq t_1^{(n+1)}, t_2^{(n)} \leq t_2^{(n+1)}, \dots, t_n^{(n)} \leq t_n^{(n+1)}. \tag{328}$$

Then, we define $x_n$, $x_j^{(n)}$ and $X_n^{(n)}$ as

$$x_n = \sum_{i=1}^{n} t_i^{(n)}, \quad n = 1, 2, \dots; \tag{329}$$

$$x_j^{(n)} = \sum_{i=1}^{j} t_i^{(n)}, \quad j = 1, 2, \dots, n; \tag{330}$$

$$X_n^{(n)} = \left( \frac{x_1^{(n)}}{x_n^{(n)}}, \frac{x_2^{(n)}}{x_n^{(n)}}, \dots, \frac{x_n^{(n)}}{x_n^{(n)}} \right), \quad n = 1, 2, \dots. \tag{331}$$

Clearly $x_n^{(n)} = x_n$ and using (328) we see that

$$x_j = \sum_{i=1}^{j} t_i^{(j)} \le \sum_{i=1}^{j} t_i^{(n)} = x_j^{(n)}, \quad j = 1, 2, \dots, n$$

which implies

$$F(X_n^{(n)}, x) \le F(X_n, x) \text{ for all } x \in [0,1], \quad n = 1, 2, \dots. \tag{332}$$

Selecting a sequence of indices $n_k$, $k = 1, 2, \dots$, such that $F(X_{n_k}, x) \to g(x)$ and $F(X_{n_k}^{(n_k)}, x) \to \tilde{g}(x)$ for all $x \in [0,1]$, we have

$$\tilde{g}(x) \le g(x) \text{ for all } x \in [0,1]. \tag{333}$$

**The case $\underline{d} = 0$**

[12, p. 12]: In the case $\underline{d} = 0$ the Algorithm implies $\lim_{n\to\infty} D_n = \infty$ since if $D_n = \text{const.} = c$, then $t_1^{(n)}, t_2^{(n)}, \dots, t_n^{(n)}$ satisfy (A) and $d_g = \frac{1}{\alpha(c-1)+1} \ge \frac{1}{c} > 0$. Note that, in the opposite direction, $\lim_{n\to\infty} D_n = \infty$ need not imply $\underline{d} = 0$, see the Construction.

The following theorem we shall formulate for the case (B), since the case (A) gives the same result, putting $\gamma = 0$ and $s_k = 0$.

**Theorem 148** ([12, Th. 3]). *Let $x_n$, $n = 1, 2, \dots$, be an increasing sequence of positive integers such that $\underline{d} = 0$ and let $t_1^{(n)}, t_2^{(n)}, \dots, t_n^{(n)}$ be a sequence produced by Algorithm. For a selected sequence of indices $n_k$, $k = 1, 2, \dots$, assume that*

(i) *$F(X_{n_k}, x) \to g(x)$ and $F(X_{n_k}^{(n_k)}, x) \to \tilde{g}(x)$ for all $x \in [0,1]$;*

(ii) *$t_1^{(n_k)} = \dots = t_{m_k}^{(n_k)} = D_{n_k} > t_{m_k+1}^{(n_k)} = \dots = t_{m_k+s_k}^{(n_k)} = D_{n_k} - 1$
$\ge t_{m_k+s_k+1}^{(n_k)} = \dots = t_{n_k}^{(n_k)} = 1$;*

(iii) *$\frac{m_k}{n_k} \to \alpha$;*

(iv) *$\frac{s_k}{n_k} \to \gamma$.*

*Then we have $\tilde{g}(x) \le g(x)$ for all $x \in [0,1]$, where*

(a) *If $\alpha + \gamma > 0$ then $d_g = 0$ and $\tilde{g}(x) = x(\alpha + \gamma)$ for all $x \in [0,1]$.*

(b) *If $\alpha + \gamma = 0$ and $\frac{m_k+s_k}{n_k} D_{n_k} \to \infty$ then $d_g = 0$ and $\tilde{g}(x) = 0$ for all $x \in (0,1)$.*

(c) *If $\alpha + \gamma = 0$ and $\frac{m_k+s_k}{n_k} D_{n_k} \to \delta$, $0 < \delta < \infty$, then $d_g = \frac{1}{\delta+1}$ and*

$$\tilde{g}(x) = \begin{cases} 0 & \text{if } x < y_2 = \frac{\delta}{\delta+1}, \\ x(\delta+1) - \delta & \text{if } y_2 \le x \le 1. \end{cases}$$

(d) *If* $\alpha+\gamma=0$ *and* $\frac{m_k+s_k}{n_k}D_{n_k}\to\delta=0$*, then* $d_g=1$ *and* $\tilde{g}(x)=x$.

### 6.2.15 $g(x)\in G(X_n)$ with constant intervals

**Theorem 149** ([174]). *Assume that* $\underline{d}>0$. *If there exists an interval* $(u,v)\subset[0,1]$ *such that every* $g\in G(X_n)$ *has a constant value on* $(u,v)$ *(may be different), then every* $g\in G(X_n)$ *has infinitely many intervals with constant values such that* $g$ *increases at their endpoints.*

*Proof.* Since

$$x_i<xx_m\Longleftrightarrow x_i<\left(x\frac{x_m}{x_n}\right)x_n,$$

then we have (22)

$$F(X_m,x)=\frac{n}{m}F\left(X_n,x\frac{x_m}{x_n}\right),$$

for every $m\le n$ and $x\in[0,1)$. Using the Helly selection principle, we can select a subsequence $(m_k,n_k)$ of the sequence $(m,n)$ such that $F(X_{n_k})\to g(x)$, $F(X_{m_k})\to\tilde{g}(x)$ as $k\to\infty$; furthermore $x_{m_k}/x_{n_k}\to\beta$ and $n_k/m_k\to\alpha$, but $\alpha$ may be infinity. Assuming $\beta>0$ and $g(\beta-0)>0$, we have $\alpha<\infty$ and (271)

$$\tilde{g}(x)=\alpha g(x\beta)\text{ a.e. on }[0,1].$$

Thus, if $\tilde{g}(x)$ has a constant value on $(u,v)$, then $g(x)$ must be constant on the interval $(u\beta,v\beta)$. Furthermore, if $\underline{d}>0$, then for every $g\in G(X_n)$ we have (276)

$$(\underline{d}/\overline{d})x\le g(x)\le(\overline{d}/\underline{d})x$$

for every $x\in[0,1]$. Thus, there exists a sequence $\beta_k\in(0,1)$ such that $\beta_k\searrow 0$ and $g(x)$ increases in $\beta_k$, $g(\beta_k)>0$, $k=1,2,\dots$. For such $\beta_k$, $g(x)$, applying the Helly principle, we can find sequences $\alpha_k$ and $\tilde{g}_k(x)\in G(X_n)$ such that

$$\tilde{g}_k(x)=\alpha_k g(x\beta_k)$$

a.e. on $[0,1]$. Every $\tilde{g}_k(x)$ has a constant value on the interval $(u,v)$, hence, $g(x)$ must be constant on the intervals $(u\beta_k,v\beta_k)$ for $k=1,2,\dots$. □

### 6.2.16 Transformation of $X_n$ by $1/x \bmod 1$

The mapping $1/x \bmod 1$ transforms the block $X_n$ to the block

$$Z_n = \left(\frac{x_n}{x_1}, \frac{x_n}{x_2}, \dots, \frac{x_n}{x_n}\right) \bmod 1.$$

For example, the block sequence $X_n = \left(\frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n}\right), \quad n = 1, 2, \dots$ which is u.d. is transformed to the block sequence

$$Z_n = \left(\frac{n}{1}, \frac{n}{2}, \dots, \frac{n}{n}\right) \bmod 1, \quad n = 1, 2, \dots$$

which has a.d.f.

$$g(x) = \int_0^1 \frac{1 - t^x}{1 - t} \mathrm{d}t = \sum_{n=1}^{\infty} \frac{x}{n(n + x)} = \gamma_0 + \frac{\Gamma'(1 + x)}{\Gamma(1 + x)},$$

where $\gamma_0$ is Euler's constant. This was proved by G. Pólya, (see I.J. Schoenberg [143]). The following theorem, which generalizes [92, p.56, Th. 7.6] describes a relation between $G(X_n)$ and $G(Z_n)$.

**Theorem 150** ([66, Th. 7])**.** *If every $g(x) \in G(X_n)$ is continuous on $[0, 1]$, then*

$$G(Z_n) = \left\{ \tilde{g}(x) = \sum_{n=1}^{\infty} g(1/n) - g(1/(n + x)); g(x) \in G(X_n) \right\}.$$

*Proof.* For $f(x) = 1/x \bmod 1$ we have $f^{-1}([0, t)) = \cup_{i=1}^{\infty}(1/(t+i), 1/i]$. Thus $F(Z_n, t) = \sum_{i=1}^{\infty}(F(X_n, 1/i) - F(X_n, 1/(t + i)))$.
$1^0$. Assume that $F(X_{n_k}, x) \to g(x)$, where $g(x)$ is everywhere continuous on $[0, 1]$. Thus

$$\sum_{i=1}^{K}(F(X_{n_k}, 1/i) - F(X_{n_k}, 1/(t + i))) \to \sum_{i=1}^{K}(g(1/i) - g(1/(t + i))),$$

$$\sum_{i=K+1}^{\infty} (F(X_{n_k}, 1/i) - F(X_{n_k}, 1/(t + i))) \le F(X_{n_k}, 1/(K + 1))$$
$$\to g(1/(K + 1)) \to 0.$$

Thus $F(Z_{n_k}, t) \to \tilde{g}(t) = \sum_{i=1}^{\infty}(g(1/i) - g(1/(t + i)))$ for $t \in [0, 1]$.
$2^0$. Assume that $F(Z_{n_k}, t) \to \tilde{g}(t)$ weakly. From $n_k$ there can be selected $n'_k$ such that $F(X_{n'_k}, x) \to g(x)$. Assuming continuity of $g(x)$, we apply $1^0$. □

### 6.2.17 Construction $H \subset G(X_n)$ of d.f.s

Basic open problem is that characterize a nonempty set $H$ of d.f.s for which there exists an increasing sequence of positive integers $x_n$ such that $G(X_n) = H$. In [11] we found integer sequence $1 \le x_1 < x_2 < \dots$ such that the piecewise linear function $h_2(x)$ defined in (23) belongs to $G(X_n)$. Now we are going to extend this construction. In [12] is proved:

**Theorem 151.** *Let $H$ be a nonempty set of d.f.s defined on $[0,1]$. Then there exists an integer sequence $1 \le x_1 < x_2 < \dots$ such that $H \subset G(X_n)$.*

*Proof.* The proof contains the following 5 steps.

$1^0$. To the set $H$ it can be constructed a sequence of continuous strictly increasing piecewise linear functions $h_n(x)$, $n = 1, 2, \dots$, such that every $f(x) \in H$ is a weak limit $h_{n_k}(x) \to f(x)$.

$2^0$. For every $h(x)$ possessing at points $\beta_1 = 0 < \beta_2 < \dots < \beta_{s-1} < \beta_s = 1$ the values $\alpha_1 = 0 < \alpha_2 < \dots < \alpha_{s-1} < \alpha_s = 1$, respectively, and being linear in each interval $[\beta_i, \beta_{i+1}]$, we can define a sequence of integer intervals $[m_k^{(1)}, n_k]$, $k = 1, 2, \dots$, and their divisions

$$m_k^{(1)} < m_k^{(2)} < \dots < m_k^{(s-1)} < m_k^{(s)} < n_k$$

in which we can define integers

$$x_{m_k^{(1)}} < x_{m_k^{(2)}} < \dots < x_{m_k^{(s-1)}} < x_{m_k^{(s)}} < x_{n_k}$$

such that for $i = 1, 2, \dots, s$ we have

(i) $\frac{x_{m_k^{(i)}}}{x_{n_k}} \to \beta_i$,

(ii) $\frac{m_k^{(i)}}{n_k} \to \alpha_i$,

(iii) $x_{m_k^{(i)}} - x_{m_k^{(i-1)}} \ge m_k^{(i)} - m_k^{(i-1)}$,

(iv) $x_{n_k} - x_{m_k^{(s)}} \ge n_k - m_k^{(s)}$.

For other $n \in [m_k^{(1)}, n_k]$ we define $x_n$ linearly, i.e. for $n \in [m_k^{(i-1)}, m_k^{(i)}]$ we put

(v)

$$x_n = x_{m_k^{(i-1)}} + \left[ \left(n - m_k^{(i-1)}\right) \frac{x_{m_k^{(i)}} - x_{m_k^{(i-1)}}}{m_k^{(i)} - m_k^{(i-1)}} \right].$$

Directly from (i), (ii) and (v) it follows that

$$\frac{\#\left\{n \in [m_k^{(1)}, n_k]; \frac{x_n}{x_{n_k}} < x\right\}}{n_k} \to h(x) \tag{334}$$

for $x \in (0,1)$ as $k \to \infty$.

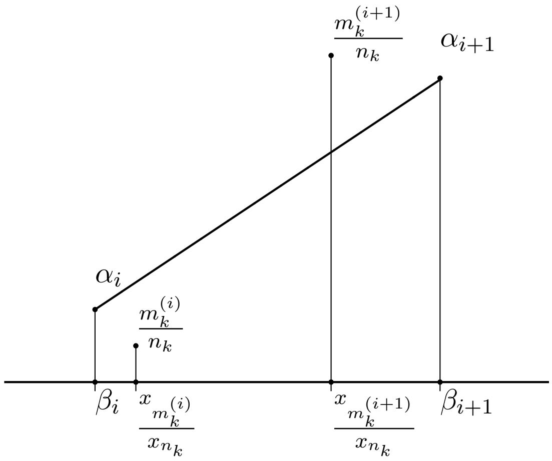



Figure: A part of graph of $h(x)$ and (i)-(ii) properties.

Note that, in this step, the intervals $[m_k^{(1)}, n_k]$, $k = 1, 2, \ldots$, can intersect. For necessity of pairwise disjointness we use the next step.

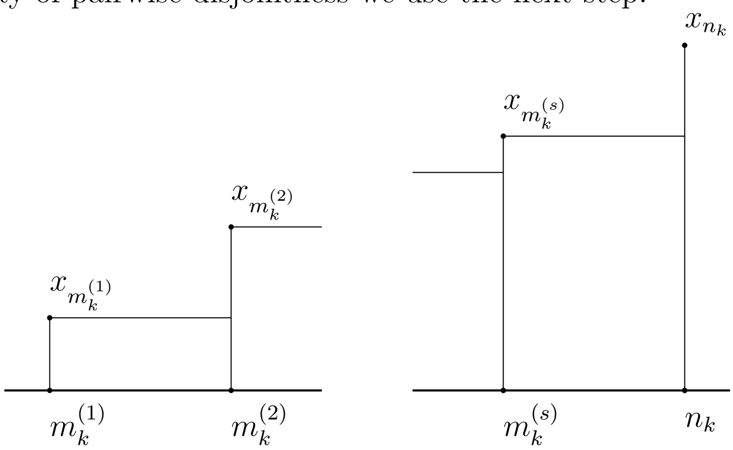



Figure: (iii)-(iv) properties.

$3^0$. One solution $[m_k^{(1)}, n_k]$, $k = 1, 2, \dots$ in $2^0$ gives infinitely many solutions by the following: Let $A_k < B_k$ be two positive integer sequences. Replace $[m_k^{(1)}, n_k]$ by $[A_k m_k^{(1)}, A_k n_k]$ with division

$$A_k m_k^{(1)} < A_k m_k^{(2)} < \cdots < A_k m_k^{(s-1)} < A_k m_k^{(s)} < A_k n_k$$

and define the values of $x_n$ as

$$x_{A_k m_k^{(i)}} = B_k x_{m_k^{(i)}},$$

$i = 1, 2, \dots, s$ and $x_{A_k n_k} = B_k x_{n_k}$. Then the limit (i) and (ii) again hold

$$\frac{x_{A_k m_k^{(i)}}}{x_{A_k n_k}} = \frac{B_k x_{m_k^{(i)}}}{B_k x_{n_k}} \to \beta_i, \quad \frac{A_k m_k^{(i)}}{A_k n_k} \to \alpha_i.$$

Also (iii) and (iv) hold, since

$$\begin{aligned} x_{A_k m_k^{(i)}} - x_{A_k m_k^{(i-1)}} &= B_k x_{m_k^{(i)}} - B_k x_{m_k^{(i-1)}} \\ &\geq B_k (m_k^{(i)} - m_k^{(i-1)}) \geq A_k m_k^{(i)} - A_k m_k^{(i-1)}. \end{aligned}$$

$4^0$. Let $h_i(x)$, $i = 1, 2, \dots$ be a dense set of d.f. in $H$ and for $h_i(x) = h(x)$ rewrite the interval $[m_k^{(1)}, n_k]$ in $2^0$ as $[m_k^{(1,i)}, n_k^{(i)}]$. Order these intervals to infinite matrix $\mathbb{A}$

$$\begin{array}{l} [m_1^{(1,1)}, n_1^{(1)}], [m_2^{(1,1)}, n_2^{(1)}], \dots, [m_k^{(1,1)}, n_k^{(1)}], \dots \\ [m_1^{(1,2)}, n_1^{(2)}], [m_2^{(1,2)}, n_2^{(2)}], \dots, [m_k^{(1,2)}, n_k^{(2)}], \dots \\ \dots \\ [m_1^{(1,i)}, n_1^{(i)}], [m_2^{(1,i)}, n_2^{(i)}], \dots, [m_k^{(1,i)}, n_k^{(i)}], \dots \\ \dots \end{array}$$

and reorder it to a linear sequence by diagonals, i.e. to

$$[m_1^{(1,1)}, n_1^{(1)}], [m_1^{(1,2)}, n_1^{(2)}], [m_2^{(1,1)}, n_2^{(1)}], \dots$$

and denote it as a new sequence $[m_k^{(1)}, n_k]$, $k = 1, 2, \dots$. Since these intervals can intersect we use in $3^0$ suitable $A_k < B_k$, $k = 1, 2, \dots$ such that the resulting sequence is disjoint and

(vi) $x_{m_{k+1}^{(1)}} - x_{n_k} \geq m_{k+1}^{(1)} - n_k$,

(vii) $x_{m_1^{(1)}} \geq m_1^{(1)}$.

For $n$ which are not in the intervals $[m_k^{(1)}, n_k]$, $k = 1, 2, \dots$ we can define $x_n$ linearly. Now, if from $n_k$, $k = 1, 2, \dots$ we select $n'_k$ corresponding to $i$th line of $\mathbb{A}$, then $F(X_{n'_k}, x) \to h_i(x)$ for $x \in [0, 1]$.

$5^0$. Finally, we give a solution of (i)-(iv) in $2^0$. We start with increasing sequence of indices $n_k$, $k = 1, 2, \dots$, and let $\lambda > 1$ and put (integer parts are omitted)

$x_{n_k} = \lambda n_k$,

$x_{m_k^{(i)}} = \beta_i \lambda n_k$,

$m_k^{(i)} = \alpha_i n_k$.

For (iv) we need

$$\begin{aligned} x_{m_k^{(i)}} - x_{m_k^{(i-1)}} =& \beta_i \lambda n_k - \beta_{i-1} \lambda n_k = \lambda(\beta_i - \beta_{i-1}) n_k \\ & \geq m_k^{(i)} - m_k^{(i-1)} = (\alpha_i - \alpha_{i-1}) n_k \end{aligned}$$

which gives assumption

$\lambda > \max \frac{\alpha_i - \alpha_{i-1}}{\beta_i - \beta_{i-1}}$. □

**Notes 27.** By Theorem 151 there exists an integer sequence $1 \leq x_1 < x_2 < \dots$ such that $G(X_n)$ contains all d.f.s. Especially, for every sequence $y_n \in [0, 1)$, $n = 1, 2, \dots$, there exists an $X_n$ such that $G(y_n) \subset G(X_n)$.

### 6.2.18 Examples

**Example 57.** [173]: Put $x_n = p_n$, the $n$th prime and denote

$$X_n = \left( \frac{2}{p_n}, \frac{3}{p_n}, \dots, \frac{p_{n-1}}{p_n}, \frac{p_n}{p_n} \right).$$

The sequence of blocks $X_n$ is u.d. and therefore the ratio sequence $p_m/p_n$, $m = 1, 2, \dots, n$, $n = 1, 2, \dots$ is u.d. in $[0, 1]$. This generalizes a result of A. Schinzel (cf. W. Sierpiński (1964, p. 155)). Note that from u.d. of $X_n$ applying for the $L^2$ discrepancy of $X_n$ we get the following limit

$$\lim_{n \to \infty} \frac{1}{n^2 p_n} \sum_{i,j=1}^{n} |p_i - p_j| = \frac{1}{3}.$$

**Example 58.** [173, Ex. 11.1]: Let $\gamma$, $\delta$, and $a$ be given real numbers satisfying $1 \le \gamma < \delta \le a$. Let $x_n$ be an increasing sequence of all integer points lying in the intervals

$$(\gamma, \delta), (\gamma a, \delta a), \dots, (\gamma a^k, \delta a^k), \dots.$$

Then $G(X_n) = \{g_t(x); t \in [0,1]\}$, where $g_t(x)$ has constant values

$$g_t(x) = \frac{1}{a^i(1 + t(a-1))} \text{ for } x \in \frac{(\delta, a\gamma)}{a^{i+1}(t\delta + (1-t)\gamma)}, \quad i = 0, 1, 2, \dots$$

and on the component intervals it has a constant derivative

$$g_t'(x) = \frac{t\delta + (1-t)\gamma}{(\delta - \gamma)(\frac{1}{a-1} + t)} \text{ for } x \in \frac{(\gamma, \delta)}{a^{i+1}(t\delta + (1-t)\gamma)}, \quad i = 0, 1, 2, \dots$$

$$\text{and } x \in \left(\frac{\gamma}{t\delta + (1-t)\gamma}, 1\right),$$

where

$$F(X_{n_k}, x) \to g_t(x) \text{ for } n_k \text{ for which } x_{n_k} = [a^k\gamma + ta^k(\delta - \gamma)] \tag{335}$$

Here we write $(xz, yz) = (x, y)z$ and $(x/z, y/z) = (x, y)/z$. From it follows that the set $G(X_n)$ has the following properties:

(i) Every $g \in G(X_n)$ is continuous.

(ii) Every $g \in G(X_n)$ has infinitely many intervals with constant values , i.e. with $g'(x) = 0$, and in the infinitely many complement intervals it has a constant derivative $g'(x) = c$, where $\frac{1}{\overline{d}} \le c \le \frac{1}{\underline{d}}$ and for lower $\underline{d}$ and upper $\overline{d}$ asymptotic density of $x_n$ we have $\underline{d} = \frac{(\delta-\gamma)}{\gamma(a-1)}$, $\overline{d} = \frac{(\delta-\gamma)a}{\delta(a-1)}$.

(iii) The graph of every $g \in G(X_n)$ lies in the intervals $\left[\frac{1}{a}, 1\right] \times \left[\frac{1}{a}, 1\right] \cup \left[\frac{1}{a^2}, \frac{1}{a}\right] \times \left[\frac{1}{a^2}, \frac{1}{a}\right] \cup \dots$ Moreover, the graph $g$ in $\left[\frac{1}{a^k}, \frac{1}{a^{k-1}}\right] \times \left[\frac{1}{a^k}, \frac{1}{a^{k-1}}\right]$ is similar to the graph of $g$ in $\left[\frac{1}{a^{k+1}}, \frac{1}{a^k}\right] \times \left[\frac{1}{a^{k+1}}, \frac{1}{a^k}\right]$ with coefficient $\frac{1}{a}$. Using the parametric expression, it can be written for all $x \in \left(\frac{1}{a^{i+1}}, \frac{1}{a^i}\right)$ that $g_t(x) = \frac{g_t(a^i x)}{a^i}$, $\quad i = 0, 1, 2, \dots$.

(iv) $G(X_n)$ is connected and the upper distribution function $\overline{g}(x) = g_0(x) \in G(X_n)$ and the lower distribution function $\underline{g}(x) \notin G(X_n)$. The graph of $\underline{g}(x)$ on $\left[\frac{1}{a}, 1\right] \times \left[\frac{1}{a}, 1\right]$ coincides with the graph of $y(x) = \left(1 + \frac{1}{\underline{d}}\left(\frac{1}{x} - 1\right)\right)^{-1}$ on $\left[\frac{\gamma}{\delta}, 1\right]$, further, on $\left[\frac{1}{a}, \frac{\gamma}{\delta}\right]$ we have $\underline{g}(x) = \frac{1}{a}$.

(v) $G(X_n) = \left\{ \frac{g_0(x\beta)}{g_0(\beta)}; \beta \in \left[\frac{1}{a}, \frac{\delta}{a\gamma}\right] \right\}$.

For the proofs of i-v we only note: Assume that $x_n \in a^k(\gamma, \delta)$, $i, i+1, i+2, \dots \in a^j(\gamma, \delta)$ for some $j < k$, and let $F(X_n, x) \to g(x)$ for some sequence of $n$. Then $g(x)$ has a constant derivative in the intervals containing $\frac{i}{x_n}, \frac{i+1}{x_n}, \frac{i+2}{x_n}, \dots$, since

$$\frac{\frac{1}{n}}{\frac{i+1}{x_n} - \frac{i}{x_n}} = \frac{x_n}{n},$$

and thus $\frac{x_n}{n}$ must be convergent to $g'(x)$, so $\frac{1}{\overline{d}} \le g'(x) \le \frac{1}{\underline{d}}$. For $x_n = [ta^k\delta + (1-t)a^k\gamma]$ we can find

$$\begin{aligned} g'(x) = \lim_{n\to\infty} \frac{x_n}{n} &= \lim_{k\to\infty} \frac{a^k(t\delta + (1-t)\gamma)}{\sum_{j=0}^{k-1} a^j(\delta - \gamma) + a^k(t\delta + (1-t)\gamma) - a^k\gamma} \\ &= \frac{t\delta + (1-t)\gamma}{(\delta - \gamma)\left(\frac{1}{a-1} + t\right)}. \end{aligned}$$

Using Theorem 61 and [11, Ex. 3] we shall add the following properties moreover:

(vi) By Definition 8 of the local asymptotic density $d_g$ and by (335) for $g(x) = g_t(x)$ we have

$$d_{g_t} = \lim_{k\to\infty} \frac{n_k}{x_{n_k}} = \lim_{k\to\infty} \frac{\sum_{i=0}^{k-1} a^i(\delta - \gamma) + ta^k(\delta - \gamma)}{a^k\gamma + ta^k(\delta - \gamma)} = \frac{(\delta - \gamma)(1 + t(a-1))}{(a-1)(\gamma + t(\delta - \gamma))} \tag{336}$$

and for $t = 0$ we have $d_{g_0} = \underline{d}$ and for $t = 1$ we have $d_{g_1} = \overline{d}$ and we see

$$g_t'(x) = \frac{1}{d_{g_t}} \tag{337}$$

for $x$ with the constant derivative of $g_t(x)$.

(vii) For the function $h_{1,g}(x)$ defined in (308), putting $g(x) = g_t(x)$, we have:

$$\frac{\underline{d}}{d_{g_t}} = \frac{\gamma + t(\delta - \gamma)}{\gamma(1 + t(a-1))}, \frac{1 - d_{g_t}}{1 - \underline{d}} = \frac{\gamma}{\gamma + t(\delta - \gamma)}, \frac{\underline{d}}{d_{g_t}} \frac{1 - d_{g_t}}{1 - \underline{d}} = \frac{1}{1 + t(a-1)}.$$

Then

$$h_{1,g_t}(x) = \begin{cases} x\frac{\underline{d}}{d_{g_t}}, & \text{for } x \in \left(0, \frac{\gamma}{\gamma + t(\delta - \gamma}\right), \\ g_t(x) = x\frac{1}{d_{g_t}} + 1 - \frac{1}{d_{g_t}}, & \text{for } x \in \left(\frac{\gamma}{\gamma + t(\delta - \gamma}, 1\right), \end{cases} \tag{338}$$

see the following Fig.

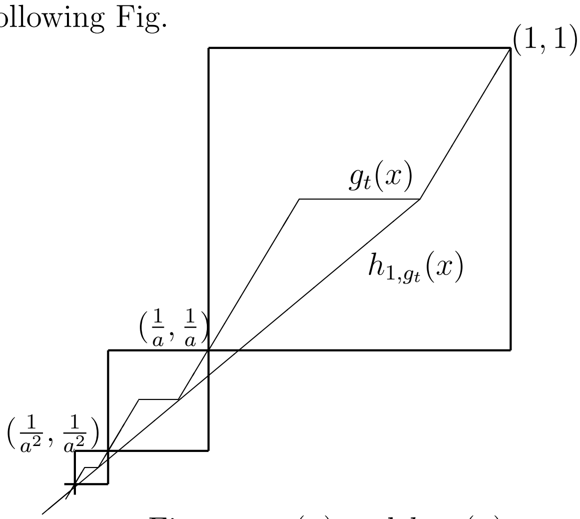



Figure : $g_t(x)$ and $h_{1,g_t}(x)$.

(viii) In proof of the upper bound (311) we have proved that $1-\int_0^1 h_{1,g}(x)\mathrm{d}x$ is maximal for $d_g = \min(\sqrt{\underline{d}}, \overline{d})$. Let $t_0 \in [0,1]$ be such that $d_{g_{t_0}} = \min(\sqrt{\underline{d}}, \overline{d})$. $t_0$ can be computed by inverse formula to (336)

$$t = \frac{d_{g_t}(a-1)\gamma - (\delta-\gamma)}{(\delta-\gamma)(a-1)(1-d_{g_t})}. \tag{339}$$

(ix) Let $P(t)$ be the area in $\left[\frac{1}{a},1\right] \times \left[\frac{1}{a},1\right]$ bounded by the graph of $g_t(x)$. Then

$$\begin{aligned}\int_0^1 g_t(x)\mathrm{d}x =& P(t)\frac{1}{1-\frac{1}{a^2}} + \frac{1}{a+1}\\ =& \frac{1}{2} + \frac{1}{2}.\frac{1}{(a+1)}.\frac{(\gamma a-\delta)}{(1+t(a-1))(\gamma+t(\delta-\gamma))}\\ &+\frac{1}{2}.\frac{t(\delta-\gamma a)}{(1+t(a-1))(\gamma+t(\delta-\gamma))}\end{aligned} \tag{340}$$

and since $g_0(x) = \overline{g}(x)$ we have that $\max_{t\in[0,1]} \int_0^1 g_t(x)\mathrm{d}x$ is attained at $t=0$. Using derivative of $P(t)$ it can be see that the $\min_{t\in[0,1]} \int_0^1 g_t(x)\mathrm{d}x$ is attained at $t=1$. It also follows from the fact that for $x_{n+1} = x_n + 1$ we have

$$\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{x_i}{x_{n+1}} - \frac{1}{n}\sum_{i=1}^{n}\frac{x_i}{x_n}$$

$$= \frac{1}{n+1} - \left(\frac{1}{x_n+1} + \frac{1}{n+1}.\frac{1}{1+\frac{1}{x_n}}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\frac{x_i}{x_n}\right) > 0$$

because $c_1(x) \notin G(X_n)$ and thus $\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}\frac{x_i}{x_n} < 1$. Now, denoting the index $n_k$ for $x_{n_k} = [a^k\delta]$, the lim sup of $\frac{1}{n}\sum_{i=1}^{n}\frac{x_i}{x_n}$ is attained over $n = n_k$, $k = 0, 1, 2, \dots$ and for such $n_k$ we have $F(X_{n_k}, x) \to g_1(x)$ for $x \in [0, 1]$.

(x) Thus we have

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}\frac{x_i}{x_n} = 1 - \int_0^1 g_0(x)\mathrm{d}x = \frac{1}{2} - \frac{1}{2}.\frac{1}{(a+1)}\left(\frac{\gamma a - \delta}{\gamma}\right), \tag{341}$$

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}\frac{x_i}{x_n} = 1 - \int_0^1 g_1(x)\mathrm{d}x = \frac{1}{2} + \frac{1}{2}.\frac{1}{(a+1)}\left(\frac{\gamma a - \delta}{\delta}\right). \tag{342}$$

The upper bound (311) coincide with the maximal value of $1 - \int_0^1 h_{1,g}(x)\mathrm{d}x$ attained for $d_g = \min(\sqrt{\underline{d}}, \overline{d})$. Since $1 - \int_0^1 g_1(x)\mathrm{d}x$ is maximal for all $1 - \int_0^1 g_t(x)\mathrm{d}x$, $t \in [0, 1]$ and $1 - \int_0^1 g_1(x)\mathrm{d}x \le 1 - \int_0^1 h_{1,g_1}(x)\mathrm{d}x$ then the upper bound (342) satisfies (311).

(xi) Using explicit formulas

$$\underline{d} = \frac{(\delta - \gamma)}{\gamma(a-1)}, \quad \overline{d} = \frac{(\delta - \gamma)a}{\delta(a-1)} \tag{343}$$

for asymptotic densities we see again that (341) and (342) satisfy (310) and (311), respectively, in Theorem 143.

**Example 59.** [66, Ex. 2]: Let $x_n$ and $y_n$, $n = 1, 2, \dots$, be two strictly increasing sequences of positive integers such that for the related block sequences $X_n = \left(\frac{x_1}{x_n}, \dots, \frac{x_n}{x_n}\right)$ and $Y_n = \left(\frac{y_1}{y_n}, \dots, \frac{y_n}{y_n}\right)$, we have singleton $G(X_n) = \{g_1(x)\}$ and $G(Y_n) = \{g_2(x)\}$. Furthermore, let $n_k$, $k = 1, 2, \dots$, be an increasing sequence of positive integers such that $N_k = \sum_{i=1}^{k} n_i$ satisfies $\frac{n_k}{N_k} \to 1$. Denote by $z_n$ the following increasing sequence of positive integers composed by blocks (here we use the notation $a(b, c, d, \dots) = (ab, ac, ad, \dots)$)

$$(x_1, \dots, x_{n_1}), x_{n_1}(y_1, \dots, y_{n_2}), x_{n_1}y_{n_2}(x_1, \dots, x_{n_3}), x_{n_1}y_{n_2}x_{n_3}(y_1, \dots, y_{n_4}), \dots$$

Then the sequence of blocks $Z_n = \left(\frac{z_1}{z_n}, \dots, \frac{z_n}{z_n}\right)$ has the set of d.f.s

$$G(Z_n) = \{g_1(x), g_2(x), c_0(x)\} \cup \{g_1(xy_n); n = 1, 2, \dots\}$$

$$\cup \{g_2(xx_n); n = 1, 2, \dots\}$$
$$\cup \left\{ \frac{1}{1+\alpha} c_0(x) + \frac{\alpha}{1+\alpha} g_1(x); \alpha \in [0, \infty) \right\}$$
$$\cup \left\{ \frac{1}{1+\alpha} c_0(x) + \frac{\alpha}{1+\alpha} g_2(x); \alpha \in [0, \infty) \right\},$$

where $g_1(xy_n) = 1$ if $xy_n \geq 1$, similarly for $g_2(xx_n)$.

*Proof.* For every $n = 1, 2, \dots$ there exists an integer $k$ such that

$$N_{k-1} < n \leq N_k$$

(here $N_0 = 0$). Put $n' = n - N_{k-1}$. For every $n$ we have

$$z_n = \begin{cases} x_{n_1} y_{n_2} \dots x_{n_{k-1}} y_{n'} & \text{if } k \text{ is even,} \\ x_{n_1} y_{n_2} \dots y_{n_{k-1}} x_{n'} & \text{if } k \text{ is odd.} \end{cases}$$

Firstly we assume that $k$ is even. Then $Z_n$ has the form

$$Z_n =$$
$$\left( \dots, \frac{x_{n_1} y_{n_2} \dots y_{n_{k-2}} (x_1, \dots, x_{n_{k-1}})}{x_{n_1} y_{n_2} \dots x_{n_{k-1}} y_{n'}}, \frac{x_{n_1} y_{n_2} \dots x_{n_{k-1}} (y_1, \dots, y_{n'})}{x_{n_1} y_{n_2} \dots x_{n_{k-1}} y_{n'}} \right) =$$
$$\left( \dots, \frac{1}{x_{n_{k-1}} y_{n'}} \left( \frac{y_1}{y_{n_{k-2}}}, \dots, \frac{y_{n_{k-2}}}{y_{n_{k-2}}} \right), \frac{1}{y_{n'}} \left( \frac{x_1}{x_{n_{k-1}}}, \dots, \frac{x_{n_{k-1}}}{x_{n_{k-1}}} \right), \left( \frac{y_1}{y_{n'}}, \dots, \frac{y_{n'}}{y_{n'}} \right) \right)$$

and thus for $x > \frac{1}{x_{n_{k-1}}}$ we have

$$F(Z_n, x) = \frac{N_{k-2} + n_{k-1} F(X_{n_{k-1}}, xy_{n'}) + n' F(Y_{n'}, x)}{N_{k-1} + n'}$$
$$= \frac{N_{k-2}}{N_{k-1} + n'} + \frac{\frac{n_{k-1}}{N_{k-1}}}{1 + \frac{n'}{N_{k-1}}} F(X_{n_{k-1}}, xy_{n'}) + \frac{1}{1 + \frac{N_{k-1}}{n'}} F(Y_{n'}, x).$$

If $n \to \infty$, then the first term tends to zero. If $F(Z_n, x) \to g(x)$ for some sequence of $n$, we can select a subsequence of $n$'s such that $\frac{n'}{N_{k-1}} \to \alpha$ for some $\alpha \in [0, \infty)$, or $\frac{n'}{N_{k-1}} \to \infty$. For such $n'$ we distinguish the following cases:

(a) If $n' = \text{constant}$, then

$$\frac{\frac{n_{k-1}}{N_{k-1}}}{1+\frac{n'}{N_{k-1}}} F(X_{n_{k-1}}, xy_{n'}) \to g_1(xy_{n'})(\text{ here } g_1(xy_{n'}) = 1 \text{ for } xy_{n'} > 1)$$

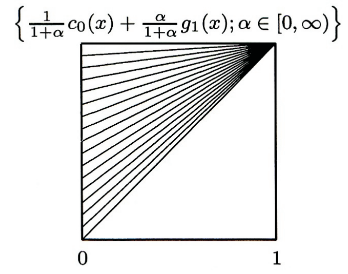
$$\frac{1}{1+\frac{N_{k-1}}{n'}} F(Y_{n'}, x) \to 0$$

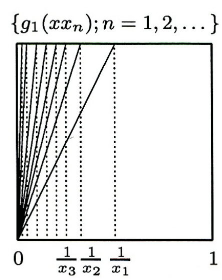
and thus $F(Z_n, x) \to g_1(xy_{n'})$.
(b) If $n' \to \infty$, then $F(X_{n_{k-1}}, xy_{n'}) \to 1$; precisely $F(X_{n_{k-1}}, xy_{n'}) \to c_0(x)$.
(b1) If $\frac{n'}{N_{k-1}} \to 0$, then $F(Z_n, x) \to c_0(x)$.
(b2) If $\frac{n'}{N_{k-1}} \to \alpha \in (0, \infty)$, then $F(Z_n, x) \to \frac{1}{1+\alpha} c_0(x) + \frac{\alpha}{1+\alpha} g_2(x)$.
(b3) If $\frac{n'}{N_{k-1}} \to \infty$, then $F(Z_n, x) \to 0 + g_2(x)$.
For $k$-odd we use a similar computation. $\square$

Now, identify $x_n = y_n$ and select $x_n$ such that $g_1(x) = x$ (e.g., $x_n = n$ or $x_n = p_n$, the $n$th prime) and put $n_k = 2^{k^2}$ for $k = 1, 2, \dots$. Then the set of all d.f.s

$$\begin{aligned} G(Z_n) = &\{g_1(x), c_0(x)\} \cup \{g_1(xx_n); n = 1, 2, \dots\} \\ &\cup \left\{ \frac{1}{1+\alpha} c_0(x) + \frac{\alpha}{1+\alpha} g_1(x); \alpha \in [0, \infty) \right\} \end{aligned}$$

is disconnected, as can be seen in the following Figures




**Example 60.** In [55] is proved that $\frac{x_n}{x_{n+1}} \to 1$ does not imply that $G(X_n)$ is a singleton. This is a negative answer to the Problem 1.9.2 in [166].

Let $a_k$, $n_k$, $k = 1, 2, \dots$, and $x_n$, $n = 1, 2, \dots$ be three increasing integer sequences and $h_1 < h_2$ be two positive integers. Assume that
(i) $\frac{n_k}{n_{k+1}} \to 0$ for $k \to \infty$;
(ii) $\frac{a_k}{n_{k+1}} \to 0$ for $k \to \infty$;
(iii) for odd $k$ we have
$a_k^{h_2} \le x_{n_k} = (a_{k-1} + n_k - n_{k-1})^{h_1} \le (a_k + 1)^{h_2}$ and
$x_i = (a_k + i - n_k)^{h_2}$ for $n_k < i \le n_{k+1}$;
(iv) for even $k$ we have
$a_k^{h_1} \le x_{n_k} = (a_{k-1} + n_k - n_{k-1})^{h_2} \le (a_k + 1)^{h_1}$ and
$x_i = (a_k + i - n_k)^{h_1}$ for $n_k < i \le n_{k+1}$.
Then $\frac{x_n}{x_{n+1}} \to 1$ and the set $G(X_n)$ of all distribution functions of the sequence of blocks $X_n$ is $G(X_n) = G_1 \cup G_2 \cup G_3 \cup G_4$, where
$G_1 = \{x^{\frac{1}{h_2}}.t; t \in [0, 1]\}$,
$G_2 = \{x^{\frac{1}{h_2}}(1 - t) + t; t \in [0, 1]\}$,
$G_3 = \{\max(0, x^{\frac{1}{h_1}} - (1 - x^{\frac{1}{h_1}})u); u \in [0, \infty)\}$ and
$G_4 = \{\min(1, x^{\frac{1}{h_1}}.v); v \in [1, \infty)\}$.
In [173, Th. 5.2, p. 762] it is proved that the condition $\frac{x_n}{x_{n+1}} \to 1$ implies the connectivity of $G(X_n)$

*Proof.* **1.** Firstly we prove that for any $h_1 < h_2$ the sequences $a_k, n_k, x_n$ satisfying (i)–(iv) exist:

For $i = 1, \dots, n_1$ we put $x_i = i^{h_1}$ and then we find $a_1$ such that $a_1^{h_2} \le x_{n_1} \le (a_1+1)^{h_2}$. If we have selected, for an odd step $k$, all $a_i$, $i = 1, 2, \dots, k-1$, $x_i$, $i = 1, 2, \dots, n_k$, then we find $a_k$ such that $a_k^{h_2} \le x_{n_k} < (a_k + 1)^{h_2}$, and then we put $x_i = (a_k + i - n_k)^{h_2}$ for $n_k < i \le n_{k+1}$, where we choose $n_{k+1}$ sufficiently large to satisfy the limits (i) and (ii). For an even step $k$ we proceed similarly replacing $h_2$ by $h_1$.

**2.** In contrary to the independence of $a_k$ and $n_{k+1}$ we have

$$\frac{a_k}{n_k^{\frac{h_1}{h_2}}} \to 1 \text{ for odd } k \to \infty, \quad \frac{a_k}{n_k^{\frac{h_2}{h_1}}} \to 1 \text{ for even } k \to \infty. \tag{344}$$

This follows from (iii) and (iv), directly, e.g. from (iii) we have

$$\frac{a_k^{h_2}}{n_k^{h_1}} < \left(\frac{a_{k-1}}{n_k} + 1 - \frac{n_{k-1}}{n_k}\right)^{h_1} < \frac{(a_k+1)^{h_2}}{n_k^{h_1}}.$$

As an application of (344) we have

$$\frac{a_k}{n_k} \to 0 \text{ for odd } k \to \infty, \quad \frac{a_k}{n_k} \to \infty \text{ for even } k \to \infty. \tag{345}$$

**3.** Now we prove $\frac{x_i}{x_{i+1}} \to 1$ as $i \to \infty$. Let $i \in (n_k, n_{k+1})$ and let e.g. $k$ be odd. Then by (iii)

$$\frac{x_i}{x_{i+1}} = \left(1 - \frac{1}{a_k + i + 1 - n_k}\right)^{h_2} > \left(1 - \frac{1}{a_k}\right)^{h_2}$$

and for $i = n_k$ again

$$\frac{x_{n_k}}{x_{n_k+1}} > \frac{a_k^{h_2}}{(a_k+1)^{h_2}} > \left(1 - \frac{1}{a_k}\right)^{h_2}$$

which implies the limit 1 as odd $k \to \infty$. Similarly for even $k$.

**4.** Let $N \in [n_k, n_{k+1}]$ be an integer sequence (we shall omit the index in $N_k$) for $k \to \infty$. For $x \in (0,1)$ we have

$$\begin{aligned} F(X_N, x) &= \frac{\#\{1 \le i \le n_{k-1}; \frac{x_i}{x_N} < x\}}{N} + \frac{\#\{n_{k-1} < i \le n_k; \frac{x_i}{x_N} < x\}}{N} \\ &+ \frac{\#\{n_k < i \le N; \frac{x_i}{x_N} < x\}}{N} = o(1) + \frac{A}{N} + \frac{B}{N}. \end{aligned} \tag{346}$$

To compute $\frac{A}{N}$ for odd $k$ we use

$$\frac{x_i}{x_N} = \frac{(a_{k-1} + i - n_{k-1})^{h_1}}{(a_k + N - n_k)^{h_2}} < x \Longleftrightarrow i - n_{k-1} < x^{\frac{1}{h_1}} (a_k + N - n_k)^{\frac{h_2}{h_1}} - a_{k-1}$$

and we have

$$\frac{A}{N} = \frac{\min(n_k - n_{k-1}, \max(0, [x^{\frac{1}{h_1}} (a_k + N - n_k)^{\frac{h_2}{h_1}} - a_{k-1}]))}{N}. \tag{347}$$

Similarly, for even $k$

$$\frac{A}{N} = \frac{\min(n_k - n_{k-1}, \max(0, [x^{\frac{1}{h_2}} (a_k + N - n_k)^{\frac{h_1}{h_2}} - a_{k-1}]))}{N}. \tag{348}$$

For $\frac{B}{N}$ and odd $k$ we use

$$\frac{x_i}{x_N} = \left(\frac{a_k + i - n_k}{a_k + N - n_k}\right)^{h_2} < x \Longleftrightarrow i - n_k < x^{\frac{1}{h_2}} (a_k + N - n_k) - a_k$$

which gives

$$\frac{B}{N} = \frac{\min(N - n_k, \max(0, [x^{\frac{1}{h_2}}(a_k + N - n_k) - a_k]))}{N}. \tag{349}$$

Similarly, for even $k$ we have

$$\frac{B}{N} = \frac{\min(N - n_k, \max(0, [x^{\frac{1}{h_1}}(a_k + N - n_k) - a_k]))}{N}. \tag{350}$$

In the following we will distinguish three cases

$$\frac{n_k}{N} \to t > 0, \quad \frac{n_k}{N} \to 0 \text{ and } \frac{N}{n_{k+1}} \to 0, \quad \text{ and } \frac{N}{n_{k+1}} \to t > 0.$$

**5.** Now, let $\frac{n_k}{N} \to t > 0$ as $k \to \infty$.

a) Assume that $k$ is odd and compute the limit of $\frac{A}{N}$ by (347). We have $\frac{n_k - n_{k-1}}{N} \to t$ and if $t < 1$ we see

$$x^{\frac{1}{h_1}} \left( \frac{a_k}{N^{\frac{h_1}{h_2}}} + \frac{N}{N^{\frac{h_1}{h_2}}} \left(1 - \frac{n_k}{N}\right) \right)^{\frac{h_2}{h_1}} - \frac{a_{k-1}}{N} \to \infty$$

since $\frac{N}{N^{\frac{h_1}{h_2}}}$ for $h_1 < h_2$ is unbounded and by (344)

$$\frac{a_k}{N^{\frac{h_1}{h_2}}} = \frac{a_k}{n_k^{\frac{h_1}{h_2}}} \left(\frac{n_k}{N}\right)^{\frac{h_1}{h_2}} \to t^{\frac{h_1}{h_2}}.$$

is bounded. Thus, for $0 < t < 1$, we have

$$\frac{A}{N} \to t \text{ for odd } k \to \infty. \tag{351}$$

a1) Let for the moment $t = 1$. We have $\frac{a_k}{n_k^{\frac{h_1}{h_2}}} \to 1$ and

$$x^{\frac{1}{h_1}} \left( \frac{a_k}{N^{\frac{h_1}{h_2}}} + \frac{N - n_k}{N^{\frac{h_1}{h_2}}} \right)^{\frac{h_2}{h_1}} - \frac{a_{k-1}}{N} \to x^{\frac{1}{h_1}}(1 + u)^{\frac{h_2}{h_1}}$$

assuming the limit $\frac{N-n_k}{N^{\frac{h_1}{h_2}}} \to u$, where $u \in [0, \infty)$ can be arbitrary. Put $v = (1+u)^{\frac{h_2}{h_1}}$. Thus for $t = 1$ and corresponding $v \in [1, \infty)$ we have

$$\frac{A}{N} \to \min(1, x^{\frac{1}{h_1}} v) \text{ for odd } k \to \infty. \tag{352}$$

If $\frac{N-n_k}{N^{\frac{h_1}{h_2}}} \to \infty$, then

$$\frac{A}{N} \to 1 \text{ for odd } k \to \infty. \tag{353}$$

b) Now, again $0 < t \leq 1$. For even $k$ in (348) we have

$$x^{\frac{1}{h_2}} \left( \frac{a_k}{N^{\frac{h_2}{h_1}}} + \frac{N}{N^{\frac{h_2}{h_1}}} \left(1 - \frac{n_k}{N}\right) \right)^{\frac{h_1}{h_2}} - \frac{a_{k-1}}{N} \to x^{\frac{1}{h_2}}.t$$

since by (344)

$$\frac{a_k}{N^{\frac{h_2}{h_1}}} = \frac{a_k}{n_k^{\frac{h_2}{h_1}}} \left(\frac{n_k}{N}\right)^{\frac{h_2}{h_1}} \to t^{\frac{h_2}{h_1}}.$$

Thus

$$\frac{A}{N} \to x^{\frac{1}{h_2}}.t \text{ for even } k \to \infty. \tag{354}$$

c) For the limit $\frac{B}{N}$ as odd $k \to \infty$ we compute (349) by using $\frac{N-n_k}{N} \to 1-t$ and

$$x^{\frac{1}{h_2}} \left(\frac{a_k}{N} + 1 - \frac{n_k}{N}\right) - \frac{a_k}{N} \to x^{\frac{1}{h_2}}(1-t)$$

since by (345) we have $\frac{a_k}{N} = \frac{a_k}{n_k}\frac{n_k}{N} \to 0$. Thus

$$\frac{B}{N} \to x^{\frac{1}{h_2}}(1-t) \text{ for odd } k \to \infty. \tag{355}$$

d) Again by (345), for even $k$ we have $\frac{a_k}{N} = \frac{a_k}{n_k}\frac{n_k}{N} \to \infty$, then (assuming $x < 1$)

$$x^{\frac{1}{h_1}} \left(\frac{a_k}{N} + 1 - \frac{n_k}{N}\right) - \frac{a_k}{N} \to -\infty.$$

Thus

$$\frac{B}{N} \to 0 \text{ for even } k \to \infty. \tag{356}$$

e) Summing up (351), (354), (355) and (356) we find, for every $x \in (0,1)$,

$$F(X_N, x) \to \begin{cases} x^{\frac{1}{h_2}}(1-t)+t, & \text{for odd } k \to \infty, \\ x^{\frac{1}{h_2}}.t, & \text{for even } k \to \infty \end{cases} \tag{357}$$

for $\frac{n_k}{N} \to t$, $0 < t < 1$. For $\frac{n_k}{N} \to t = 1$, $\frac{N-n_k}{N^{\frac{h_1}{h_2}}} \to u$ and $v = (1+u)^{\frac{h_2}{h_1}}$ we have applying (352)

$$F(X_N, x) \to \min(1, x^{\frac{1}{h_1}}.v) \quad \text{for odd } k \to \infty, \tag{358}$$

and for $\frac{N-n_k}{N^{\frac{h_1}{h_2}}} \to \infty$ we have

$$F(X_N, x) \to c_0(x) \quad \text{for odd } k \to \infty, \tag{359}$$

where $c_0(x) = 1$ for $x \in (0,1)$.

**6.** In the case $\frac{n_k}{N} \to 0$ and $\frac{N}{n_{k+1}} \to 0$ we have $\frac{A}{N} = o(1)$ and then it suffices to compute the limit $\frac{B}{N}$ by (349) or (350).

a) Assume that odd $k \to \infty$. Since $\frac{N-n_k}{N} \to 1$ and by (345) we have $\frac{a_k}{N} = \frac{a_k}{n_k}\frac{n_k}{N} \to 0$ and thus

$$x^{\frac{1}{h_2}}\left(\frac{a_k}{N} + 1 - \frac{n_k}{N}\right) - \frac{a_k}{N} \to x^{\frac{1}{h_2}}. \tag{360}$$

b) Assume that even $k \to \infty$. In this case (by (344) and (ii)) we have

$$\frac{a_k}{N} = \frac{a_k}{n_k^{\frac{h_2}{h_1}}}\frac{n_k^{\frac{h_2}{h_1}}}{N}, \quad \frac{a_k}{n_k^{\frac{h_2}{h_1}}} \to 1, \quad \frac{a_k}{n_{k+1}} \to 0, \quad \text{then } \frac{n_k^{\frac{h_2}{h_1}}}{n_{k+1}} \to 0.$$

Thus, for any $u \in [0,\infty)$ we can find a subsequence of $N$ such that

$$\frac{n_k^{\frac{h_2}{h_1}}}{N} \to u. \tag{361}$$

Then

$$x^{\frac{1}{h_1}}\left(\frac{a_k}{N} + 1 - \frac{n_k}{N}\right) - \frac{a_k}{N} \to x^{\frac{1}{h_1}} - (1 - x^{\frac{1}{h_1}})u. \tag{362}$$

c) Summing up (748) and (362) we find for every $x \in (0,1)$

$$F(X_N, x) \to \begin{cases} x^{\frac{1}{h_2}} & \text{for odd } k \to \infty, \\ \max(0, x^{\frac{1}{h_1}} - (1 - x^{\frac{1}{h_1}})u) & \text{for even } k \to \infty \end{cases} \tag{363}$$

for $\frac{n_k}{N} \to 0$, $\frac{N}{n_{k+1}} \to 0$ and for $u \in (0,\infty)$ satisfying (361) if $k$ is even. If $\frac{n_k^{\frac{h_2}{h_1}}}{N} \to \infty$ then

$$F(X_N, x) \to c_1(x) \quad \text{for even } k \to \infty, \tag{364}$$

where $c_1(x) = 0$ for $x \in (0,1)$.

**7.** Finally, let $\frac{N}{n_{k+1}} \to t > 0$. Then $\frac{a_k}{N} \to 0$, because (ii) $\frac{a_k}{n_{k+1}} \to 0$. Computing the limit $\frac{B}{N}$ by (349) or (350) we find

$$F(N_N, x) \to \begin{cases} x^{\frac{1}{h_2}}, & \text{for odd } k \to \infty, \\ x^{\frac{1}{h_1}}, & \text{for even } k \to \infty. \end{cases} \tag{365}$$

**8.** Now, assume that $F(X_N, x) \to g(x)$ for some sequence of $N \in [n_k, n_{k+1}]$, i.e. $g(x) \in G(X_n)$. Then we can find subsequence of $N$ (denoting again as $N$) such that $\frac{n_k}{N}$, $\frac{N-n_k}{N^{\frac{h_1}{h_2}}}$, $\frac{N}{n_{k+1}}$, and $\frac{n_k^{\frac{h_2}{h_1}}}{N}$ converge. Consequently $g(x)$ is contained in the collection of (357), (358), (359), (360), (363), (364) and (365).

Thus the proof is finished. $\square$

### 6.2.19 Block sequence $A_n = (1/q_n, a_2/q_n, \dots, a_{\varphi(q_n)}/q_n)$ of reduced rational numbers

Let $q_n$ be a an infinite sequence of positive integer numbers. Denote

$$A_n = \left(\frac{1}{q_n}, \frac{a_2}{q_n}, \dots, \frac{a_{\varphi(q_n)}}{q_n}\right),$$

where $\mathrm{g.c.d(a_i, q_n)} = 1$. In [154] we have proved: Denote
$d_{ij} = (q_i, q_j)$ is a common greatest divisor $q_i$ and $q_j$,
$q_{ij} = \frac{q_i q_j}{d_{ij}^2}$,
$p$ is a prime,
$v(n) = \#\{p; p|n\}$,
$\mu$ is the Möbius funkction,
$\varphi$ is the Euler's funkction.
For every $N = \sum_{i=1}^{n} \varphi(q_i)$, $n = 1, 2, \dots$, we have $L^2$ discrepancy

$$N^2 \int_0^1 \left(\frac{A([0,x); N; A_n)}{N} - x\right)^2 \mathrm{d}x = N^2 D_N^{(2)}$$

$$= \frac{1}{12} \sum_{i,j=1}^{n} \frac{2^{v(d_{ij})}}{q_{ij}} \prod_{\substack{p|q_i q_j \\ p\nmid d_{ij}}} (1-p) \prod_{\substack{p|d_{ij} \\ p\nmid q_{ij}}} \left(1 - \frac{1}{p}\right) \prod_{\substack{p|d_{ij} \\ p|q_{ij}}} \left(1 - \frac{p}{2}\left(1 + \frac{1}{p^2}\right)\right)$$

$$= \frac{1}{2\pi^2} \sum_{h=1}^{\infty} \frac{1}{h^2} \left| \frac{\phi(q_i)}{\phi\left(\frac{q_i}{(h,q_i)}\right)} \mu\left(\frac{q_i}{(h,q_i)}\right) \right|^2 .$$

For $A_1$ this formula is implicitly in E. Space [150] and explicitly in H. Delange [36].

### 6.2.20 Block sequence $A_n = (1/q_n, 2/q_n, \dots, q_n/q_n)$ of non-reduced rational numbers

Let $q_n$ be an infinite sequence of positive integer numbers. Denote

$$A_n = \left(\frac{1}{q_n}, \frac{2}{q_n}, \dots, \frac{q_n}{q_n}\right).$$

For $N$ of the form $N = N_n + k$ where $N_n = \sum_{i=1}^n q_i$ and $0 \le k < q_{n+1}$ in [130] for $L^2$ discrepancy we have proved

$$\begin{aligned}
N^2 D_N^{(2)} =& \frac{1}{4}n^2 + \frac{1}{12}\sum_{i,j=1}^n \frac{(q_i,q_j)^2}{q_iq_j} + \left(\frac{k}{q_{n+1}}\right)^2\left(\frac{1}{3}k^2+\frac{1}{2}k+\frac{1}{6}\right)\\
&+\left(\frac{k}{q_{n+1}}\right)\left(-\frac{2}{3}k^2-\frac{1}{2}k+\frac{1}{6}\right)+\frac{1}{3}k^2+\frac{1}{2}kn+\frac{1}{6}k\sum_{i=1}^n\frac{1}{q_i}\\
&+2\int_0^{k/q_{n+1}}\{xq_{n+1}\}\left(\sum_{i=1}^n\{xq_i\}\right)\mathrm{d}x - 2q_{n+1}\int_0^{k/q_{n+1}} x\left(\sum_{i=1}^n\{xq_i\}\right)\mathrm{d}x\\
&-2k\int_{k/q_{n+1}}^1\left(\sum_{i=1}^n\{xq_i\}\right)\mathrm{d}x.
\end{aligned}$$

In proof we have used the following formulas

$$\begin{aligned}
\int_0^{k/b}\{xb\}\{xa\}\mathrm{d}x =& \frac{1}{b}\left(\frac{ak}{3b}-\frac{k}{2ba}(a-1)\left(\frac{2}{3}a+\frac{1}{6}\right)+\sum_{s=0}^{a-1}\sum_{i=0}^{k-1}\frac{2s+1}{2a^2}\left\{\frac{s+ia}{b}\right\}\right),\\
\int_0^t x\{xa\}\mathrm{d}x =& \frac{t^2}{4}+\frac{t}{12a}-\frac{\{ta\}^3}{2a^2}+\frac{t\{ta\}^2}{2a}-\frac{t\{ta\}}{2a}+\frac{\{ta\}^2}{4a^2}-\frac{\{ta\}}{12a^2},\\
\int_0^t\{xa\}\mathrm{d}x =& \frac{t}{2}+\frac{\{ta\}^2}{2a}+\frac{\{ta\}}{2a}.
\end{aligned}$$

The block sequence $A_n$ was investigated S. Knapowski [86] who proved the sufficiency of

$$\lim_{n\to\infty}\frac{q_{n+1}}{q_1+q_2+\cdots+q_n}=0. \tag{366}$$

for the u.d. of $A_n$, $n = 1, 2, \dots$. In [130] is proved that (366) is also satisfactory and we have the following conditions:

(i) If increasing $q_n$ satisfies $\lim_{n\to\infty} q_{n-1}/q_n = 1$, then $q_n$ satisfies (366); contrary if increasing $q_n$ satisfies (366), then $\limsup_{n\to\infty} q_{n-1}/q_n = 1$.

(ii) Increasing sequence $q_n$ with $q_n = o(n^2)$ satisfies (366).

(iii) Increasing $q_n$ with positive upper asymptotic density implies (366).

(iv) If $q_n$ satisfies (366), then $\lim_{n\to\infty} n^{-1}\log(q_1+q_2+\cdots+q_n) = 0$.

(v) If the sequence $q_n$ increase and its subsequence $q_{n_k}$ satisfies (366), then $q_n$ also satisfies (366).

(vi) If two increasing sequences $p_n$ and $q_n$ satisfies (366), then also $p_n + q_n$ and the convolution $p_1 d_n + p_2 q_{n-1} + \cdots + p_n q_1$ satisfy (366).

(vii) If two increasing sequences $p_n$ a $q_n$ satisfy $p_n = O(n^{3/2})$ and $q_n = o(n^{3/2})$, then $p_n q_n$ satisfies (366).

(viii) Let $p(x)$ be a polynomial with integers coefficients, with a positive leading coefficient. Then increasing $q_n$ and $p(n)q_n$ satisfy (366) simultaneously.

(ix) Let $q_n$ be increasing linear recurrence with the characteristic polynomial $Q(x)$. Then $q_n$ satisfies (366) if and only if (a) every root of $Q(x)$ is a root of 1; (b) $Q(1) = 0$ and the multiplicity of 1 is at least 2 and it is strongly greater that a multiplicity of any other root of $Q(x)$.

(x) Consequence. Increasing linear recurring sequence $q_n$ satisfies (366) if and only if $\lim_{n\to\infty} q_{n-1}/q_n = 1$.

## 6.3 The sequence $\varphi(n)/n$, $n \in (k, k+N]$

In this section we describe result in [10].
Many papers have been devoted to the study of the distribution of the sequence

$$\frac{\varphi(n)}{n}, \quad n = 1, 2, 3, \dots,$$

where $\varphi$ denotes the classical Euler totient function. I. J. Schoenberg [145], [146] established, among other results, that this sequence has a continuous and strictly increasing a.d.f. and P. Erdős [45] showed that this function is singular (*i.e.*, the derivative exists almost everywhere on $[0, 1]$ and is zero, see [171, p. 2–191]). Recall that the a.d.f. $g_0(x)$ of $\varphi(n)/n$, $n = 1, 2, 3 \dots$, is defined as

$$g_0(x) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} c_{[0,x)}\left(\frac{\varphi(n)}{n}\right), \qquad \text{for any } x \in [0, 1],$$

where $c_{[0,x)}(t)$ denotes the characteristic function of the subinterval $[0, x)$ of $[0, 1]$. An explicit construction of $g_0(x)$ can be found in B. A. Venkov [186].

For any interval $(k, k+N]$ define the step distribution function

$$F_{(k,k+N]}(x) = \frac{1}{N} \sum_{k<n\le k+N} c_{[0,x)}\left(\frac{\varphi(n)}{n}\right) \ (x \in [0,1)) \text{ and } F_{(k,k+N]}(1) = 1.$$

In this part, convergence properties of $F_{(k_n,k_n+N_n]}$ are investigated for sequences of intervals $(k_m, k_m + N_m]$, $m = 1, 2, 3, \dots$ using and mixing mainly two methods. The first one designed as the P. Erdős's approach introduces a parameter $t$ to separate the prime divisors of integers into those greater that $t$ and the others. The second one associated to the name of H. Davenport, takes also his foundation from the works of S. Ramanujan [133], P. Erdős [44, 47], B.A. Venkov [186], and many other people, is related to the notion of primitive $x$-abundant number introduced about the divisor function.

The initial source of this section is the following result of P. Erdős [46]: if

$$\lim_{m\to\infty} \frac{\log\log\log k_m}{N_m} = 0$$

(for given increasing subsequences $k_m$ and $N_m$ of integers) then

$$\lim_{m\to\infty} F_{(k_m,k_m+N_m]}(x) = g_0(x), \quad \text{for every } x \in [0,1]. \tag{367}$$

In the opposite direction, P. Erdős complete his theorem by constructing sequences $k_m$ and $N_m$ such that $\lim_m \frac{\log\log\log k_m}{N_m} = \frac{1}{2}$ and the sequence of distribution functions $F_{(k_m,k_m+N_m]}$ do not converges in distribution to $g_0$.

In the sequel, for short, the indexation by $m$ should be omitted but having in mind that $N_m$ and $k_m$, both go to infinity. In that case we write simply $k, N \to \infty$ if the constraints on these sequences are unambiguous.

In Section 6.3.1, a necessary and sufficient condition to have (367) is given, that depends on divisors $d$ of $n$, $d > N$, with $n \in (k, k+N]$. In Section 6.3.2, we analyze the Erdős approach and improve his result by exhibiting some error terms. In Section 6.3.3, examples of sequences of intervals $(k, k+N]$ $(k, N \to \infty)$ are given such that $\lim_{N\to\infty} \frac{\log\log\log k}{N} = \infty$ but (367) still holds. Next, in Section 6.3.4, we analyze the H. Davenport's method and find a necessary and sufficient condition such that $F_{(k,k+N]}(x)$ converges to a given distribution function $g(x)$ (as $N \to \infty$). Finally, applying Schinzel–Wang's Theorem [142] in Section 6.3.5, we show that asymptotic distribution $F_{(k,k+N]}(x) \to g(x)$ $(k, N \to \infty)$, can have the form $g(x) = \tilde{g}\left(\frac{x}{\alpha}\right)$ $(x \in [0,1])$, where $\tilde{g}(x)$ is an arbitrary given distribution function and $\alpha$ is a related constant depending on $\tilde{g}(x)$.

### 6.3.1 A necessary and sufficient condition

**Theorem 152.** *For any two increasing sequences of natural numbers $N_m$ and $k_m$, the limit* (367) *holds if and only if for every positive integer $s$,*

$$\lim_{m\to\infty} \frac{1}{N_m} \sum_{k_m<n\le k_m+N_m} \sum_{\substack{d>N_m \\ d\,|\,n}} \Phi_s(d) = 0, \tag{368}$$

*where*

$$\Phi_s(d) := \prod_{\substack{p\,|\,d \\ (p \text{ prime})}} \left( \left(1-\frac{1}{p}\right)^s - 1 \right) \tag{369}$$

*for any square-free integer $d$, $\Phi_s(1) := 1$ and $\Phi_s(d) := 0$ otherwise.*

*Proof.* By applying Weyl's limit relation (see [171, p. 1–12, Th. 1.8.1.1]) we get (367) if and only if, for all positive integers $s$,

$$\lim_{m\to\infty} \frac{1}{N_m} \sum_{k_m<n\le k_m+N_m} \left(\frac{\varphi(n)}{n}\right)^s = \int_0^1 x^s \mathrm{d}g_0(x). \tag{370}$$

By a general theorem of H. Delange (see [35, Théorème 2]) we derive that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left(\frac{\varphi(n)}{n}\right)^s = \prod_p \left(1-\frac{1}{p}+\frac{1}{p}\left(1-\frac{1}{p}\right)^s\right). \tag{371}$$

Now we use the equality $\sum_{d\,|\,n} \Phi_s(d) = \left(\frac{\varphi(n)}{n}\right)^s$ to expand

$$\frac{1}{N} \sum_{k<n\le k+N} \left(\frac{\varphi(n)}{n}\right)^s.$$

To this aim, we write

$$\begin{aligned}
\sum_{k<n\le k+N} \sum_{d|n} \Phi_s(d) &= \sum_{d=1}^{k+N} \Phi_s(d) \left( \left[\frac{k+N}{d}\right] - \left[\frac{k}{d}\right] \right) \\
&= \sum_{d=1}^{k+N} N\frac{\Phi_s(d)}{d} + \sum_{d=1}^{k+N} \Phi_s(d) \left( \left\{\frac{k}{d}\right\} - \left\{\frac{k+N}{d}\right\} \right),
\end{aligned}$$

where $[x]$ denote the integer part of $x$ and $\{x\}$ the fractional part of $x$. Since

$$\left\{\frac{k}{d}\right\} - \left\{\frac{k+N}{d}\right\} = \begin{cases} -\left\{\frac{N}{d}\right\} & \text{if } \left\{\frac{k}{d}\right\} + \left\{\frac{N}{d}\right\} < 1, \\ 1 - \left\{\frac{N}{d}\right\} & \text{otherwise,} \end{cases} \tag{372}$$

the summation up to $k+N$ can be reduced to $N$ to get

$$\begin{aligned} \frac{1}{N}\sum_{k<n\leq k+N}\left(\frac{\varphi(n)}{n}\right)^s = \sum_{d=1}^{N}\frac{\Phi_s(d)}{d} + \frac{1}{N}\sum_{d=1}^{N}\Phi_s(d)\left(\left\{\frac{k}{d}\right\} - \left\{\frac{k+N}{d}\right\}\right) \\ + \frac{1}{N}\sum_{\substack{N<d\leq k+N \\ \left\{\frac{k}{d}\right\}+\frac{N}{d}\geq 1}}\Phi_s(d). \end{aligned} \tag{373}$$

Let us prove that

$$\sum_{\substack{N<d\leq k+N \\ \left\{\frac{k}{d}\right\}+\frac{N}{d}\geq 1}}\Phi_s(d) = \sum_{j=1}^{N}\sum_{\substack{d\mid k+j \\ d>N}}\Phi_s(d) \tag{374}$$

for any positive integers $s$, $k$ and $N$ by using the following theorem:

**Theorem 153.** *Let $d > N$, then $\left\{\frac{k}{d}\right\} + \frac{N}{d} \geq 1$ if and only if there exists $1 \leq j \leq N$ such that*

$$d \mid k+j,$$

*and in that case, $j$ is unique.*

*Proof.* Let us start with any integer $d$, that is to say without assuming $d > N$. Let $k = m_d d + k_d$ and $N = n_d d + N_d$ be Euclidean divisions, so that $m_d$, $k_d$, $n_d$ and $N_d$ are integers satisfying $0 \leq k_d < d$, $m_d \geq 0$, $0 \leq N_d < d$ and $n_d \geq 0$. Readily

$$\left\{\frac{k}{d}\right\} + \left\{\frac{N}{d}\right\} \geq 1 \Longleftrightarrow k_d + N_d \geq d.$$

Now we prove the lemma in four steps:

1. If $k_d + N_d \geq d$, then $k_d > 0$, $N_d > 0$ and there exists $0 \leq i_d < N_d$ such that $k_d = d - N_d + i_d$. In particular $k = (m_d+1)d - (N_d - i_d)$ with $0 < N_d - i_d \leq N_d$. Hence $d \mid k+j$ with $j = N_d - i_d$.

2. Reciprocally, assume that $d \mid k+j$ for an integer $j$ with $1 \leq j \leq N_d$. Write $j$ as $j = N_d - i_d$ with integers $0 \leq i_d < N_d$. From the above notations,

$k + j = m_d d + k_d + N_d - i_d$, hence $d|k_d + N_d - i_d$, implying $k_d + N_d \geq d$ as expected.

3. If there exist $1 \leq j_1 \leq N_d$ and $1 \leq j_2 \leq N_d$, such that $d \,|\, k + j_1$ and $d \,|\, k + j_2$, then $d \,|\, j_1 - j_2$, but $|j_1 - j_2| \leq N_d < d$, hence $j_1 = j_2$.

4. Now we assume $d > N$ so that $N_d = N$; that ends the proof. $\square$

Applying Theorem 153 in (373) we obtain the following basic equality

$$\frac{1}{N} \sum_{k<n\leq k+N} \left(\frac{\varphi(n)}{n}\right)^s = \sum_{d=1}^{N} \frac{\Phi_s(d)}{d} + \frac{1}{N} \sum_{d=1}^{N} \Phi_s(d) \left(\left\{\frac{k}{d}\right\} - \left\{\frac{k+N}{d}\right\}\right) + \frac{1}{N} \sum_{k<n\leq k+N} \sum_{d>N,\, d\,|\,n} \Phi_s(d)\,. \tag{375}$$

Clearly, $|\Phi_s(d)| \leq \frac{s^{\omega(d)}}{d}$, if $d$ is square free, where $\omega(d)$ denote the number of different primes which divide $d$ and successively, from A. G. Postnikov [131, p. 361–363 or English trans. p. 264–266],

$$\sum_{d=1}^{N} |\Phi_s(d)| \leq (1 + \log N)^s, \tag{376}$$

$$\sum_{d=N+1}^{\infty} \frac{|\Phi_s(d)|}{d} \leq \frac{3^s (1 + \log N)^s}{N}, \tag{377}$$

$$\sum_{d=1}^{\infty} \frac{\Phi_s(d)}{d} = \prod_p \left(1 - \frac{1}{p} + \frac{1}{p}\left(1 - \frac{1}{p}\right)^s\right). \tag{378}$$

Consequently, Theorem 152 follows from (371), (375) and the above relations. $\square$

**Notes 28.** Using (375) and

$$\frac{1}{N} \sum_{k<n\leq k+N} \sum_{\substack{d>N \\ d\,|\,n}} \Phi_s(d) + \frac{1}{N} \sum_{k<n\leq k+N} \sum_{\substack{d\leq N \\ d\,|\,n}} \Phi_s(d) = \frac{1}{N} \sum_{k<n\leq k+N} \left(\frac{\varphi(n)}{n}\right)^s$$

we obtain

$$\frac{1}{N} \sum_{k<n\leq k+N} \sum_{\substack{d\leq N \\ d\,|\,n}} \Phi_s(d) = \sum_{d=1}^{N} \frac{\Phi_s(d)}{d} + \mathcal{O}\left(\frac{(1 + \log N)^s}{N}\right),$$

the error term being independent of $k$ and thus, when the integer $N$ goes to infinity, the left hand side of this equality converges to $\prod_p \left(1 - \frac{1}{p} + \frac{1}{p}\left(1 - \frac{1}{p}\right)^s\right)$ uniformly with respect to $k$.

### 6.3.2 The Erdős' approach

For any positive integer $n$ and real number $t \geq 2$, set

$$n(t) := \prod_{\substack{p|n \\ p\leq t}} p, \quad n'(t) := \prod_{\substack{p|n \\ p>t}} p, \text{ and } P(t) := \prod_{p\leq t} p, \tag{379}$$

where $p$ are primes and the empty product is 1. P. Erdős in [46] proved (without any explicit error term and for $s = 1$) the following theorem:

**Theorem 154.** *For every positive integers $k$, $N$ and for $t = N$, the equality*

$$\frac{1}{N} \sum_{k<n\leq k+N} \left(\frac{\varphi(n(t))}{n(t)}\right)^s = \frac{1}{N} \sum_{n=1}^{N} \left(\frac{\varphi(n)}{n}\right)^s + \mathcal{O}\left(\frac{3^s(1+\log N)^s}{N}\right) \tag{380}$$

*holds for all integers $s \geq 1$*

*Proof.* As above, from the definition of $\Phi_s$, we have for any $t \geq 2$

$$\begin{aligned}
\sum_{k<n\leq k+N} \left(\frac{\varphi(n(t))}{n(t)}\right)^s &= \sum_{k<n\leq k+N} \sum_{d\,|\,n(t)} \Phi_s(d) \\
&= \sum_{d\,|\,P(t)} \Phi_s(d) \left(\left\lfloor \frac{k+N}{d} \right\rfloor - \left\lfloor \frac{k}{d} \right\rfloor\right) \\
&= N \sum_{d\,|\,P(t)} \frac{\Phi_s(d)}{d} + \sum_{d\,|\,P(t)} \Phi_s(d) \left(\left\{\frac{k}{d}\right\} - \left\{\frac{k+N}{d}\right\}\right).
\end{aligned}$$

Observe that

$$\sum_{d|P(t)} |\Phi_s(d)| \leq \sum_{d|P(t)} \frac{s^{\omega(d)}}{d} = \prod_{p\leq t}\left(1+\frac{s}{p}\right)$$

and using the classical estimate

$$\left(\prod_{p\leq t}\left(1-\frac{1}{p}\right)\right)^{-1} \leq (e^{\gamma}\log t)\big(1+c(\log t)^{-2}\big)\big)$$

with an absolute constant $c > 0$ (see J.B. Rosser and L. Schoenfeld L. [137] for explicit value of $c$) we get

$$\prod_{p\leq t}\left(1+\frac{s}{p}\right) \;\leq\; \prod_{p\leq t}\left(1-\frac{1}{p^2}\right)^s \prod_{p\leq t}\left(1-\frac{1}{p}\right)^{-s}$$

$$\leq \quad (3/4)^s e^{s(\gamma + c(\log t)^{-2})} (\log t)^s \,.$$

In particilar, there exists an integer $t_0 \geq 2$ (which is explicit) such that

$$\sum_{d|P(t)} |\Phi_s(d)| \leq 3^s (\log t)^s \,.$$

for any $t \geq t_0$ and $s \geq 1$.

Now, due to the multiplicativity of $n \mapsto \Phi_s(n)/n$,

$$\sum_{d|P(t)} \frac{\Phi_s(d)}{d} = \prod_{p \leq t} \left( 1 + \frac{\left(1 - \frac{1}{p}\right)^s - 1}{p} \right)$$

and from [131, p. 363, or english trans. p. 264 and p. 265] one has the quantitative form of the above result of Schur

$$\frac{1}{N} \sum_{n=1}^{N} \left( \frac{\varphi(n)}{n} \right)^s = \prod_p \left( 1 - \frac{1}{p} + \frac{1}{p} \left( 1 - \frac{1}{p} \right)^s \right) + \mathcal{O} \left( \frac{3^s (1 + \log N)^s}{N} \right)$$

where the constant involved by the big $\mathcal{O}$ is absolute and also (see (377)),

$$\left| 1 - \prod_{p > N} \left( 1 - \frac{1}{p} + \frac{1}{p} \left( 1 - \frac{1}{p} \right)^s \right) \right| \leq \sum_{n > N} \frac{|\Phi_s(n)|}{n} \leq \frac{3^s (1 + \log N)^{s-1}}{N} \,.$$

Consequently, for all integers $s \geq 1$ and $N \geq 2$,

$$\left| \prod_{p \leq N} \left( 1 - \frac{1}{p} + \frac{1}{p} \left( 1 - \frac{1}{p} \right)^s \right) - \prod_p \left( 1 - \frac{1}{p} + \frac{1}{p} \left( 1 - \frac{1}{p} \right)^s \right) \right|$$
$$\leq (3/4) \frac{3^s (1 + \log N)^{s-1}}{N} .$$

Taking into account all these bounds leads to (380)). $\square$

In his work, Erdős used implicitly the following theorem:

**Theorem 155.** *For every two increasing sequences of integers $k_m$ and $N_m$ and for $t = N_m$ if*

$$\lim_{m \to \infty} \Bigg( \prod_{k_m < n \leq k_m + N_m} \frac{\varphi(n'(t))}{n'(t)} \Bigg)^{\frac{1}{N_m}} = 1 \Longrightarrow \lim_{m \to \infty} F_{(k_m, k_m + N_m]}(x) = g_0(x) \tag{381}$$

*holds for all $x \in [0, 1]$.*

*Proof.* We used the following theorem:

**Theorem 156.** *Let $x_n$, $n = 1, 2, \dots$, be a sequence in the interval $(0, 1)$. All the following limits are equivalent:*

(i) $\frac{1}{N}\sum_{n=1}^{N} x_n \to 1$,

(ii) $\frac{1}{N}\sum_{n=1}^{N} x_n^s \to 1$,

(iii) $\frac{1}{N}\sum_{n=1}^{N} \log x_n \to 0$,

(iv) $\left(\prod_{n=1}^{N} x_n\right)^{1/N} \to 1$,

*were $N \to \infty$. Every of the limits* (i)-(iv) *implies that $x_n \to 1$ converges statistically, i.e. for every $\varepsilon > 0$ and $A_\varepsilon = \{n \le N; x_n \le 1 - \varepsilon\}$ we have*

$$\frac{\#A_\varepsilon}{N} \to 0.$$

*Proof.* The equivalence (iii) and (iv) is clear, and for (i)-(iii) we follow [171]. Define the step distribution function $F_N(x)$ as

$$F_N(x) = \frac{\#\{n \le N; x_n \in [0, x)\}}{N}.$$

By Riemann-Stiltjes integration for every continuous function $f(x)$ on $[0, 1]$ we have $\frac{1}{N}\sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)\mathrm{d}F_N(x)$. By Helly theorem, if $F_N(x) \to g(x)$, then, as $N \to \infty$,

$$\int_0^1 f(x)\mathrm{d}F_N(x) \to \int_0^1 f(x)\mathrm{d}g(x).$$

Thus the limits (i), (ii) and (iii) separately imply

(i) $\int_0^1 x\mathrm{d}g(x) = 1$,

(ii) $\int_0^1 x^s\mathrm{d}g(x) = 1$,

(iii) $\int_0^1 \log x\mathrm{d}g(x) = 0$ (precise $\int_0^1 \log x\mathrm{d}g(x) = \lim_{\varepsilon\to 0}\int_\varepsilon^1 \log x\mathrm{d}g(x)$).

Any of (i)-(iii) implies that $g(x) = c_1(x)$-the d.f. has step 1 at $x = 1$. This is equivalent that the sequence $x_n$, $n = 1, 2, \dots$, statistically converge to 1. Note that for every continuous function $f(x)$ we have the ordinary limit

$$\frac{1}{N}\sum_{n=1}^{N} f(x_n) = \int_0^1 F(x)\mathrm{d}F_N(x) \to \int_0^1 f(x)\mathrm{d}c_1(x) = f(1)$$

and the statistical limit $f(x_n) \to f(1)$. □

Now, we shall return to proof of Theorem 156. The equivalence of limits (i)–(iv) we apply to the blocks sequence

$$x_n = \frac{\varphi(n'(t))}{n'(t)}, \quad n \in (k, k+N], \quad t = N.$$

Then by (i)$\Longleftrightarrow$(ii) we have

$$\frac{1}{N}\sum_{k<n\le k+N}\left(\frac{\varphi(n'(t))}{n'(t)}\right) \to 1 \Longleftrightarrow \frac{1}{N}\sum_{k<n\le k+N}\left(\frac{\varphi(n'(t))}{n'(t)}\right)^s \to 1$$

and assuming (ii) then for

$$A_\varepsilon = \left\{n \in (k, k+N]; \left(\frac{\varphi(n'(t))}{n'(t)}\right)^s < 1-\varepsilon\right\}$$

we have $\frac{\#A_\varepsilon}{N} \to 0$ for every $\varepsilon > 0$. Now, in

$$\frac{\varphi(n)}{n} = \frac{\varphi(n(t))}{n(t)}\frac{\varphi(n'(t))}{n'(t)}$$

we replace $\frac{\varphi(n'(t))}{n'(t)}$ by $1-\varepsilon$, then we see that

$$\frac{1}{N}\sum_{k<n\le k+N}\left(\frac{\varphi(n)}{n}\right)^s \ge \left(\frac{1}{N}\sum_{k<n\le k+N}\left(\frac{\varphi(n(t))}{n(t)}\right)^s\right)(1-\varepsilon) - (1-\varepsilon)\frac{\#A_\varepsilon}{N}. \tag{382}$$

and the other hand

$$\frac{1}{N}\sum_{k<n\le k+N}\left(\frac{\varphi(n)}{n}\right)^s \le \frac{1}{N}\sum_{k<n\le k+N}\left(\frac{\varphi(n(t))}{n(t)}\right)^s.$$

The Theorem 154 implies

$$(1-\varepsilon)\int_0^1 x^s \mathrm{d}g_0(x) \le \lim_{N\to\infty}\frac{1}{N}\sum_{k<n\le k+N}\left(\frac{\varphi(n)}{n}\right)^s \le \int_0^1 x^s \mathrm{d}g_0(x).$$

Thus we have

**Theorem 157.** *For every two sequences $k$ and $N$ and $t = N$ we have*

$$\frac{1}{N}\sum_{k<n\le k+N}\frac{\varphi(n'(t))}{n'(t)} \to 1 \Longrightarrow F_{(k,k+N]}(x) \to g_0(x) \tag{383}$$

*for all $x \in [0,1]$.*

Finally, Theorem 155 follows from (iv). □

**Notes 29.** In (381) we have only implication, since the right-hand side of (382) has the following precise form

$$(1-\varepsilon)\left(\frac{1}{N}\sum_{k<n\le k+N}\left(\frac{\varphi(n(t))}{n(t)}\right)^s\right) - (1-\varepsilon)\left(\frac{1}{N}\sum_{k<n\le k+N, n\in A_\varepsilon}\left(\frac{\varphi(n(t))}{n(t)}\right)^s\right)$$

and it can be $\left(\frac{1}{N}\sum_{k<n\le k+N,n\in A_\varepsilon}\left(\frac{\varphi(n(t))}{n(t)}\right)^s\right) \to 0$ and $\frac{\#A_\varepsilon}{N} \to \delta > 0$.

Finally Erdős proved the following theorem but we give here a more readable proof for the convenience of the reader.

**Theorem 158.** *For any increasing sequences of integers $k_m$ and $N_m$ such that*

$$\lim_{m\to\infty}\frac{\log\log\log k_m}{N_m} = 0 \tag{384}$$

*one has*

$$\lim_{m\to\infty} F_{(k_m,k_m+N_m]}(x) = g_0(x) \tag{385}$$

*for all $x \in [0,1]$.*

*Proof.* The basic fact is that for $t = N$ the integers $n'(t)$ such that $k < n \le k+N$ are pairwise relatively prime, because the interval $(k, k+N]$ cannot contain two different integers divisible by the same prime number $p > N$. Set

$$M'(k,N,t) := \prod_{k<n\le k+N} n'(t) \tag{386}$$

but use the notation $M'(t)$ for short and let $x(k, N)$ be defined such that the number of prime numbers $p$, $N < p \leq x$, is equal to $\omega(M'(t))$, where $t = N$. From the classical Mertens' formula

$$\prod_{p \leq y}\Big(1 - \frac{1}{p}\Big) = \frac{e^{-\gamma}}{\log y}\Big(1 + \mathcal{O}\Big(\frac{1}{\log y}\Big)\Big) \tag{387}$$

(see [106, p. 259, VII. 29] for example) we get

$$\frac{\varphi(M'(t))}{M'(t)} \geq \prod_{N<p\leq x}\Big(1 - \frac{1}{p}\Big) \geq c_1 \frac{\log N}{\log x}$$

for a constant $c_1 > 0$. Therefore, for any increasing sequences $k_m$ and $N_m$, if $\big(\frac{\log N_m}{\log x(k_m,N_m)}\big)^{1/N_m}$ converges to 1, then the corresponding sequence $\big(\frac{\varphi(M'(N_m))}{M'(N_m)}\big)^{1/N_m}$ converges also to 1. Having in mind the Landau inequalities

$$\log 2 \leq \liminf_{x\to\infty} \frac{1}{x}\sum_{p\leq x}\log p \leq \limsup_{x\to\infty} \frac{1}{x}\sum_{p\leq x}\log p \leq 2\log 2 \tag{388}$$

(see [97, p. 83]) we conclude there exist suitable absolute positive constants $c_2$, $c_3$ such that

$$e^{c_2 x(k,N) - c_3 N} \leq \prod_{N<p\leq x(k,N)} p$$

and, after considering the inequalities

$$\prod_{N<p\leq x(k,N)} p \leq (k+1)(k+2)\dots(k+N) < (k+N)^N,$$

we obtain $x(k, N) < c_4 N \log(k + N)$ with $c_4 > 0$.

Consequently, if the sequence $\big(\frac{\log N_m}{\log(N_m \log(k_m+N_m))}\big)^{1/N_m}$ converges to 1, the same is true for the sequence $\big(\frac{\log N_m}{\log x(k_m,N_m)}\big)^{1/N_m}$, hence the corresponding sequence $\big(\frac{\varphi(M'(t))}{M'(t)}\big)^{1/N_m}$ also converge to 1 and so, $F_{(k_m,k_m+N_m]}(x)$ converges to $g_0(x)$ for all $x \in [0, 1]$ by Theorem 155. The proof ends by noticing that

$$\lim_{m\to\infty} \frac{1}{N_m}\Big(\log \frac{\log N_m}{\log(N_m \log(k_m + N_m))}\Big) = 0 \Longleftrightarrow \lim_{m\to\infty} \frac{\log\log\log k_m}{N_m} = 0.$$

$\square$

**Notes 30.** Assume that $P(t)\,|\,k$, where $P(t) = \prod_{p\le t} p$ and $t = N$. As in (379), we introduce for divisors $d$ of $n$ the integers

$$d(t) = \prod_{\substack{p\,|\,d\\ p\le t}} p \text{ and } d'(t) = \prod_{\substack{p\,|\,d,\\ p>t}} p.$$

Since $d(t)\,|\,n$, $n = k + j$ with $j \le N$ and $d(t)\,|\,k$, it follows that $d(t) \le N$. Hence, if $d > N$ one has $d'(t) > 1$. Therefore

$$\begin{aligned}\sum_{\substack{d>N\\ d\,|\,n}} \Phi_s(d) &= \sum_{d\,|\,n(t)} \Phi_s(d) \sum_{\substack{d'\,|\,n'(t)\\ d'\neq 1}} \Phi_s(d')\\ &= \left(\frac{\varphi(n(t))}{n(t)}\right)^s \left(\left(\frac{\varphi(n'(t))}{n'(t)}\right)^s - 1\right)\end{aligned}$$

leading to

$$\left|\sum_{\substack{d>N\\ d\,|\,n}} \Phi_s(d)\right| \le 1 - \left(\frac{\varphi(n'(t))}{n'(t)}\right)^s.$$

If for all $s = 1, 2, 3, \ldots$ one has

$$\lim_{m\to\infty} \frac{1}{N_m} \sum_{k_m<n\le k_m+N_m} \left(\frac{\varphi(n'(t))}{n'(t)}\right)^s = 1$$

for a given subsequence of integers $k_m$ and $N_m$ with $P(N_m)\,|\,k_m$. By Theorem 152 we conclude to (367), but in fact Theorem 157 gives the same conclusion without such a constraint on $k_m$.

Notice that due to $\frac{\varphi(M'(k,N,t))}{M'(k,N,t)} \le \frac{\varphi(n'(t))}{n'(t)}$ for $k < n \le k + N$ (with $M'(k, N, t) = \prod_{k<n\le k+N} n'(t)$ as above in (386)) one obtains

**Theorem 159.** *If the sequence* $\frac{\varphi(M'(k_m,N_n,N_m))}{M'(k_m,N_n,N_m)}$ *converges to 1 for increasing sequences of integers* $k_m$ *and* $N_m$*, then the sequence of distribution functions* $F_{(k_m,k_m+N_m]}$ *converges to the d.f.* $g_0$*.*

To end this section we prove the following quantitative version of Theorem 152.

**Theorem 160.** *For any positive integers* $k$*,* $N$ *and* $s$*, we have*

$$\begin{aligned}\frac{1}{N}\sum_{k<n\le k+N}\sum_{\substack{d>N\\ d\,|\,n}} \Phi_s(d) =& \frac{1}{N}\sum_{k<n\le k+N}\left(\frac{\varphi(n)}{n}\right)^s - \frac{1}{N}\sum_{n=1}^{N}\left(\frac{\varphi(n)}{n}\right)^s\\ &+ \mathcal{O}\Big(\frac{(1+\log N)^s}{N}\Big)\end{aligned} \tag{389}$$

*and the constant in the big* $\mathcal{O}$ *can be chosen equal to 2.*

*Proof.* Let $t = N$. Notice that

$$\sum_{\substack{d>N \\ d\,|\,n}} \Phi_s(d) = \sum_{\substack{d>N \\ d\,|\,n(t)}} \Phi_s(d) + \sum_{\substack{d\,|\,n(t)n'(t) \\ d'(t)\neq 1}} \Phi_s(d)$$

and the second sum is equal to $\left(\frac{\varphi(n(t)}{n(t)}\right)^s \left(\left(\frac{\varphi(n'(t))}{n'(t)}\right)^s - 1\right)$. Summing from $k+1$ to $k+N$ gives

$$\frac{1}{N} \sum_{k<n\leq k+N} \sum_{\substack{d>N \\ d|n}} \Phi_s(d) = \frac{1}{N} \sum_{k<n\leq k+N} \sum_{\substack{d>N \\ d|n(t)}} \Phi_s(d) + \frac{1}{N} \sum_{k<n\leq k+N} \left(\frac{\varphi(n)}{n}\right)^s - \frac{1}{N} \sum_{k<n\leq k+N} \left(\frac{\varphi(n(t))}{n(t)}\right)^s. \tag{390}$$

Now, successively

$$\begin{aligned}
\frac{1}{N} \sum_{k<n\leq k+N} \sum_{d\,|\,n(t)} \Phi_s(d) &= \frac{1}{N} \sum_{k<n\leq k+N} \left(\frac{\varphi(n(t))}{n(t)}\right)^s \\
&= \frac{1}{N} \sum_{k<n\leq k+N} \sum_{\substack{d\leq N \\ d\,|\,n(t)}} \Phi_s(d) + \frac{1}{N} \sum_{k<n\leq k+N} \sum_{\substack{d>N \\ d\,|\,n(t)}} \Phi_s(d) \\
&= \sum_{d=1}^{N} \frac{\Phi_s(d)}{d} + \frac{1}{N} \sum_{d=1}^{N} \Phi_s(d) \left(\left\{\frac{k}{d}\right\} - \left\{\frac{k+N}{d}\right\}\right) \\
&\qquad + \frac{1}{N} \sum_{k<n\leq k+N} \sum_{\substack{d>N \\ d\,|\,n(t)}} \Phi_s(d) \\
&= \sum_{d=1}^{N} \frac{\Phi_s(d)}{d} + \frac{1}{N} \sum_{k<n\leq k+N} \sum_{\substack{d>N \\ d\,|\,n(t)}} \Phi_s(d) + \mathcal{O}\left(\frac{(1+\log N)^s}{N}\right)
\end{aligned}$$

and after inserting

$$\sum_{d=1}^{N} \frac{\Phi_s(d)}{d} = \sum_{n=1}^{N} \left(\frac{\varphi(n)}{n}\right)^s + \mathcal{O}\left(\frac{(1+\log)^s}{N}\right),$$

which can be obtained from (375) with $k = 0$, we get

$$\frac{1}{N}\sum_{k<n\le k+N}\sum_{\substack{d>N\\ d|n(t)}}\Phi_s(d) = \frac{1}{N}\sum_{k<n\le k+N}\Big(\frac{\varphi(n(t))}{n(t)}\Big)^s - \sum_{n=1}^{N}\Big(\frac{\varphi(n)}{n}\Big)^s$$
$$+\,\mathcal{O}\Big(\frac{(1+\log)^s}{N}\Big).$$

Inserting this equality in (390) gives (389). Finally, notice that the error term comes from the bound (376) used twice. □

### 6.3.3 Examples

To show that his assumption in Theorem 158 is optimal, Erdős gave the following example.

**Example 61.** Take $t$ large enough to write $P(t) = \prod_{p\le t} p$ as the product of $N$ numbers $A_1, A_2, \dots, A_N$ such that

(i) $A_i$, $i = 1, \dots, N$, are relatively prime,

(ii) $\frac{\varphi(A_i)}{A_i} < \frac{1}{2}$ for $i = 1, \dots, N$,

(iii) if $p$ is the maximal prime in $A_i$, then for $A_i' = A_i/p$ one has $\frac{\varphi(A_i')}{A_i'} > \frac{1}{2}$.

The part (iii) implies $\frac{\varphi(A_i)}{A_i} > \frac{1}{4}$ and thus

$$\left(\frac{1}{4}\right)^N < \prod_{p\le t}\left(1-\frac{1}{p}\right) = \frac{\varphi(A_1)}{A_1}\cdots\frac{\varphi(A_N)}{A_N} < \left(\frac{1}{2}\right)^N.$$

From it, applying (387), we find $N < c_1 \log\log t$. By Chinese theorem there exists $k_0 < A_1 \dots A_N$ such that $k_0 \equiv -i \pmod{A_i}$ for $i = 1, \dots, N$. Put $k = k_0 + A_1 \dots A_N$, then we have

$$e^{c_2 t} < P(t) = A_1 \dots A_N < k$$

which implies $t < c_3 \log k$ and $\log\log t < c_4 \log\log\log k$. Thus

$$\frac{\log\log\log k}{N} > \frac{1}{c_1 c_4}\frac{\log\log t}{\log\log t}.$$

Furthermore, for these $k$ and $N$, the sequence of d.f.s $F_{(k,k+N]}(x)$ does not converge to $g_0(x)$ due to (ii) that gives

$$\frac{1}{N}\sum_{k<n\le k+N}\frac{\varphi(n)}{n} < \frac{1}{2} < \frac{1}{N}\sum_{n=1}^{N}\frac{\varphi(n)}{n} = \frac{6}{\pi^2} + O\left(\frac{\log N}{N}\right).$$

**Example 62.** In Example 61, replace in (ii) the ratio $1/2$ by $1/N$ and use the corresponding definition of the $A_n$ as above. Then, by the Chinese theorem, for every $N$ we can find $k$ such that $A_n|k+n$, $n=1,\dots,N$, and consequently

$$\frac{1}{N}\sum_{k<n\le k+N}\left(\frac{\varphi(n)}{n}\right)\le\frac{1}{N}\sum_{n=1}^{N}\left(\frac{\varphi(A_n)}{A_n}\right)\le\frac{1}{N}.$$

Now select sequences of such integers $k$ and $N$, but with a distribution function $g(x)$ such that $\lim_{k,N\to\infty}F_{(k,k+N]}(x)=g(x)$ a.e. in $[0,1]$. By this construction, we obtain $\int_0^1 x\mathrm{d}g(x)=0$. Therefore $g(x)$ is the the Heaviside distribution function (jump 1 at $x=0$).

In the next example we construct sequences of integers $k$, $N$, for which (367) holds but $\lim_{N\to\infty}\frac{\log\log\log k}{N}=\infty$.

**Example 63.** For any integer $N\ge 1$, let $x=x(N)$ be a real number, $x>N$, that will be chosen later but very large with respect to $N$ (like $x(N)=e^{e^{e^N}}$ for example). Let $k:=\prod_{p\le x}p$, (where $p$ are primes), consider the interval $(k,N+k]$ and define $M^*:=\prod_{x<p\le x+y(x)}p$, where $y(x)$ is chosen such that $M^*$ has the same number of prime divisors than the product $M'(k,N,t)$ ($t=N$) defined in (386). Presently, if a prime number $p$ verifies $p>N$ and $p|k+j$ with $j\le N$ then $p>x$. Thus, $\frac{\varphi(M^*)}{M^*}\le\frac{\varphi(M'(t))}{M'(t)}$ and to satisfy the assumption of Theorem 159 it suffices that the ratio $\frac{\varphi(M^*)}{M^*}=\prod_{x<p\le x+y(x)}\left(1-\frac{1}{p}\right)$ converge to 1 as $x$ tends to infinity. According to the Mertens' formula it is equivalent to have

$$\lim_{x\to\infty}\frac{\log\left(1+\frac{y(x)}{x}\right)}{\log x}=0\,. \tag{391}$$

The inequalities

$$M^*\le M'=\prod_{k<n\le k+N}n'(t)\le(k+N)^N\le(2k)^N$$

lead to $\sum_{x<p\le x+y(x)}\log p\le 2N\sum_{p\le x}\log p$ and thus

$$\sum_{p\le x+y(x)}\log p\le(2N+1)\sum_{p\le x}\log p. \tag{392}$$

Using (388) in (392), we see that for any $\varepsilon > 0$, there exists $x_0$ such that $x \geq x_0(\varepsilon)$ implies

$$(\log 2 - \varepsilon)(x + y(x)) \leq (2N + 1)(2\log 2 + \varepsilon)x\,,$$

so that $\frac{y(x)}{x} \leq cN$ for a positive constant $c$. Therefore, (391) holds and consequently (367) holds also, if we chose $x = x(N) \geq e^N$. Since $k(N) = \prod_{p\leq x(N)} p \geq e^{c_1 x(N)}$, by taking $x(N) = e^{e^{e^N}}$ the limit

$$\lim_{N\to\infty} \frac{\log\log\log k}{N} = \infty$$

holds as expected.

### 6.3.4 $x$-numbers

Let $f : \mathbf{N} \to (0, 1]$ be a multiplicative function *i.e.* $f(1) = 1$ and $f(mn) = f(m)f(n)$ if $(m, n) = 1$. Assume that $0 < f(n) \leq 1$ for all $n$; it is useful to introduce for any $x \in (0, 1)$ the increasing sequence $a_k(x)$ of all integers $a$ such that $f(a) \leq x$ but $f(d) > x$ for every divisor $d$ of $a$, $d \neq a$. In the case $f(n) = n/\sigma(n)$ (where $\sigma(n)$ is the sum of divisors of $n$) such an integer $a$ is classically called primitive $x$-abundant number. In 1933, H. Davenport [34] using this notion proved that the sequence $n/\sigma(n)$ has a d.f. and found an explicit construction of it. In addition he gave sufficient conditions for $f$ to have a d.f. These conditions are easily verified for both sequences $n/\sigma(n)$ and $\varphi(n)/n$.

B.A. Venkov applied the same method in his paper [186] but for the sequence of ratios $\frac{\varphi(n)}{n}$. Following him, we introduce, for convenience, the definition of *$x$-numbers* (also called *primitive $x$-numbers* in [131]), that is to say integers $a > 0$ such that $\frac{\varphi(a)}{a} \leq x$ and for every $d\,|\,a$ but $d \neq a$ one has $\frac{\varphi(d)}{d} > x$. We denote by $A(x)$ the set of all $x$-numbers ordering in increase magnitude *i.e.*,

$$a_1(x) < a_2(x) < a_3(x) < \dots$$

From now on, the sequence $p_1, p_2, p_3, \dots$ denotes the increasing sequence of all prime numbers. From the above definitions we get the following properties.

(i) Every $x$-number is square-free.

(ii) Every square-free $a$ is an $x$-number for some $x$. Concretely, if $a = q_1 q_2 \dots q_m$ with $q_1 < q_2 < \cdots < q_m$, all prime numbers, then $a$ is $x$-number for every $x$ in the interval $\left[\prod_{i=1}^{m}\left(1-\frac{1}{q_i}\right), \prod_{i=1}^{m-1}\left(1-\frac{1}{q_i}\right)\right)$.

(iii) For every $i < j$ we have $a_i(x) \nmid a_j(x)$.

(iv) Let $p_s$ be the $s$-th prime number and choose $x \in \left[1-\frac{1}{p_s}, 1\right)$. Then $a_1(x) = p_1 = 2$, $a_2(x) = p_2 = 3$ and so on, $a_s(x) = p_s$. Furthermore, if $x < 1 - \frac{1}{p_{s+1}}$ then for every $j > s$, the integer $a_j(x)$ cannot be a prime and $p_i \nmid a_j(x)$ for $i = 1, 2, \dots, s$.

*Proof.* By (ii), prime numbers $p_1, p_2, \dots, p_s$ are $x$-numbers for $x \geq 1 - \frac{1}{p_s}$. If for some $j$ we have $p_1 \leq a_j(x) \leq p_s$ and $p \,|\, a_j(x)$, $p$ prime, then $p \leq p_s$ and $a_j(x) = p$, since $pq \,|\, a_j(x)$ with $q > 1$ contradicts (iii).

Now, $x < 1 - \frac{1}{p_{s+1}}$ implies that $p_{s+1}$ and any $p_k > p_s$ is not an $x$-number and by (iii) $p_i \nmid a_j(x)$ for $i = 1, \dots, s$. □

(v) If $x \in \left[\prod_{i=1}^{s}\left(1-\frac{1}{p_i}\right), \prod_{i=1}^{s-1}\left(1-\frac{1}{p_i}\right)\right)$, then $a_1(x) = \prod_{i=1}^{s} p_i$.

*Proof.* By contradiction. The integer $a = \prod_{i=1}^{s} p_i$ is an $x$-number, hence $a_1(x) \leq a$. Assume that $a_1(x) < a$ and let $a_1(x) = p_{i_1} p_{i_2} \dots p_{i_k}$ with $i_1 < i_2 < \cdots < i_k$, then $k < s$. By definition,

$$x \in \left[\prod_{j=1}^{k}\left(1-\frac{1}{p_{i_j}}\right), \prod_{i=1}^{k-1}\left(1-\frac{1}{p_{i_j}}\right)\right)$$

hence $\prod_{j=1}^{k}\left(1-\frac{1}{p_{i_j}}\right) < \prod_{i=1}^{s-1}\left(1-\frac{1}{p_i}\right)$ which implies $k > s-1$, a contradiction. □

(vi) For every positive integer $n$ and every $x \in (0,1)$ we have

$$\frac{\varphi(n)}{n} \leq x \Longleftrightarrow \exists\, i \in \mathbb{N}\, (a_i(x) | n).$$

(vii) Assume that $0 < x < x' < 1$. Then for every $x$-number $a_i(x)$ there exists an $x'$-number $a_j(x')$ such that $a_j(x') | a_i(x)$. This property follows from (vi) and the fact that for $n = a_i(x)$ one has $\frac{\varphi(n)}{n} < x'$.

(viii) Let $[b_1, \dots, b_j]$ denote the least common multiple of the integers $b_1, \dots, b_j$, then the asymptotic density of the set

$$\{n \in \mathbb{N}; a_m(x)|n, a_1(x) \nmid n, a_2(x) \nmid n, \dots, a_{m-1}(x) \nmid n\}$$

is given by

$$A_m(x) = \frac{1}{a_m(x)} + \sum_{u=1}^{m-1} \sum_{1 \le j_1 < j_2 < \cdots < j_u < m} \frac{(-1)^u}{[a_{j_1}(x), \dots, a_{j_u}(x), a_m(x)]} .$$

(ix) Define

$$B_n(x) = \{a \in \mathbb{N};\ a \,|\, n \text{ and } \exists\, i \in \mathbb{N}\ (a = a_i(x))\}. \tag{393}$$

Applying (vi) we see that [47]

$$F_{(k,k+N]}(x) = \frac{\#\{n \in (k, k+N]; B_n(x) \neq \emptyset\}}{N}. \tag{394}$$

(x) Directly from (vi) and (ix) we have by B.A. Venkov [186] (see also H. Davenport [34]) the theorem:

**Theorem 161.** *The a.d.f. $g_0(x)$ of the sequence $\frac{\varphi(n)}{n}$, $n = 1, 2, 3 \dots$, can be expressed by*

$$g_0(x) = \sum_{m=1}^{\infty} A_m(x) . \tag{395}$$

*Proof.* In fact, the right hand side of (395) is the asymptotic density of all integers $n$ divisible by some $x$-number. $\square$

In this part we prove that the asymptotic distribution function $g(x)$ in (367) cannot be arbitrary. The proof of the following theorem combines Erdős' Lemma 154 and (395).

**Theorem 162.** *Assume that $\lim_{m\to\infty} F_{(k_m,k_m+N_m]}(x) = g(x)$ for all $x \in [0, 1]$. Then $g_0(x) \le g(x)$ for all $x \in [0, 1]$.*

[47] In this paper we have defined $F_{(k,k+N]}(x) = \frac{1}{N}\sum_{k<n\le k+N} c_{[0,x)}\left(\frac{\varphi(n)}{n}\right)$ but in this part, due to the definition of $x$-number, we use $c_{[0,x]}$ in place of $c_{[0,x)}$.

*Proof.* Set

$$R^{(1)}_{(k,k+N]}(x) := \frac{\#\{n \in (k,k+N]; B_n(x) \neq \emptyset, \exists\, a \in B_n(x)\big(\forall\, p\,(p\,\text{prime and}\, p|a \Rightarrow p \leq N)\big)\}}{N}, \tag{396}$$

$$R^{(2)}_{(k,k+N]}(x) := \frac{\#\{n \in (k,k+N]; B_n(x) \neq \emptyset, \forall\, a \in B_n(x)\big(\exists\, p\,(p\,\text{prime}, p|a\,\text{and}\, p > N)\big)\}}{N}. \tag{397}$$

By (394),

$$F_{(k,k+N]}(x) = R^{(1)}_{(k,k+N]}(x) + R^{(2)}_{(k,k+N]}(x). \tag{398}$$

The monotonicity of $R^{(1)}_{(k,k+N]}(x)$ ($x \in [0,1]$) follows from (vii) and then for the d.f.s $F_{(k,k+N]}(x)$ and $R^{(1)}_{(k,k+N]}(x)$ we can apply Helly selection principle to exhibit a subsequence of the intervals $(k_m, k_m+N_m]$, still denoted $(k_m, k_m+N_m]$, such that for every $x \in (0,1)$ we have both $\lim_{m\to\infty} F_{(k_m,k_m+N_m]}(x) = g(x)$ and $\lim_{m\to\infty} R^{(1)}_{(k_m,k_m+N_m]}(x) = g^{(1)}(x)$ for a suitable d.f. $g^{(1)}(x)$. Therefore, we also have the limit

$$\lim_{m\to\infty} R^{(2)}_{(k_m,k_m+N_m]}(x) = g^{(2)}(x) = g(x) - g^{(1)}(x).$$

Now we prove the equality

$$g^{(1)}(x) = g_0(x) \tag{399}$$

for all $x$, that is to say

$$g(x) = g_0(x) + g^{(2)}(x). \tag{400}$$

For the sequence $\frac{\varphi(n(t))}{n(t)}$, $n \in (k,k+N]$, $n(t) = \prod_{\substack{p|n\\ p\leq t}} p$, where $t = N$, define

$$\tilde{F}_{(k,k+N]}(x) := \frac{\#\{n \in (k,k+N]; \frac{\varphi(n(t))}{n(t)} \leq x\}}{N}. \tag{401}$$

By property (vi), if $\frac{\varphi(n(t))}{n(t)} \leq x$, then there exists $x$-number $a_i(x)$ such that $a_i(x) \,|\, n(t)$. Since $n(t) \,|\, n$ it follows that $a_i(x)|n$ and furthermore for all prime numbers $p$, $p \,|\, a_i(x)$ implies $p \le t$ $(= N)$. Reciprocally, if $a_i(x)|n$ and for all prime numbers $p$, $p \,|\, a_i(x)$ implies $p \le t$, then $a_i(x) \,|\, n(t)$ and $\frac{\varphi(n(t))}{n(t)} \le x$. Thus

$$\tilde{F}_{(k,k+N]}(x) = R^{(1)}_{(k,k+N]}(x) \tag{402}$$

and consequently, $\tilde{F}_{(k,k+N]}(x) \to g^{(1)}(x)$ too. By Erdős' Theorem 154

$$\int_0^1 x^s \mathrm{d}g^{(1)}(x) = \int_0^1 x^s \mathrm{d}g_0(x)$$

for $s = 1, 2, 3 \ldots$ and thus $g^{(1)}(x) = g_0(x)$ for $x \in (0,1)$ a.e.

□

**Theorem 163.** *For every d.f. $g(x)$ such that*

$$\lim_{m\to\infty} F_{(k_m,k_m+N_m]}(x) = g(x)$$

*a.e. on $[0,1]$ (with $k_m, N_m \to \infty$), there exists a constant $c_1$ such that*

$$\int_0^1 x^s \mathrm{d}g(x) \le \int_0^1 x^s \mathrm{d}g_0(x) \le \frac{c_1}{\log(s+1)}, \tag{403}$$

*for every positive integer $s$.*

*Proof.* The first inequality in (403) follows from Lemma 154, since $\left(\frac{\varphi(n)}{n}\right)^s \le \left(\frac{\varphi(n(t))}{n(t)}\right)^s$. It also follows from Theorem 162, because $\int_0^1 x^s \mathrm{d}g(x) \le \int_0^1 x^s \mathrm{d}g_0(x)$ is equivalent to $\int_0^1 x^{s-1} g(x)\mathrm{d}x \ge \int_0^1 x^{s-1} g_0(x)\mathrm{d}x$. The second inequality in (403) was proved by B.A. Venkov [186, Theorem 3] in the form

$$\lim_{s\to\infty} \left( \int_0^1 x^s \mathrm{d}g_0(x) \right) \log s = e^{-\gamma},$$

where $\gamma$ is the Euler's constant. □

**Theorem 164.** *For every $\alpha \in (0,1)$ there exists a sequence of intervals $(k_m, k_m+N_m]$ $(k_m, N_m \to \infty)$ such that $F_{(k_m,k_m+N_m]}(x)$ converges to a distribution function $g(x)$ with $g(x) = 1$ for $\alpha \le x \le 1$.*

*Proof.* Let $\alpha \in (0,1)$ be fixed and let $p_s$ be the greatest prime number $p_i$ verifying $\left(1-\frac{1}{p_i}\right) \leq \alpha$. The $\alpha$-numbers being square free, we can select a subsequence of them $a_{s_1}(\alpha) < a_{s_2}(\alpha) < a_{s_3}(\alpha) < \dots$ pairwise co-prime. By Chinese theorem, there exists a positive integer $k$ such that $k+i \equiv 0 \pmod{a_{s_i}(\alpha)}$ for $i = 1, \dots, N$. Therefore

$$\#\{n \in (k, k+N]; B_n(\alpha) \neq \emptyset\} = N$$

and thus, by (394),

$$F_{(k,k+N]}(\alpha) = 1. \tag{404}$$

□

**Notes 31.** If $1 - \frac{1}{p_s} \leq x$, then readily $1 \leq g_0(x) + \prod_{i=1}^{s}(1 - p_i^{-1})$ since the second term of this sum is the density of natural numbers coprime to $p_1 \cdots p_s$. So,

$$g_0(x) \geq 1 - \prod_{p \leq \frac{1}{1-x}} \left(1 - \frac{1}{p}\right) \geq 1 - \frac{c_2}{\log\left(\frac{1}{1-x}\right)} \tag{405}$$

for all $x \in (0,1)$. This inequality was first proved by B.A. Venkov [186]. He also proved

(i) $\lim_{\substack{x \to 1 \\ x<1}} (1 - g_0(x)) \log \frac{1}{1-x} = e^{-\gamma}$.

(ii) $\lim_{\substack{x \to 0 \\ x>0}} (x \log\log \frac{1}{g_0(x)} = e^{-\gamma}$.

(iii) Let $p$ be a prime number. If $1 - \frac{1}{p} \leq x$, then

$$\frac{1}{p} = \sum_{n=0}^{\infty} (-1)^n (p-1)^n g_0\left(x\left(1 - \frac{1}{p}\right)^n\right).$$

(iv) The function $g_0(x)$ at every value $x = \frac{\varphi(n)}{n}$, $n = 1, 2, 3, \dots$, has an infinite left derivative.

### 6.3.5 Schinzel–Wang theorem

A. Schinzel and Y. Wang [142] proved that for every fixed integer $N$ the $(N-1)$-dimensional sequence

$$\left(\frac{\varphi(k+2)}{\varphi(k+1)}, \frac{\varphi(k+3)}{\varphi(k+2)}, \dots, \frac{\varphi(k+N)}{\varphi(k+N-1)}\right), \quad k = 1, 2, 3 \dots \tag{406}$$

is dense in $[0,\infty)^{N-1}$. Thus, for any given $N$-tuple $(\alpha_1,\alpha_2,\dots,\alpha_{N-1})$ in $[0,\infty)^{N-1}$ we can select an increasing sequence of integers $k_m$, such that the sequence of $N$-tuples

$$\left(\frac{\varphi(k_m+2)}{\varphi(k_m+1)},\frac{\varphi(k_m+3)}{\varphi(k_m+2)},\dots,\frac{\varphi(k_m+N)}{\varphi(k_m+N-1)}\right)$$

converge to $(\alpha_1,\alpha_2,\dots,\alpha_{N-1})$. Using the factorization

$$\frac{\varphi(k+n)}{k+n}=\frac{\varphi(k+n)}{\varphi(k+n-1)}\frac{\varphi(k+n-1)}{\varphi(k+n-2)}\cdots\frac{\varphi(k+2)}{\varphi(k+1)}\frac{\varphi(k+1)}{k+1}\frac{k+1}{k+n} \tag{407}$$

we can chose integers $k_m$ such that the sequence of ratios $\frac{\varphi(k_m+1)}{k_m+1}$ converges, say to $\alpha$, hence

$$\lim_{m\to\infty}\left(\frac{\varphi(k_m+1)}{k_m+1},\frac{\varphi(k_m+2)}{k_m+2},\dots,\frac{\varphi(k_m+N)}{k_m+N}\right)$$
$$=(\alpha,\alpha\alpha_1,\alpha\alpha_1\alpha_2,\dots,\alpha\alpha_1\alpha_2\dots\alpha_{N-1}).$$

In the following we apply the above fact but for an infinite sequence $\alpha_n$, $n=1,2,3\dots$.

**Theorem 165.** *Let $\tilde{g}(x)$ be an arbitrary d.f. There exists $\alpha\in(0,1]$ and a sequence of intervals $(k_m,k_m{+}N_m]$ such that the sequence of d.f.s $F_{(k_m,k_m+N_m]}(x)$ converges to a d.f. $g(x)$ such that for a.e. $x\in[0,1)$ one has*

$$g(x)=\begin{cases}\tilde{g}\left(\frac{x}{\alpha}\right) & \text{if } x\in[0,\alpha),\\ 1 & \text{if } x\in[\alpha,1].\end{cases} \tag{408}$$

*Proof.* For arbitrary d.f. $\tilde{g}(x)$ there exists a sequence $\alpha_n$, $n=1,2,\dots$ in $(0,\infty)$ such that for every $n=1,2,3,\dots$ we have $\alpha_1\alpha_2\dots\alpha_n\in(0,1)$ and the sequence

$$\alpha_1\alpha_2\dots\alpha_n,\quad n=1,2,\dots \tag{409}$$

has a.d.f. $\tilde{g}(x)$. Now, using density of (406), for an arbitrary sequence $\varepsilon(N)$ with $\varepsilon(N)>0$ and $\varepsilon(N)\to 0$, there exist integers $k=k(N)$ such that

$$\left|\frac{\varphi(k+2)}{\varphi(k+1)}\frac{\varphi(k+3)}{\varphi(k+2)}\cdots\frac{\varphi(k+n)}{\varphi(k+n-1)}-\alpha_1\alpha_2\dots\alpha_{n-1}\right|<\varepsilon(N) \tag{410}$$

for every $n = 2, \dots, N$ and

$$\left| \frac{k+1}{k+N} - 1 \right| < \varepsilon(N). \tag{411}$$

From the sequence of couples $(k(N), N)$, $N = 1, 2, 3, \dots$, we select a subsequence $(k', N')$, $k' = k(N')$, such that

$$\frac{\varphi(k'+1)}{k'+1} \to \alpha \text{ as } N' \to \infty, \tag{412}$$

for some $\alpha$ in $(0, 1]$, but $\alpha$ cannot be arbitrary. Then, from (410), (411) and (412) there exists a sequence of positive real numbers $\varepsilon'(N')$ that tends to 0 as $N'$ go to infinity along a subsequence of integers such that

$$\left| \frac{\varphi(k'+n)}{k'+n} - \alpha\alpha_1 \dots \alpha_{n-1} \right| < \varepsilon'(N') \tag{413}$$

for $n = 1, \dots, N'$.

Now we use the following fact: let $x_n$ and $y_n$ in $[0, 1)$ for $n = 1, 2, \dots, N$ and define on $[0, 1]$ the step d.f.s

$$F_N^{(1)}(x) := \frac{1}{N} \sum_{n=1}^{N} c_{[0,x)}(x_n), \;\; F_N^{(2)}(x) := \frac{1}{N} \sum_{n=1}^{N} c_{[0,x)}(y_n).$$

By triangular inequality one has

$$\begin{aligned} \int_0^1 \left| F_N^{(1)}(x) - F_N^{(2)}(x) \right| \mathrm{d}x &\le \frac{1}{N} \sum_{n=1}^{N} \int_0^1 |c_{[0,x)}(x_n) - c_{[0,x)}(y_n)| \mathrm{d}x \\ &= \frac{1}{N} \sum_{n=1}^{N} |x_n - y_n| . \end{aligned} \tag{414}$$

Choose

$$x_n = \frac{\varphi(k'+n)}{k'+n} \quad \text{and} \quad y_n = \alpha\alpha_1 \dots \alpha_{n-1}$$

for $n = 1, \dots, N'$. By construction of $y_n$, the sequence of d.f. $F_{N'}^{(2)}(x)$ converges to $\tilde{g}\left(\frac{x}{\alpha}\right)$ and from (413) and (414) the distribution function $F_{N'}^{(1)}(x)$, that is to say $F_{(k',k'+N']}(x)$, converges along a subsequence of integers $N'$ to $g(x)$ almost every where and so, $g(x)$ satisfies (408). $\square$

**Notes 32.** As we already have mentioned, the value of $\alpha$ in (412) cannot be arbitrary. Applying (403) we see that

$$\int_0^\alpha x^s \mathrm{d}\tilde{g}\left(\frac{x}{\alpha}\right) = \alpha^s \int_0^1 x^s \mathrm{d}\tilde{g}(x) \le \int_0^1 x^s \mathrm{d}g_0(x), \tag{415}$$

for every positive integer $s$. Recall that for $s=1$ we have the classical result

$$\int_0^1 x \mathrm{d}g_0(x) = \frac{6}{\pi^2}$$

and more generally, we have (371). Consequently, with the d.f. $\tilde{g}(x) = x^2$ on $[0,1]$ and $s=1$ we obtain $\alpha \le \frac{9}{\pi^2}$ and the case where $\tilde{g}(x)$ is the step d.f. with jump 1 at $x=1$ gives the inequality $\alpha \le \frac{6}{\pi^2}$. It can be compared with Theorem 164 in which $\alpha$ is arbitrary but $g(x)$ is a special d.f.

### 6.3.6 Additional property of $\varphi(n)/n$

As in above, let $g_0(x)$ be the a.d.f. of the sequence $(\varphi(n)/n)$, $n = 1, 2, \dots$. It is proved that $g_0(x)$ is singular. In the connection of the body $\mathbf{\Omega}$ of the form

$$\left(\int_0^1 g(x)\mathrm{d}x, \int_0^1 xg(x)\mathrm{d}x, \int_0^1 g^2(x)\mathrm{d}x\right),$$

where $g(x)$ run the set of all d.f.s, the point corresponding to $g_0(x)$

$$X^{(0)} = (X_1^{(0)}, X_2^{(0)}, X_3^{(0)}) := \left(\int_0^1 g_0(x)\mathrm{d}x, \int_0^1 xg_0(x)\mathrm{d}x, \int_0^1 g_0^2(x)\mathrm{d}x\right)$$

is an interior point of $\mathbf{\Omega}$. Using the following (416) we have

$$X_1^{(0)} = 1 - \frac{6}{\pi^2}, \quad X_2^{(0)} = \frac{1}{2} - \frac{1}{2}\prod_{p-prime}\left(1 - \frac{2}{p^2} + \frac{1}{p^3}\right) = 0.285\dots.$$

Intersect $\mathbf{\Omega}$ by straight line $(X_1^{(0)}, X_2^{(0)}, X_3)$ and then we find boundary [48]

$$0.250\dots = \min X_3 \le X_3^{(0)} \le \max X_3 = 0.307\dots.$$

[48] The point $(X_1^{(0)}, X_2^{(0)})$ lies in the projections $\Pi_4$ and $\Pi_5$ to the plane $X_1 \times X_2$. Then we use expression of $\Pi_4$ and $\Pi_5$ by $X_3$ and then we find

$$\min X_3 = \frac{4(X_1^{(0)})^3}{9(X_1^{(0)} - X_2^{(0)})} \text{ a } \max X_3 = 2X_1^{(0)} - 1 + \frac{(1 - X_1^{(0)})^3}{1 - 2X_2^{(0)}}.$$

For better estimation we need better computation of

$$\int_0^1 g_0^2(x)\mathrm{d}x.$$

We shall transform it to compute the limit

$$\lim_{N\to\infty}\frac{1}{N^2}\sum_{m,n=1}^{N}\left|\frac{\varphi(m)}{m}-\frac{\varphi(n)}{n}\right|$$

*Proof.* Applying H. Delange [35] we can compute all moments of $g_0(x)$ ($s = 1, 2, \dots$) as

$$\begin{aligned}\int_0^1 x^s \mathrm{d}g_0(x) &= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\left(\frac{\phi(n)}{n}\right)^s \\ &= \prod_{p-prime}\left(1-\frac{1}{p}+\frac{1}{p}\left(1-\frac{1}{p}\right)^s\right). \end{aligned} \tag{416}$$

Using

$$\int_0^1\int_0^1 |x-y|\mathrm{d}g(x)\mathrm{d}g(y) = 2\left(\int_0^1 g(x)\mathrm{d}x - \int_0^1 g^2(x)\mathrm{d}x\right)$$

and

$$\int_0^1 x\mathrm{d}g_0(x) = 1-\int_0^1 g_0(x)\mathrm{d}x = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\frac{\phi(n)}{n} = \prod_p\left(1-\frac{1}{p^2}\right) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}$$

we find

$$\int_0^1 g_0^2(x)\mathrm{d}x = 1-\frac{6}{\pi^2}-\frac{1}{2}\lim_{N\to\infty}\frac{1}{N^2}\sum_{m,n=1}^{N}\left|\frac{\phi(m)}{m}-\frac{\phi(n)}{n}\right|.$$

□

### 6.3.7 Upper bound of $\lim_{N\to\infty}\frac{1}{N^2}\sum_{m,n=1}^{N}\left|\frac{\phi(m)}{m}-\frac{\phi(n)}{n}\right|$.

Expressing the $L^2$ discrepancy in the form

$$\int_0^1\left(\frac{A([0,x),\omega_N)}{N}-\tilde{g}(x)\right)^2\mathrm{d}x =$$

$$=1+\int_0^1\tilde{g}^2(x)\mathrm{d}x-2\int_0^1\tilde{g}(x)\mathrm{d}x+\frac{2}{N}\sum_{n=1}^{N}\int_0^{x_n}\tilde{g}(x)\mathrm{d}x$$
$$-\frac{1}{N}\sum_{n=1}^{N}x_n-\frac{1}{2N^2}\sum_{m,n=1}^{N}|x_m-x_n|. \tag{417}$$

Applying

$$\left(\frac{1}{N}\sum_{n=1}^{N}f(x_n)-\int_0^1 f(x)\mathrm{d}\tilde{g}(x)\right)^2\le\int_0^1(F_N(x)-\tilde{g}(x))^2\mathrm{d}x\int_0^1(f'(x))^2\mathrm{d}x \tag{418}$$

to $L^2$ discrepancy (417) we find

$$\frac{1}{N^2}\sum_{m,n=1}^{N}|x_m-x_n|\le 2\Bigg(1+\int_0^1\tilde{g}^2(x)\mathrm{d}x-2\int_0^1\tilde{g}(x)\mathrm{d}x+\frac{2}{N}\sum_{n=1}^{N}\int_0^{x_n}\tilde{g}(x)\mathrm{d}x$$
$$-\frac{1}{N}\sum_{n=1}^{N}x_n-\frac{\left(\frac{1}{N}\sum_{n=1}^{N}f(x_n)-\int_0^1 f(x)\mathrm{d}\tilde{g}(x)\right)^2}{\int_0^1(f'(x))^2\mathrm{d}x}\Bigg). \tag{419}$$

It hold for arbitrary index $N$, for arbitrary sequence $x_n$ in $[0,1]$, for arbitrary d.f. $\tilde{g}(x)$ and arbitrary $f:[0,1]\to\mathbb{R}$ with continuous derivative $f'(x)$ on $[0,1]$. Putting $\tilde{g}(0)=0$ and $\tilde{g}(x)=1$ for $x\in(0,1]$, $f(x)=\sum_{s=1}^{k}a_s x^s$ and $x_n=\varphi(n)/n$, then

$$\int_0^1(f'(x))^2\mathrm{d}x=\sum_{i,j=1}^{k}a_ia_j\frac{ij}{i+j-1},$$

and (419) has the form

$$\lim_{N\to\infty}\frac{1}{N^2}\sum_{m,n=1}^{N}\left|\frac{\phi(m)}{m}-\frac{\phi(n)}{n}\right|\le 2\prod_{p}\left(1-\frac{1}{p^2}\right)-$$

$$-2\frac{\sum_{i,j=1}^{k}a_ia_j\prod_{p}\left(1-\frac{1}{p}+\frac{1}{p}\left(1-\frac{1}{p}\right)^i\right)\prod_{p-prime}\left(1-\frac{1}{p}+\frac{1}{p}\left(1-\frac{1}{p}\right)^j\right)}{\sum_{i,j=1}^{k}a_ia_j\frac{ij}{i+j-1}}.$$

For $k=1$ we have the limit

$$\leq 2\left(\prod_{p-prime}\left(1-\frac{1}{p^2}\right)-\left(\prod_{p-prime}\left(1-\frac{1}{p^2}\right)\right)^2\right)=2\frac{6}{\pi^2}\left(1-\frac{6}{\pi^2}\right)=0.476\dots.$$

#### 6.3.8 Lover bound of $\lim_{N\to\infty}\frac{1}{N^2}\sum_{m,n=1}^N\left|\frac{\phi(m)}{m}-\frac{\phi(n)}{n}\right|$.

To do this we use (see [160])

$$\left|\frac{1}{N}\sum_{n=1}^N f(x_n)-f\left(\frac{1}{N}\sum_{n=1}^N x_n\right)\right|\leq\frac{c}{N^2}\sum_{m,n=1}^N|x_m-x_n|,$$

with holds for arbitrary $N$, for arbitrary $x_n\in[0,1]$ assuming $|f'(x)|\leq c$ for $x\in[0,1]$. Putting $f(x)=x^2$ and using moments (416) for $s=1,2$ then we find

$$\frac{1}{2}\left(\frac{6}{\pi^2}\right)^2\left(\prod_{p-prime}\left(1+\frac{1}{(p+1)^2(p-1)}\right)-1\right)\leq\lim_{N\to\infty}\frac{1}{N^2}\sum_{m,n=1}^N\left|\frac{\varphi(m)}{m}-\frac{\varphi(n)}{n}\right|.$$

using the first term $p=2$ we have

$$\frac{2}{\pi^4}\leq\lim_{N\to\infty}\frac{1}{N^2}\sum_{m,n=1}^N\left|\frac{\varphi(m)}{m}-\frac{\varphi(n)}{n}\right|:=L.$$

For $L$ we have $L\in[0.021,0.392]$.

**Notes 33.** J.-Ch. Schlage-Puchta (2009) have a method which gives $L\in[0.27425,0.274465]$.

## 6.4 Benford's law (continuation of 3.9)

### 6.4.1 D.f. of sequences involving logarithm

D.f.s of $\log_b x_n \bmod 1$ which we need in (86) can be computed by Theorem 48. We repeat it:

**Theorem 166** ([169]). *Let the real-valued function $f(x)$ be strictly increasing for $x\geq 1$ and let*
*$f^{-1}(x)$ be its inverse function and*

$F_N(x) = \frac{1}{N}\#\{n \le N; f(n) \bmod 1 \in [0,x)\}$.
*Assume that*

(i) $\lim_{x\to\infty} f'(x) = 0$,

(ii) $\lim_{k\to\infty} f^{-1}(k+1) - f^{-1}(k) = \infty$,

(iii) $\lim_{k\to\infty} \frac{f^{-1}(k+w(k))}{f^{-1}(k)} = \psi(w)$ *for every sequence* $w(k) \in [0,1]$ *for which* $\lim_{k\to\infty} w(k) = w$, *where this limit defines the function* $\psi : [0,1] \to [1, \psi(1)]$,

(iv) $\psi(1) > 1$.

*Then for the sequence* $f(n) \bmod 1$, $n = 1, 2, \dots,$ *we have*

$$G(f(n) \bmod 1) = \left\{g_w(x) = \frac{1}{\psi(w)}\frac{\psi(x)-1}{\psi(1)-1} + \frac{\min(\psi(x), \psi(w)) - 1}{\psi(w)}; w \in [0,1]\right\}. \tag{420}$$

*Now, if* $f(N_i) \bmod 1$ *is a subsequence* $f(n) \bmod 1$ *such that* $f(N_i) \bmod 1 \to w$ *then* $F_{N_i}(x) \to g_w(x)$ *for every* $x \in [0,1]$.

Similar Theorem 53 valid also for $f(p_n)$, where $p_n$ are primes. We repeat it:

**Theorem 167** ([122]). *Let* $F_N(x) = \frac{1}{N}\#\{n \le N; f(p_n) \bmod 1 \in [0,x)\}$ *for* $x \in [0,1]$, *where* $p_n$ *is the increasing sequence of all primes. Assume* (i)-(iv) *from Theorem* 166. *Then the sequence* $f(p_n) \bmod 1$, $n = 1, 2, \dots,$ *has*

$$G(f(p_n) \bmod 1) = G(f(n) \bmod 1).$$

*Now, if* $f(p_{N_i}) \bmod 1$ *is a subsequence* $f(p_n) \bmod 1$ *such that* $f(p_{N_i}) \bmod 1 \to w$ *then* $F_{N_i}(x) \to g_w(x)$ *for every* $x \in [0,1]$.

A proof of Theorem 166 and 167 can be found in p. 76.

### 6.4.2 B.L. for natural numbers

Applying Theorem 166 to the sequence

$f(n) = \log_b n^r$, $n = 1, 2, \dots$ we have

$f^{-1}(x) = b^{\frac{x}{r}}$,

$\lim_{k\to\infty} \frac{f^{-1}(k+w)}{f^{-1}(k)} = \frac{b^{\frac{k+w}{r}}}{b^{\frac{k}{r}}} = b^{\frac{w}{r}} = \psi(w)$,

$G(\log_b n^r \bmod 1) = \left\{g_w(x) = \frac{1}{b^{\frac{w}{r}}}\frac{b^{\frac{x}{r}}-1}{b^{\frac{1}{r}}-1} + \frac{\min(b^{\frac{x}{r}}, b^{\frac{w}{r}})-1}{b^{\frac{w}{r}}}; w \in [0,1]\right\}$.

If $\lim_{i\to\infty}\{f(N_i)\} = \lim_{i\to\infty}\{\log_b(N_i^r)\} = w$, then we have

$$\lim_{i\to\infty} \frac{\#\{n \le N_i;\ \text{first } s \text{ digits of } n^r \text{ are } k_1k_2\dots k_s\}}{N_i}$$

$$= g_w\big(\log_b k_1 \cdot k_2k_3\ldots(k_s+1)\big) - g_w\big(\log_b k_1.k_2k_3\ldots k_s\big). \qquad (421)$$

Some examples of $w$ :

**1.** Assume that $N_i = b^i$ and $r$ is a positive integer then $\lim_{i\to\infty}\{\log_b(b^{ir})\} = 0 = w$ and thus for (421) we have

$$=\frac{b^{(\log_b k_1\cdot k_2\ldots(k_s+1))/r}-1}{b^{1/r}-1} - \frac{b^{(\log_b k_1\cdot k_2\ldots k_s)/r}-1}{b^{1/r}-1}$$
$$=\frac{(k_1.k_2k_3\ldots(k_s+1))^{(1/r)} - (k_1.k_2k_3\ldots k_s))^{(1/r)}}{b^{1/r}-1}$$
$$=\frac{1}{b^{r-1}}\frac{1}{b-1} \text{ if } r = 1.$$

**2.** Put $N_i = [b^{\frac{i+w}{r}}]$, where $i$ is a positive integer. The $\lim_{i\to\infty}\{\log_b(N_i^r)\} = w$. Proof.

We have $N_i = [b^{\frac{i+w}{r}}] = b^{\frac{i+w'}{r}}$ where $w' < w$ and $\{\log_b(N_i^r)\} = w'$. Further $|b^{\frac{i+w'}{r}} - b^{\frac{i+w}{r}}| = |w' - w|.b^{\frac{i+x}{r}}.\log b\frac{1}{r}$.

Since $|b^{\frac{i+w'}{r}} - b^{\frac{i+w}{r}}| < 1$, then $|w' - w| \to 0$.

Thus $\lim_{i\to\infty}\{\log_b(N_i^r)\} = w$.

### 6.4.3 B.L. for primes

Applying Theorem 123 for the sequence

$f(p_n) = \log_b p_n^r$, $n = 1, 2, \ldots$, $p_n$ is the $n$th prime we have

$f(x) = \log_b x^r$,

$G(\log_b p_n^r \bmod 1) = \big\{g_w(x) = \frac{1}{b^{\frac{w}{r}}}\frac{b^{\frac{x}{r}}-1}{b^{\frac{1}{r}}-1} + \frac{\min(b^{\frac{x}{r}}, b^{\frac{w}{r}})-1}{b^{\frac{w}{r}}}; w \in [0,1]\big\}$.

If $\{f(p_{N_i})\} = \{\log_b(p_{N_i}^r)\} \to w$ then

$$\lim_{i\to\infty}\frac{\#\{n \le N_i;\ \text{first } s \text{ digits of } p_n^r = k_1k_2\ldots k_s\}}{N_i}$$
$$= g_w\big(\log_b k_1 \cdot k_2k_3\ldots(k_s+1)\big) - g_w\big(\log_b k_1.k_2k_3\ldots k_s\big).$$

Some examples of $w$:

**1.** If $N_i = \pi(b^{\frac{i+w}{r}})$, then $\lim_{i\to\infty}\{\log_b p_{\pi(N_i)}^r)\} = w$.

Proof. Denote $F_N(x) = \frac{1}{N}\#\{n \le N; f(p_n) \bmod 1 \in [0,x)\}$. As we see in (436)

$$F_N(x) = \frac{\sum_{k=0}^{K-1}(\pi(f^{-1}(k+x)) - \pi(f^{-1}(k)))}{N} +$$

$$+\frac{\min(\pi(f^{-1}(K+x)),\pi(f^{-1}(K+w)))-\pi(f^{-1}(K))}{N}$$
$$+\frac{O(K)}{N}+\frac{O(f^{-1}(0))}{N},$$

where $f^{-1}(K+w)=p_N$ but for $F_{\pi(N)}(x)$ we have $f^{-1}(K+w)=N$.

**2.** If $N_i=b^{b^i}$ and $r=1$, then $\{\log_b p_{N_i}\}\to\{\log_b\log b\}$. Proof.

$p_n=n\left(\log n+\log\log n-1+o\left(\frac{\log\log n}{\log n}\right)\right)$

$\log_b p_n=\log_b n+\log_b\log n\left(1+\frac{\log\log n-1}{\log n}+o\left(\frac{\log\log n}{\log^2 n}\right)\right)\Big)$

$\log_b p_n=\log_b n+\log_b\log n+\log_b\left(1+\frac{\log\log n-1}{\log n}+o\left(\frac{\log\log n}{\log^2 n}\right)\right)\Big)$

If $n=b^i$, then

$\{\log_b p_n\}=\{\log_b\log b^i\}+o(1)=\{\log_b i+\log_b\log b\}$

for sufficiently large $n$. Thus

If $N_i=b^{b^i}$ then $\{\log_b p_{N_i}\}\to\{\log_b\log b\}$.

### 6.4.4 The same B.L. for natural and prime numbers

From Section 6.4.2 and 6.4.3 follows that the sequences

$\log_b n^r \bmod 1$, $n=1,2,\dots,$

$\log_b p_n^r \bmod 1$, $n=1,2,\dots,$

have the same distribution functions

$g_w(x)=\frac{1}{b^{\frac{w}{r}}}\frac{b^{\frac{x}{r}}-1}{b^{\frac{1}{r}}-1}+\frac{\min(b^{\frac{x}{r}},b^{\frac{w}{r}})-1}{b^{\frac{w}{r}}}$; $w\in[0,1]$.

Since $\lim_{r\to\infty}\frac{b^{\frac{x}{r}}-1}{b^{\frac{1}{r}}-1}=x$ then $\lim_{r\to\infty}g_w(x)=x$.

Thus, as $r\to\infty$ the sequences $n^r$ and $p_n^r$ tends to B.L.

This is qualitative proof of results in [42].

### 6.4.5 Rate of convergence of $F_N(x)$ of the sequence $\log_b n^r \bmod 1$

All the following results are from Y. Ohkubo and O. Strauch[123]:

**Theorem 168.** *Let $N,b$ be positive integers, $b>1$, $r>0$, $w_0\in[0,1]$. Denote*

$$F_N(x)=\frac{\#\{n\le N;\log_b(n^r)\bmod 1\in[0,x)\}}{N},$$

$$
\begin{aligned}
g_{w_0}(x) &= \frac{1}{b^{\frac{w_0}{r}}} \cdot \frac{b^{\frac{x}{r}} - 1}{b^{\frac{1}{r}} - 1} + \frac{\min(b^{\frac{x}{r}}, b^{\frac{w_0}{r}}) - 1}{b^{\frac{w_0}{r}}},\\
K &= [r \log_b N], w = \{r \log_b N\}.
\end{aligned}
$$

*Then for every $x \in [0,1]$ we have*

$$
|F_N(x) - g_{w_0}(x)| \leq \frac{|w - w_0|}{r} \cdot \log b \cdot b^{\frac{1}{r}} \cdot (b^{\frac{1}{r}} + 1) + \frac{3}{N} + \frac{r \log_b N}{N}. \tag{422}
$$

*Proof.* Firstly we repeat a proof of (29).

For a positive integer $N$ define

- $K_N = [f(N)]$, abbreviating $K_N = K$,
- $w_N = \{f(N)\}$, abbreviating $w_N = w$,
- $A_N([x,y)) = \#\{n \leq N; f(n) \in [x,y)\}$,
- $F_N(x) = \frac{\#\{n \leq N; f(n) \bmod 1 \in [0,x)\}}{N}$.

Clearly $f^{-1}(K+w) = N$ and for every $x \in [0,1]$ and $F_N(x)$ in (106) we have

$$
\begin{aligned}
F_N(x) = &\frac{\sum_{k=0}^{K-1} A_N([k, k+x))}{N} + \\
&+ \frac{A_N([K, K+x) \cap [K, K+w))}{N} + \frac{O(A_N([1, f^{-1}(0))))}{N}
\end{aligned}
$$

From monotonicity of $f(x)$ it follows $A_N([x,y)) = f^{-1}(y) - f^{-1}(x) + \theta$, where $|\theta| \leq 1$. Thus

$$
\begin{aligned}
F_N(x) = &\frac{\sum_{k=0}^{K-1} (f^{-1}(k+x) - f^{-1}(k))}{N} + \\
&+ \frac{\min(f^{-1}(K+x), f^{-1}(K+w)) - f^{-1}(K)}{N} + \frac{O(K)}{N} + \frac{O(f^{-1}(0))}{N},
\end{aligned} \tag{423}
$$

where

$$
O(K) \leq K + 1 \text{ and } O(f^{-1}(0)) \leq f^{-1}(0). \tag{424}
$$

The assumption (ii) implies $1/(f^{-1}(k+1) - f^{-1}(k)) \to 0$ which together with Cauchy-Stolz (other name is Stolz-Cesàro, see [171, p. 4–7]) lemma implies that $K/f^{-1}(K) \to 0$ and thus $K/N \to 0$. Furthermore we can express the first term of $F_N(x)$ in (423) as

$$\frac{\sum_{k=0}^{K-1}(f^{-1}(k+x)-f^{-1}(k))}{\sum_{k=0}^{K-1}(f^{-1}(k+1)-f^{-1}(k))}\cdot\frac{f^{-1}(K)-f^{-1}(0)}{f^{-1}(K+w)} \tag{425}$$

and the second term as

$$\frac{\min\left(\frac{f^{-1}(K+x)}{f^{-1}(K)},\frac{f^{-1}(K+w)}{f^{-1}(K)}\right)-1}{\frac{f^{-1}(K+w)}{f^{-1}(K)}}. \tag{426}$$

Using the assumption (ii) and (iii) the Cauchy-Stolz lemma implies

$$\lim_{K\to\infty}\frac{\sum_{k=0}^{K-1}(f^{-1}(k+x)-f^{-1}(k))}{\sum_{k=0}^{K-1}(f^{-1}(k+1)-f^{-1}(k))}=\lim_{k\to\infty}\frac{f^{-1}(k+x)-f^{-1}(k)}{f^{-1}(k+1)-f^{-1}(k)}=\frac{\psi(x)-1}{\psi(1)-1}, \tag{427}$$

where $K=[f(N)]$, $N=1,2,\dots$. Now, for increasing subsequence $N_i$ of indices $N$, denote $K_i=[f(N_i)]$ and $w_i=\{f(N_i)\}$. If $w_i\to w$, then by (iii) $f^{-1}(K_i)/f^{-1}(K_i+w_i)\to 1/\psi(w)$, and $f^{-1}(K_i+x)/f^{-1}(K_i)\to\psi(x)$. Thus (425), (426), (427) imply (420)

$$F_{N_i}(x)\to g_w(x)=\frac{1}{\psi(w)}\cdot\frac{\psi(x)-1}{\psi(1)-1}+\frac{\min(\psi(x),\psi(w))-1}{\psi(w)}$$

for all $x\in[0,1)$.

In the following we prove a quantitative form of (420). Put

- $K=[f(N)]$,
- $w=\{f(N)\}$,
- $N=f^{-1}(K+w)$.

Then

$$\begin{aligned}
&F_N(x)-g_{w_0}(x)=\\
&+\left(\frac{\sum_{k=0}^{K-1}(f^{-1}(k+x)-f^{-1}(k))}{\sum_{k=0}^{K-1}(f^{-1}(k+1)-f^{-1}(k))}\cdot\frac{f^{-1}(K)-f^{-1}(0)}{f^{-1}(K+w)}-\frac{\psi(x)-1}{\psi(1)-1}\cdot\frac{1}{\psi(w_0)}\right)\\
&+\left(\frac{\min\left(\frac{f^{-1}(K+x)}{f^{-1}(K)},\frac{f^{-1}(K+w)}{f^{-1}(K)}\right)-1}{\frac{f^{-1}(K+w)}{f^{-1}(K)}}-\frac{\min(\psi(x),\psi(w_0))-1}{\psi(w_0)}\right)\\
&+\left(\frac{O(K)}{N}+\frac{O(f^{-1}(0))}{N}\right)
\end{aligned}$$

$$=(I)\left(\frac{\sum_{k=0}^{K-1}(f^{-1}(k+x)-f^{-1}(k))}{\sum_{k=0}^{K-1}(f^{-1}(k+1)-f^{-1}(k))}-\frac{\psi(x)-1}{\psi(1)-1}\right)\left(\frac{f^{-1}(K)-f^{-1}(0)}{f^{-1}(K+w)}\right) \tag{428}$$

$$+(II)\left(\frac{f^{-1}(K)-f^{-1}(0)}{f^{-1}(K+w)}-\frac{1}{\psi(w_0)}\right)\left(\frac{\psi(x)-1}{\psi(1)-1}\right) \tag{429}$$

$$+(III)\left(\min\left(\frac{f^{-1}(K+x)}{f^{-1}(K)},\frac{f^{-1}(K+w)}{f^{-1}(K)}\right)-\min(\psi(x),\psi(w_0))\right)\left(\frac{1}{\frac{f^{-1}(K+w)}{f^{-1}(K_i)}}\right) \tag{430}$$

$$+(IV)\left(\psi(w_0)-\frac{f^{-1}(K+w)}{f^{-1}(K)}\right)\left(\frac{\min(\psi(x),\psi(w_0))-1}{\frac{f^{-1}(K+w)}{f^{-1}(K)}\cdot\psi(w_0)}\right) \tag{431}$$

$$+(V)\left(\frac{O(K)}{N}+\frac{O(f^{-1}(0))}{N}\right). \tag{432}$$

Every second term in (I)=(428), (II)=(429), (III)=(430) and (IV)=(431) is bounded, precisely

$$\left(\frac{f^{-1}(K)-f^{-1}(0)}{f^{-1}(K+w)}\right)\le 1,\quad \left(\frac{\psi(x)-1}{\psi(1)-1}\right)\le 1,$$
$$\left(\frac{1}{\frac{f^{-1}(K+w)}{f^{-1}(K)}}\right)\le 1,\quad \left(\frac{\min(\psi(x),\psi(w_0))-1}{\frac{f^{-1}(K+w)}{f^{-1}(K)}\cdot\psi(w_0)}\right)\le \psi(1)-1.$$

Now put $f(x)=\log_b(x^r)$. This function satisfies Theorem 166 and

$$f^{-1}(x)=b^{\frac{x}{r}},$$
$$\lim_{k\to\infty}\frac{f^{-1}(k+x)}{f^{-1}(k)}=\frac{b^{\frac{k+x}{r}}}{b^{\frac{k}{r}}}=b^{\frac{x}{r}}=\psi(x),$$
$$g_{w_0}(x)=\frac{1}{b^{\frac{w_0}{r}}}\cdot\frac{b^{\frac{x}{r}}-1}{b^{\frac{1}{r}}-1}+\frac{\min(b^{\frac{x}{r}},b^{\frac{w_0}{r}})-1}{b^{\frac{w_0}{r}}}$$
$$f^{-1}(0)=b^{\frac{0}{r}}=1,\psi(1)=b^{\frac{1}{r}},$$
$$K=[r\log_b N],\omega=\{r\log_b N\},N=b^{\frac{K+w}{r}}.$$

Then for first terms (I)=(428), (II)=(429), (III)=(430) and (IV)=(431) we have:

$$(I)\left|\frac{\sum_{k=0}^{K-1}(f^{-1}(k+x)-f^{-1}(k))}{\sum_{k=0}^{K-1}(f^{-1}(k+1)-f^{-1}(k))}-\frac{\psi(x)-1}{\psi(1)-1}\right|$$

$$= \left| \frac{\sum_{k=0}^{K-1}(b^{\frac{k+x}{r}} - b^{\frac{k}{r}})}{\sum_{k=0}^{K-1}(b^{\frac{k+1}{r}} - b^{\frac{k}{r}})} - \frac{b^{\frac{x}{r}} - 1}{b^{\frac{1}{r}} - 1} \right| = 0,$$

$$(II) \left| \frac{f^{-1}(K) - f^{-1}(0)}{f^{-1}(K+w)} - \frac{1}{\psi(w_0)} \right| = \left| \frac{b^{\frac{K}{r}} - 1}{b^{\frac{K+w}{r}}} - \frac{1}{b^{\frac{w_0}{r}}} \right| \leq \left| \frac{1}{b^{\frac{w}{r}}} - \frac{1}{b^{\frac{w_0}{r}}} \right| + \frac{1}{N},$$

$$(III) \left| \min\left( \frac{f^{-1}(K+x)}{f^{-1}(K)}, \frac{f^{-1}(K+w)}{f^{-1}(K)} \right) - \min(\psi(x), \psi(w_0)) \right|$$

$$= |\min(\psi(x), \psi(w)) - \min(\psi(x), \psi(w_0))|$$

$$\leq |\psi(w) - \psi(w_0)| = |b^{\frac{w}{r}} - b^{\frac{w_0}{r}}| (\text{ a proof see in appendix}), \tag{433}$$

$$(IV) \left| \psi(w_0) - \frac{f^{-1}(K+w)}{f^{-1}(K)} \right| (\psi(1) - 1) = |b^{\frac{w}{r}} - b^{\frac{w_0}{r}}|(b^{\frac{1}{r}} - 1),$$

$$(V) \left( \frac{O(K)}{N} + \frac{O(f^{-1}(0))}{N} \right) \leq \frac{K + 1 + f^{-1}(0)}{N} \leq \frac{r \log_b N + 2}{N}.$$

In the end of proof we use

$$|b^{\frac{w}{r}} - b^{\frac{w_0}{r}}| \leq \frac{|w - w_0|}{r} \cdot \log b \cdot b^{\frac{1}{r}}.$$

□

**Example 64.** Let $r > 0$, $w = \{r \log_b N\}$. Then for $D = d_1 d_2 \cdots d_s$ the [49] fraction

$\frac{1}{N}\#\{1 \leq n \leq N; \text{the leading block of } s \text{ digits of } n^r \text{ is equal to } D\}$

it can be approximated by

$$g_w\left( \log_b \left( \frac{D+1}{b^{s-1}} \right) \right) - g_w\left( \log_b \left( \frac{D}{b^{s-1}} \right) \right) \tag{434}$$

with the error term

$$2\left( \frac{3}{N} + \frac{r \log_b N}{N} \right) = O_{b,r}\left( \frac{\log N}{N} \right). \tag{435}$$

[49] Previously, we have used $K = k_1 k_2 \ldots k_s$.

*Proof.* We have

$$\frac{1}{N}\#\{1 \le n \le N; \text{the leading block of } s \text{ digits of } n^r \text{ is equal to } K\}$$
$$= F_N\left(\log_b\left(\frac{D+1}{b^{s-1}}\right)\right) - F_N\left(\log_b\left(\frac{D}{b^{s-1}}\right)\right).$$

Applying Theorem 168 with $w_0 = w$, we have
$|F_N(x) - g_w(x)| \le \frac{3}{N} + \frac{r\log_b N}{N}$. □

For example: Let $r > 0$ and $i$ be a positive integer. By (434), (435) for $D = d_1 d_2 \cdots d_s$ the fraction
$\frac{1}{b^i}\#\{1 \le n \le b^i; \text{the leading block of } s \text{ digits of } n^r \text{ is equal to } D\}$
it can be approximated by

$$\frac{\left(\frac{D+1}{b^{s-1}}\right)^{\frac{1}{r}} - \left(\frac{D}{b^{s-1}}\right)^{\frac{1}{r}}}{b^{\frac{1}{r}} - 1}$$

with the error term

$$\frac{2ri+6}{b^i}.$$

### 6.4.6 Rate of convergence of $F_N(x)$ for $\log_b p_n^r \bmod 1$ with primes $p_n$

Similar method it can be used to approximate $|F_N(x) - g_w(x)|$ for

$$F_N(x) = \frac{\#\{n \le N; \log_b({p_n}^r) \bmod 1 \in [0,x)\}}{N},$$
$$g_w(x) = \frac{1}{b^{\frac{w}{r}}} \cdot \frac{b^{\frac{x}{r}} - 1}{b^{\frac{1}{r}} - 1} + \frac{\min(b^{\frac{x}{r}}, b^{\frac{w}{r}}) - 1}{b^{\frac{w}{r}}},$$
$$K = [r\log_b p_N], w = \{r\log_b p_N\}.$$

We have

$$f(p_n) < x \Longleftrightarrow, 0 \le f(p_n) - k < x \Longleftrightarrow$$
$$k \le f(p_n) < k + x \Longleftrightarrow f^{-1}(k) \le p_n < f^{-1}(k+x).$$

Let $\pi(x) = \#\{p \le x; p-\text{ prime }\}$. Using $\pi(x) = \#\{p < x; p-\text{ prime }\} + O(1)$, then we have

$$
\begin{aligned}
F_N(x) =&\frac{\sum_{k=0}^{K-1}(\pi(f^{-1}(k+x))-\pi(f^{-1}(k)))}{N}+\\
&+\frac{\min(\pi(f^{-1}(K+x)),\pi(f^{-1}(K+w)))-\pi(f^{-1}(K))}{N}\\
&+\frac{O(K)}{N}+\frac{O(f^{-1}(0))}{N}.
\end{aligned} \tag{436}
$$

As in (428), (429), (430), (431), (432), then we have

$$|F_N(x)-g_{w_0}(x)|\le$$

$$(I)\left|\frac{\sum_{k=0}^{K-1}(\pi(f^{-1}(k+x))-\pi(f^{-1}(k)))}{\sum_{k=0}^{K-1}(\pi(f^{-1}(k+1))-\pi(f^{-1}(k)))}-\frac{\psi(x)-1}{\psi(1)-1}\right| \tag{437}$$

$$+(II)\left|\frac{\pi(f^{-1}(K))-\pi(f^{-1}(0))}{\pi(f^{-1}(K+w))}-\frac{1}{\psi(w_0)}\right| \tag{438}$$

$$+(III)\left|\min\left(\frac{\pi(f^{-1}(K+x))}{\pi(f^{-1}(K))},\frac{\pi(f^{-1}(K+w))}{\pi(f^{-1}(K))}\right)-\min(\psi(x),\psi(w_0))\right| \tag{439}$$

$$+(IV)\left|\psi(w_0)-\frac{\pi(f^{-1}(K+w))}{\pi(f^{-1}(K))}\right|(\psi(1)-1) \tag{440}$$

$$+(V)\left(\frac{O(K)}{N}+\frac{O(f^{-1}(0))}{N}\right). \tag{441}$$

But the following upper bounds in (437), (438), (439), (440), and (441) are very unclear. Now in the following we use Prime number theorem in the form

$$\pi(x)=\frac{x}{\log x}\theta(x),\quad \theta(x)\to 1,\ \text{if } x\to\infty. \tag{442}$$

Using (see Wikipedia, Prime number theorem, p. 12)

$$\frac{x}{\log x+2}<\pi(x)<\frac{x}{\log x-4} \tag{443}$$

for $x\ge 55$ then we have

$$\frac{\log x}{\log x+2}<\theta(x)<\frac{\log x}{\log x-4}. \tag{444}$$

In our case
$f(x) = \log_b x^r$,
$f^{-1}(x) = b^{\frac{x}{r}}$,
$\psi(x) = b^{\frac{x}{r}}$,
$g_{w_0}(x) = \frac{1}{b^{\frac{w_0}{r}}} \cdot \frac{b^{\frac{x}{r}}-1}{b^{\frac{1}{r}}-1} + \frac{\min(b^{\frac{x}{r}}, b^{\frac{w_0}{r}})-1}{b^{\frac{w_0}{r}}}$.

For an upper bound of (I)=(437) we use the inequality proved later

$$(I)\left|\frac{\sum_{k=0}^{K-1}(\pi(f^{-1}(k+x))-\pi(f^{-1}(k)))}{\sum_{k=0}^{K-1}(\pi(f^{-1}(k+1))-\pi(f^{-1}(k)))} - \frac{\psi(x)-1}{\psi(1)-1}\right|$$
$$\leq \Psi_{k_1} + \left|\frac{\pi(f^{-1}(k_1))-\pi(f^{-1}(0))}{\pi(f^{-1}(K))-\pi(f^{-1}(0))}\right|, \tag{445}$$

where $K \geq k_1$, $\Psi_{k_1} = \sup_{k\geq k_1} \Phi_k$ and

$$\Phi_k = \left|\frac{\pi(f^{-1}(k+x))-\pi(f^{-1}(k))}{\pi(f^{-1}(k+1))-\pi(f^{-1}(k))} - \frac{\psi(x)-1}{\psi(1)-1}\right|. \tag{446}$$

Applying (442) to (446) we have

$$\Phi_k = \left|\frac{b^{\frac{x}{r}}\left(\left(\frac{k}{k+x}\right)\frac{\theta(b^{\frac{k+x}{r}})}{\theta(b^{\frac{k}{r}})}-1\right)+b^{\frac{x}{r}}-1}{b^{\frac{1}{r}}\left(\left(\frac{k}{k+1}\right)\frac{\theta(b^{\frac{k+1}{r}})}{\theta(b^{\frac{k}{r}})}-1\right)+b^{\frac{1}{r}}-1} - \frac{b^{\frac{x}{r}}-1}{b^{\frac{1}{r}}-1}\right| \tag{447}$$

For (II)=(438) we have

$$(II)\left|\frac{\pi(b^{\frac{K}{r}})-\pi(1)}{\pi(b^{\frac{K+w}{r}})} - \frac{1}{b^{\frac{w_0}{r}}}\right| = \left|\frac{1}{b^{\frac{w}{r}}}\Big(\frac{K+w}{K}\Big)\frac{\theta(b^{\frac{K}{r}})}{\theta(b^{\frac{K+w}{r}})} - \frac{1}{b^{\frac{w_0}{r}}}\right|$$
$$\leq \frac{1}{b^{\frac{w}{r}}}\left|\Big(\frac{K+w}{K}\Big)\frac{\theta(b^{\frac{K}{r}})}{\theta(b^{\frac{K+w}{r}})} - 1\right| + \left|\frac{1}{b^{\frac{w}{r}}} - \frac{1}{b^{\frac{w_0}{r}}}\right| \tag{448}$$

For (III)=(439) we use

$$(III) \leq \left|\min\left(\frac{\pi(f^{-1}(K+x))}{\pi(f^{-1}(K))}, \frac{\pi(f^{-1}(K+w))}{\pi(f^{-1}(K))}\right) - \min(\psi(x),\psi(w))\right|$$
$$+ |\min(\psi(x),\psi(w)) - \min(\psi(x),\psi(w_0))| \tag{449}$$

$$\le \max\left(\left|\frac{\pi(f^{-1}(K+x))}{\pi(f^{-1}(K))}-\psi(x)\right|,\left|\frac{\pi(f^{-1}(K+w))}{\pi(f^{-1}(K))}-\psi(w)\right|\right)$$
$$+|\psi(w)-\psi(w_0)|$$
$$=\max\left(b^{\frac{x}{r}}\left|\left(\frac{K}{K+x}\right)\frac{\theta(b^{\frac{K+x}{r}})}{\theta(b^{\frac{K}{r}})}-1\right|,b^{\frac{w}{r}}\left|\left(\frac{K}{K+w}\right)\frac{\theta(b^{\frac{K+w}{r}})}{\theta(b^{\frac{K}{r}})}-1\right|\right)$$
$$+|b^{\frac{w}{r}}-b^{\frac{w_0}{r}}|. \tag{450}$$

For (IV)=(732) we have

$$(IV)\left|b^{\frac{w_0}{r}}-\frac{\pi(b^{\frac{K+w}{r}})}{\pi(b^{\frac{K}{r}})}\right|(b^{\frac{1}{r}}-1)$$
$$\le\left(\left|b^{\frac{w_0}{r}}-b^{\frac{w}{r}}\right|+b^{\frac{w}{r}}\left|\left(\frac{K}{K+w}\right)\frac{\theta(b^{\frac{K+w}{r}})}{\theta(b^{\frac{K}{r}})}-1\right|\right)(b^{\frac{1}{r}}-1). \tag{451}$$

For V=(344) we have

$$(V)\left(\frac{O(K)}{N}+\frac{O(f^{-1}(0))}{N}\right)=\frac{K+1+\pi(f^{-1}(0))}{N}=\frac{K+1}{N}$$
$$=\frac{[f(p_N)]+1}{N}\le\frac{r\log_b p_N+1}{N}.$$

In all (I)=(447), (II)=(448), (III)=(450) and (IV)=(451) we have common factors $\left|\left(\frac{K}{K+w}\right)\frac{\theta(b^{\frac{K+w}{r}})}{\theta(b^{\frac{K}{r}})}-1\right|$ our $\left|\left(\frac{K+w}{K}\right)\frac{\theta(b^{\frac{K}{r}})}{\theta(b^{\frac{K+w}{r}})}-1\right|$. Assuming $\log b-\frac{4r}{k+x}>0$ then (444) implies

$$\frac{\log b-\frac{4r}{k}}{\log b+\frac{2r}{k+x}}<\frac{\theta(b^{\frac{k+x}{r}})}{\theta(b^{\frac{k}{r}})}<\frac{\log b+\frac{2r}{k}}{\log b-\frac{4r}{k+x}}$$

and we have

$$\frac{-\frac{x}{r}\log b-6}{\frac{k+x}{r}\log b+2}<\left(\frac{k}{k+x}\right)\frac{\theta(b^{\frac{k+x}{r}})}{\theta(b^{\frac{k}{r}})}-1<\frac{-\frac{x}{r}\log b+6}{\frac{k+x}{r}\log b-4}$$

which gives

$$\left|\left(\frac{k}{k+x}\right)\frac{\theta(b^{\frac{k+x}{r}})}{\theta(b^{\frac{k}{r}})}-1\right|\le\frac{\log b+6r}{k\log b-4r} \tag{452}$$

or

$$\left|\left(\frac{k+x}{k}\right)\frac{\theta(b^{\frac{k}{r}})}{\theta(b^{\frac{k+x}{r}})}-1\right|\leq\frac{\log b+6r}{k\log b-4r}. \tag{453}$$

Applying (452) and (453) to (I)=(447) we find

$$\Phi_{k_1}\leq\quad\frac{2b^{\frac{1}{r}}\frac{\log b+6r}{k_1\log b-4r}}{b^{\frac{1}{r}}-1-1-b^{\frac{1}{r}}\frac{\log b+6r}{k_1\log b-4r}}. \tag{454}$$

In this case (454) is decreasing with respect to $k$ and in (15) it can be using $\Phi_{k_1}$. Then for approximate (I) we need approximate second term in (445):

$$\left|\frac{\pi(f^{-1}(k_1))-\pi(f^{-1}(0))}{\pi(f^{-1}(K))-\pi(f^{-1}(0))}\right|=\frac{\pi(b^{\frac{k_1}{r}})}{\pi(b^{\frac{K}{r}})}=\frac{K}{k_1}\frac{1}{b^{\frac{K-k_1}{r}}}\frac{\theta(b^{\frac{k_1}{r}})}{\theta(b^{\frac{K}{r}})}, \tag{455}$$

Also for (I)=(445), (II)=(448), (III)=(450) and (IV)=(451) then we find

**Theorem 169.**

$$\begin{aligned}
&|F_N(x)-g_{w_0}(x)|\leq\\
&(I)\quad\frac{2b^{\frac{1}{r}}\frac{\log b+6r}{k_1\log b-4r}}{b^{\frac{1}{r}}-1-b^{\frac{1}{r}}\frac{\log b+6r}{k_1\log b-4r}}+\frac{1}{b^{\frac{K-k_1}{r}}}.\frac{K\log b+2r}{k_1\log b-4r}\\
&+(II)\quad\frac{1}{b^{\frac{w}{r}}}.\frac{\log b+6r}{K\log b-4r}+\left|\frac{1}{b^{\frac{w_0}{r}}}-\frac{1}{b^{\frac{w}{r}}}\right|\\
&+(III)\quad b^{\frac{1}{r}}.\frac{\log b+6r}{K\log b-4r}+|b^{\frac{w_0}{r}}-b^{\frac{w}{r}}|\\
&+(IV)\quad b^{\frac{w}{r}}(b^{\frac{1}{r}}-1)\frac{\log b+6r}{K\log b-4r}+(b^{\frac{1}{r}}-1).|b^{\frac{w_0}{r}}-b^{\frac{w}{r}}|\\
&+(V)\quad\frac{r\log_b p_N+1}{N}.
\end{aligned}$$

**Notes 34. Proof of the inequality** (445)**.** Denote $\Psi_{k_1}=\sup_{k\geq k_1}\Phi_k$ and

$$\Phi_k=\left|\frac{\pi(f^{-1}(k+x))-\pi(f^{-1}(k))}{\pi(f^{-1}(k+1))-\pi(f^{-1}(k))}-\frac{\psi(x)-1}{\psi(1)-1}\right|.$$

For $k\geq k_1$ we have

$$-\Psi_{k_1}\leq\frac{\pi(f^{-1}(k+x))-\pi(f^{-1}(k))}{\pi(f^{-1}(k+1))-\pi(f^{-1}(k))}-\frac{\psi(x)-1}{\psi(1)-1}\leq\Psi_{k_1}.$$

Then

$$\pi(f^{-1}(k+x)) - \pi(f^{-1}(k)) \leq \left(\Psi_{k_1} + \frac{\psi(x)-1}{\psi(1)-1}\right)(\pi(f^{-1}(k+1)) - \pi(f^{-1}(k))),$$

$$(\pi(f^{-1}(k+1)) - \pi(f^{-1}(k)))\left( - \Psi_{k_1} + \frac{\psi(x)-1}{\psi(1)-1}\right) \leq \pi(f^{-1}(k+x)) - \pi(f^{-1}(k)),$$

Thus we have

$$-\Psi_{k_1} + \frac{\psi(x)-1}{\psi(1)-1} \leq \frac{\sum_{k=k_1}^{K-1}(\pi(f^{-1}(k+x)) - \pi(f^{-1}(k)))}{\sum_{k=k_1}^{K-1}(\pi(f^{-1}(k+1)) - \pi(f^{-1}(k)))} \leq \Psi_{k_1} + \frac{\psi(x)-1}{\psi(1)-1}.$$

Abbreviated

$$A = \sum_{k=k_1}^{K-1}(\pi(f^{-1}(k+x)) - \pi(f^{-1}(k)));$$
$$B = \sum_{k=k_1}^{K-1}(\pi(f^{-1}(k+1)) - \pi(f^{-1}(k)));$$
$$C = \sum_{k=0}^{k_1-1}(\pi(f^{-1}(k+x)) - \pi(f^{-1}(k)));$$
$$D = \sum_{k=0}^{k_1-1}(\pi(f^{-1}(k+1)) - \pi(f^{-1}(k)));$$

Then

$$\frac{A+C}{B+D} - \frac{A}{B} = \frac{D\left(\frac{C}{D} - \frac{A}{B}\right)}{B+D}, \text{ where}$$

$$0 \leq \frac{C}{D} \leq 1, 0 \leq \frac{A}{B} \leq 1, D = \pi(f^{-1}(k_1)) - \pi(f^{-1}(0)), B + D = \pi(f^{-1}(K)) - \pi(f^{-1}(0)).$$

From it

$$\left|\frac{A+C}{B+D} - \frac{A}{B}\right| \leq \frac{\pi(f^{-1}(k_1)) - \pi(f^{-1}(0))}{\pi(f^{-1}(K)) - \pi(f^{-1}(0))} \text{ and using}$$
$$\left|\frac{A+C}{B+D} - \frac{\psi(x)-1}{\psi(1)-1}\right| \leq \left|\frac{A+C}{B+D} - \frac{A}{B}\right| + \left|\frac{A}{B} - \frac{\psi(x)-1}{\psi(1)-1}\right|$$

this implies (445).

**Proof of the inequality** (433) **and** (449)**.**

$$A := |\min(\psi(x), \psi(w)) - \min(\psi(x), \psi(w_0))| \leq |\psi(w) - \psi(w_0)|$$

in (433) and (449).

We compute all cases:
$x < w < w_0$; $A = x - x$,
$x < w_0 < w$; $A = x - x$,
$w_0 < x < w$; $A = |x - w_0| < |w_0 - w|$
$w_0 < w < x$; $A = |w - w_0|$,
$w < w_0 < x$; $A = |w - w_0|$,
$w < x < w_0$; $A = |w - x| < w_0 - w|$.

### 6.4.7 Discrepancy $D_N$ of the sequence $\log_b n^r \bmod 1$

We have the sequence $\{\log_b 1^r\}, \{\log_b 2^r\}, \{\log_b 3^r\}, \dots \{\log_b N^r\}$. Put $F_N(x) = \frac{1}{N}\#\{n \le N; \{\log_b n^r\} \in [0, x)\}$. Then discrepancy is defined as $D_N = \sup_{x\in[0,1]} |F_N(x) - x|$. Clearly we have

$$D_N \le \sup_{x\in[0,1]} |F_N(x) - g_{w_0}(x)| + |g_{w_0}(x) - x|. \tag{456}$$

For the first part of (456) we use Theorem 168. Now we put $w_0 = 0$ and for the second part we need found upper bound of

$$x - g_0(x) = x - \frac{b^{\frac{x}{r}} - 1}{b^{\frac{1}{r}} - 1}.$$

By Lagrange theorem

$$b^{\frac{x}{r}} - 1 = (x - 0)b^{\frac{x_1}{r}}\frac{\log b}{r}, \quad x_1 \in (0, x),$$
$$b^{\frac{1}{r}} - 1 = (1 - 0)b^{\frac{x_2}{r}}\frac{\log b}{r}, \quad x_2 \in (0, 1).$$

Thus

$$x - g_0(x) = x\big(1 - b^{\frac{x_1 - x_2}{r}}\big) = x(0 - (x_1 - x_2))b^{\frac{x_3}{r}}\frac{\log b}{r}, \quad x_3 \in (x_1, x_2).$$

The upper bound is

$$|x - g_0(x)| < 1.b^{\frac{1}{r}}\frac{\log b}{r}$$

and applying Theorem 168 we have

$$D_N \le \frac{|w - 0|}{r} \cdot \log b \cdot b^{\frac{1}{r}} \cdot (b^{\frac{1}{r}} + 1) + \frac{3}{N} + \frac{r\log_b N}{N} + b^{\frac{1}{r}}\frac{\log b}{r}. \tag{457}$$

Thus is of the type $O(1/r)$ as in [42].

### 6.4.8 Benford's law of $x_n$ and properties of $G(x_n)$

In this Section we characterize u.d. of $\log_b x_n \bmod 1$ by d.f.s in $G(x_n)$. We proceed from [13].

**Theorem 170.** *Let $x_n$, $n = 1, 2, \dots$, be a sequence in $(0, 1)$ and $G(x_n)$ be the set of all d.f.s of $x_n$. Assume that every d.f. $g(x) \in G(x_n)$ is continuous at $x = 0$. Then the sequence $x_n$ satisfies B.L. in the base $b$ if and only if for every $g(x) \in G(x_n)$ we have*

$$x = \sum_{i=0}^{\infty} \left( g\left(\frac{1}{b^i}\right) - g\left(\frac{1}{b^{i+x}}\right) \right) \quad \textit{for } x \in [0, 1]. \tag{458}$$

*Proof.* Firstly we note that the sequences $\log_b x_n \bmod 1$ and $-\log_b x_n \bmod 1$ are u.d. simultaneously and in the following we study $-\log_b x_n \bmod 1$, $n = 1, 2, \dots$. Now, considering the function $-\log_b x$ defined on the domain $(0, 1]$. The domain $(0, 1]$ is divided into infinitely many parts $\left(\frac{1}{b^{i+1}}, \frac{1}{b^i}\right]$, for $i = 0, 1, 2, \dots$ and on each part $\left(\frac{1}{b^{i+1}}, \frac{1}{b^i}\right]$ denote

$$f_i(x) = -\log_b x \bmod 1 = -\log_b x - i.$$

Then, for every term $x_n \in (0, 1)$ and $x \in [0, 1]$ we have

$$0 \le -\log_b x_n \bmod 1 < x \iff \exists i \in \{0, 1, 2, \dots\} \Big( x_n \in f_i^{-1}([0, x)) \Big), \tag{459}$$

where $f_i^{-1}([0, x)) = \left(\frac{1}{b^{i+x}}, \frac{1}{b^i}\right]$.

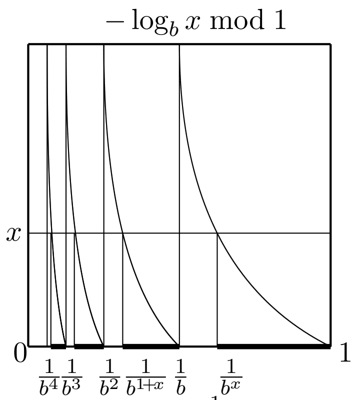



Figure: Intervals $f_i^{-1}([0, x))$, $i = 0, 1, 2, \dots$

Following (2), denote two types of step d.f.s

$$F_N^{(1)}(x) = \frac{\#\{n \le N; -\log_b x_n \bmod 1 \in [0,x)\}}{N},$$
$$F_N^{(2)}(x) = \frac{\#\{n \le N; x_n \bmod 1 \in [0,x)\}}{N}.$$

Then (459) has the form

$$F_N^{(1)}(x) = \sum_{i=0}^{\infty} F_N^{(2)}\left(\frac{1}{b^i}\right) - F_N^{(2)}\left(\frac{1}{b^{i+x}}\right). \tag{460}$$

Moreover, by definition of $F_N^{(2)}(x)$ we see

$$\sum_{i=K}^{\infty} F_N^{(2)}\left(\frac{1}{b^i}\right) - F_N^{(2)}\left(\frac{1}{b^{i+x}}\right) \le F_N^{(2)}\left(\frac{1}{b^K}\right). \tag{461}$$

Now, assume that d.f. $g(x) \in G(x_n)$, i.e. there exist a sequence of indices $N_1 < N_2 < \dots$ such that $\lim_{N_k\to\infty} F_{N_k}^{(2)}(x) = g(x)$ for every $x \in [0,1]$. By Helly theorem from $N_k$ can be select $N_k'$ such that also $\lim_{N_k'\to\infty} F_{N_k'}^{(1)}(x) = \tilde{g}(x)$. Summing up (460) and (461) we find

$$\left|\tilde{g}(x) - \sum_{i=0}^{K-1}\left(g\left(\frac{1}{b^i}\right) - g\left(\frac{1}{b^{i+x}}\right)\right)\right| \le g\left(\frac{1}{b^K}\right) \tag{462}$$

for $x \in [0,1]$. The continuity implies $g\left(\frac{1}{b^K}\right) \to 0$ and we have

$$\tilde{g}(x) = x \Longleftrightarrow x = \sum_{i=0}^{\infty}\left(g\left(\frac{1}{b^i}\right) - g\left(\frac{1}{b^{i+x}}\right)\right). \tag{463}$$

Conclusion of proof: If $x_n$ satisfies B.L., i.e. $\tilde{g}(x) = x$, then (458) holds for every $g(x) \in G(x_n)$. Vice verse (458) implies $\tilde{g}(x) = x$ for every d.f. of $-\log x_n \bmod 1$, since we can starting with $\lim_{N_k\to\infty} F_{N_k}^{(1)}(x) = \tilde{g}(x)$ and then select $N_k'$ such that $\lim_{N_k'\to\infty} F_{N_k'}^{(2)}(x) = g(x)$. □

**Notes 35.** The continuity of $g(x)$ at $x = 0$ is necessary, since the sequence $x_n = \alpha^n$, $n = 1, 2, \dots$, $0 < \alpha < 1$, $\log_b \alpha$ is irrational, satisfies B.L. in base $b$, but $G(x_n) = \{c_0(x)\}$, where d.f. $c_0(x)$ defined as $c_0(x) = 1$ for $x \in (0,1]$, does not satisfy (458).

The following criterion of continuity directly follows from [90, Th. 2.1]

**Theorem 171.** *Every d.f. $g(x) \in G(x_n)$ is continuous at $x = 0$ if and only if for every subsequence $x_{n_k} \to 0$ $(k \to \infty)$ the asymptotic density $d(n_k) = 0$.*

In the following Examples 65, 66 and 67 we shall find some partial solutions of the equation (458). It is motivated by the following construction of sequences $x_n$ satisfying B.L.: Let $g(x)$ be the d.f. continuous at $x = 0$, which solves (458). Then by [92, p. 140, Th. 4.4] there exists a sequence $x_n \in (0, 1)$ with a.d.f. $g(x)$ and by Theorem 170, this $x_n$ satisfies B.L. Note that if $g_1(x), \dots, g_k(x)$ are solutions of (458) and $\alpha_1 + \dots + \alpha_k = 1$, $\alpha_i \geq 0$, $i = 1, 2, \dots, k$, then $\alpha_1 g_1(x) + \dots + \alpha_k g_k(x)$ also satisfies (458). Finally, note that by [171, p. 1–9], if $H$ is nonempty, closed and connected set of solutions of (458) and continuous at $x = 0$, then there exists a sequence $x_n \in (0, 1)$, $n = 1, 2, \dots$, such that $H = G(x_n)$ and this $x_n$ satisfies B.L.

**Example 65.** We can find a solution $g(x)$ satisfying (458) such that we choice increasing $g(x)$ on the interval $\left[0, \frac{1}{b}\right]$, $g(0) = 0$, and then $g(x)$ on $\left(\frac{1}{b}, 1\right]$ must satisfy

$$g\left(\frac{1}{b^x}\right) = \sum_{i=1}^{\infty} \left( g\left(\frac{1}{b^i}\right) - g\left(\frac{1}{b^{i+x}}\right) \right) + 1 - x. \tag{464}$$

Such defined $g : [0, 1] \to [0, 1]$ will be d.f. if the following conditions hold:

(i) $g(x)$ is nondecreasing for $x \in \left(\frac{1}{b}, 1\right]$,

(ii) $g\left(\frac{1}{b}\right) \leq \lim_{x \to 1-0} g\left(\frac{1}{b^x}\right)$,

(iii) $\lim_{x \to 0} g\left(\frac{1}{b^x}\right) \leq 1$.

We assume further that $g(x)$ is differentiable on $\left(\frac{1}{b}, 1\right]$. Then by differentiating (458) , the condition (i) is equivalent to

$$g'\left(\frac{1}{b^x}\right) = b^x - \sum_{i=1}^{\infty} g'\left(\frac{1}{b^{i+x}}\right) \cdot \frac{1}{b^i} \geq 0. \tag{465}$$

Now, we put $g(x) = x$ on the interval $\left[0, \frac{1}{b}\right]$. Then (465) has the form

$$g'\left(\frac{1}{b^x}\right) = b^x - \sum_{i=1}^{\infty} \frac{1}{b^i} = b^x - \frac{1}{b-1} \geq 0.$$

For (ii) we compute (464) and we find

$$g\left(\frac{1}{b^x}\right)=\sum_{i=1}^{\infty}\left(\left(\frac{1}{b^i}\right)-g\left(\frac{1}{b^{i+x}}\right)\right)+1-x=1-x+\left(1-\frac{1}{b^x}\right)\frac{1}{b-1}. \quad (466)$$

Thus $g(x)=x$ for $x\in\left[0,\frac{1}{b}\right]$ satisfies (458) because

(i) $g'\left(\frac{1}{b^x}\right)=b^x-\frac{1}{b-1}>0$,

(ii) $g\left(\frac{1}{b}\right)\le\lim_{x\to1}g\left(\frac{1}{b^x}\right)=\lim_{x\to1}1-x+\left(1-\frac{1}{b^x}\right)\frac{1}{b-1}=\frac{1}{b}$,

(iii) $\lim_{x\to0}g\left(\frac{1}{b^x}\right)=1$.

Replacing $\frac{1}{b^x}$ by $x$ in (464) we find $g(x)$ for $x\in\left[\frac{1}{b},1\right]$ which satisfies (i)–(iii) and thus we have the following solution $g(x)$ of (458)

$$g(x)=\begin{cases}x & \text{if } x\in\left[0,\frac{1}{b}\right],\\ 1+\log_b x+(1-x)\frac{1}{b-1} & \text{if } x\in\left[\frac{1}{b},1\right].\end{cases} \quad (467)$$

**Definition 9.** For every term $x_n$ of a sequence, $x_n=0.00\dots0a_1a_2\dots$, where $a_1$ is the first nonzero digit of $x_n$, define:
$\tilde{x}_n=0.a_1a_2\dots$, i.e. the first zero digits are omitted.
$x_n^*=0.0a_1a_2\dots$, i.e. we add in the first position a zero digit.

Note that if one of three sequences $x_n$, $\tilde{x}_n$, and $x_n^*$ satisfies B.L. then all three sequences $x_n$, $\tilde{x}_n$, and $x_n^*$ satisfy B.L. simultaneously. [50]

**Example 66.** By definition of B.L. the sequence $x_n\in(0,1)$, $n=1,2,\dots$, and $\tilde{x}_n$, $n=1,2,\dots$, satisfy B.L. simultaneously. Since $\tilde{x}_n\in\left[\frac{1}{b},1\right)$ every $\tilde{g}(x)\in G(\tilde{x}_n)$ is continuous in $x=0$ and thus $x_n$ satisfies B.L. if and only if for every $\tilde{g}(x)\in G(\tilde{x}_n)$ the equation (458) holds. Since $\tilde{g}(x)=0$ for $x\in\left[0,\frac{1}{b}\right]$ the equation (458) has the form $x=1-\tilde{g}\left(\frac{1}{b^x}\right)+0$, for $x\in[0,1]$ and thus

$$\tilde{g}(x)=\begin{cases}0 & \text{if } x\in\left[0,\frac{1}{b}\right],\\ 1+\frac{\log x}{\log b} & \text{if } x\in\left[\frac{1}{b},1\right].\end{cases} \quad (468)$$

[50] For a sequence $x_n\in(0,1)$ the set of all d.f.s $G(x_n)$ does not depend on a base $b$ in which $x_n$ are expressed, but the sequence $\tilde{x}_n$ is depend.

**Example 67.** A.d.f. $g^*(x)$ of $x_n^*$ can be computed by $g^*(x) = \tilde{g}(f^{-1}(x))$, where $f(x) = \frac{x}{b}$. Thus we have

$$g^*(x) = \begin{cases} 0 & \text{if } x \in \left[0, \frac{1}{b^2}\right], \\ 2 + \frac{\log x}{\log b} & \text{if } x \in \left[\frac{1}{b^2}, \frac{1}{b}\right], \\ 1 & \text{if } x \in \left[\frac{1}{b}, 1\right] \end{cases} \tag{469}$$

again for $g^*(x)$ the equation (458) holds. Similarly, putting $x_n^{**} = 0.00a_1a_2\ldots$, the a.d.f. of $x_n^{**}$, $n = 1, 2, \ldots$, is

$$g^{**}(x) = \begin{cases} 0 & \text{if } x \in \left[0, \frac{1}{b^3}\right], \\ 3 + \frac{\log x}{\log b} & \text{if } x \in \left[\frac{1}{b^3}, \frac{1}{b^2}\right], \\ 1 & \text{if } x \in \left[\frac{1}{b^2}, 1\right] \end{cases} \tag{470}$$

and $g^{**}(x)$ again satisfies (458) again.

A.d.f $\tilde{g}(x)$ of the sequence $\tilde{x}_n$, $n = 1, 2, \ldots$, can be computed directly from a.d.f $g(x)$ of $x_n$, $n = 1, 2, \ldots$.

**Theorem 172.** *Assume that $x_n \in (0, 1)$ has an a.d.f. $g(x)$ continuous at $x = 0$. Then the sequence $\tilde{x}_n$ has a.d.f. $\tilde{g}(x)$ of the form*

$$\tilde{g}(x) = \sum_{i=0}^{\infty} \left( g\left(\frac{x}{b^i}\right) - g\left(\frac{1}{b^{i+1}}\right) \right) \tag{471}$$

*for $x \in \left[\frac{1}{b}, 1\right]$ and $\tilde{g}(x) = 0$ for $x \in \left[0, \frac{1}{b}\right]$.*

*Proof.* Every $t \in \left[\frac{1}{b^{i+1}}, \frac{1}{b^i}\right)$ has the first $i$ digits equal to zero and thus $tb^i \in \left[\frac{1}{b}, 1\right)$. Thus, for every $x \in \left[\frac{1}{b}, 1\right)$ we have unique $t = \frac{x}{b^i}$ such that $t \in \left[\frac{1}{b^{i+1}}, \frac{1}{b^i}\right)$. Moreover

$$\frac{\#\{n \le N; x_n \in \left[\frac{1}{b^{i+1}}, \frac{x}{b^i}\right)\}}{N} \to g\left(\frac{x}{b^i}\right) - g\left(\frac{1}{b^{i+1}}\right) \tag{472}$$

and the expression (471) follows from the continuity of $g(x)$ at $x = 0$. □

**Example 68.** Let $x_n$, $n = 1, 2, \ldots$, be a sequence in $(0, 1)$ with the a.d.f. $g(x)$ defined by (467). By Theorem 170 the sequence $x_n$ satisfies B.L. in the base $b$. By Theorem 172 if $x \in \left[\frac{1}{b}, 1\right]$ then we have

$$\tilde{g}(x) = \sum_{i=0}^{\infty} \left( g\left(\frac{x}{b^i}\right) - g\left(\frac{1}{b^{i+1}}\right) \right)$$

$$= g(x) - \frac{1}{b} + \left(x - \frac{1}{b}\right)\frac{1}{b} + \left(x - \frac{1}{b}\right)\frac{1}{b^2} + \dots$$
$$= 1 + \log_b x + (1-x)\frac{1}{b-1} - \frac{1}{b} + \left(x - \frac{1}{b}\right)\frac{1}{b-1}$$

which gives $1 + \log_b x$ equal to (468) which we are expecting.

From Example 66 directly follows

**Theorem 173.** *The sequence $x_n \in (0,1)$, $n = 1, 2, \dots$, satisfies B.L. in the base $b$ if and only if the sequence $\tilde{x}_n$, $n = 1, 2, \dots$, has an a.d.f of the form* (468)*. Moreover, if the sequence $x_n \in (0,1)$ has a.d.f.* (468)*, then this a.d.f. has also $\tilde{x}_n$.*

The following theorem simplified functional equation (458) in Theorem 170.

**Theorem 174.** *Assume that every d.f. $g(x) \in G(x_n)$ is continuous at $x = 0$. Then $x_n$ satisfies B.L. in the base $b$ if and only if*

$$1 + \log_b x = \sum_{i=0}^{\infty}\left(g\left(\frac{x}{b^i}\right) - g\left(\frac{1}{b^{i+1}}\right)\right) \text{ for } x \in \left[\frac{1}{b}, 1\right] \tag{473}$$

*for every d.f. $g(x) \in G(x_n)$.*

*Proof.* Assume d.f. $g(x) \in G(x_n)$, $g(x)$ is continuous in $x = 0$ and $F_{N_i}(x) \to g(x)$ as $i \to \infty$. Similarly as in Theorem 172 the sequence $\tilde{x}_n$ has d.f. $\tilde{g}(x)$ of the form (471) and for the same sequence of indices $N_i$. It follows from Theorem 173 that $\tilde{g}(x)$, for every sequence of indices $N_i$, must have the form (468), which gives (473).

An alternative proof follows directly from (458), transforming $\frac{1}{b^x} \to x$. $\square$

As we can see in the following Examples 69 and 70 there exist integer sequences satisfying strong B.L. for arbitrary base $b$, but it does not hold for sequences $x_n \in (0,1)$.

**Example 69.** By [171, p. 2–117, 2.12.14], the sequence

$$\alpha n \log^\tau n \bmod 1, \quad \alpha \neq 0, 0 < \tau \leq 1, \tag{474}$$

is u.d. From this follows that $x_n = n^n$ satisfies strong B.L. for arbitrary integer base $b$, because $\log_b n^n = n \log n \frac{1}{\log b}$. In this case $\frac{1}{x_n} \in (0,1)$ has the a.d.f. $c_0(x)$, which is discontinuous at $x = 0$.

**Example 70.** By [171, p. 2–117, 2.12.15], the sequence

$$\alpha n^2 \log^\tau n \bmod 1, \quad \alpha \neq 0, 0 < \tau \leq 1, \tag{475}$$

is u.d. From this follows that $x_n = n^{n^2}$ satisfies B.L. for arbitrary integer base $b$. Again, the reciprocal sequence $\frac{1}{x_n}$ has the a.d.f. $c_0(x)$.

**Example 71.** The sequence $x_n = b^{\{n\alpha\}}$, $n = 1, 2, \dots$, where $\alpha$ is an irrational, satisfies strong B.L. in the base $b$, but not strong in the base $b^r$ for $r \geq 2$, since

$$\log_{b^r} b^{\{n\alpha\}} = \frac{\{n\alpha\}}{r} \in [0, 1/r).$$

Moreover, $x_n$ does not satisfy B.L. in the base $b^r$ of order 1, because every d.f. $g(x)$ of $\{n\alpha\}/r$ is constant on $[1/r, 1]$ and for $x = 0.(b-1)(b-1)\dots(b-1)$ we have $1 - x = 1/b^r$. Thus $g(x) \neq x$ and the criterion in Theorem 44 does not hold.

In the following example we can see that there exists a sequence $x_n \in (0, 1)$, which satisfies B.L. with respect to a finite set of different bases.

**Example 72.** Let $x_n \in (0, 1)$ be a sequence with a.d.f. $g_1(x)$ of the form (468), i.e.

$$g_1(x) = \begin{cases} 0 & \text{if } x \in \left[0, \frac{1}{b_1}\right], \\ 1 + \log_{b_1} x & \text{if } x \in \left[\frac{1}{b_1}, 1\right]. \end{cases} \tag{476}$$

Then by Example 22 this sequence satisfies strong B.L. with respect to $b_1$. Now we compute all basis $b$ for which $x_n$ satisfies B.L. simultaneously. Every such $b$ must satisfy (473) with $g(x) = g_1(x)$ and for $x \in \left[\frac{1}{b}, 1\right]$. This gives inequality (480) of the form

$$1 + \log_b x \leq g_1(x) \leq 1 + \log_b x + g_1\left(\frac{1}{b}\right).$$

a) Let $b_1 < b$.
Then for $x \in \left[\frac{1}{b}, \frac{1}{b_1}\right]$ we have $g_1(x) = 0$, which contradicts to (480).
b) Let $b < b_1$.
Let $k$ be the first positive integer such that

$$\frac{1}{b^k} \leq \frac{1}{b_1} < \frac{1}{b^{k-1}}.$$

Then (473) has the form

$$1+\log_b x = \sum_{i=0}^{k-1}\left(g_1\left(\frac{x}{b^i}\right) - g_1\left(\frac{1}{b^{i+1}}\right)\right), \tag{477}$$

where

$$g_1\left(\frac{x}{b^i}\right) - g_1\left(\frac{1}{b^{i+1}}\right) = \left(1+\log_{b_1}\frac{x}{b^i}\right) - \left(1+\log_{b_1}\frac{1}{b^{i+1}}\right)$$

for $i = 1, 2, \dots, k-2$. For $i = k-1$ we have $g_1\left(\frac{1}{b^k}\right) = 0$ and

$$g_1\left(\frac{x}{b^{k-1}}\right) = 1+\log_{b_1}\frac{x}{b^{k-1}} \text{ for } x \in \left[\frac{b^{k-1}}{b_1}, 1\right].$$

Thus (477) has the form

$$1+\log_b x = \log_{b_1}\left(x.\frac{x}{b}\cdots\frac{x}{b^{k-1}}\right) - \log_{b_1}\left(\frac{1}{b}\cdots\frac{1}{b^{k-1}}\right)+1 = 1+\log_{b_1} x^k \tag{478}$$

and the equation (478) holds if and only if $\frac{\log x}{\log b} = \frac{k\log x}{\log b_1}$, which is equivalent $b_1 = b^k$. In the case $b_1 = b^k$ we have $x \in \left[\frac{b^{k-1}}{b_1}, 1\right] = \left[\frac{1}{b}, 1\right]$ and thus $x_n$, $n = 1, 2, \dots$, satisfies B.L. also for bases $b$, which are integer roots of $b_1$.

Similar result we find also for d.f of the type (467).

**Example 73.** Let $x_n$, $n = 1, 2, \dots$, be a sequence in $(0, 1)$ with a.d.f. $g(x)$ defined by (467) for $b = b_1$, i.e.

$$g_1(x) = \begin{cases} x & \text{if } x \in \left[0, \frac{1}{b_1}\right], \\ 1+\log_{b_1} + (1-x)\frac{1}{b_1-1} & \text{if } x \in \left[\frac{1}{b_1}, 1\right]. \end{cases} \tag{479}$$

Then by Theorem 170 the sequence $x_n$ satisfies B.L. in the base $b_1$. In the following we find all other basis $b$ for which $x_n$ satisfies B.L. again. Every such $b$ must satisfy (473) with $g(x) = g_1(x)$ and for $x \in \left[\frac{1}{b}, 1\right]$.

a) Assume that $b_1 < b$. Then for $x \in \left[\frac{1}{b}, 1\right]$ we have $\frac{x}{b^i}, \frac{1}{b^{i+1}} \in \left[0, \frac{1}{b_1}\right]$ for $i = 1, 2, \dots$ and (104) has the form

$$1+\log_b x = g_1(x) - g_1\left(\frac{1}{b}\right) + \sum_{i=1}^{\infty}\left(\frac{x}{b^i} - \frac{1}{b^{i+1}}\right)$$

and for $x \in \left[\frac{1}{b}, \frac{1}{b_1}\right]$ we have $g_1(x) - g_1\left(\frac{1}{b}\right) = x - \frac{1}{b}$ and

$$1 + \log_b x = \left(x - \frac{1}{b}\right)\frac{b}{b-1},$$

a contradiction.

b) Let $b < b_1$ and let $k$ be integer such that $\frac{1}{b^k} \le \frac{1}{b_1} < \frac{1}{b^{k-1}}$. In every cases $x \in \left[\frac{1}{b}, 1\right]$ implies $\frac{x}{b^i} \in \left[\frac{1}{b^{i+1}}, \frac{1}{b^i}\right]$. In the case b) we have:

(i) In the intervals $\left[\frac{1}{b^{i+1}}, \frac{1}{b^i}\right]$, $i = 1, 2, \dots, k-2$, d.f. $g_1(x)$ has the form $g_1(x) = 1 + \log_{b_1} x + (1-x)\frac{1}{b_1 - 1}$.

(ii) The same form of $g_1(x)$ we have also in $\left[\frac{1}{b_1}, \frac{1}{b^{k-1}}\right]$.

(iii) In the intervals $\left[\frac{1}{b^{i+1}}, \frac{1}{b^i}\right]$, $i = k, k+1, \dots$, the d.f. $g_1(x) = x$.

Using (i)–(iii) for $g_1\left(\frac{x}{b^i}\right)$, $g_1\left(\frac{1}{b^{i+1}}\right)$, $g_1\left(\frac{x}{b^{k-1}}\right)$, and $g_1\left(\frac{1}{b^k}\right)$, we find (473) in the form

$$\begin{aligned}
1 + \log_b(x) &= \log_{b_1} x\frac{x}{b}\cdots\frac{x}{b^{k-2}} - \log_{b_1}\frac{1}{b}\cdots\frac{1}{b^{k-1}} \\
&\quad - \frac{x}{b_1 - 1}\left(1 + \frac{1}{b} + \cdots + \frac{1}{b^{k-2}}\right) + \frac{1}{b_1 - 1}\left(\frac{1}{b} + \cdots + \frac{1}{b^{k-1}}\right) \\
&\quad + \left(1 + \log_{b_1}\frac{x}{b^{k-1}} + \left(1 - \frac{x}{b^{k-1}}\right)\frac{1}{b_1 - 1}\right) - \frac{1}{b^k} \\
&\quad + \frac{x}{b^k}\frac{1}{1 - \frac{1}{b}} - \frac{1}{b^{k+1}}\frac{1}{1 - \frac{1}{b}}.
\end{aligned}$$

Summing this we have, for $\frac{x}{b^{k-1}} \in \left[\frac{1}{b_1}, \frac{1}{b^{k-1}}\right]$,

$$1 + \log_b x = 1 + \log_{b_1} x^k + (1-x)\frac{b(b^k - 1)}{(b_1 - 1)(b-1)b^k} - (1-x)\frac{b}{(b-1)b^k},$$

which holds only if $b_1 = b^k$. In this case the assumption $\frac{x}{b^{k-1}} \in \left[\frac{1}{b_1}, \frac{1}{b^{k-1}}\right] \Leftrightarrow x \in \left[\frac{b^{k-1}}{b_1}, 1\right] \Leftrightarrow x \in \left[\frac{1}{b}, 1\right]$ and (473) holds. Thus the sequence $x_n$ satisfies B.L. also for the base $b = \sqrt[k]{b_1}$.

The Example 72 and 73 give an impulse to the following theorem:

**Theorem 175.** *For a sequence $x_n \in (0,1)$, $n = 1, 2, \dots$, assume that every d.f. $g(x) \in G(x_n)$ is continuous at $x = 0$. Then there exist only finitely many* *different integer bases $b$ for which the sequence $x_n$ satisfies B.L. simultaneously. Moreover, if the sequence $x_n$ satisfies B.L. in base $b$, and for some $k = 1, 2, \dots$ there exists $k$th integer root $\sqrt[k]{b}$, then $x_n$ satisfies B.L. also in the base $\sqrt[k]{b}$.*

*Proof.* $1^0$. Assume that $x_n$ satisfies B.L. in the base $b$ and and let $g(x) \in G(x_n)$. Then by Theorem 174 for $x \in \left[\frac{1}{b}, 1\right]$ we have (473). Rewrite it in the form

$$g(x) = 1 + \log_b x + g\left(\frac{1}{b}\right) - \sum_{i=1}^{\infty}\left(g\left(\frac{x}{b^i}\right) - g\left(\frac{1}{b^{i+1}}\right)\right)$$

this gives

$$1 + \log_b x \le g(x) \le 1 + \log_b x + g\left(\frac{1}{b}\right). \tag{480}$$

If the sequence $x_n$ satisfies B.L. for infinitely many $b$, then (480) implies $g(x) = 1$ for $x \in (0, 1]$, a contradiction. Quantitatively, if $g(x_0) < 1$, then either $x_0 < \frac{1}{b}$, i.e., $b < \frac{1}{x_0}$, or $\frac{1}{b} \le x_0$ and by (480) we have $1+\log_b x_0 \le g(x_0)$, then $1 - g(x_0) \le \frac{-\log x_0}{\log b}$, which gives

$$b \le \max\left(\frac{1}{x_0}, e^{\frac{-\log x_0}{1-g(x_0)}}\right). \tag{481}$$

$2^0$. Now, assume that $x_n$ satisfies B.L. in the base $b$, i.e. the sequence $\log_b x_n \bmod 1$ is u.d. Applying well known Weyl's criterion we have the following chain of implications.

(i) $\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i h \log_b x_n} \to 0$ as $N \to \infty$ and for every $h = 1, 2, \dots$, thus

(ii) $\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i h k \log_b x_n} \to 0$ as $N \to \infty$ and for every $h, k = 1, 2, \dots$ and then we obtain

(iii) $x_n^k$ satisfies B.L. in base $b$ again.

(iv) Thus, if $g(x) \in G(x_n)$ then $g(\sqrt[k]{x}) \in G(x_n^k)$ and vice versa, because $x_n^k < x \Leftrightarrow x_n < \sqrt[k]{x}$.

(v) Furthermore, for every d.f. $g(x)$ continuous at $x = 0$, the $g(x)$ and $g(\sqrt[k]{x})$, $k = 1, 2, \dots$, solve (473) simultaneously.

Thus (473) implies

$$1 + \log_b x = \sum_{i=0}^{\infty}\left(g\left(\sqrt[k]{\frac{x}{b^i}}\right) - g\left(\sqrt[k]{\frac{1}{b^{i+1}}}\right)\right), \text{ for } x \in \left[\frac{1}{b}, 1\right]. \tag{482}$$

For technical necessity using

$$\sum_{i=0}^{\infty}\left(g\left(\frac{1}{b^i}\right)-g\left(\frac{1}{b^{i+1}}\right)\right)=1$$

we rewrite (473) to the form

$$1+\log_b x = 1-\sum_{i=0}^{\infty}\left(g\left(\frac{1}{b^i}\right)-g\left(\frac{x}{b^i}\right)\right) \text{ for } x\in\left[\frac{1}{b},1\right] \tag{483}$$

and (482) to

$$1+\log_b x = 1-\sum_{i=0}^{\infty}\left(g\left(\sqrt[k]{\frac{1}{b^i}}\right)-g\left(\sqrt[k]{\frac{x}{b^i}}\right)\right), \text{ for } x\in\left[\frac{1}{b},1\right]. \tag{484}$$

Let for integer $k$ we have integer root $\sqrt[k]{b}$. Using
(i) $1+\log_b x = 1+\log_{\sqrt[k]{b}}\sqrt[k]{x}$,
(ii) $g\left(\sqrt[k]{\frac{1}{b^i}}\right)-g\left(\sqrt[k]{\frac{x}{b^i}}\right)=g\left(\frac{1}{(\sqrt[k]{b})^i}\right)-g\left(\frac{\sqrt[k]{x}}{(\sqrt[k]{b})^i}\right)$,
(iii) $x\in\left[\frac{1}{b},1\right]\Leftrightarrow\sqrt[k]{x}\in\left[\frac{1}{\sqrt[k]{b}},1\right]$
and exchanging $\sqrt[k]{x}\to x$, then (484) has the form

$$1+\log_{\sqrt[k]{b}} x = 1-\sum_{i=0}^{\infty}\left(g\left(\frac{1}{(\sqrt[k]{b})^i}\right)-g\left(\frac{x}{(\sqrt[k]{b})^i}\right)\right). \tag{485}$$

By Theorem 174 the sequence $x_n$ satisfies B.L. in the base $\sqrt[k]{b}$. $\square$

### 6.4.9 Two-dimensional Benford's law

Let $x_n>0$, $y_n>0$, $n=1,2,\dots$ and $b$ is an integer base,
$K_1=k_1^{(1)}k_2^{(1)}\dots k_{r_1}^{(1)}$ in base $b$,
$K_2=k_1^{(2)}k_2^{(2)}\dots k_{r_2}^{(2)}$ in base $b$,
$u_1=\log_b\left(\frac{K_1}{b^{r_1-1}}\right)$,
$u_2=\log_b\left(\frac{K_1+1}{b^{r_1-1}}\right)$,
$v_1=\log_b\left(\frac{K_2}{b^{r_2-1}}\right)$,
$v_2=\log_b\left(\frac{K_2+1}{b^{r_2-1}}\right)$.
By (81) we have

first $r_1$ digits (starting a non-zero digit) of $x_n = K_1 \Longleftrightarrow \{\log_b x_n\} \in [u_1, u_2)$,
first $r_2$ digits (starting a non-zero digit) of $y_n = K_2 \Longleftrightarrow \{\log_b y_n\} \in [v_1, v_2)$.
Denote [51]

$$F_N(x,y) = \frac{\#\{n \le N; \{\log_b x_n\} < x \text{ and } \{\log_b y_n\} < y\}}{N}.$$

**Theorem 176.** *Let $g(x,y) \in G(\{\log_b x_n\}, \{\log_b y_n\})$ and $\lim_{k\to\infty} F_{N_k}(x,y) = g(x,y)$ for $(x,y) \in [0,1]^2$. Then*

$$\lim_{k\to\infty} \frac{\#\{n \le N_k;\ \textit{first } r_1 \textit{ digits of } x_n = K_1 \textit{ and } \textit{first } r_2 \textit{ digits of } y_n = K_2}{N_k}$$
$$= g(u_2, v_2) + g(u_1, v_1) - g(u_2, v_1) - g(u_1, v_2). \tag{486}$$

**Example 74.** We have (see (757))

$$G(\{\log_b n\}, \{\log_b(n+1)\})$$
$$= \left\{ g_u(x,y) = \frac{b^{\min(x,y)} - 1}{b-1}\frac{1}{b^u} + \frac{b^{\min(x,y,u)} - 1}{b^u}; u \in [0,1] \right\}.$$

Another, by Sklar's theorem 209

$$g_u(x,y) = \min(g_u(x), g_u(y)), \text{ where}$$
$$g_u(x) = \frac{b^x - 1}{b-1} \cdot \frac{1}{b^u} + \frac{b^{\min(x,u)} - 1}{b^u}.$$

Thus

$$\lim_{k\to\infty} \frac{\#\{n \le N_k;\ \text{first } r_1 \text{ digits of } x_n = K_1 \text{ and } \text{first } r_2 \text{ digits of } y_n = K_2}{N_k}$$
$$= g_u(u_2, v_2) + g_u(u_1, v_1) - g_u(u_2, v_1) - g_u(u_1, v_2). \tag{487}$$

If $K_1 = K_2$ then $= g_u(u_2) - g_u(u_1)$. It can be found directly.

In the following examples we use known examples of statistical independent sequences:

Let $x_n \in [0,1)$, $n = 1, 2, \dots$, be an u.d. sequence. Then

(I) $x_n$ and $\log_b n \bmod 1$ are statistically independent (Theorem 92 according to G. Rauzy (1973) [135] also see [171, p. 2–27, 2.3.6.]).

[51] In the following the sentence "*starting a non-zero digit*" we will not mention.

(II) $x_n$ and $\log_b(n \log n)$ mod 1 are statistically independent (Theorem 93 according to Y. Ohkubo (2011) [122]).

(III) $x_n$ and $\log_b p_n$ mod 1 are statistically independent (Theorem 94 according to Y. Ohkubo (2011) [122]).

**Example 75.** Let $x_n \in [0,1)$, $n = 1, 2, \dots$, be u.d. sequence. Then

$$G(x_n, \{\log_b n\}) = \{g_u(x,y) = x.g_u(y); u \in [0,1]\},$$

where

$$g_u(x) = \frac{b^x - 1}{b - 1} \cdot \frac{1}{b^u} + \frac{b^{\min(x,u)} - 1}{b^u}$$

and $F_{N_k}(x,y) \to g_u(x,y)$ if $\{\log_b N_k\} \to u$.

**Example 76.** Let $x_n \in [0,1)$, $n = 1, 2, \dots$, be u.d. sequence. Then

$$G(x_n, \{\log_b p_n\}) = \{g_u(x,y) = x.g_u(y); u \in [0,1]\},$$

where

$$g_u(x) = \frac{b^x - 1}{b - 1} \cdot \frac{1}{b^u} + \frac{b^{\min(x,u)} - 1}{b^u}$$

and $F_{N_k}(x,y) \to g_u(x,y)$ if $\{\log_b N_k\} \to u$.

**Example 77.** We have

$$G(\{\log_b F_n\}, \{\log_b p_n\}) = \{x.g_u(y); u \in [0,1]\}$$

and let $\{\log_b N_k\} \to u$. Then

$$\lim_{k\to\infty} \frac{\#\{n \le N_k; F_n \text{ has the first } r_1 \text{ digits} = K_1 \text{ and } p_n \text{ has the first } r_2 \text{ digits} = K_2\}}{N_k}$$
$$= u_2 g_u(v_2) + u_1 g_u(v_1) - u_2 g_u(v_1) - u_1 g_u(v_2), \tag{488}$$

where

$$u_1 = \log_b\left(\frac{K_1}{b^{r_1-1}}\right), u_2 = \log_b\left(\frac{K_1+1}{b^{r_1-1}}\right),$$
$$v_1 = \log_b\left(\frac{K_2}{b^{r_2-1}}\right), v_2 = \log_b\left(\frac{K_2+1}{b^{r_2-1}}\right),$$
$$g_u(x) = \frac{b^x - 1}{b - 1} \cdot \frac{1}{b^u} + \frac{b^{\min(x,u)} - 1}{b^u}.$$

This Section 6.4.9 has been inspired by F. Luca and P. Stanica [101]:

**Theorem 177.** *There exists infinite many $n$ such that Fibonacci number $F_n$ starts with digits $K_1$ and $\varphi(F_n)$ starts with digits $K_2$ in the base $b$ representation. Here $K_1$ and $K_2$ are arbitrary and $\varphi(x)$ is the Euler function.*

In the following we see that this claim is equivalent that the sequence

$$(\log_b F_n, \log_b \varphi(F_n)) \bmod 1, \quad n = 1, 2, \dots,$$

is everywhere dense in $[0,1]^2$, but the authors use the following method:

(i) By the first author $\varphi(F_n)/F_n$ is dense in $[0,1]$. Thus, for an interval $I$ with arbitrary small length and containing $K_2/K_1$, there exists $\varphi(F_a)/F_a \in I$.

(ii) Then $\varphi(F_{ap})/F_{ap} \in I$ for all sufficiently large prime $p$.

(iii) There exists infinitely many primes $p$ such that $F_{ap}$ starts with $K_1$.

(iv) Finally, multiplying $I$ by $F_{ap}$ they find $\varphi(F_{ap})$ which starts with $K_2$.

**Notes 36.** The $3x+1$ function $T(n)$ is given by $T(n) = \frac{3n+1}{2}$ if $n$ is odd, and $T(n) = \frac{n}{2}$ if $n$ is even. J.C. Lagarias and K. Soundararajan [95] shows that for most initial values $n$ the sequence $x_k = T^{(k)}(n)$, $k = 1, 2, \dots, N$ approximately satisfy B.L. in the sense that the discrepancy of the sequence $\log_b(x_k)$, $k = 1, 2, \dots, N$ is small.

The $3x+1$ Conjecture asserts when started from any positive integer $n$, some iterate $T^{(k)}(n) = 1$.

## 6.5 Gauss-Kuzmin theorem and $g(x) = g_f(x)$

Denote

$$\begin{aligned}
&f(x) = 1/x \bmod 1,\\
&g_f(x) = \sum_{n=1}^{\infty} g\left(\frac{1}{n}\right) - g\left(\frac{1}{n+x}\right) \text{ for d.f. } g(x), x \in [0,1],\\
&g_f(x) = \int_{f^{-1}([0,x))} 1.\mathrm{d}g(x),\\
&(g_{f^n})_f(x) = g_{f^{n+1}}(x),\\
&g_0(x) = \frac{\log(1+x)}{\log 2}.
\end{aligned}$$

The problem is to find all solutions $g(x)$ of the functional equation

$$g(x) = g_f(x) \text{ for } x \in [0,1], \text{ where} \tag{489}$$

$$g_f(x) = \sum_{n=1}^{\infty} g\left(\frac{1}{n}\right) - g\left(\frac{1}{n+x}\right) \text{ for } x \in [0,1]. \tag{490}$$

*Proof of* (490). We have
$f^{-1}([0,x)) = \cup_{k=1}^{\infty}\big(f_k^{-1}(x), \frac{1}{k}\big)$, where
$f_k^{-1}(x) = \frac{1}{k+x}$ and thus
$g_f(x) = \sum_{k=1}^{\infty}\big(g(f_k^{-1}(0)) - g(f_k^{-1}(x))\big)$ for $x \in [0,1]$.

□

The following is known:

(I) $g_0(x)$ satisfies (489).

**Theorem 178** (Gauss-Kuzmin)**.** *If $g(x) = x$, then $g_{f^n}(x) \to g_0(x)$ and the rate of convergence is $O(q^{\sqrt{n}})$, $0 < q < 1$.*

(II) Theorem 178 was proved by R. Kuzmin [94] and for a starting function $g(x)$ for which

(i) $0 < g'(x) < M$ and

(ii) $|g''(x)| < \mu$.

Thus, if $g(x)$ satisfies (i), (ii), and (489), then $g_f(x) = g_0(x)$.

(III) Theorem 178 it was inspired by Gauss, who conjectured $m_n(x) \to g_0(x)$, where $m_n(x) = |\{\alpha \in [0,1]; 1/r_n(\alpha) < x\}|$ and for continued fraction expansion $\alpha = [a_0(\alpha); a_1(\alpha), a_2(\alpha), \dots]$, $r_n(\alpha) = [a_{n+1}(\alpha); a_{n+2}(\alpha), \dots,]$. In this case $m_n(x) = g_{f^n}(x)$ for $g(x) = x$, since $f(1/r_n(\alpha)) = 1/r_{n+1}(\alpha)$.

Theorem 178 can be extended to

**Theorem 179** (P. Lévy [99])**.**

$$g_{f^n}(x) = \log(1+x)/\log 2 + O(\theta^n) \quad (0 < \theta < 1)$$

*for every starting d.f. $g(x)$ two-times differentiable and $f(x) = (1/x) \bmod 1$.*

**Theorem 180** (E.A. Wirsing [195])**.**

$$g_{f^n}(x) = \log(1+x)/\log 2 + O(\theta^n) + (-\lambda)^n \Psi(x) + O(x(1-x)\theta^n),$$

*where d.f. $g(x)$ is two-times differentiable, the constants $0 < \theta < \lambda = 0.303\dots$ are independent on $g(x)$ and $\Psi$ is a function defined on $[0,1]$ independent on $g(x)$ with continuous second derivatives, $\Psi(0) = \Psi(1) = 0$. The $O$ constant depend on $g(x)$. Again $f(x) = (1/x) \bmod 1$.*

(IV) For starting point $x_0 \in [0,1]$ we define the iterate sequence $x_n$ as

$$x_1 = f(x_0), x_2 = f(f(x_0)), x_3 = f(f(f(x_0))), \dots$$

Then every $g(x) \in G(x_n)$ solve (489). For example, the sequence $x_1 = 1/r_1, x_2 = 1/r_2, \dots$ for $\frac{\sqrt{5}-1}{2} = [0; 1, 1, 1, \dots]$ produces solution $g(x) = c_{\frac{\sqrt{5}-1}{2}}(x)$.

(V) If $g_1(x)$ and $g_2(x)$ solve the equation $g(x) = g_f(x)$ and $g_1(x) = g_2(x)$ for $x \in [0, 1/2)$ then $g_1(x)$ and $g_2(x)$ coincide on the whole interval $[0,1]$.

**Notes 37.** The above method can be used to proving uniqueness of $g = g_f$ also for other function as $f(x) = 1/x \bmod 1$. In M.Y. Goh and E. Schmutz [60] and in generalization of P. Schatte [141] is proved

$$g_{f^n}(x) = g_0(x) + O(\theta^n), \quad (0 < \theta < 1)$$

where
$f = g_0^{-1} \circ f_0 \circ g_0$;
$f_0$ is pairwise monotone uniform distribution preserving function (see Section 12.3);
$g_0$ is d.f., $g_0 \in C^2([0,1])$ and $g_0'(x) \geq \delta > 0$.
For example, if

$$f(x) = \begin{cases} 2x & \text{if } x \in [0, 1/2], \\ 2 - 2x & \text{if } x \in [1/2, 1]. \end{cases}$$

then the functional equation $g(x) = g_f(x)$ i.e.
$g(x) = g(x/2) + 1 - g(1 - (x/2))$
has a unique solution $g(x) = x$ on d.f. in $C^2([0,1])$. [52]

**Notes 38.** For two-dimensional function

$$f(x,y) = (1/x - [1/x], ([1/x] + y)^{-1})$$

H. Nakada [112] proved the solution of $g = g_f$ in the form

$$g(x,y) = \log(1 + xy)/\log 2.$$

K. Dajani and C. Kraaikap [33] proved an iteration

$$g_{f^n}(x,y) = \log(1 + xy)/\log 2 + O(\theta^n)$$

for starting $g(x,y) = xy$.

[52] $C^2([0,1])$ is the set of all $f : [0,1] \to \mathbb{R}$ with continuous second derivative.

**Theorem 181.** *Let*
$f(x) = \frac{1}{x} \bmod 1$ *and*
$x_i(t) = \frac{t}{(i-1)t+1}$ *where* $i = 1, 2, \ldots$ *and*
$g_f(x) = \sum_{n=1}^{\infty} g\left(\frac{1}{n}\right) - g\left(\frac{1}{n+x}\right)$ *for* $x \in [0,1]$.
*Then*

$$\int_0^1 (g(x) - \tilde{g}(x))^2 \mathrm{d}x = \int_{1/2}^1 \frac{1}{t^2} \left( \sum_{i=1}^{\infty} (g(x_i(t)) - \tilde{g}(x_i(t)) \right)^2 \mathrm{d}t \tag{491}$$

*Proof.* By the proof in (490) we have
$g_f(x) = \sum_{i=1}^{\infty} \left( g(f_i^{-1}(0)) - g(f_i^{-1}(x)) \right)$ for $x \in [0,1]$. Thus

$$\begin{aligned}
&\int_0^1 (g(x) - \tilde{g}(x))^2 \mathrm{d}x \\
&= \int_0^1 \left( \sum_{i=1}^{\infty} \left( g(f_i^{-1}(0)) - g(f_i^{-1}(x)) \right) - \sum_{i=1}^{\infty} \left( \tilde{g}(f_i^{-1}(0)) - \tilde{g}(f_i^{-1}(x)) \right) \right)^2 \mathrm{d}x \\
&= \int_0^1 \left( \sum_{i=1}^{\infty} \left( g(f_i^{-1}(x)) - \tilde{g}(f_i^{-1}(x)) \right) \right)^2 \mathrm{d}x \\
&= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_0^1 \left( \left( g(f_i^{-1}(x)) - \tilde{g}(f_i^{-1}(x)) \right) \left( g(f_j^{-1}(x)) - \tilde{g}(f_j^{-1}(x)) \right) \right) \mathrm{d}x
\end{aligned} \tag{492}$$

Here we assume

$$\sum_{i=1}^{\infty} \left( g(f_i^{-1}(0)) - \tilde{g}(f_i^{-1}(0)) \right) = 0. \tag{493}$$

Applying substitution
$f_i^{-1}(x) = \frac{1}{x+i} = y$,
$x = \frac{1}{y} - i$,
$\mathrm{d}x = -\frac{1}{y^2} \mathrm{d}y$,
$f_j^{-1}(x) = \frac{1}{x+j}$
$f_j^{-1}\left(\frac{1}{y} - i\right) = \frac{y}{(j-i)y+1}$
to the one part of (492) we find

$$\int_0^1 \left( \left( g(f_i^{-1}(x)) - \tilde{g}(f_i^{-1}(x)) \right) \left( g(f_j^{-1}(x)) - \tilde{g}(f_j^{-1}(x)) \right) \right) \mathrm{d}x =$$

$$\int_{1/i}^{1/(i+1)} \big(g(y)-\tilde{g}(y)\big)\Big(-\frac{1}{y^2}\Big)\Big(g\big(\frac{y}{(j-i)y+1}\big)-\tilde{g}\big(\frac{y}{(j-i)y+1}\big)\Big)\mathrm{d}y. \tag{494}$$

Using substitution
$y=\frac{t}{(i-1)t+1}=x_i(t)$
$\frac{y}{(j-i)y+1}=x_j(t)$
$\frac{1}{y^2}\mathrm{d}y=\frac{1}{t^2}\mathrm{d}t$
on (494) then we find

$$=\int_{1/2}^{1}\big(g(x_i(t))-\tilde{g}(x_i(t))\big)\frac{1}{t^2}\big(g(x_j(t))-\tilde{g}(x_j(t))\big)\mathrm{d}t \tag{495}$$

which gives (491).

□

## 6.6 Uniformly maldistributed sequences (continuation of 3.7)

Here we repeat and extend the results of Section 3.7.

Let $x_n$, $n=1,2,\dots$, be a given sequence.

The sequence $x_n$ is said to be *uniformly maldistributed* (u.m.) if for every nonempty proper subinterval $I\subset[0,1]$ we have both

$$\liminf_{N\to\infty}\frac{1}{N}\#\{n\le N; x_n\in I\}=0 \text{ and } \limsup_{N\to\infty}\frac{1}{N}\#\{n\le N; x_n\in I\}=1.$$

The definition of uniform maldistribution is due to G. Myerson [109] , who mentioned that the first condition is superfluous, and showed the following three properties:

The sequence $x_n=\{\log\log n\}$ of fractional parts of the iterated logarithm is u.m.

Let $M_k$, $k=1,2,\dots$ be a sequence of natural numbers with $\lim_{k\to\infty}(M_1+\cdots+M_k)/M_k=1$, and let $y_n$, $n=1,2,\dots$ by dense in $[0,1)$. Finally, let $x_n$ be defined by $x_n=y_k$ if $\sum_{i=1}^{k-1}M_i\le n<\sum_{i=1}^{k}M_i$. Then $x_n$ is u.m.

**Theorem 182.** *The sequence $x_n$ is u.m. if and only if*

$$\liminf_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}f(x_n)=\min f,\quad \limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}f(x_n)=\max f$$

*for every function $f$ continuous on* $[0,1]$.

In the following we comment results from [160]:
It is easily seen that

**Theorem 183.** *The sequence $x_n$ is u.m. if and only if*

$$\{c_\alpha(x);\alpha\in[0,1]\}\subset G(x_n).$$

Thus, in the theory of u.m. we need not consider d.f.s other than one-jump d.f. $c_\alpha(x)$ which has a jump of size 1 at $\alpha$. This suggests the definition

**Definition 10.** The sequence $x_n$ is said to be *uniformly maldistributed in the strict sense* (u.m.s.) if $G(x_n)=\{c_\alpha(x);\alpha\in[0,1]\}$.

Let us consider the following moment problem

$$\int_0^1\int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y)=0$$

in d.f. $g:[0,1]\to[0,1]$, where $F(x,y)$ is continuous on the unit square $[0,1]^2$. As before we denote $G(F)$ as the set of all solutions $g(x)$ of this moment problem. By Theorem 126 necessary and sufficient conditions for the set inclusion $G(x_n)\subset G(F)$ is that $\lim\limits_{N\to\infty}\frac{1}{N^2}\sum_{m,n=1}^N F(x_m,x_n)=0$.

By Theorem 58 for any sequence $x_n\in[0,1)$ we have

$$G(x_n)\subset\{c_\alpha(x);\alpha\in[0,1]\}\iff\lim_{N\to\infty}\frac{1}{N^2}\sum_{m,n=1}^N|x_m-x_n|=0$$

and if $G(x_n)\subset\{c_\alpha(x);\alpha\in[0,1]\}$, then $G(x_n)=\{c_\alpha(x);\alpha\in I\}$, where $I$ is a closed subinterval

$$I=\left[\liminf_{N\to\infty}\frac{1}{N}\sum_{n=1}^N x_n,\ \limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^N x_n\right],$$

with the length $|I|=\limsup\limits_{M,N\to\infty}\frac{1}{MN}\sum_{m=1}^M\sum_{n=1}^N|x_m-x_n|$. The following assertion is evident from the preceding.

**Theorem 184.** *The sequence $x_n \in [0,1)$ is u.m.s. if and only if*

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{m,n=1}^{N} |x_m - x_n| = 0 \text{ and } \limsup_{M,N\to\infty} \frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} |x_m - x_n| = 1,$$

*or alternatively*

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_n - \liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_n = 1.$$

**Example 78.** Let $\{x\}$ be the fractional part of $x$. It is shown Theorem 48, (100) ([27, p. 58]) that, for $x_n = \{\log n\}$, $n = 1, 2, \dots,$

$$G(x_n) = \left\{ \frac{1}{e^u} \frac{e^x - 1}{e - 1} + \frac{e^{\min(x,u)} - 1}{e^u}; u \in [0,1] \right\}.$$

Starting with $x_n = \{\log\log n\}$ all the sequences $\{\log\log\dots\log n\}$, $n = n_0, n_0+1, \dots,$ have

$$G(x_n) = \{c_\alpha(x); \alpha \in [0,1]\} \cup \{h_\alpha(x); \alpha \in [0,1]\}$$

and thus are u.m.

*Proof.* For the first iterated logarithm we chose an index-sequence $N_k$, with $N_k = [\exp\exp(k+\alpha)]$. Then we have $\lim_{k\to\infty} F_{N_k}(x) = c_\alpha(x)$. For $N_k = [\exp\exp(k+\varepsilon_k)]$, where $\varepsilon_k \to 0$ such that $(\exp\exp(k+\varepsilon_k))/(\exp\exp k) \to \beta$, we have $\lim_{k\to\infty} F_{N_k}(x) = h_\alpha(x)$, where $\alpha = (\beta-1)/\beta$.

On the other hand, let $\lim_{n\to\infty} F_{N_n}(x) = g(x)$. Then $N_n = \exp\exp(k_n + \varepsilon_n)$, where $k_n = [\log\log N_n]$, $\varepsilon_n = \{\log\log N_n\}$, and the sequence $\varepsilon_n$ cannot have different limit points.

The same distribution function are of course obtained if we replace $\log\log t$ by $\log\dots\log t$ and $\exp\exp t$ by $\exp\dots\exp t$ in the above limits. □

By Example 24: Let $x_n$, $n = 1, 2, \dots$ be defined as

$$x_n = \left\{ 1 + (-1)^{\left[\sqrt{[\sqrt{\log_2 n}]}\right]} \left\{ \sqrt{\left[\sqrt{\log_2 n}\right]} \right\} \right\},$$

were $[x]$ denotes the integral part and $\{x\}$ the fractional part of $x$. Then $G(x_n) = \{c_\alpha(x); \alpha \in [0,1]\}$.

From [183, Satz 9] follows the fact that any everywhere dense sequence in $[0,1)$ can be rearranged to a sequence $x_n$ for which $G(x_n)$ is the set of all distribution functions $g : [0,1] \to [0,1]$. Application of Theorem 183 to $x_n$ gives immediately that this $x_n$ is u.m.

**Theorem 185.** *Suppose the continuous function $f(x)$ which map $[0,1]$ onto itself has bounded derivative $|f'| \le c$. Let $x_n \in [0,1)$ be u.m.s. Then so is $f(x_n)$, $n = 1, 2, \dots$.*

*Proof.* This follows from criterion in Theorem 184, since it can be easily verified that

$$\frac{1}{N^2} \sum_{m,n=1}^{N} |f(x_m) - f(x_n)| \le \frac{c}{N^2} \sum_{m,n=1}^{N} |x_m - x_n|,$$

and

$$\left| \frac{1}{N} \sum_{n=1}^{N} f(x_n) - f\left( \frac{1}{N} \sum_{n=1}^{N} x_n \right) \right| \le \frac{c}{N^2} \sum_{m,n=1}^{N} |x_m - x_n|.$$

□

**Notes 39.** Consider a given sequence $x_n \in [0,1)$, $n = 1, 2, \dots$. For every continuous function $f : [0,1] \to \mathbb{R}$ the set of limit points of the sequence $\frac{1}{N} \sum_{n=1}^{N} f(x_n)$, $N = 1, 2, \dots$ coincides with the set

$$\left\{ \int_0^1 f(x) \mathrm{d}g(x); g \in G(x_n) \right\}.$$

Moreover, this set constitutes a subinterval of $[\min f, \max f]$. If for d.f. $g(x)$ we have $\max f = \int_0^1 f(x)\mathrm{d}g(x)$, then $g(x)$ has a special structure e.g. $\int_{[0,1]\setminus f^{-1}(\max f)} 1.\mathrm{d}g(x) = 0$. Thus we can find information about $G(x_n)$ whenever

$$\max f \in \left\{ \int_0^1 f(x) \mathrm{d}g(x); g \in G(x_n) \right\}.$$

Similarly for $\min f$. Using this we can rewrite Theorem 182 in the form

$$\{c_\alpha(x); \alpha \in [0,1]\} \subset G(x_n) \Longleftrightarrow \left\{ \int_0^1 f(x) \mathrm{d}g(x); g \in G(x_n) \right\} = [\min f, \max f]$$
$$\text{for every continuous } f : [0,1] \to \mathbb{R}.$$

Note that we do not consider all continuous $f$, but only such $f$ with $f^{-1}(\max f)$ consisting of only one point. This suggests the following result.

**Theorem 186.** *The sequence* $x_n \in [0,1)$ *is u.m. if and only if*

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f_\alpha(x_n) = 1$$

*for every* $\alpha \in (0,1)$. *Here* $f_\alpha(x)$ *are triangular functions defined by*

$$f_\alpha(x) = \begin{cases} x/\alpha, & \text{if } x \in [0,\alpha), \\ (1-x)/(1-\alpha), & \text{if } x \in [\alpha,1]. \end{cases}$$

### 6.6.1 Applications of u.m. sequences

A quasi-Monte Carlo method for the approximate evaluation of the extreme values of a function was proposed by H. Niederreiter [128]. He shows that the error between the approximate value $\max_{1\le n\le N} f(x_n)$ and the correct value $\max_{x\in[0,1]} f(x)$ can be estimated in terms of *dispersion* of $x_1, x_2, \ldots, x_N$ and the *modulus of continuity* of $f$ (cf. [119, Theorem 1]). The dispersion of a finite sequence $x_1, x_2, \ldots, x_N$ is defined as

$$d_N = \max_{x\in[0,1]} \min_{1\le n\le N} |x - x_n|;$$

it is a measure of the denseness of a sequence.

The dispersion is related to the well-known *discrepancy* of $x_1, x_2, \ldots, x_N$, which is a measure of the *uniform distribution* of a sequence (cf.[119, Theorem 3]). An analogous relation between u.m.s. and denseness of a sequence can be derived on the basis of Theorem 184 and the following bound of dispersion (see [160]).

**Theorem 187.** *For a given finite sequence* $x_1, x_2, \ldots, x_N$ *in* $(0,1)$ *define the numbers* $A_N$ *and* $B_N$ *by*

$$A_N = \min_{1\le M\le N} \frac{1}{M}\sum_{n=1}^{M} x_n, \ \ B_N = \max_{1\le M\le N} \frac{1}{M}\sum_{n=1}^{M} x_n.$$

*Assume that these* min *and* max *are attained in* $M_1$ *and* $M_2$, *respectively, and denote* $C_N$ *by*

$$C_N = \max_{\min(M_1,M_2)\le M\le \max(M_1,M_2)} \frac{1}{M^2}\sum_{m,n=1}^{M} |x_m - x_n|.$$

*Then,*

$$d_N \leq \max(A_N, 1-B_N, 2C_N)$$

*and the limit* $\lim_{N\to\infty} \max(A_N, 1-B_N, 2C_N) = 0$ *characterize u.m.s.*

*Proof.* Let $x_i < x_j$ be neighbouring points from $x_1, x_2, \dots, x_N$ (i.e. $(x_i, x_j) \cap \{x_1, x_2, \dots, x_N\} = \emptyset$) and consider the mean $(x_i + x_j)/2$. We distinguish two cases.

$1^o$ We either have $(x_i + x_j)/2 \in [0, A_N] \cup [B_N, 1]$ and then $x_j - x_i \leq 2\max(A_N, 1 - B_N)$, or

$2^o$ $(x_i+x_j)/2 \in (A_N, B_N)$. If we assume that $M_1 < M_2$ (the case $M_2 < M_1$ is completely similar), then there is an integer $M'$ such that $M', M'+1 \in [M_1, M_2]$ and

$$\frac{1}{M'}\sum_{n=1}^{M'} x_n \leq \frac{x_i + x_j}{2} \leq \frac{1}{M'+1}\sum_{n=1}^{M'+1} x_n.$$

There are four possibilities for the situation of $x_i, x_j$, $\frac{1}{M'}\sum_{n=1}^{M'} x_n (= A)$, and $\frac{1}{M'+1}\sum_{n=1}^{M'+1} x_n (= B)$,

(i) either $x_i, x_j \in [A, B]$, or

(ii) $A \in (x_i, x_j)$ and $B \notin (x_i, x_j)$, or

(iii) $B \in (x_i, x_j)$ and $A \notin (x_i, x_j)$, or

(iv) $A, B \in (x_i, x_j)$.

There is a simple relationship between the configurations (i)–(iv) and the bound of $x_j - x_i$ which we shall establish by using the following properties:

For any finite sequence $x_1, x_2, \dots, x_M$, we have

$$\left|\frac{1}{M}\sum_{n=1}^{M} x_n - \frac{1}{M+1}\sum_{n=1}^{M+1} x_n\right| \leq \frac{2}{(M+1)^2}\sum_{m,n=1}^{M+1} |x_m - x_n|. \tag{496}$$

Assuming $\frac{1}{M}\sum_{n=1}^{M} x_n \in (u, v) \subset [0,1]$ and $(u,v) \cap \{x_1, x_2, \dots, x_M\} = \emptyset$, we have

$$\min\left(\frac{1}{M}\sum_{n=1}^{M} x_n - u,\ v - \frac{1}{M}\sum_{n=1}^{M} x_n\right) \leq \frac{1}{M^2}\sum_{m,n=1}^{M} |x_m - x_n|. \tag{497}$$

It follows from the above inequalities that $x_j - x_i \leq 2C_N$ for (i), and $x_j - x_i \leq 4C_N$ for (ii),(iii) and (iv).

Since (496) is evident, we shall begin with proving (497).

Integration by parts shows that every continuous $f$ with piecewise continuous derivative $f'$

$$\left|\frac{1}{M}\sum_{n=1}^{M} f(x_n) - \int_0^1 f(x)\mathrm{d}c_\alpha(x)\right| = \left|\int_0^1 (F_M(x) - c_\alpha(x))f'(x)\mathrm{d}x\right|. \quad (498)$$

Now let us take as $f$ the following function

$$f(x) = \begin{cases} 0, & \text{if } x \in [0,u), \\ x - u, & \text{if } x \in [u,\alpha), \\ \frac{\alpha-u}{v-\alpha}(v-x), & \text{if } x \in [\alpha,v), \\ 0, & \text{if } x \in [v,1], \end{cases}$$

where $\alpha \in (u,v)$ and $\alpha - u = \min(\alpha - u, v - \alpha)$. Then, after the application of Cauchy inequality, we obtain

$$\min(\alpha - u, v - \alpha) \leq 2\int_0^1 (F_M(x) - c_\alpha(x))^2 \mathrm{d}x.$$

By an easy computation, we evaluate

$$\int_0^1 (F_M(x) - c_\alpha(x))^2 \mathrm{d}x = \frac{1}{M}\sum_{n=1}^{M} |x_n - \alpha| - \frac{1}{2M^2}\sum_{m,n=1}^{M} |x_m - x_n|. \quad (499)$$

Setting $\alpha = \frac{1}{M}\sum_{n=1}^{M} x_n$ we thus conclude (497). □

### 6.6.2 The multidimensional u.m.

The following Theorems are from P.J. Grabner, O. Strauch and R.F. Tichy [63]:

**Theorem 188.** *Let* $M_i, i = 1, 2, \ldots$ *be a sequence of positive numbers satisfying* $\lim_{k\to\infty}\sum_{i=1}^{k-1} M_i/M_k = 0$. *For a given sequence* $\mathbf{y}_k, k = 1, 2, \ldots$ *in* $[0,1]^s$, *let the sequence* $\mathbf{x}_n, n = 1, 2, \ldots$ *in* $[0,1]^s$ *be constructed by* $\mathbf{x}_n = \mathbf{y}_k$

*for* $\sum_{i=1}^{k-1} M_i \le n < \sum_{i=1}^{k} M_i$. *Finally, let* $\mathbf{H} \subset [0,1]^s \times [0,1]^s$ *denote the set of all limit points of the sequence* $(\mathbf{y}_{k-1}, \mathbf{y}_k), k = 2, 3, \dots$. *Then*

$$G(\mathbf{x}_n) = \{tc_{\boldsymbol{\alpha}}(\mathbf{x}) + (1-t)c_{\boldsymbol{\beta}}(\mathbf{x}) : t \in [0,1], (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathbf{H}\}$$

*where*

$$c_{\boldsymbol{\alpha}}(\mathbf{x}) = \begin{cases} 1, & \text{for } \mathbf{x} \in [\boldsymbol{\alpha}, \mathbf{1}], \\ 0, & \text{otherwise.} \end{cases}$$

*Proof.* For $N = \sum_{i=1}^{k-1} M_i + \theta_k M_k$, $0 \le \theta_k < 1$, we have

$$A([\mathbf{0}, \mathbf{x}); N; \mathbf{x}_n) = B([\mathbf{0}, \mathbf{x}), (\mathbf{y}_{k-1}, \mathbf{y}_k)) + o(N),$$

where

$$B([\mathbf{0}, \mathbf{x}), (\mathbf{y}_{k-1}, \mathbf{y}_k)) = \begin{cases} M_{k-1}, \text{ if } \mathbf{y}_{k-1} \in [\mathbf{0}, \mathbf{x}) \text{ and } \mathbf{y}_k \notin [\mathbf{0}, \mathbf{x}), \\ \theta_k M_k, \text{ if } \mathbf{y}_{k-1} \notin [\mathbf{0}, \mathbf{x}) \text{ and } \mathbf{y}_k \in [\mathbf{0}, \mathbf{x}), \\ M_{k-1} + \theta_k M_k, \text{ if } \mathbf{y}_{k-1}, \mathbf{y}_k \in [\mathbf{0}, \mathbf{x}) \text{ and} \\ 0, \text{ if } \mathbf{y}_{k-1}, \mathbf{y}_k \notin [\mathbf{0}, \mathbf{x}). \end{cases}$$

Assume that $A([\mathbf{0}, \mathbf{x}), \omega_N)/N \to g(\mathbf{x})$, $\mathbf{x} \in [0,1]^s$ for selected sequences of indices $N = \sum_{i=1}^{k-1} M_i + \theta_k M_k$, $k = k(N)$. Then we can further chose $N$ such that $(\mathbf{y}_{k-1}, \mathbf{y}_k) \to (\boldsymbol{\alpha}, \boldsymbol{\beta})$, $M_{k-1}/(M_{k-1} + \theta_k M_k) \to t$, and $\theta_k M_k/(M_{k-1} + \theta_k M_k) \to (t-1)$, for some $\boldsymbol{\alpha}, \boldsymbol{\beta}$, and $t$. Thus $g(\mathbf{x}) = tc_{\boldsymbol{\alpha}}(\mathbf{x}) + (1-t)c_{\boldsymbol{\beta}}(\mathbf{x})$.

On the other hand we can construct a sequence $N_k = \sum_{i=1}^{k-1} M_i + \theta_k M_k$ satisfying $M_{k-1}/(M_{k-1} + \theta_k M_k) \to t$ for any $t \in [0,1]$ provided that $(\mathbf{y}_{k-1}, \mathbf{y}_k) \to (\boldsymbol{\alpha}, \boldsymbol{\beta})$. Then $A([\mathbf{0}, \mathbf{x}), \omega_{N_k})/N_k \to tc_{\boldsymbol{\alpha}}(\mathbf{x}) + (1-t)c_{\boldsymbol{\beta}}(\mathbf{x})$. □

This theorem has the following consequences:

**Theorem 189.** *For a given* $\boldsymbol{H} \subset [0,1]^s$ *suppose that there exist a sequence* $\mathbf{y}_k$, $k = 1, 2, \dots$ *in* $[0,1]^s$ *such that*

(i) $\mathbf{H}$ *coincides with the set of limit points of* $\sigma$,

(ii) $\lim_{k\to\infty} \mathbf{y}_k - \mathbf{y}_{k-1} = \mathbf{0}$.

*Then there exists a sequence* $\mathbf{x}_n$, $n = 1, 2, \dots$ *in* $[0,1]^s$

$$G(\mathbf{x}_n) = \{c_{\boldsymbol{\alpha}}(\mathbf{x}) : \boldsymbol{\alpha} \in \mathbf{H}\}.$$

*Proof.* According to (i) and (ii), we have $\mathbf{H}_\sigma = \{(\boldsymbol{\alpha}, \boldsymbol{\alpha}) : \boldsymbol{\alpha} \in \mathbf{H}\}$. Hence, Theorem 189 is a consequence of Theorem 188. □

**Theorem 190.** *Let* $\mathbf{H} \subset [0,1]^s$ *be such that there exists a continuous function* $\boldsymbol{\phi}; [0,1] \to [0,1]^s$, *for which* $\mathbf{H} = \{\boldsymbol{\phi}(t); t \in [0,1]\}$. *Then there exists a sequence* $\mathbf{x}_n$, $n = 1, 2, \dots$ *in* $[0,1]^s$ *such that* $G(\mathbf{x}_n) = \{c_{\boldsymbol{\alpha}}(\mathbf{x}) : \boldsymbol{\alpha} \in \mathbf{H}\}$.

*Proof.* For the proof, we note that (i) and (ii) from Theorem 189 hold for $\mathbf{y}_k = \boldsymbol{\phi}(y_k)$, $k = 1, 2, \dots$, where

$$y_k = \left\{(-1)^{[\sqrt{k}]}\sqrt{k}\right\}.$$

Here $[x]$ denotes the integral part and $\{x\}$ the fractional part of $x$. The density of $y_k$, $k = 1, 2, \dots$ and $\lim_{k\to\infty} y_k - y_{k-1} = 0$ are proved (in a more general form) in Theorem 191, below. □

**Theorem 191.** [63], [171, 3.13.1., p. 3–49]. *Let* $p_1, \dots, p_s$ *be mutually co-prime positive integers and* $j > 1$. *Then the set of all d.f.s of the* $s$*–dimensional sequence*

$$\mathbf{x}_n = \left((-1)^{[[\log^{(j)} n]^{1/p_1}]}[\log^{(j)} n]^{1/p_1}, \dots, (-1)^{[[\log^{(j)} n]^{1/p_s}]}[\log^{(j)} n]^{1/p_s}\right) \bmod 1 \tag{500}$$

*is*

$$G(\mathbf{x}_n) = \{c_{\boldsymbol{\alpha}}(\mathbf{x}); \boldsymbol{\alpha} \in [0,1]^s\}, \tag{501}$$

*Proof.* The line of construction of the sequence (500) is the same as in Theorem 188. Precisely, for $\exp^{(j)} k < n < \exp^{(j)}(k+1)$, ($\exp^{(j)} k = \exp \dots \exp k$) we have $\mathbf{x}_n = \mathbf{y}_k$, where

$$\mathbf{y}_k = \left((-1)^{[k^{1/p_1}]}k^{1/p_1}, \dots, (-1)^{[k^{1/p_s}]}k^{1/p_s}\right) \mod 1. \tag{502}$$

Thus, in order to show (501) it suffices to prove that

(i) $\mathbf{y}_k$, $k = 1, 2, \dots$ is dense in $[0,1]^s$, and

(ii) $\lim_{k\to\infty} \mathbf{y}_k - \mathbf{y}_{k-1} = \mathbf{0}$.

Condition (i) follows from Propositions 3,4 and Remark 2 in [63], see:

**Notes 40.**

**Definition 11.** Let $m_1, \dots, m_s$ be integers $\geq 2$. A sequence $(\mathbf{x}_n) = (x_{n1}, \dots, x_{ns})$ in $\mathbb{R}^s$ is called $(m_1, \dots, m_s)$-uniformly distributed, if the sequence $(x_{n1}, \dots, x_{ns}, [x_{n1}] \mod m_1, \dots, [x_{ns}] \mod m_s)$ is uniformly distributed in $\mathbb{R}^s/\mathbb{Z}^s \times \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_s}$.

**Theorem 192** ([63], Proposition 3)**.** *A sequence $\mathbf{x}_n$ in $\mathbb{R}^s$ is $(m_1, \dots, m_s)$-uniformly distributed, if and only if $(x_{n1}/m_1, \dots, x_{ns}/m_s)$ is uniformly distributed modulo* 1.

**Theorem 193** ([63], Corollary 3)**.** *Let $1, \alpha_1, \dots, \alpha_s$ be linearly independent over the rationals and set $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_s)$. Then the sequence $(n\boldsymbol{\alpha})$ is $(m_1, \dots, m_s)$-uniformly distributed for any choice of $(m_1, \dots, m_s)$.*

**Theorem 194** ([63], Remark 2)**.** *If a sequence $(x_{n1}, \dots, x_{ns})$ is $(2, \dots, 2)$-uniformly distributed, then $((-1)^{[x_{n1}]}x_{n1}, \dots, (-1)^{[x_{ns}]}x_{ns})$ is u.d.* mod1.

**Theorem 195** ([63], Proposition 4)**.** *Let $a_i, b_i, c_i$ be real numbers with $a_i, b_i \neq 0$ $(i = 1, \dots, s)$ and $0 < u_i < 1$ and let $v_i$ be given such that $0 < u_1v_1 < u_2v_2 < \cdots < u_sv_s$ and $u_iv_i \notin \mathbb{Z}$ for all $i = 1, \dots, s$. Set*

$$x_{ni} = (a_i[b_in^{v_i}] + c_i)^{u_i} \,,$$

*and $\mathbf{x}_n = (x_{n1}, \dots, x_{ns})$. Then the $s$-dimensional sequence $\mathbf{x}_n$, $n = 1, 2, \dots$ is u.d.* mod 1.

Condition (ii) follows from the expression

$$y_{ki} = \begin{cases} \left\{k^{1/p_i}\right\} \text{ for } k \in \cup_{n=0}^{\infty}\left[(2n)^{p_i}, (2n+1)^{p_i}\right) \\ 1 - \left\{k^{1/p_i}\right\} \text{ for } k \in \cup_{n=0}^{\infty}\left[(2n+1)^{p_i}, (2n+2)^{p_i}\right). \end{cases}$$

$y_{ki}$, for $k \in \left[(2n)^{p_i}, (2n+1)^{p_i}\right)$, increases from 0 to $\left((2n+1)^{p_i} - 1\right)^{1/p_i} - 2n$ $(\to 1$ as $n \to \infty)$ with differences $y_{ki} - y_{(k-1)i} = k^{1/p_i} - (k-1)^{1/p_i}$ $(\to 0$ as $k \to \infty)$, and, for $k \in \left[(2n+1)^{p_i}, (2n+2)^{p_i}\right)$, $y_{ki}$ decreases from 1 to $1 - \left(\left((2n+2)^{p_i} - 1\right)^{1/p_i} - (2n+1)\right)$ $(\to 0$ as $n \to \infty)$ with differences $y_{ki} - y_{(k-1)i} = (k-1)^{1/p_i} - k^{1/p_i}$ $(\to 0$ as $k \to \infty)$. Thus $\lim_{k\to\infty} y_{ki} - y_{(k-1)i} = 0$. □

**Theorem 196.** [63], [171, 3.13.2., p. 3–49]. *Let $p_1, \dots, p_s$ be mutually coprime positive integers and $j > 1$. Then the set of all d.f.s of the $s$–dimensional sequence*

$$\mathbf{x}_n = \left([\log^{(j)} n]^{1/p_1}, \dots, [\log^{(j)} n]^{1/p_s}\right) \bmod 1.$$

*is*

$$G(\mathbf{x}_n) = \left\{tc_{\boldsymbol{\alpha}}(\mathbf{x}) + (1-t)c_{\boldsymbol{\beta}}(\mathbf{x})\right\},$$

*where $t \in [0,1]$, $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_s), \boldsymbol{\beta} = (\beta_1, \dots, \beta_s) \in [0,1]^s$, and if $\alpha_i \neq \beta_i$ then $\alpha_i = 1, \beta_i = 0$ for $i = 1, \dots, s$.*

*Proof.* As in Theorem 188, for $\exp^{(j)} k < n < \exp^{(j)}(k+1)$, we have $\mathbf{x}_n = \mathbf{y}_k$, where

$$\mathbf{y}_k = \left(k^{1/p_1}, \dots, k^{1/p_s}\right) \mod 1.$$

First we show that the sequence $(\mathbf{y}_{k-1}, \mathbf{y}_k), k = 2, 3, \dots$ has two types of limit points:

(i) $(\boldsymbol{\alpha}, \boldsymbol{\alpha})$, where $\boldsymbol{\alpha} \in [0,1]^s$ is arbitrary, and

(ii) $(\boldsymbol{\alpha}, \boldsymbol{\beta})$, where $\boldsymbol{\alpha} \neq \boldsymbol{\beta} \in [0,1]^s$ and $\alpha_i \neq \beta_i \Rightarrow \alpha_i = 1, \beta_i = 0$ for $i = 1, \dots, s$.

Proof:

$1^o$. Let us assume $k \neq n^{p_i}$, for $n = 1, 2, \dots$ and $i = 1, 2, \dots, s$. Then there exist positive integers $n_1, \dots, n_s$ such that $k \in \left(n_1^{p_1}, (n_1+1)^{p_1}\right) \cap \dots \cap \left(n_s^{p_s}, (n_s+1)^{p_s}\right)$, and so,

$$y_{k_i} - y_{(k-1)_i} = \left(n_i^{p_i} + j\right)^{1/p_i} - \left(n_i^{p_i} + j - 1\right)^{1/p_i} \to 0,$$

as $k \to \infty$. By applying [92, Th. 3.5] to $\mathbf{y}_k$ one shows that $\mathbf{y}_k, k = 1, 2, \dots$ is u.d. in $[0,1]^s$. But the $k$ with $k = n^{p_i}$ have zero density, and so we derive that the sequence $(\mathbf{y}_{k-1}, \mathbf{y}_k), k = 2, 3, \dots,\ k \neq n^{p_i}$, for $n = 1, 2, \dots$ and $i = 1, 2, \dots, s$ has limit points of type (i).

$2^o$. Now we take $k = n_i^{p_i}$, for $i \in I$, and assume that $k \neq n^{p_j}$, for $n = 1, 2, \dots$ and $j \in \{1, 2, \dots, s\} \setminus I$. Then $y_{k_i} = 0$ and

$$y_{(k-1)_i} = \left(n_i^{p_i} - 1\right)^{1/p_i} - \left(n_i - 1\right) \to 1.$$

Put $\prod_{i \in I} p_i = A$ and $\{1, 2, \dots, s\} \backslash I = \{j_1 \dots, j_l\}$. Because of the assumption $A/p_{j_1}, \dots, A/p_{j_l}$ are pairwise different and nonintegers. Thus, using [92, Th. 3.5], the sequence $\left((n^A)^{1/p_{j_1}}, \dots, (n^A)^{1/p_{j_l}}\right), n = 1, 2, \dots$ is u.d. $\mod 1$. This property remains, if we restrict $n$ to be $\neq m^{p_{j_i}}$, for $m = 1, 2, \dots$ and $i = 1, \dots, l$. This shows that the sequence $(\mathbf{y}_{k-1}, \mathbf{y}_k), k = 2, 3, \dots,$ where $k = n^A$, and $n \neq m^{p_j}$ for $m = 1, 2, \dots$ and $j \in \{1, 2, \dots, s\} \setminus I$, has a limit point of type (ii) with coordinates $\alpha_i = 1$ and $\beta_i = 0$ for $i \in I$, and $\alpha_j = \beta_j$ for $j \in \{1, 2, \dots, s\} \setminus I$ for arbitrary $\alpha_j \in [0,1]$. Finally, we apply Theorem 188, and the proof is complete. $\square$

**Example 79.** [171, 3.13.3., p. 3–50]. Let $1, \alpha_1, \dots, \alpha_s$ be linearly independent over the rationals. Then the set $G(\mathbf{x}_n)$ of all d.f.s of the sequence

$$\mathbf{x}_n = (\alpha_1 \log\log n, \dots, \alpha_s \log\log n) \bmod 1$$

satisfies

$$G(\mathbf{x}_n) \supset \left\{c_{\boldsymbol{\alpha}}(\mathbf{x}); \boldsymbol{\alpha} \in [0,1]^s\right\}.$$

In other words, the sequence is u.m. This example was given by G. Myerson [109].

**Notes 41.** G. Myerson [109] notes that if the sequence $x_n$, $n = 1, 2, \dots$, in $[0,1)$ is u.m. then the $s$ dimensional sequence

$$\mathbf{x}_n = (x_{n+1}, x_{n+2}, \dots, x_{n+s})$$

does not u.m. but for every interval $\emptyset \neq \mathbf{I} \subset [0,1]^s$ we have

$$\limsup_{N\to\infty} \frac{A(\mathbf{I}; N; \mathbf{x}_n)}{N} \geq \frac{1}{s}.$$

# 7 The multi-dimensional d.f.s

## 7.1 The two-dimensional d.f.s.: basic results

In the multi–dimensional case we can proceed in a manner similar to the one–dimensional one. A basic notion is a differential:

$$\begin{aligned}
&\mathrm{d}_x\mathrm{d}_y F(x,y) \\
&= F(x+\mathrm{d}x, y+\mathrm{d}y) + F(x,y) - F(x+\mathrm{d}x, y) - F(x, y+\mathrm{d}y), \qquad (503)\\
&\mathrm{d}_x\mathrm{d}_y\mathrm{d}_z F(x,y,z) \\
&= F(x+\mathrm{d}x, y+\mathrm{d}y, z+\mathrm{d}z) - F(x,y,z) \\
&+ F(x+\mathrm{d}x, y, z) + F(x, y+\mathrm{d}y, z) + F(x, y, z+\mathrm{d}z) \\
&- F(x+\mathrm{d}x, y+\mathrm{d}y, z) - F(x, y+\mathrm{d}y, z+\mathrm{d}z) - F(x+\mathrm{d}x, y, z+\mathrm{d}z) \qquad (504)
\end{aligned}$$

Note that

$$\mathrm{d}_x\mathrm{d}_y F(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y}\mathrm{d}x\mathrm{d}y, \quad \mathrm{d}_x\mathrm{d}_y\mathrm{d}_z F(x,y,z) = \frac{\partial^3 F(x,y)}{\partial x \partial y \partial z}\mathrm{d}x\mathrm{d}y\mathrm{d}z,$$

if the needs partial derivatives exist.

Now we starting with $s = 2$.

Let $(x_n, y_n)$, $n = 1, 2, \dots$, be two-dimensional sequence in the unit square $[0,1)^2$. Define a step d.f. $F_N(x,y)$ of $(x_1, y_1), \dots, (x_N, y_N)$ by

$$F_N(x,y) = \frac{1}{N}\#\{n \leq N; (x_n, y_n) \in [0,x) \times [0,y)\}$$

and put $F_N(1,1) = 1$. We have

(i) $F_N(x+\Delta_x, y+\Delta_y) + F_N(x,y) - F_N(x+\Delta_x, y) - F_N(x, y+\Delta_y)$

$= \frac{1}{N}\#\{n \le N; (x_n, y_n) \in [x, x+\Delta_x) \times [y, y+\Delta_y)\} \ge 0;$

for $[x, x+\Delta_x) \times [y, y+\Delta_y) \subset [0,1)^2$.

(ii) $F_N(0,0) = F_N(x,0) = F_N(0,y) = 0$ for $x, y \in [0,1]$.

Since a two-dimensional d.f. $g(x,y)$ is a limit of $F_N(x,y)$ we have the following definition:

**Definition 12.** A two-dimensional function $g : [0,1]^2 \to [0,1]$ is d.f. if

(i) $\mathrm{d}g(x,y) = g(x+\mathrm{d}x, y+\mathrm{d}y) + g(x,y) - g(x+\mathrm{d}x, y) - g(x, y+\mathrm{d}y) \ge 0$, for $(x,y) \in (0,1)^2$ and $\mathrm{d}x, \mathrm{d}y \ge 0$.

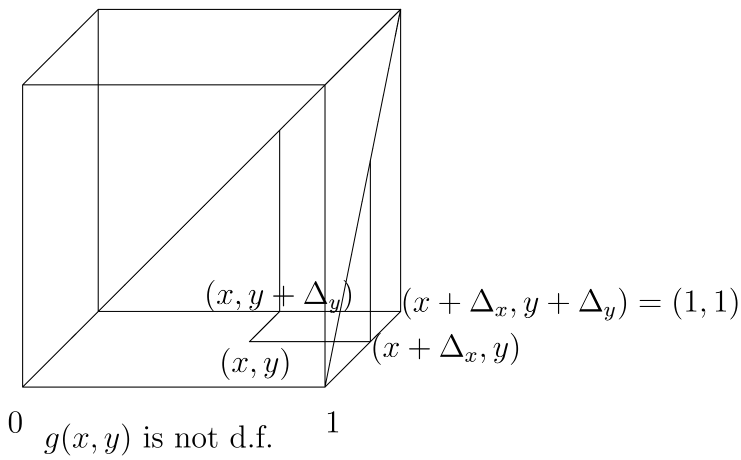
(ii) $g(0,0) = g(x,0) = g(0,y) = 0$ for $x, y \in [0,1]$.

From (i) and (ii) directly follows that $g(x,y)$ is non-decreasing for every variable $x$ and $y$.

**Example 80.** Define $g(x,y) = 0$ for $(x,y) \in [0,1)^2$ and $g(x,1) = x$ and $g(1,y) = y$. Then $g(x,y)$ is not d.f. since the differential

$$g(x+\Delta_x, y+\Delta_y) + g(x,y) - g(x, y+\Delta_y) - g(x+\Delta_x, y) = 1 - x - y < 0$$

for $x, y, \Delta_x, \Delta_y$ given by Figure:




$g(x,y)$ is not d.f.

Other examples can be found in R.G. Nelsen [114, p. 6] (1999), c.f.

**Example 81.** For $g(x,y)=\max(x,y)$ we have $\mathrm{d}_x\mathrm{d}_y g(x,y)=-\mathrm{d}x$ for $x=y$, $\mathrm{d}x=\mathrm{d}y>0$, thus by (i) it is not d.f.

**Example 82.** For $g(x,y)=(2x-1)(2y-1)$ we have $\mathrm{d}_x\mathrm{d}_y g(x,y)=4\mathrm{d}x\mathrm{d}y$ for $x,y\in(0,1)$, but (ii) not holds and thus it is not d.f.

Y. Ohkubo [122] generalized Theorem 10 to the following two-dimensional form. [53]

**Theorem 197.** *Let $(x_n,y_n)$ and $(x_n',y_n')$, $n=1,2,\dots$ be two-dimensional sequences. Assume:*

(i) $||x_n-x_n'||\to 0$ *and* $||y_n-y_n'||\to 0$.

(ii) *Every d.f.* $g(x,y)\in G((x_n,y_n))$ *is continuous in* $(0,0)$,$(0,1)$, $(1,0)$ *and* $(1,1)$.

*Then* $G((x_n,y_n))=G((x_n',y_n'))$.

*Proof.* For $g_1(x,y)\in G((x_n,y_n))$ there exists $N_k$, $k=1,2,\dots$ and $g_2(x,y)\in G((x_n',y_n'))$ such that $F_{N_k}(x,y)\to g_1(x,y)$ and $F'_{N_k}(x,y)\to g_2(x,y)$, where $F_N(x,y)$ and $F'_N(x,y)$ are step function defined for $(x_n,y_n)$ and $(x_n',y_n')$, respectively. By Riemann-Stieltjes integration we have

$$\int_0^1\int_0^1 e^{2\pi i(hx+ky)}\mathrm{d}_x\mathrm{d}_y F_{N_k}(x,y)=\frac{1}{N_k}\sum_{n=1}^{N_k}e^{2\pi i(hx_n+ky_n)},$$
$$\int_0^1\int_0^1 e^{2\pi i(hx+ky)}\mathrm{d}_x\mathrm{d}_y F'_{N_k}(x,y)=\frac{1}{N_k}\sum_{n=1}^{N_k}e^{2\pi i(hx_n'+ky_n')},$$

Put
$$A_n=|e^{2\pi i(hx_n+ky_n)}-e^{2\pi i(hx_n'+ky_n')}|$$
for $n=1,2,\dots$. If $A_n\to 0$, then
$$\left|\frac{1}{N_k}\sum_{n=1}^{N_k}e^{2\pi i(hx_n+ky_n)}-\frac{1}{N_k}\sum_{n=1}^{N_k}e^{2\pi i(hx_n'+ky_n')}\right|\to 0,$$
which implies, by Helly theorem,
$$\int_0^1\int_0^1 e^{2\pi i(hx+ky)}\mathrm{d}_x\mathrm{d}_y g_1(x,y)=\int_0^1\int_0^1 e^{2\pi i(hx+ky)}\mathrm{d}_x\mathrm{d}_y g_2(x,y)\tag{505}$$

[53] He also describe multidimensional form.

for every integer $h$ and $k$. The two-dimensional Weierstrass theorem on approximation of continuous functions by trigonometric polynomials follows that (505) implies: For every continuous $F(x,y)$ on $[0,1]^2$ for which $F(0,0) = F(0,1) = F(1,0) = F(1,1)$ we have

$$\int_0^1 \int_0^1 F(x,y)\mathrm{d}g_1(x,y) = \int_0^1 \int_0^1 F(x,y)\mathrm{d}g_2(x,y). \tag{506}$$

Note that by a classical inequality

$$\begin{aligned}
&A_n \le 2\pi|hx_n + ky_n - (hx'_n + ky'_n)|,\\
&A_n \le 2\pi(h|x_n - x'_n| + k|y_n - y'_n|),\\
&A_n \le 2\pi(h\{x_n - x'_n\} + k\{y_n - y'_n\}),\\
&A_n \le 2\pi(h|\{x_n\} - \{x'_n\}| + k|\{y_n\} - \{y'_n\}|),\\
&A_n \le 2\pi(h||x_n - x'_n|| + k||y_n - y'_n||),\\
&A_n \le 2\pi(h|x_n - x'_n - A| + k|y_n - y'_n - B|)
\end{aligned} \tag{507}$$

for $A_n \to 0$ it can be used not only assumption (i) but also arbitrary from (507).

In the following we modify Ohkubo's proof by applying (541). For $i = 1,2$ we have

$$\begin{aligned}
&\int_0^1 \int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g_i(x,y) = F(1,1) - \int_0^1 g_i(1,y)\mathrm{d}_y F(1,y)\\
&- \int_0^1 g_i(x,1)\mathrm{d}_x F(x,1) + \int_0^1 \int_0^1 g_i(x,y)\mathrm{d}_x\mathrm{d}_y F(x,y).
\end{aligned}$$

Applying Theorem 10, the assumption (i) implies that

(j) $g_1(x,1) = g_2(x,1)$ and

(jj) $g_1(1,y) = g_2(1,y)$

in the points $(x,y)$ of continuity. Then (506) implies

$$\int_0^1 \int_0^1 g_1(x,y)\mathrm{d}_x\mathrm{d}_y F(x,y) = \int_0^1 \int_0^1 g_2(x,y)\mathrm{d}_x\mathrm{d}_y F(x,y). \tag{508}$$

Now, we define $F(x,y)$ as

$$F(x,y) = \begin{cases} 1 & \text{if } (x,y) \in [x_1,x_2] \times [y_1,y_2],\\ 0 & \text{otherwise.} \end{cases} \tag{509}$$

Note that the function $F(x,y)$ is not continuous, but its little change gives a continuous function. For $F(x,y)$ defined by (508) we have

$$\big[\mathrm{d}_x\mathrm{d}_y F(x,y)\big]_{(x_1,y_1)} = 1,\ \big[\mathrm{d}_x\mathrm{d}_y F(x,y)\big]_{(x_2,y_2)} = 1,$$
$$\big[\mathrm{d}_x\mathrm{d}_y F(x,y)\big]_{(x_1,y_2)} = -1,\ \big[\mathrm{d}_x\mathrm{d}_y F(x,y)\big]_{(x_2,y_1)} = -1.$$

This implies by (508)

$$\begin{aligned} &g_1(x_1,y_1)+g_1(x_2,y_2)-g_1(x_1,y_2)-g_1(x_1,y_1)\\ &=g_2(x_1,y_1)+g_2(x_2,y_2)-g_2(x_1,y_2)-g_2(x_1,y_1) \end{aligned} \tag{510}$$

for arbitrary $(x_1,y_1)$ and $(x_2,y_2)$ in $[0,1]^2$. Putting $(x_1,y_1)=(x,y)$ and $(x_2,y_1)=(1,1)$ the (510) implies

$$g_1(x,y)=g_2(x,y)$$

for $(x,y)\in[0,1]^2$. □

**Theorem 198.** *Let* $(x_1,y_1)$ *and* $(x_2,y_2)$ *be any points in* $[0,1]^2$ *and* $g(x,y)$ *be a d.f. Then*

$$|g(x_2,y_2)-g(x_1,y_1)|\le|g(x_2,1)-g(x_1,1)|+|g(1,y_2)-g(1,y_1)| \tag{511}$$

*Proof.* R.B. Nelson [114, p.7]: From the triangle inequality
$|g(x_2,y_2)-g(x_1,y_1)|\le|g(x_2,y_2)-g(x_1,y_2)|+|g(x_1,y_2)-g(x_1,y_1)|$.
If $x_1\le x_2$, then $g(x_2,y_2)-g(x_1,y_1)\le g(x_2,1)-g(x_1,1)$. An analogous inequality holds when $x_2\le x_1$. Thus
$|g(x_2,y_2)-g(x_1,y_1)|\le|g(x_2,1)-g(x_1,1)|$.
Similarly for any $y_1,y_2$. □

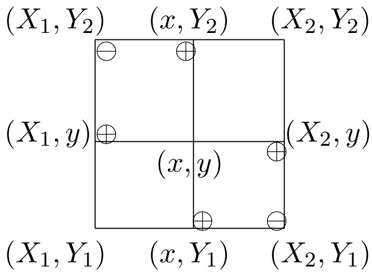
**Theorem 199.** *Every two-dimensional d.f.* $g(x,y)$ *satisfies*

$$g(x,y)\le\min(g(x,Y_2)+g(X_1,y)-g(X_1,Y_2),g(x,Y_1)+g(X_2,y)-g(X_2,Y_1)), \tag{512}$$

$$g(x,y)\ge\max(g(x,Y_2)+g(X_2,y)-g(X_2,Y_2),g(x,Y_1)+g(X_1,y)-g(X_1,Y_1)) \tag{513}$$

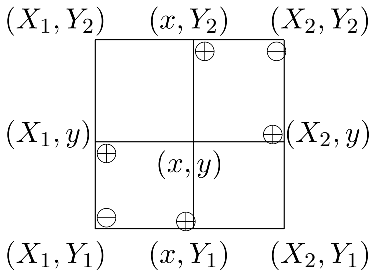
*for every* $(x,y)\in[X_1,X_2]\times[Y_1,Y_2]$.

*Proof.* Schematically, for minimum




Here, if in point $(u, v)$ we have add $\oplus$ or $\ominus$ then in $g(x, y)$ we add $+g(u, v)$ or $-g(u, v)$, respectively.

Schematically for maximum




$\square$

## 7.2 The multidimensional d.f.s

Let $\mathbf{x} = (x_1, \dots, x_s)$, $x_i \in [0, 1)$, $i = 1, \dots, s$ and $\mathbf{x}_n = (x_{n,1}, \dots, x_{n,s}) \in \mathbb{R}^s$, $n = 1, 2, \dots$

• Define the *$s$–dimensional step d.f.* $F_N(\mathbf{x})$ of the sequence $\mathbf{x_n}$ as

(i) $F_N(\mathbf{x}) = \frac{1}{N}\#\{n \le N; \{x_{n,1}\} \in [0, x_1), \dots, \{x_{n,s}\} \in [0, x_s)\}$,

(ii) $F_N(\mathbf{x}) = 0$ for every $\mathbf{x}$ having a vanishing coordinate,

(iii) $F_N(\mathbf{1}) = \mathbf{1}$,

(iv) $F_N(1, \dots, 1, x_{i_1}, 1, \dots, 1, x_{i_2}, 1, \dots, 1, x_{i_l}, 1 \dots, 1) = F_N(x_{i_1}, x_{i_2}, \dots, x_{i_l})$ for every restricted $l$–dimensional face sequence $(x_{n,i_1}, x_{n,i_2}, \dots, x_{n,i_l})$ of $\mathbf{x}_n$ for $l = 1, 2, \dots, s$.

Then

• If $f : [0,1]^s \to \mathbb{R}$ is continuous, again

$$\frac{1}{N}\sum_{n=1}^{N} f(\mathbf{x}_n \bmod 1) = \int_{[0,1]^s} f(\mathbf{x})\mathrm{d}F_N(\mathbf{x}).$$

**Definition 13.** An $s$-dimensional function $g : [0,1]^s \to [0,1]$ is d.f. if

(i) $g(\mathbf{1}) = 1$,

(ii) $g(\mathbf{0}) = 0$, and also $g(\mathbf{x}) = 0$ for every $\mathbf{x}$ with a vanishing coordinate,

(iii) $g(\mathbf{x})$ is non–decreasing, i.e.

$$\Delta(g, J) = \sum_{\varepsilon_1=1}^{2} \cdots \sum_{\varepsilon_s=1}^{2} (-1)^{\varepsilon_1+\cdots+\varepsilon_s} g(x_{\varepsilon_1}^{(1)}, \dots, x_{\varepsilon_s}^{(s)}) \geq 0 \tag{514}$$

for every interval $J = [x_1^{(1)}, x_2^{(1)}] \times [x_1^{(2)}, x_2^{(2)}] \times \cdots \times [x_1^{(s)}, x_2^{(s)}] \subset [0,1]^s$.

(iv) For every $l = 1, 2, \dots, s-1$ the *$l$–dimensional marginal d.f.* $g(1, \dots, 1, x_{i_1}, 1, \dots, 1, x_{i_2}, 1, \dots, 1, x_{i_l}, 1 \dots, 1)$ of $g$ in variables $(x_{i_1}, x_{i_2}, \dots, x_{i_l}) \in (0,1)^l$, is d.f.

• The *differential* $\mathrm{d}g(\mathbf{x})$ of $g(\mathbf{x})$ at the point $\mathbf{x} = (x_1, \dots, x_s)$ is defined by $\mathrm{d}g(\mathbf{x}) = \Delta(g, J)$, where $J = [x_1, x_1 + \mathrm{d}x_1] \times \cdots \times [x_s, x_s + \mathrm{d}x_s]$, and $g(\mathbf{x})$ is non–decreasing if and only if $\mathrm{d}g(\mathbf{x}) \geq 0$ for every $\mathbf{x} \in [0,1)^s$.

**Example 83.** Putting $F(x_1, x_2, \dots, x_s) = \max(x_1, x_2, \dots, x_s)$, $x_1 = \cdots = x_s = x$, then we have

$$\begin{aligned}
\mathrm{d}F(x, \dots, x) &= (-1)^{1+1+\cdots+1}x + \sum_{\substack{\varepsilon_1=1 \\ (\varepsilon_1,\dots,\varepsilon_s)\neq(1,\dots,1)}}^{2} \cdots \sum_{\varepsilon_s=1}^{2} (-1)^{\varepsilon_1+\cdots+\varepsilon_s}(x + \mathrm{d}x) \\
&= \sum_{\varepsilon_1=1}^{2} \cdots \sum_{\varepsilon_s=1}^{2} (-1)^{\varepsilon_1+\cdots+\varepsilon_s}(x + \mathrm{d}x) - (-1)^{1+1+\cdots+1}\mathrm{d}x \\
&= (-1)^{s+1}\mathrm{d}x.
\end{aligned}$$

• If $g$ is an $s$-dimensional d.f. then $\int_{[0,1]^s} \mathrm{d}g(\mathbf{x}) = 1$.
• We shall identify two d.f.s $g(\mathbf{x})$ and $\widetilde{g}(\mathbf{x})$ if:

(i) $g(\mathbf{x}) = \widetilde{g}(\mathbf{x})$ at every common point $\mathbf{x} \in (0,1)^s$ of continuity, and

(ii) $g(1,\dots,1,x_{i_1},1,\dots,1,x_{i_2},1,\dots,1,x_{i_l},1\dots,1) =$
$= \widetilde{g}(1,\dots,1,x_{i_1},1,\dots,1,x_{i_2},1,\dots,1,x_{i_l},1\dots,1)$
at every common point $(x_{i_1},x_{i_2},\dots,x_{i_l}) \in (0,1)^l$ of continuity in every $l$–dimensional marginal d.f. of $g$ and $\widetilde{g}$, $l = 1,2,\dots,s-1$.

• The $s$–dimensional d.f. $g(\mathbf{x})$ is a d.f. of the sequence $\mathbf{x}_n$ mod 1 if

(i) $g(\mathbf{x}) = \lim_{k\to\infty} F_{N_k}(\mathbf{x})$ for all continuity points $\mathbf{x} \in (0,1)^s$ of $g$ (the so–called *weak limit*) and,

(ii) $g(1,\dots,1,x_{i_1},1,\dots,1,x_{i_2},1,\dots,1,x_{i_l},1\dots,1) =$
$= \lim_{k\to\infty} F_{N_k}(x_{i_1},x_{i_2},\dots,x_{i_l})$
weakly over $(0,1)^l$ and every $l$–dimensional marginal sequence of $\mathbf{x}_n$ for $l = 1,2,\dots,s-1$, and for a suitable sequence of indices $N_1 < N_2 < \dots$.

• The Second Helly theorem (see Theorem 2) shows that the weak limit[54] $F_{N_k}(\mathbf{x}) \to g(\mathbf{x})$ implies

$$\int_{[0,1]^s} f(\mathbf{x})\mathrm{d}F_{N_k}(\mathbf{x}) \to \int_{[0,1]^s} f(\mathbf{x})\mathrm{d}g(\mathbf{x})$$

for every continuous $f : [0,1]^s \to \mathbb{R}$.
• $G(\mathbf{x}_n \bmod 1)$ is the set of all d.f.s of $\mathbf{x}_n$ mod 1.

**Theorem 200.** *$G(\mathbf{x}_n \bmod 1)$ is again a non–empty, closed and connected set, and either it is a singleton or it has infinitely many elements.*

Proof can be found in R. Winkler (1997) (cf. [193, p. 1–9]). Note that the connection is in the weak topology on which is metrizable by the metric

$$d(g_1,g_2) = \left(\int_{\mathbf{0}}^{\mathbf{1}} (g_1(\mathbf{x}) - g_2(\mathbf{x}))^2 \mathrm{d}\mathbf{x}\right)^{1/2}.$$

[54] that is (i), and (ii) above are fulfilled

## 7.3 $L^2$ discrepancies

The following discrepancies are known, see [92, Chap. 2], [38, Chap. 1] and [171, p. 1–40, 1.9; p. 1–71, 1.11.4].

For sequences $x_n \in [0,1)$, $n = 1,2,\dots$, $y_n \in [0,1)$, $n = 1,2,\dots$, $z_n \in [0,1)$, $n = 1,2,\dots$, $(x_n, y_n)$, $n = 1,2,\dots$, and $(x_n, y_n, z_n)$, $n = 1,2,\dots$ we denote

$F_N^{(1)}(x)$ - the step d.f. of the sequence $x_n$, $n = 1,2,\dots$;

$F_N^{(2)}(y)$ - the step d.f. of the sequence $y_n$, $n = 1,2,\dots$.

$F_N^{(3)}(z)$ - the step d.f. of the sequence $z_n$, $n = 1,2,\dots$.

$F_N(x,y)$ - the step d.f. of the sequence $(x_n, y_n)$, $n = 1,2,\dots$;

$F_N(x,y,z)$ - the step d.f. of the sequence $(x_n, y_n, z_n)$, $n = 1,2,\dots$;

We use the following $L^2$ discrepancies:

(i) $\int_0^1 (F_N^{(1)}(x) - x)^2 \mathrm{d}x$ - the $L^2$-discrepancy of $x_1, \dots, x_N$;

(ii) $\int_0^1 \int_0^1 (F_N(x,y) - xy)^2 \mathrm{d}x\mathrm{d}y$ - the $L^2$ discrepancy of $(x_1, y_1), \dots, (x_N, y_N)$;

(iii) $\int_0^1 \int_0^1 \int_0^1 (F_N(x,y,z) - xyz)^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z$ - the $L^2$ discrepancy of $(x_1, y_1, z_1), \dots, (x_N, y_N, z_N)$;

(iv) $\int_0^1 \int_0^1 (F_N(x,y) - F_N^{(1)}(x) F_N^{(2)}(y))^2 \mathrm{d}x\mathrm{d}y$ - the $L^2$ discrepancy of statistical independence of $x_1, \dots, x_N$ and $y_1, \dots, y_N$.

(v) $\int_0^1 \int_0^1 \int_0^1 (F_N(x,y,z) - F_N^{(1)}(x) F_N^{(2)}(y) F_N^{(3)}(z))^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z$ - the $L^2$ discrepancy of statistical independence of $x_1, \dots, x_N$, $y_1, \dots, y_N$, and $z_1, \dots, z_N$.

(vi) $\int_0^1 \int_0^1 \int_0^1 (F_N(x,y,z) - F_N(x,y) F_N^{(3)}(z))^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z$ - the $L^2$ discrepancy of statistical independence of $(x_1, y_1) \dots, (x_N, y_N)$ and $z_1, \dots, z_N$.

Every $L^2$ discrepancy can be expressed as arithmetic mean of relevant functions, see [158]. In the cases (i)–(iv) we use

$$F_0(x,y) = \frac{1}{3} - \frac{1-x^2}{2} - \frac{1-y^2}{2} + (1 - \max(x,y)); \tag{515}$$

$$F_0((x,y),(u,v)) = \left(\frac{1}{3}\right)^2 - \frac{1-x^2}{2}\frac{1-y^2}{2} - \frac{1-u^2}{2}\frac{1-v^2}{2} + (1 - \max(x,u))(1 - \max(y,v)); \tag{516}$$

$$F_0((x,y,z),(u,v,w)) = \left(\frac{1}{3}\right)^3 - \frac{1-x^2}{2}\frac{1-y^2}{2}\frac{1-z^2}{2} - \frac{1-u^2}{2}\frac{1-v^2}{2}\frac{1-w^2}{2}$$

$$+ (1-\max(x,u))(1-\max(y,v))(1-\max(z,w)); \tag{517}$$

$$F_1((x,y),(u,v),(s,t),(z,w)) = (1-\max(x,u))(1-\max(y,v))$$
$$+ (1-\max(x,s))(1-\max(v,w)) - 2(1-\max(x,s))(1-\max(y,w)). \tag{518}$$

We have

(i) $\int_0^1 (F_N^{(1)}(x)-x)^2 \mathrm{d}x = \frac{1}{N^2}\sum_{m,n=1}^{N} F_0(x_m,x_n) =$

$= \int_0^1\int_0^1 F_0(x,y)\mathrm{d}F_N^{(1)}(x)\mathrm{d}F_N^{(1)}(y);$

(ii) $\int_0^1\int_0^1 (F_N(x,y)-xy)^2\mathrm{d}x\mathrm{d}y = \frac{1}{N^2}\sum_{m,n=1}^{N} F_0((x_m,y_m),(x_n,y_n)) =$

$= \int_0^1\int_0^1\int_0^1\int_0^1 F_0((x,y),(u,v))\mathrm{d}F_N(x,y)\mathrm{d}F_N(u,v);$

(iii) $\int_0^1\int_0^1\int_0^1 (F_N(x,y,z)-xyz)^2\mathrm{d}x\mathrm{d}y\mathrm{d}z =$

$= \frac{1}{N^2}\sum_{m,n=1}^{N} F_0((x_m,y_m,z_m),(x_n,y_n,z_n)) =$

$= \int_0^1\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 F_0((x,y,z),(u,v,w))\mathrm{d}F_N(x,y,z)\mathrm{d}F_N(u,v,w);$

(iv) $\int_0^1\int_0^1 (F_N(x,y)-F_N^{(1)}(x)F_N^{(2)}(y))^2\mathrm{d}x\mathrm{d}y$

$= \frac{1}{N^4}\sum_{m,n,k,l=1}^{N} F_1((x_m,y_m),(x_n,y_n),(x_k,y_k),(x_l,y_l)) =$

$= \int_0^1\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 F_1((x,y),(u,v),(s,t),(z,w))\cdot$

$\cdot\, \mathrm{d}(F_N(x,y)-F_N^{(1)}(x)F_N^{(2)}(y))\mathrm{d}(F_N(u,v)-F_N^{(1)}(u)F_N^{(2)}(v)).$

From (i)–(iv), Helly and Lebesgue theorems imply

(i) $\int_0^1 (g(x)-x)^2\mathrm{d}x = \int_0^1\int_0^1 F_0(x,y)\mathrm{d}g(x)\mathrm{d}g(y);$

(ii) $\int_0^1 \int_0^1 (g(x,y) - xy)^2 \mathrm{d}x\mathrm{d}y =$

$= \int_0^1 \int_0^1 \int_0^1 \int_0^1 F_0((x,y),(u,v)) \mathrm{d}g(x,y)\mathrm{d}g(u,v);$

(iii) $\int_0^1 \int_0^1 \int_0^1 (g(x,y,z) - xyz)^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z =$

$= \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 F_0((x,y,z),(u,v,w)) \mathrm{d}g(x,y,z)\mathrm{d}g(u,v,w);$

(iv) $\int_0^1 \int_0^1 (g(x,y) - g(x,1)g(1,y))^2 \mathrm{d}x\mathrm{d}y$

$= \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 F_1((x,y),(u,v),(s,t),(z,w))\cdot$

$\cdot\, \mathrm{d}(g(x,y) - g(x,1)g(1,y))\mathrm{d}(g(u,v) - g(u,1)g(1,v)),$

for arbitrary d.f.s $g(x)$, $g(x,y)$ and $g(x,y,z)$.

Similarly, we can compute $L^2$ discrepancy of statistical independence of coordinate sequences of $(x_n, y_n, z_n)$, $n = 1, 2, \dots, N$, i.e.,

$$\text{(v)} \int_0^1 \int_0^1 \int_0^1 (F_N(x,y,z) - F_N^{(1)}(x)F_N^{(2)}(y)F_N^{(3)}(z))^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

We apply the method described in (29) and (30) from [158, Th. 3]: Denote the set $X_{m,n,k} \subset [0,1]^3$ as

$$X_{m,n,k} = \{(x,y,z) \in [0,1]^3; (x_m, y_n, z_k) \in [0,x) \times [0,y) \times [0,z)\}$$

and $c_X(x,y,z)$ is the characteristic function of $X \subset [0,1]^3$. Then

$$\int_0^1 \int_0^1 \int_0^1 \left( \frac{1}{N} \sum_{n=1}^N c_{X_{n,n,n}}(x,y,z) - \frac{1}{N^3} \sum_{m,n,k=1}^N c_{X_{m,n,k}}(x,y,z) \right)^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$= \frac{1}{N^2} \sum_{n,n'=1}^N |X_{n,n,n} \cap X_{n',n',n'}| + \frac{1}{N^6} \sum_{m,n,k,m',n',k'=1}^N |X_{m,n,k} \cap X_{m',n',k'}|$$

$$- \frac{2}{N^4} \sum_{m,m',n',k'=1}^N |X_{m,m,m} \cap X_{m',n',k'}|.$$

Because

$$|X_{m,n,k} \cap X_{m',n',k'}| = (1 - \max(x_m, x_{m'}))(1 - \max(y_n, y_{n'}))(1 - \max(z_k, z_{k'}))$$
$$= (1 - \max(x, x'))(1 - \max(y, y'))(1 - \max(z, z'))\mathrm{d}F_N(x, y, z)\mathrm{d}F_N(x', y', z')$$

for $(x, y, z) = (x_m, y_n, z_k)$, $(x', y', z') = (x_{m'}, y_{n'}, z_{k'})$, we have

$$\int_0^1 \int_0^1 \int_0^1 (F_N(x, y, z) - F_N^{(1)}(x) F_N^{(2)}(y) F_N^{(3)}(z))^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z = \tag{519}$$

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 (1 - \max(x, u))(1 - \max(y, v))(1 - \max(z, w)) \cdot \tag{520}$$

$$\cdot\, \mathrm{d}(F_N(x, y, z) - F_N^{(1)}(x) F_N^{(2)}(y) F_N^{(3)}(z)) \mathrm{d}(F_N(u, v, w) - F_N^{(1)}(u) F_N^{(2)}(v) F_N^{(3)}(w)) \tag{521}$$

which gives (v). Proof of (vi) is similar. Note that using, for the left hand side of (521), the Lebesgue theorem of dominant convergence and in the right the Helly second theorem we find that for every d.f. $g(x, y, z)$, $g_1(x)$, $g_2(y)$, $g_3(z)$ we have

$$\int_0^1 \int_0^1 \int_0^1 (g(x, y, z) - g_1(x) g_2(y) g_3(z))^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z =$$
$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 (1 - \max(x, u))(1 - \max(y, v))(1 - \max(z, w)) \cdot$$
$$\cdot\, \mathrm{d}(g(x, y, z) - g_1(x) g_2(y) g_3(z)) \mathrm{d}(g(u, v, w) - g_1(u) g_2(v) g_3(w)),$$

and also similarly

$$\int_0^1 \int_0^1 \int_0^1 (g(x, y, z) - g_1(x, y, z))^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z =$$
$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 (1 - \max(x, u))(1 - \max(y, v))(1 - \max(z, w)) \cdot$$
$$\cdot\, \mathrm{d}(g(x, y, z) - g_1(x, y, z)) \mathrm{d}(g(u, v, w) - g_1(u, v, w)). \tag{522}$$

This gives the following generalization

## 7.4 Multidimensional generalization of $L^2$ discrepancy

For $\mathbf{x} = (x_1, \dots, x_s)$ and $\mathbf{y} = (y_1, \dots, y_s)$ denote

$$(\mathbf{1} - \max(\mathbf{x}, \mathbf{y})) = (1 - \max(x_1, y_1)) \dots (1 - \max(x_s, y_s)) \tag{523}$$

and $\mathbf{0} = (0, \dots, 0)$ and $\mathbf{1} = (1, \dots, 1)$. Using [158] the (522) can be generalized to

$$\int_{\mathbf{0}}^{\mathbf{1}} (g_1(\mathbf{x}) - g_2(\mathbf{x}))^2 \mathrm{d}\mathbf{x} = \int_{\mathbf{0}}^{\mathbf{1}} \int_{\mathbf{0}}^{\mathbf{1}} (\mathbf{1} - \max(\mathbf{x}, \mathbf{y})) \mathrm{d}(g_1(\mathbf{x}) - g_2(\mathbf{x})) \mathrm{d}(g_1(\mathbf{y}) - g_2(\mathbf{y})) \tag{524}$$

directly, for every two d.f.s $g_1(\mathbf{x})$ and $g_2(\mathbf{x})$ defined in $[0,1]^s$. Now, divide the vector $\mathbf{x} = (x_1, \dots, x_s)$ into two face vectors $\mathbf{x}^{(1)} = (x_{i_1}, \dots, x_{i_l})$ and $\mathbf{x}^{(2)} = (x_{j_1}, \dots, x_{j_k})$, $l + k = s$. Similarly divide

the $s$-dimensional sequence $\mathbf{x}_n$, $n = 1, 2, \dots$ in $[0,1)^s$ with step d.f. $F_N(\mathbf{x}) = F_N(\mathbf{x}^{(1)}, \mathbf{x}^{(2)})$ into two face sequences

$l$-dimensional $\mathbf{x}_n^{(1)}$, $n = 1, 2, \dots$, with step d.f. $F_N(\mathbf{x}^{(1)}, \mathbf{1})$, and

$k$-dimensional $\mathbf{x}_n^{(2)}$ $n = 1, 2, \dots$, with step d.f. $F_N(\mathbf{1}, \mathbf{x}^{(2)})$.

Using (524) we see that the $L^2$ discrepancy (with respect to $g(\mathbf{x})$) and the statistical $L^2$ discrepancy have the following similar structures

$$\int_{\mathbf{0}}^{\mathbf{1}} (F_N(\mathbf{x}) - g(\mathbf{x}))^2 \mathrm{d}\mathbf{x} \tag{525}$$

$$= \int_{\mathbf{0}}^{\mathbf{1}} \int_{\mathbf{0}}^{\mathbf{1}} (\mathbf{1} - \max(\mathbf{x}, \mathbf{y})) \mathrm{d}(F_N(\mathbf{x}) - g(\mathbf{x})) \mathrm{d}(F_N(\mathbf{y}) - g(\mathbf{y})), \tag{526}$$

$$\begin{aligned} &\int_{\mathbf{0}}^{\mathbf{1}} (F_N(\mathbf{x}) - F_N(\mathbf{x}^{(1)}, \mathbf{1}) F_N(\mathbf{1}, \mathbf{x}^{(2)}))^2 \mathrm{d}\mathbf{x} = \int_{\mathbf{0}}^{\mathbf{1}} \int_{\mathbf{0}}^{\mathbf{1}} (\mathbf{1} - \max(\mathbf{x}, \mathbf{y})) \cdot \\ &\cdot \mathrm{d}(F_N(\mathbf{x}) - F_N(\mathbf{x}^{(1)}, \mathbf{1}) F_N(\mathbf{1}, \mathbf{x}^{(2)})) \mathrm{d}(F_N(\mathbf{y}) - F_N(\mathbf{y}^{(1)}, \mathbf{1}) F_N(\mathbf{1}, \mathbf{y}^{(2)})) = \\ &= \int_{\mathbf{0}}^{\mathbf{1}} \int_{\mathbf{0}}^{\mathbf{1}} \int_{\mathbf{0}}^{\mathbf{1}} \int_{\mathbf{0}}^{\mathbf{1}} (\mathbf{1} - \max(\mathbf{x}^{(1)}, \mathbf{y}^{(1)}))(\mathbf{1} - \max(\mathbf{x}^{(2)}, \mathbf{y}^{(2)})) \cdot \\ &\cdot \mathrm{d}(F_N(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) - F_N(\mathbf{x}^{(1)}, \mathbf{1},) F_N(\mathbf{1}, \mathbf{x}^{(2)})) \cdot \\ &\cdot \mathrm{d}(F_N(\mathbf{y}^{(1)}, \mathbf{y}^{(2)}) - F_N(\mathbf{y}^{(1)}, \mathbf{1},) F_N(\mathbf{1}, \mathbf{y}^{(2)})). \end{aligned} \tag{527}$$

Expressing $L^2$ discrepancy (526) as

$$\int_{\mathbf{0}}^{\mathbf{1}} (F_N(\mathbf{x}) - g(\mathbf{x}))^2 \mathrm{d}\mathbf{x} =$$

$$= \int_{\mathbf{0}}^{\mathbf{1}} \int_{\mathbf{0}}^{\mathbf{1}} \Big[ \big( F_N(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) - F_N(\mathbf{x}^{(1)}, \mathbf{1}) F_N(\mathbf{1}, \mathbf{x}^{(2)}) \big) +$$

$$+ \big( F_N(\mathbf{x}^{(1)}, \mathbf{1}) - g(\mathbf{x}^{(1)}, \mathbf{1}) \big) F_N(\mathbf{1}, \mathbf{x}^{(2)}) +$$

$$+ g(\mathbf{x}^{(1)}, \mathbf{1}) \big( F_N(\mathbf{1}, \mathbf{x}^{(2)}) - g(\mathbf{1}, \mathbf{x}^{(2)}) \big) +$$

$$+ \big( g(\mathbf{x}^{(1)}, \mathbf{1}) g(\mathbf{1}, \mathbf{x}^{(2)}) - g(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) \big) \Big]^2 \mathrm{d}\mathbf{x}^{(1)} \mathrm{d}\mathbf{x}^{(2)}$$

the Cauchy inequality implies

$$\sqrt{\int_{\mathbf{0}}^{\mathbf{1}} (F_N(\mathbf{x}) - g(\mathbf{x}))^2 \mathrm{d}\mathbf{x}} \leq$$

$$\leq \sqrt{\int_{\mathbf{0}}^{\mathbf{1}} \int_{\mathbf{0}}^{\mathbf{1}} \big( F_N(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) - F_N(\mathbf{x}^{(1)}, \mathbf{1}) F_N(\mathbf{1}, \mathbf{x}^{(2)}) \big)^2 \mathrm{d}\mathbf{x}^{(1)} \mathrm{d}\mathbf{x}^{(2)}} +$$

$$+ \sqrt{\int_{\mathbf{0}}^{\mathbf{1}} \int_{\mathbf{0}}^{\mathbf{1}} \big( g(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) - g(\mathbf{x}^{(1)}, \mathbf{1}) g(\mathbf{1}, \mathbf{x}^{(2)}) \big)^2 \mathrm{d}\mathbf{x}^{(1)} \mathrm{d}\mathbf{x}^{(2)}} +$$

$$+ \sqrt{\int_{\mathbf{0}}^{\mathbf{1}} \big( F_N(\mathbf{x}^{(1)}, \mathbf{1}) - g(\mathbf{x}^{(1)}, \mathbf{1}) \big)^2 \mathrm{d}\mathbf{x}^{(1)} \int_{\mathbf{0}}^{\mathbf{1}} F_N^2(\mathbf{1}, \mathbf{x}^{(2)}) \mathrm{d}\mathbf{x}^{(2)}} +$$

$$+ \sqrt{\int_{\mathbf{0}}^{\mathbf{1}} \big( F_N(\mathbf{1}, \mathbf{x}^{(2)}) - g(\mathbf{1}, \mathbf{x}^{(2)}) \big)^2 \mathrm{d}\mathbf{x}^{(2)} \int_{\mathbf{0}}^{\mathbf{1}} g^2(\mathbf{x}^{(1)}, \mathbf{1}) \mathrm{d}\mathbf{x}^{(1)}}. \qquad (528)$$

Thus in (528) we have an upper bound of the classical $L^2$ discrepancy (526) of $\mathbf{x}_1, \dots, \mathbf{x}_N$ which contains the $L^2$ discrepancy of statistical independence of partial sequences $\mathbf{x}_1^{(1)}, \dots, \mathbf{x}_N^{(1)}$ and $\mathbf{x}_1^{(2)}, \dots, \mathbf{x}_N^{(2)}$. Note that the infinite partial sequences $\mathbf{x}_n^{(1)}$, $n = 1, 2, \dots$, and $\mathbf{x}_n^{(2)}$, $n = 1, 2, \dots$ of the sequence $\mathbf{x}_n$, $n = 1, 2, \dots$ are statistically independent if and only if for every d.f. $g(\mathbf{x}) \in G(\mathbf{x}_n)$ we have

$$g(\mathbf{x}) = g(\mathbf{x}^{(1)}, \mathbf{1}) \cdot g(\mathbf{1}, \mathbf{x}^{(2)}) \qquad (529)$$

in common points of continuity od d.f.s. It can be used as a definition of independence, cf. [171, p. 1–17, 1.8.9].

By the formula (526) the $s$–dimensional $L^2$ discrepancy of $\mathbf{x}_1, \dots, \mathbf{x}_N$ in $[0, 1)^s$, where $\mathbf{x}_n = (x_{n,1}, \dots, x_{n,s})$ can be expressed in the form

$$\int_{\mathbf{0}}^{\mathbf{1}} (F_N(\mathbf{x}) - x_1 x_2 \dots x_s)^2 \mathrm{d}\mathbf{x} =$$

$$= \frac{1}{3^s} + \frac{1}{N^2}\sum_{m,n=1}^{N}\prod_{j=1}^{s}(1-\max(x_{m,j},x_{n,j})) - \frac{1}{2^{s-1}N}\sum_{n=1}^{N}\prod_{j=1}^{s}(1-x_{n,j}^2).$$

This formula can be found in T.T. Warnock (1972) (see [171, p. 1–71]).

## 7.5 A.d.f. of $(x_n, x_{n+k_1}, x_{n+k_2}, \dots, x_{n+k_{s-1}})$

In Section 8.10.1 we have proved that two, three end four-dimensional van der Corput shifted sequences have an a.d.f. It would be appropriate to prove the following theorem.

**Theorem 201.** *Let $x_n$, $n = 1, 2, \dots$, be a u.d. sequence in $[0, 1)$ and let $k_1, k_2, \dots, k_{s-1}$ be the increasing sequence of nonnegative integers. Then the sequence*

$$\mathbf{x}_n = (x_n, x_{n+k_1}, x_{n+k_2}, \dots, x_{n+k_{s-1}}), \quad n = 1, 2, \dots \tag{530}$$

*has in some cases an a.d.f., e.g. $G(\mathbf{x}_n)$ is singleton.*

*Proposal.* We are starting with

$$\mathbf{x}_n = (x_n, x_{n+1}, x_{n+2}, \dots, x_{n+s-1}), \quad n = 1, 2, \dots \tag{531}$$

and let $F_N(\mathbf{x})$ be the step d.f. of (531). Insert it to (524), then

$$\begin{aligned}
&\int_{\mathbf{0}}^{\mathbf{1}}(F_M(\mathbf{x}) - F_N(\mathbf{x}))^2 \mathrm{d}\mathbf{x}\\
=&\int_{\mathbf{0}}^{\mathbf{1}}\int_{\mathbf{0}}^{\mathbf{1}}(\mathbf{1}-\max(\mathbf{x},\mathbf{y}))\mathrm{d}(F_M(\mathbf{x}) - F_N(\mathbf{x}))\mathrm{d}(F_M(\mathbf{y}) - F_N(\mathbf{y}))\\
=&\frac{1}{M^2}\sum_{m,n=1}^{M}(\mathbf{1}-\max(\mathbf{x}_m,\mathbf{x}_n)) + \frac{1}{N^2}\sum_{m,n=1}^{M}(\mathbf{1}-\max(\mathbf{x}_m,\mathbf{x}_n))\\
&-2\frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}(\mathbf{1}-\max(\mathbf{x}_m,\mathbf{x}_n)).
\end{aligned} \tag{532}$$

Expressing $\mathbf{1} - \max(\mathbf{x}_m, \mathbf{x}_n)$ by (523) we find (532) as

$$\frac{1}{M^2}\sum_{m,n=1}^{M}(1-\max(x_m,x_n))\dots(1-\max(x_{m+s-1},x_{n+s-1}))$$

$$+\frac{1}{N^2}\sum_{m,n=1}^{N}(1-\max(x_m,x_n))\dots(1-\max(x_{m+s-1},x_{n+s-1}))$$
$$-2\frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}(1-\max(x_m,x_n))\dots(1-\max(x_{m+s-1},x_{n+s-1})). \quad (533)$$

Now, the sequence $x_n$, $n=1,2,\dots,$ has an a.d.f. if and only if the limit $M,N,\to\infty$ of

$$\frac{1}{M^2}\sum_{m,n=1}^{M}(1-\max(x_m,x_n))+\frac{1}{N^2}\sum_{m,n=1}^{N}(1-\max(x_m,x_n))$$
$$-2\frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}(1-\max(x_m,x_n)) \quad (534)$$

tends to zero. In some cases (533) is bounded by (534), then (532) tend to zero. □

Generally Theorem 201 not holds for all u.d. $x_n$. In the following we give u.d. sequence $z_n$ such that $(z_n,z_{n+1})$ not have an a.d.f.

**Example 84.** Let $n=1,2,\dots$

Let $(x_n,y_n)$ by two-dimensional sequence in $[0,1)^2$.

Assume that $G(x_n,y_n)$ coincide with the set of all copulas. Thus $x_n$ and $y_n$ are u.d.

The sequence $x_1,y_1,x_2,y_2,x_3,y_3,x_4,y_4,\dots$ we denote as $z_n$, $n=1,2,\dots$

The two-dimensional sequence $(z_n,z_{n+1})$ can be divided on two subsequences

$(x_n,y_n)$, and

$(y_n,x_{n+1})$. We have

$$\frac{A(n\le 2N_k;(z_n,z_{n+1})\in[0,x)\times[0,y))}{2N_k} \quad (535)$$
$$=\frac{A(n\le N_k;(x_n,y_n)\in[0,x)\times[0,y))}{2N_k} \quad (536)$$
$$+\frac{A(n\le N_k;(y_n,x_{n+1})\in[0,x)\times[0,y))}{2N_k}. \quad (537)$$

Assume that as $k\to\infty$

(535) tends to $g(x,y)$,
(536) tens to $(1/2)g_t(x,y)$,
(537) tens to $(1/2)\tilde{g}_t(x,y)$.
Then we find

$$g(x,y) = (1/2)g_t(x,y) + (1/2)\tilde{g}_t(x,y). \tag{538}$$

Here $g_t(x,y)$ is an arbitrary copula. The equation (538) not holds for fixed $g(x,y)$ since we can find $g_t(x,y)$ such that

$$g(x,y) + g(x,y) - g_t(x,y) \tag{539}$$

is not a copula. This (539) can be proved by using one-dimensional d.f.

**Notes 42.** Let $\beta$ be a PV-number. Using $\beta$-adic expansion of $n$ the authors [98] define Monna map $\phi_\beta(n)$ and they study the $s$-dimensional sequence

$$(\phi_\beta(k_n+n_1),\dots,\phi_\beta(k_n+n_s)), n=1,2,\dots \tag{540}$$

Assuming that the sequence of integers $k_n$, $n=1,2,\dots$ is Hartman uniformly distributed and $L^p$-good universal for a $p\in[1,\infty]$, the authors prove that the sequence (540) has an a.d.f. (Also see Section 11.7.)

## 7.6 Computation of integrals by d.f.s

In Riemann-Siltjes integral the integration by parts can be used, see:

**Theorem 202.** *For every continuous $F(x,y)$ and arbitrary d.f. $g(x,y)$ we have*

$$\begin{aligned}\int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y) &= F(1,1) - \int_0^1 g(1,y)\mathrm{d}_y F(1,y)\\ &- \int_0^1 g(x,1)\mathrm{d}_x F(x,1) + \int_0^1\int_0^1 g(x,y)\mathrm{d}_x\mathrm{d}_y F(x,y). \end{aligned}\tag{541}$$

*Proof.* Integration by parts

$$\begin{aligned}&\int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y)\\ &= \left[\int_0^1 F(x,y)\mathrm{d}_y g(x,y)\right]_{x=0}^{x=1} - \int_0^1\int_0^1 \mathrm{d}_y g(x,y)\mathrm{d}_x F(x,y)\end{aligned}$$

$$= \int_0^1 F(1,y)\mathrm{d}_y g(1,y) - \int_0^1 \int_0^1 \mathrm{d}_y g(x,y)\mathrm{d}_x F(x,y)$$
$$= [F(1,y)g(1,y)]_{y=0}^{y=1} - \int_0^1 g(1,y)\mathrm{d}_y F(1,y)$$
$$- \left[\int_0^1 g(x,y)\mathrm{d}_x F(x,y)\right]_{y=0}^{y=1} + \int_0^1 \int_0^1 g(x,y)\mathrm{d}_y \mathrm{d}_x F(x,y).$$

Here we use $g(0,y) = g(x,0) = 0$ for every $x, y \in [0,1]$. □

**Notes 43.** Main part in the above proof is integration by parts
$\int_0^1 f(x)\mathrm{d}g(x) = [f(x)g(x)]_0^1 - \int_0^1 g(x)\mathrm{d}f(x)$.
It holds for every Riemann-Stieltjes integrable functions $f(x)$, $g(x)$, which not have common points od discontinuity.

**Example 85.** Put $F(x,y) = x^\alpha y^\beta$, $g(x,y) = xy$. Then (541) has the form

$$\int_0^1 \int_0^1 x^\alpha y^\beta \mathrm{d}x\mathrm{d}y = 1 - \int_0^1 x\mathrm{d}x^\alpha - \int_0^1 y\mathrm{d}y^\beta + \int_0^1 \int_0^1 \mathrm{d}_x \mathrm{d}_y x^\alpha y^\beta$$

which gives

$$\frac{1}{\alpha+1}\frac{1}{\beta+1} = 1 - \frac{\alpha}{\alpha+1} - \frac{\beta}{\beta+1} + \alpha\beta\frac{1}{\alpha+1}\frac{1}{\beta+1}$$

what is true.

Repeating (541) we can find:

**Theorem 203.** *For every continuous $F(x,y,u,v)$ and any d.f. $g(x,y)$ we have*
$\int_0^1 \int_0^1 \int_0^1 \int_0^1 F(x,y,u,v)\mathrm{d}_x\mathrm{d}_y g(x,y)\mathrm{d}_u\mathrm{d}_v g(u,v) =$
$F(1,1,1,1) - \int_0^1 g(1,v)\mathrm{d}_v F(1,1,1,v) - \int_0^1 g(u,1)\mathrm{d}_u F(1,1,u,1)$
$+ \int_0^1 \int_0^1 g(u,v)\mathrm{d}_u\mathrm{d}_v F(1,1,u,v)$
$- \int_0^1 g(1,y)\mathrm{d}_y F(1,y,1,1) + \int_0^1 \int_0^1 g(1,y)g(1,v)\mathrm{d}_v\mathrm{d}_y F(1,y,1,v)$
$+ \int_0^1 \int_0^1 g(1,y)g(u,1)\mathrm{d}_u\mathrm{d}_y F(1,y,u,1)$
$- \int_0^1 \int_0^1 \int_0^1 g(1,y)g(u,v)\mathrm{d}_u\mathrm{d}_v\mathrm{d}_y F(1,y,u,v)$
$- \int_0^1 g(x,1)\mathrm{d}_x F(x,1,1,1) + \int_0^1 \int_0^1 g(x,1)g(1,v)\mathrm{d}_v\mathrm{d}_x F(x,1,1,v)$
$+ \int_0^1 \int_0^1 g(x,1)g(u,1)\mathrm{d}_u\mathrm{d}_x F(x,1,u,1)$
$- \int_0^1 \int_0^1 \int_0^1 g(x,1)g(u,v)\mathrm{d}_u\mathrm{d}_v\mathrm{d}_x F(x,1,u,v)$

$+\int_0^1\int_0^1 g(x,y)\mathrm{d}_y\mathrm{d}_xF(x,y,1,1)$
$-\int_0^1\int_0^1\int_0^1 g(x,y)g(1,v)\mathrm{d}_v\mathrm{d}_y\mathrm{d}_xF(x,y,1,v)$
$-\int_0^1\int_0^1\int_0^1 g(x,y)g(u,1)\mathrm{d}_u\mathrm{d}_y\mathrm{d}_xF(x,y,u,1)$
$+\int_0^1\int_0^1\int_0^1\int_0^1 g(x,y)g(u,v)\mathrm{d}_u\mathrm{d}_v\mathrm{d}_y\mathrm{d}_xF(x,y,u,v)$.

**Theorem 204.** *For every continuous $F(x,y,z)$ defined on $[0,1]^3$ and any 3-dimensional d.f. $g(x,y,z)$ we have*
$\int_0^1\int_0^1\int_0^1 F(x,y,z)\mathrm{d}_x\mathrm{d}_y\mathrm{d}_zg(x,y,z)=F(1,1,1)$
$-\int_0^1 g(1,1,z)\mathrm{d}_zF(1,1,z)-\int_0^1 g(1,y,1)\mathrm{d}_yF(1,y,1)-\int_0^1 g(x,1,1)\mathrm{d}_xF(x,1,1)$
$+\int_0^1\int_0^1 g(1,y,z)\mathrm{d}_y\mathrm{d}_zF(1,y,z)+\int_0^1\int_0^1 g(x,1,z)\mathrm{d}_x\mathrm{d}_zF(x,1,z)$
$+\int_0^1\int_0^1 g(x,y,1)\mathrm{d}_x\mathrm{d}_yF(x,y,1)-\int_0^1\int_0^1\int_0^1 g(x,y,z)\mathrm{d}_x\mathrm{d}_y\mathrm{d}_zF(x,y,z)$.

**Theorem 205.** *Assume that $F(x,y,z,u)$ is a continuous in $[0,1]^4$ and $g(x,y,z,u)$ is a 4-dimensional d.f. Then*

$$
\begin{aligned}
&\int_0^1\int_0^1\int_0^1\int_0^1 F(x,y,z,u)\mathrm{d}_x\mathrm{d}_y\mathrm{d}_z\mathrm{d}_ug(x,y,z,u)=F(1,1,1,1)\\
&-\int_0^1 g(x,1,1,1)\mathrm{d}_xF(x,1,1,1)-\int_0^1 g(1,y,1,1)\mathrm{d}_yF(1,y,1,1)\\
&-\int_0^1 g(1,1,z,1)\mathrm{d}_zF(1,1,z,1)-\int_0^1 g(1,1,1,u)\mathrm{d}_uF(1,1,1,u)\\
&+\int_0^1\int_0^1 g(x,y,1,1)\mathrm{d}_x\mathrm{d}_yF(x,y,1,1)+\int_0^1\int_0^1 g(x,1,z,1)\mathrm{d}_x\mathrm{d}_zF(x,1,z,1)\\
&+\int_0^1\int_0^1 g(1,y,z,1)\mathrm{d}_y\mathrm{d}_zF(1,y,z,1)+\int_0^1\int_0^1 g(x,1,1,u)\mathrm{d}_x\mathrm{d}_uF(x,1,1,u)\\
&+\int_0^1\int_0^1 g(1,y,1,u)\mathrm{d}_y\mathrm{d}_uF(1,y,1,u)+\int_0^1\int_0^1 g(1,1,z,u)\mathrm{d}_z\mathrm{d}_uF(1,1,z,u)\\
&-\int_0^1\int_0^1\int_0^1 g(x,y,z,1)\mathrm{d}_x\mathrm{d}_y\mathrm{d}_zF(x,y,z,1)\\
&-\int_0^1\int_0^1\int_0^1 g(1,y,z,u)\mathrm{d}_u\mathrm{d}_y\mathrm{d}_zF(1,y,z,u)\\
&-\int_0^1\int_0^1\int_0^1 g(x,1,z,u)\mathrm{d}_x\mathrm{d}_z\mathrm{d}_uF(x,1,z,u)\\
&-\int_0^1\int_0^1\int_0^1 g(x,y,1,u)\mathrm{d}_u\mathrm{d}_y\mathrm{d}_xF(x,y,1,u)
\end{aligned}
$$

$$+\int_0^1\int_0^1\int_0^1\int_0^1 g(x,y,z,u)\mathrm{d}_x\mathrm{d}_y\mathrm{d}_z\mathrm{d}_u F(x,y,z,u).$$

**Theorem 206.** *Let $F:[0,1]\to\mathbb{R}$ and $Q:[0,1]^s\to[0,1]$ be continuous functions. Then*

$$\int_{[0,1]^s} F(Q(u_1,u_2,\dots,u_s))\mathrm{d}u_1\mathrm{d}u_2\dots\mathrm{d}u_s=\int_0^1 F(x)\mathrm{d}g(x),$$

*where $g(x)$ is the a.d.f. of the sequence $Q(x_{n_1},x_{n_2},\dots,x_{n_s})$, $n_1,n_2,\dots,n_s=1,2,\dots,$ where $x_n$, $n=1,2,\dots,$ is u.d. in $[0,1]$ and $s$-dimensional sequence $(x_{n_1},x_{n_2},\dots,x_{n_s})$ is ordered such that $(x_{n_1},x_{n_2},\dots,x_{n_s})$, $n_1,n_2,\dots,n_s=1,2,\dots,N$ formed first $N^s$ terms.*

*Proof.* By generalized Weyl's limit relation

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N^s}\sum_{n_1,\dots,n_s}^{N} F(Q(x_{n_1},\dots,x_{n_s}))&=\int_{[0,1]^s}F(Q(u_1,\dots,u_s))\mathrm{d}u_1\dots\mathrm{d}u_s\\&=\int_0^1 F(x)\mathrm{d}g(x).\end{aligned}$$

□

In another direction, define

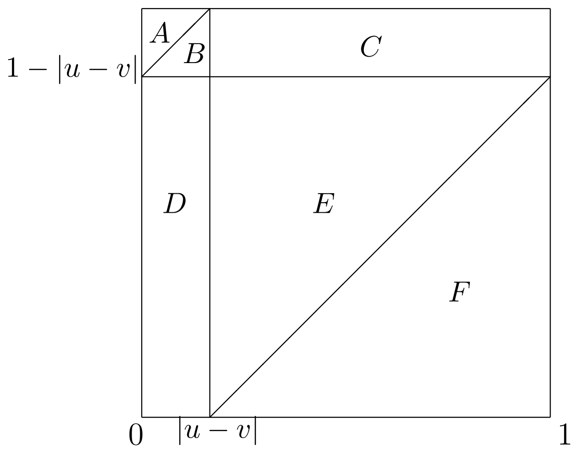



Figure: Division of $[0,1]^2$.

and

$$g_{u,v}(x,y) = \begin{cases} x & \text{if } (x,y) \in A, \\ y - (1 - |u-v|) & \text{if } (x,y) \in B, \\ x + y - 1 & \text{if } (x,y) \in C, \\ 0 & \text{if } (x,y) \in D, \\ x - |u-v| & \text{if } (x,y) \in E, \\ y & \text{if } (x,y) \in F. \end{cases} \tag{542}$$

Then we have

**Theorem 207.** *Let $K(x,y)$ be continued function defined on $[0,1]^2$. Then for every $(u,v) \in [0,1]^2$ we have*

$$\int_0^1 K(\{u+z\},\{v+z\})\mathrm{d}z = \int_0^1 \int_0^1 K(x,y)\mathrm{d}_x\mathrm{d}_y g_{u,v}(x,y). \tag{543}$$

*Proof.* For the sequence $(u+z_n, v+z_n)$ mod 1, $z_n$, $n = 1,2,\dots,$ $z_n$ is u.d. we denote a.d.f. as $g_{u,v}(x,y)$. Then

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} K(\{u+z_n, v+z_n\}) &= \int_0^1 K(\{u+z\},\{v+z\})\mathrm{d}z \\ &= \int_0^1 \int_0^1 K(x,y)\mathrm{d}_x\mathrm{d}_y g_{u,v}(x,y). \end{aligned} \tag{544}$$

By Weyl's limit relation (544) implies (717). In the following, for $g_{u,v}(x,y)$, we prove (542).

Let $0 \le u \le v \le 1$ be fixed and $x,y \in [0,1]$ be variables and define
$h_u(x) = xu$ mod 1, $h_v(y) = yv$ mod 1. Then
$g_{u,v}(x,y) = |h_u^{-1}([0,x)) \cap h_v^{-1}([0,y))|$.
Using the following graphs of $h_u(x)$

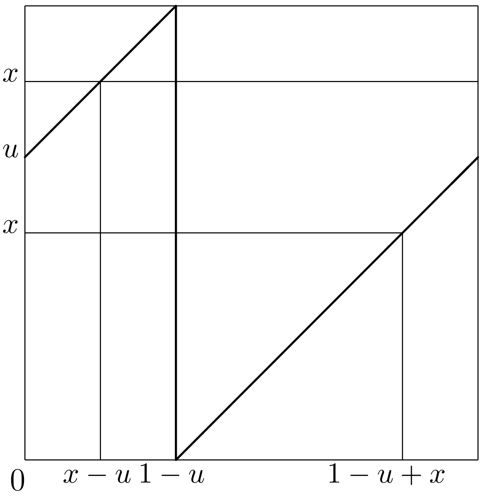



Figure: The graph of $h_u(x)$.

then we see

$$h_u^{-1}([0,x]) = \begin{cases} [1-u, 1-u+x] & \text{if } x \le u, \\ [0, x-u] \cup [1-u, 1] & \text{if } u \le x. \end{cases} \tag{545}$$

Hence

$$\begin{aligned} & g_{u,v}(x,y) \\ &= \begin{cases} \big|[1-u, 1-u+x] \cap [1-v, 1-v+y]\big| & \text{if } x \le u, y \le v, \\ \big|[1-u, 1-u+x] \cap ([0, y-v] \cup [1-v, 1])\big| & \text{if } x \le u, y > v, \\ \big|([0, x-u] \cup [1-u, 1]) \cap [1-v, 1-v+y]\big| & \text{if } x > u, y \le v, \\ \big|([0, x-u] \cup [1-u, 1]) \cap ([0, y-v] \cup [1-v, 1])\big| & \text{if } x > u, y > v. \end{cases} \end{aligned} \tag{546}$$

Now we using minimum and maximum formula for the length of intersection of two intervals $[\alpha, \beta]$ and $[\gamma, \delta]$

$$\big|[\alpha, \beta] \cap [\gamma, \delta]\big| = \max\big(\min(\beta, \delta) - \max(\alpha, \gamma), 0\big). \tag{547}$$

Insert (547) into (546) we see

$$g_{u,v}(x,y)$$
$$= \begin{cases} \max\big(\min(y, x-u+v), 0\big) & \text{if } x \le u, y \le v, \\ \max\big(\min(x, y-v-1+u), 0\big) + \max(v-u+x, 0) & \text{if } x \le u, y > v, \\ y & \text{if } x > u, y \le v, \\ \min(x-u+v, y) + \max(y-v-1+u, 0) & \text{if } x > u, y > v, \end{cases} \tag{548}$$

which implies (542). □

Note that $g_{u,v}(x,y)$ is a copula.

# 8 Copulas

We denote by $G_{s,k}$ the set of all d.f.s $g(\mathbf{x})$ on $[0,1]^s$ for which all $k$-dimensional marginal (i.e. face) d.f.s satisfy

$$g(1,\dots,1,x_{i_1},1,\dots,1,x_{i_2},1,\dots,1,x_{i_k},1,\dots,1) = x_{i_1}x_{i_2}\dots x_{i_k}.$$

For $k=1$, these d.f.s are called *copulas*, which were introduced by M. Sklar (1959)[148]. Basic properties of copulas can be found in monographs R.B. Nelsen (1999) [114], N. Balakrishnan and Chin-Diew Lai [6], and in Proccedings of EDS: P. Jaworski, F. Durante, W. Härdle and T. Rychlik [80].

Let
$$\mathbf{x}_n = (x_{n,1}, x_{n,2}, \dots, x_{n,s}), n = 1, 2, \dots,$$
be an infinite $s$-dimensional sequence in the unit cube $[0,1)^s$. We assume that, for fixed $k < s$, all $k$-dimensional marginal sequences $(x_{n,i_1},\dots,x_{n,i_k})$ are uniformly distributed (abbreviating u.d.) We introduce the following two problems:
(I) If $\mathbf{x}_n$ is not u.d., find all possible d.f.s of $\mathbf{x}_n$.
(II) Find some (possible "minimal") criterions which imply u.d. of the original sequence $\mathbf{x}_n$, $n = 1,2,\dots$. To illustrate (II) we give the following non-u.d. sequence:

**Example 86.** Let $\theta$ be an algebraic number of the degree $s$ such that for its minimal polynomial $p(x) = \sum_{i=0}^{s} a_i x^i$ we have $a_i \neq 0$ for $i = 0, 1, \dots, s$.

Define the sequence $\mathbf{x}_n = (x_{n,1}, \dots, x_{n,s})$, $n = 1, 2, \dots$, with coordinate sequences $x_{n,i} = n\theta^i \bmod 1$, $i = 1, 2, \dots, s$. Applying the well-known Kronecker theorem (cf. [DT, p. 15, Coroll. 1.20], [SP, p. 3-9, 3.4.1]) all marginal sequences with dimensions $k = 1, 2, \dots, s-1$ are u.d. but $\mathbf{x}_n$ is not.

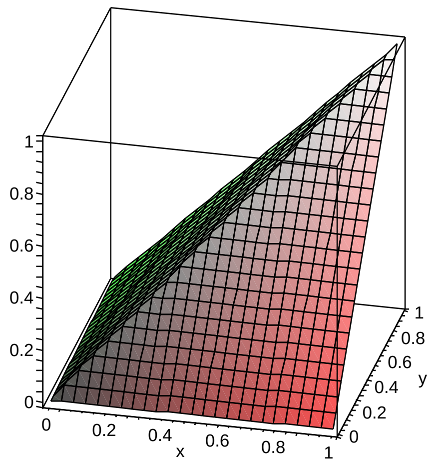



Figure 2: Graph of $g(x, y) = \min(x, y)$

## 8.1 The set of $s$-dimensional copulas $\mathbf{G}_{s,1}$

**Theorem 208** (Frécet-Hoeffding copula bounds)**.** *For any copula* $g(x_1, \dots, x_s) \in G_{s,1}$ *and any* $(x_1, \dots, x_s) \in [0, 1]^s$ *the following bounds hold:*

$$\max(1 - s + \sum_{i=1}^{s} x_i, 0) \le g(x_1, \dots, x_s) \le \min(x_1, \dots, x_s). \tag{549}$$

Here $\min(x_1, \dots, x_s)$ is always a copula. The lower bound is a copula only in two dimensions. It is only point-wise sharp, in the sense that for fixed $(u_1, \dots, u_s)$ there exists a copula $g(x_1, \dots, x_s)$ such that $g(u_1, \dots, u_s) = \max(1 - s + \sum_{i=1}^{s} u_i, 0)$, see [81] and [28].

**Theorem 209** (Sklar's theorem)**.** *A multivariete d.f.* $g(x_1, \dots, x_s)$ *with marginals* $g_i(x_i) = g(1, \dots, 1, x_i, 1, \dots, 1)$ *can be written as*

$$g(x_1, \dots, x_s) = c(g_1(x_1), \dots, g_s(x_s)), \tag{550}$$

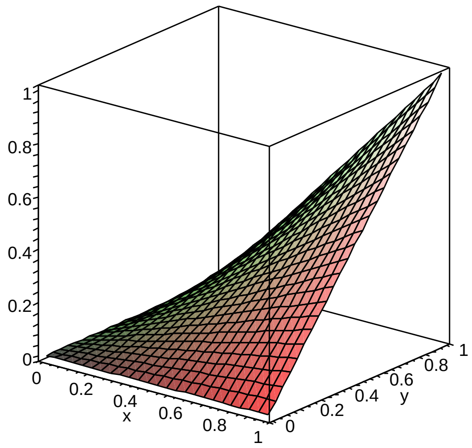



Figure 3: Graph of $g(x, y) = xy$

*where* $c(x_1, \ldots, x_s)$ *is a copula. This copula is defined unique, assuming that the marginals* $g_i$ *are continuous.*

This theorem provides the theoretical foundation for the application of copulas.

## 8.2 Dimension $s = 2$

As we denoted in Section 8, $G_{2,1}$ is the set of all two-dimensional d.f.s $g(x, y)$ defined on $[0, 1]^2$ such that their marginal d.f.s satisfy $g(x, 1) = x$ and $g(1, y) = y$. Telegraphically, to illustrate $G_{2,1}$ we give copulas:

$g_1(x, y) = xy$,

$g_2(x, y) = \min(x, y)$, [55]

$g_3(x, y) = \max(x + y - 1, 0)$,

$g_\theta(x, y) = (\min(x, y))^\theta (xy)^{1-\theta}$, where $\theta \in [0, 1]$ (Cuadras-Augé family, cf. [114, p. 12, Ex. 2.5]),

$g_4(x, y) = \frac{xy}{x+y-xy}$ (see [114, p. 19, 2.3.4]),

$\tilde{g}(x, y) = x + y - 1 + g(1 - x, 1 - y)$ for every $g(x, y) \in G_{2,1}$ (Survival copula, see [114, p. 28, 2.6.1]),

[55] $\min(x, y) \leq \frac{2xy}{x+y}$ for every $x > 0, y > 0$.

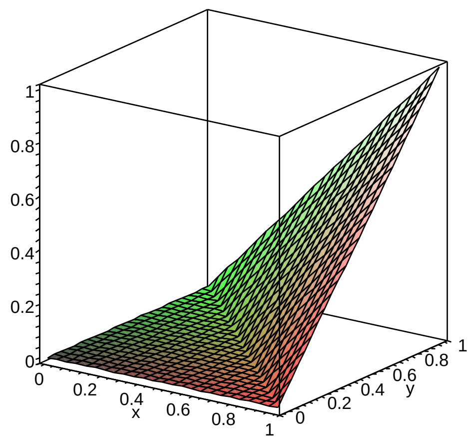



Figure 4: Graph of $g(x, y) = \max(x + y - 1, 0)$

$g_5(x, y) = \min(ya(x), xb(y))$, where $a(0) = b(0) = 0$, $a(1) = b(1) = 1$ and $a(x)/x$, $b(y)/y$ are both decreasing on $(0, 1]$ (Marshall copula, cf. [114, p. 51, Exerc. 3.3]).

## 8.3 Basic properties of $G_{2,1}$

There are some basic properties of $G_{2,1}$:

(I) $G_{2,1}$ is closed under pointwise limit and convex linear combinations.

(II) For every $g(x, y) \in G_{2,1}$ and every $(x_1, y_1), (x_2, y_2) \in [0, 1]^2$ we have

$|g(x_2, y_2) - g(x_1, y_1)| \le |x_2 - x_1| + |y_2 - y_1|$.

I.e. every copula is uniformly continuous, c.f. [115, p. 9]. Furthermore, The horizontal, vertical and diagonal sections of copula are all nondecreasing and uniformly continuous on $[0, 1]$, see [115, p. 9, Cor.2.26].

(III) For every $g(x, y) \in G_{2,1}$

$g_3(x, y) = \max(x + y - 1, 0) \le g(x, y) \le \min(x, y) = g_2(x, y)$ (Fréchet-Hoeffding bounds [114, p. 9]).

(IV) M. Sklar (1959) proved that for every d.f. $g(x, y)$ on $[0, 1]^2$ there exists $c(x, y) \in G_{2,1}$ such that $g(x, y) = c(g(x, 1), g(1, y))$ for every $(x, y) \in [0, 1]^2$. If $g(x, 1)$ and $g(1, y)$ are continuous, then $c(x, y)$ is defined unique (cf. [114, p. 15, Th. 2.3.3] and for multidimensional case see Theorem 209).

(V) For d.f. $g(x, y)$ denote the marginal $g_1(x) = g(x, 1)$ and $g_2(y) =$

$g(1,y)$ and by Sklar $g(x,y) = c(g_1(x), g_2(y))$. Then for every continuous $F(x,y)$ we have

$$\int_0^1 \int_0^1 F(x,y)\mathrm{d}g(x,y) = \int_0^1 \int_0^1 F(g_1^{(-1)}(x), g_2^{(-1)}(y))\mathrm{d}c(x,y), \tag{551}$$

see M. Hofer and M.R. Iacò [75].

Here are some new copulas:

(VI) $g_6(x,y) = \frac{1}{z_0}\min(xy, xz_0, yz_0)$ for fixed $z_0$, $0 < z_0 \le 1$.

(VII) $g_7(x,y) = \frac{1}{z_0u_0}\min(xyz_0, xyu_0, xz_0u_0, yz_0u_0)$ for fixed $z_0, u_0 \in (0,1]^2$.

In Definition 14 is a copula called shuffle of $M$. Here we give the following generalization:

(VIII) Let $f : [0,1] \to [0,1]$ be a function and denote

$$\text{graph } f = \{(x, f(x)); x \in [0,1]\}.$$

Then

$$c_{\text{graph}f}(x,y) = \left|\text{Project}_x(\text{graph } f \cap [0,x] \times [0,y))\right| \tag{552}$$

is a copula if and only if the function $f$ preserve the Lebesgue measure (called *measure preserving*)

In the books [6] and [80] copulas are studied in the position d.f.s of random variables. In the following we apply some parts of uniform distribution theory.

(IX) Let $f_1$ and $f_2$ be measure preserving functions $[0,1] \to [0,1]$ (cf. Section 12.3). Then

$$c_{f_1,f_2}(x,y) = |f_1^{-1}[0,x] \cap f_2^{-1}[0,y]| \tag{553}$$

is a copula. Furthermore, for every copula $c(x,y)$ there exist measure preserving $f_1, f_2$ such that $c(x,y) = c_{f_1,f_2}(x,y)$.

(X) If $\phi$ is a measure preserving function, then

$$c_{f_1,f_2}(x,y) = c_{f_1\circ\phi, f_2\circ\phi}(x,y). \tag{554}$$

(XI) Examples:

$c_{f,f}(x,y) = \min(x,y)$ for arbitrary measure preserving $f$.

$c_{\text{graph}f}(x,y) = c_{f_1,f}(x,y)$ if $f_1(x) = x$.

$c_{f_1,f_2}(x,y) = \min(x+y+1, 0)$ if $f_1(x) = x$ and $f_2(x) = 1-x$.

(XII) If $(x_n, y_n)$, $n = 1, 2, \dots,$ is u.d. sequence in $[0, 1)^2$ and $f_1, f_2$ are u.d.p. functions, then the sequence $(f_1(x_n), f_2(y_n))$, $n = 1, 2, \dots,$ has a.d.f. the copula $c_{f_1,f_2}(x, y)$.

(XIII) A copula $c$ is called Archimedean if it admits the representation

$$c(x_1, \dots, x_s) = \psi(\psi^{-1}(x_1) + \dots + \psi^{-1}(x_s)), \tag{555}$$

where the $k$th derivatives of generator $\psi(x)$ satisfy $(-1)^k \psi^{(k)}(x) \ge 0$ for all $x \ge 0$ and $k = 0, 1, \dots, s - 2$ and $(-1)^{s-2}\psi^{(s-2)}(x)$ is nonincreasing and convex. In the following are the most popular ones:

(a) $\frac{1-t}{e^x - t}, t \in (0, 1)$ Ali-Mikhail-Haq;

(b) $(1 + tx)^{-1/t}, t \in (0, \infty)$ Clayton;

(c) $e^{-x^{1/t}}, t \in [1, \infty)$ Gumbel;

(d) $1 - (1 - e^{-x})^{1/t}, t \in [1, \infty)$ Joe.

## 8.4 Dimension $s = 3$

Let $G_{3,2}$ be the set of all three-dimensional d.f.s $g(x, y, z)$ defined on $[0, 1]^3$ such that their two-dimensional marginals (or faces) d.f.s satisfy $g(x, y, 1) = xy$, $g(1, y, z) = yz$ and $g(x, 1, z) = xz$. E.g., the $G_{3,2}$ contains

$g_1(x, y, z) = xyz$,

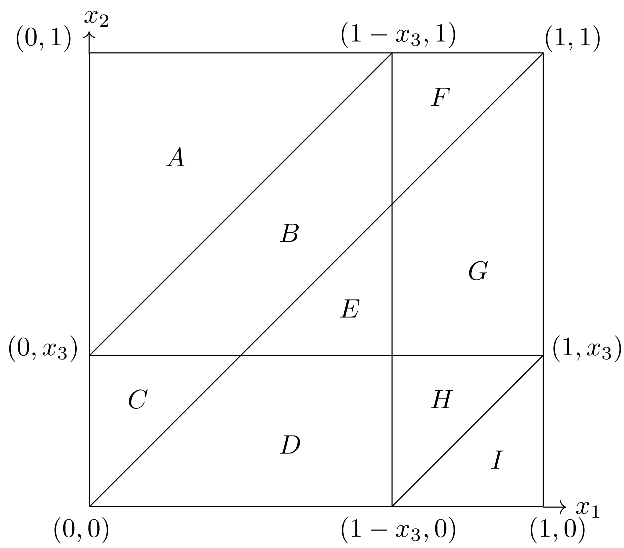
$g_2(x, y, z) = \min(xy, xz, yz)$,

$g_3(x, y, z) = \frac{1}{u_0} \min(xyz, xyu_0, xzu_0, yzu_0)$, for fixed $u_0$, $0 < u_0 \le 1$,

**Example 87.** Let $g_4(x, y, z)$ be the a.d.f. of a three-dimensional sequence $(u_n, v_n, \{u_n - v_n\})$, where two-dimensional $(u_n, v_n)$ is u.d. in $[0, 1]^2$. Applying Weyl's criterion (cf. [38, p. 14], [171, p. 3–1]) we see that also $(u_n, \{u_n - v_n\})$ and $(v_n, \{u_n - v_n\})$ are u.d., thus the marginal d.f.s are

$$g_4(1, x_2, x_3) = x_2x_3, \quad g_4(x_1, 1, x_3) = x_1x_3, \quad g_4(x_1, x_2, 1) = x_1x_2.$$

The d.f. $g_4(x_1, x_2, x_3)$ has the following explicit form: Divide the unit square $[0, 1]^2$ into regions $A, B, C, D, E, F, G, H, I$ as shown on the following Fig.




Then

$$g_4(x_1,x_2,x_3)=\begin{cases} x_1x_2, & \text{if } (x_1,x_2)\in A,\\ -\frac{1}{2}(x_1^2+x_2^2+x_3^2)+x_1x_2+x_2x_3, & \text{if } (x_1,x_2)\in B,\\ -\frac{1}{2}x_1^2+x_1x_2, & \text{if } (x_1,x_2)\in C,\\ \frac{1}{2}x_2^2, & \text{if } (x_1,x_2)\in D,\\ -\frac{1}{2}x_3^2+x_2x_3, & \text{if } (x_1,x_2)\in E,\\ -\frac{1}{2}x_2^2+x_1x_2+x_1x_3+x_2x_3-x_1-x_3+\frac{1}{2}, & \text{if } (x_1,x_2)\in F,\\ \frac{1}{2}x_1^2+x_1x_3+x_2x_3-x_1-x_3+\frac{1}{2}, & \text{if } (x_1,x_2)\in G,\\ \frac{1}{2}(x_1^2+x_2^2+x_3^2)+x_1x_3-x_1-x_3+\frac{1}{2}, & \text{if } (x_1,x_2)\in H,\\ x_1x_2+x_2x_3-x_2 & \text{if } (x_1,x_2)\in I. \end{cases}$$

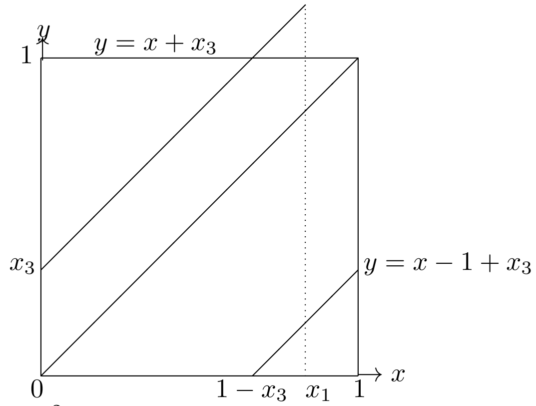
*Proof.* If $(u_n,v_n)$ is u.d. in $[0,1]^2$, then $(u_n,\{v_n-u_n\})$ is also u.d. It follows from Weyl criterion

$$\frac{1}{N}\sum_{n=1}^{N}e^{2\pi i(k_1u_n+k_2(v_n-u_n))}=\frac{1}{N}\sum_{n=1}^{N}e^{2\pi i((k_1-k_2)u_n+k_2v_n)}$$

and $(k_1,k_2)\neq(0,0)\Longleftrightarrow(k_1-k_2,k_2)\neq(0,0)$. Another proof follows directly from that

$$\{y-x\}<x_3\Longleftrightarrow\begin{cases}0\le y-x<x_3 & \text{if } y\ge x,\\ 0\le 1+y-x & \text{if } y<x.\end{cases}$$

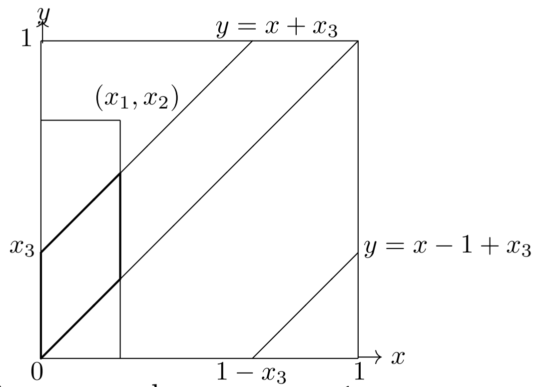
Thus the set $\{(x,y) \in [0,1]^2; \{y-x\} < x_3\}$ has the form



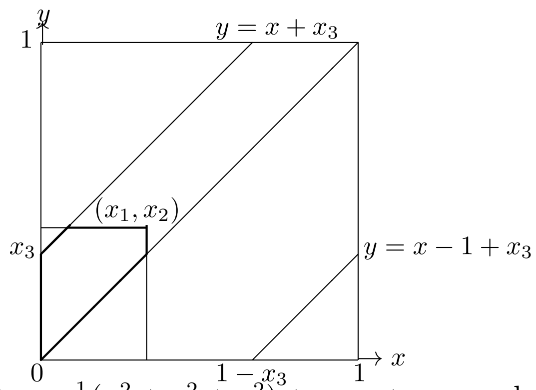

which gives $|\{(x,y) \in [0,1]^2; 0 \le x < x_1, \{y-x\} < x_3\}| = x_1x_3$. The a.d.f. $g_4(x_1,x_2,x_3)$ is the Lebesgue measure

$$g_4(x_1,x_2,x_3) = |\{(x,y,z) \in [0,1]^3; 0 \le x < x_1, 0 \le y < x_2, \{y-x\} < x_3\}|.$$

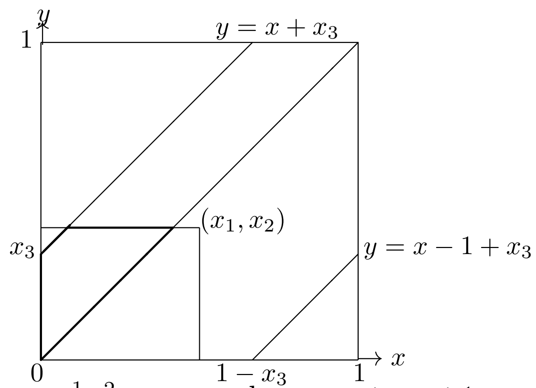
We are investigate nine cases:



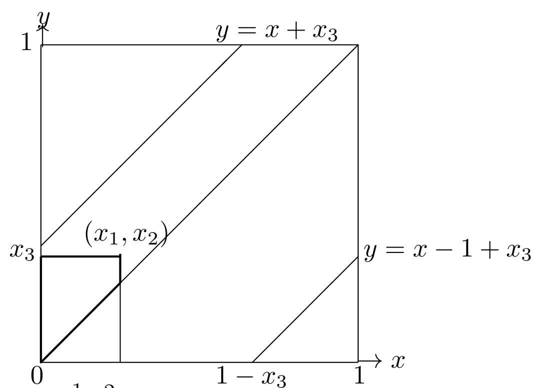

In this case $g(x_1,x_2,x_3) = x_1x_3$, where $x_1 + x_3 \le x_2$.



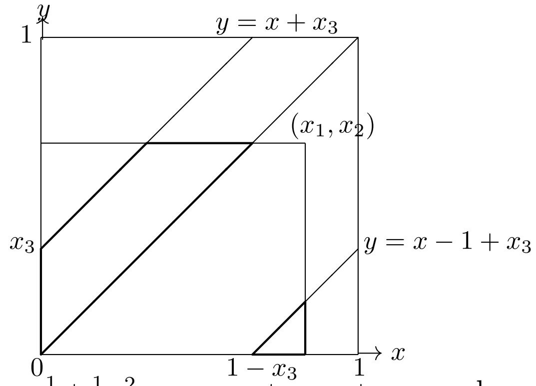

In this case $g(x_1, x_2, x_3) = -\frac{1}{2}(x_1^2 + x_2^2 + x_3^2) + x_1x_2 + x_2x_3$, where $x_1 \le x_2 \le x_1 + x_3$ and $x_1 \le 1 - x_3$.



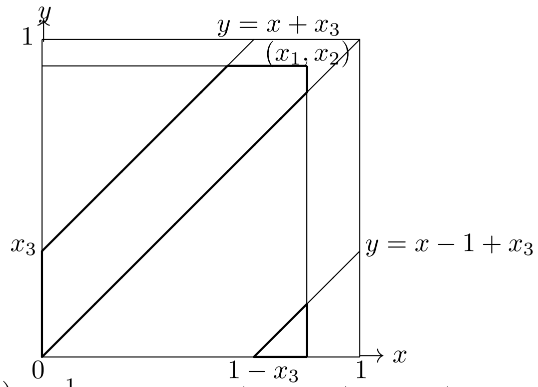

In this case $g(x_1, x_2, x_3) = -\frac{1}{2}x_3^2 + x_2x_3$, where $x_2 \le x_1 \le 1 - x_3$ and $x_3 \le x_2$.



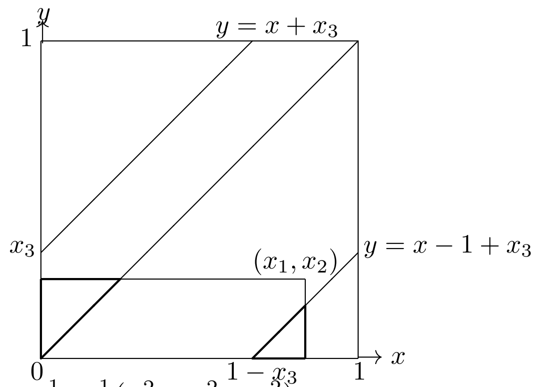

In this case $g(x_1, x_2, x_3) = -\frac{1}{2}x_1^2 + x_1x_2$, where $x_1 \le x_2 \le x_3$ and $x_1 \le 1 - x_3$.



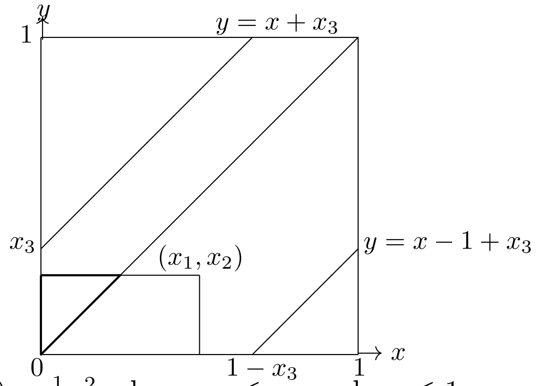

In this case $g(x_1, x_2, x_3) = \frac{1}{2} + \frac{1}{2}x_1^2 - x_1 - x_3 + x_2x_3 + x_1x_3$, where $x_3 \leq x_2 \leq x_1$ and $x_1 \geq 1 - x_3$.



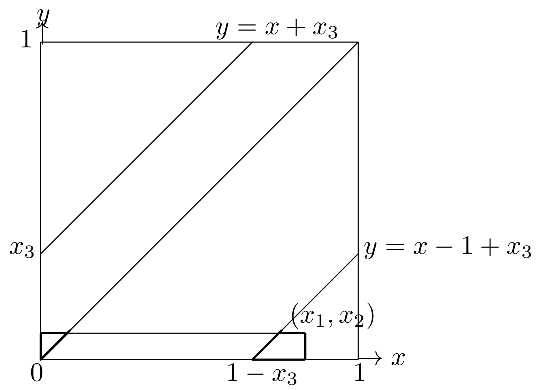

In this case $g(x_1, x_2, x_3) = \frac{1}{2} - x_1 - x_3 + x_1x_3 + x_1x_2 + x_2x_3 - \frac{1}{2}x_1^2$, where $x_1 \leq x_2$ and $x_1 \geq 1 - x_3$.




In this case $g(x_1, x_2, x_3) = \frac{1}{2} + \frac{1}{2}(x_1^2 + x_2^2 + x_3^2) - x_1 - x_3 + x_1x_3$ where $x_2 \leq x_3$ and $x_1 - x_2 \leq 1 - x_3 \leq x_1$.




In this case $g(x_1,x_2,x_3)=\frac{1}{2}x_2^2$ where $x_2\le x_3$ and $x_1\le 1-x_3$.




In this case $g(x_1,x_2,x_3)=x_1x_2+x_2x_3-x_2$ where $x_2\le x_3$ and $x_2\le x_1-(1-x_3)$.

$\square$

$G_{3,2}$ has the following properties:

• For every $g(x,y,z)\in G_{3,2}$ and fixed $z_0$, $0<z_0\le 1$ we have $\frac{1}{z_0}g(x,y,z_0)\in G_{2,1}$. Vice versa, if $g_z(x,y)$, $z\in[0,1]$ is a system of d.f.s in $G_{2,1}$ such that

(i) $g_1(x,y)=xy$;

(ii) for every $z'\le z$, we have $z'\mathrm{d}_x\mathrm{d}_y g_{z'}(x,y)\le z\mathrm{d}_x\mathrm{d}_y g_z(x,y)$ on $[0,1]^2$,

then $g(x,y,z)=zg_z(x,y)\in G_{3,2}$.

This directly follows from definition of $G_{2,1}$ and $G_{3,2}$ and that $g(x,y,z)$ is non-decreasing on $[0,1]^3$ if every $(x_1,y_1,z_1),(x_2,y_2,z_2)\in[0,1]^3$, $x_1\le x_2$, $y_1\le y_2$, $z_1\le z_2$ satisfies

$$g(x_1,y_1,z_2)+g(x_2,y_2,z_2)-g(x_1,y_2,z_2)-g(x_2,y_1,z_2)$$

$$-(g(x_1,y_1,z_1)+g(x_2,y_2,z_1)-g(x_1,y_2,z_1)-g(x_2,y_1,z_1))\geq 0.$$

## 8.5 Sequences $(x_n, y_n)$ with both u.d. $x_n$ and $y_n$

See also Section 3.10:

Let $(x_n, y_n)$, $n = 1, 2, \dots$, be a sequence in $[0,1)^2$ such that both the coordinate sequences $x_n$, $n = 1, 2, \dots$, and $y_n$, $n = 1, 2, \dots$ are u.d. Then the set $G((x_n, y_n))$ of all d.f. of $(x_n, y_n)$, $n = 1, 2, \dots$ satisfies

(i) $G((x_n, y_n)) \subset G_{2,1}$,

(ii) $G((x_n, y_n))$ is nonempty, closed and connected, and vice-versa

(iii) for every nonempty, closed and connected $H \subset G_{2,1}$, there exists a sequence $(x_n, y_n) \in [0,1)^2$ such that $G((x_n, y_n)) = H$.

This is a two-dimensional version of Theorem 200 with the metric
$\rho(g_1, g_2) = \left(\int_0^1\int_0^1 (g_1(x,y) - g_2(x,y))^2 \mathrm{d}x\mathrm{d}y\right)^{1/2}$.

Directly, from the theory of $L^2$ discrepancies [158] in Section 7.3 it follows:

**Theorem 210.** *Let $(x_n, y_n)$, $n = 1, 2, \dots$, be a sequence in $[0,1)^2$ such that both the coordinate sequences $x_n$, $n = 1, 2, \dots$, and $y_n$, $n = 1, 2, \dots$ are u.d. Then the sequence $(x_n, y_n)$, $n = 1, 2, \dots$ is u.d. if and only if one of the following conditions is satisfies:*
(i) $\lim_{N\to\infty} \frac{1}{N^2}\sum_{m,n=1}^N F_0((x_m, y_m),(x_n, y_n)) = 0$;
(ii) $\lim_{N\to\infty} \frac{1}{N^4}\sum_{m,n,k,l=1}^N F_1((x_m, y_m),(x_n, y_n),(x_k, y_k),(x_l, y_l)) = 0$.

Applying [158, Th. 4] (cf. [171, p. 1–56, 1.10.9]) for searching $G((x_n, y_n))$ the following theorem can be used.

**Theorem 211.** *Let $(x_n, y_n)$, $n = 1, 2, \dots$, be a sequence in $[0,1)^2$ for which both coordinate sequences $x_n$, $n = 1, 2, \dots$ and $y_n$, $n = 1, 2, \dots$ are u.d. Let $F(x, y, u, v, )$ be a continuous function defined on $[0,1]^4$ and assume that*

$$\lim_{N\to\infty}\frac{1}{N^2}\sum_{m,n=1}^N F(x_m, y_m, x_n, y_n) = 0.$$

*Then every d.f. $g(x,y) \in G((x_n, y_n))$ is a copula which satisfies the following equation:*

$\int_0^1\int_0^1 g(u,v)\mathrm{d}_u\mathrm{d}_v F(1,1,u,v)$
$+\int_0^1\int_0^1 g(x,y)\mathrm{d}_x\mathrm{d}_y F(x,y,1,1)$
$-\int_0^1\int_0^1\int_0^1 g(u,v)y\mathrm{d}_y\mathrm{d}_u\mathrm{d}_v F(1,y,u,v)$
$-\int_0^1\int_0^1\int_0^1 g(u,v)x\mathrm{d}_x\mathrm{d}_u\mathrm{d}_v F(x,1,u,v)$
$-\int_0^1\int_0^1\int_0^1 g(x,y)v\mathrm{d}_v\mathrm{d}_x\mathrm{d}_y F(x,y,1,v)$
$-\int_0^1\int_0^1\int_0^1 g(x,y)u\mathrm{d}_u\mathrm{d}_x\mathrm{d}_y F(x,y,u,1)$
$+\int_0^1\int_0^1\int_0^1\int_0^1 g(x,y)g(u,v)\mathrm{d}_u\mathrm{d}_v\mathrm{d}_x\mathrm{d}_y F(x,y,u,v)$
$=-F(1,1,1,1)+\int_0^1 v\mathrm{d}_v F(1,1,1,v)+\int_0^1 u\mathrm{d}_u F(1,1,u,1)$
$+\int_0^1 x\mathrm{d}_x F(x,1,1,1)+\int_0^1 y\mathrm{d}_y F(1,y,1,1)$
$-\int_0^1\int_0^1 yv\mathrm{d}_y\mathrm{d}_v F(1,y,1,v)-\int_0^1\int_0^1 yu\mathrm{d}_y\mathrm{d}_u F(1,y,u,1)$
$-\int_0^1\int_0^1 xv\mathrm{d}_x\mathrm{d}_v F(x,1,1,v)-\int_0^1\int_0^1 xu\mathrm{d}_x\mathrm{d}_u F(x,1,u,1)=0.$

*Proof.* By Helly theorem
$\lim_{k\to\infty}\frac{1}{N_k^2}\sum_{m,n=1}^{N_k} F(x_m,y_m,x_n,y_n)$
$=\int_0^1\int_0^1\int_0^1\int_0^1 F(x,y,u,v)\mathrm{d}g(x,y)\mathrm{d}g(u,v)=0$
for $F_{N_k}(x,y)\to g(x,y)$. Next, for the 4-dimensional integral we use Theorem 203 and then apply that $g(x,1)=x$ and $g(1,y)=y$. [56] □

As an application we give the following example.

**Example 88.** Let $p_n$, $n=1,2,\dots$, be the increasing sequence of all primes. For the two-dimensional sequence

$$(x_n,y_n)=\left(\frac{p_i}{p_n},\frac{i}{n}\right),\quad i=1,2,\dots,n$$

the coordinate sequences $x_n$ and $y_n$ are u.d., see [173] (cf. [171, p. 2–181, 2.19.16]). Using the well-known $p_n=n\log n+o(n\log n)$ we have that

$$\frac{1}{n}\sum_{i=1}^{n}\frac{p_i}{p_n}\frac{i}{n}\to\frac{1}{3}.$$

Putting $F(x,y,u,v)=xyuv-\frac{1}{9}$ and applying Theorem 211 we find that every d.f. $g(x,y)\in G((x_n,y_n))$ satisfies

$$\int_0^1\int_0^1 g(x,y)\mathrm{d}x\mathrm{d}y=\frac{1}{3},$$

[56] Extending one-dimensional additional assumptions of [S2, Lemma 4](cf. [SP, p. 1–56, Th. 1.10.9.1]) to the two-dimensional sequence $(x_n,y_n)$, $n=1,2,\dots$, $G((x_n,y_n))$ it can be characterized by Theorem 211.

thus the sequence $(x_n, y_n)$, $n = 1, 2, \dots$ is not u.d. in $[0,1)^2$.

**Example 89.** By Theorem 5 in [127] for $x_n = n\alpha \bmod 1$, $n = 1, 2, \dots$, $\alpha$ irrational, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |x_{n+1} - x_n| = 2\{\alpha\}(1 - \{\alpha\}). \tag{556}$$

*Proof.* Applying d.f. Because

$$\{(n+1)\alpha\} = \begin{cases} \{n\alpha\} + \{\alpha\}, & \text{if } \{n\alpha\} + \{\alpha\} < 1, \\ \{n\alpha\} + \{\alpha\} - 1, & \text{if } \{n\alpha\} + \{\alpha\} \geq 1, \end{cases} \tag{557}$$

then every point $(\{n\alpha\}, \{(n+1)\alpha\})$ lies on the line $Y = X + \{\alpha\} \bmod 1$. Using this and u.d. of $\{n\alpha\}$ we can compute a.d.f. $g(x,y)$ of the sequence $(\{n\alpha\}, \{(n+1)\alpha\})$, $n = 1, 2, \dots$ by the following Figure and similarly to (593).

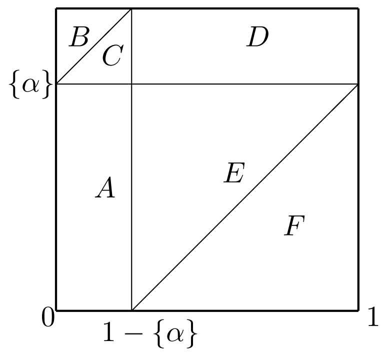



Figure: Graph of the line $Y = x + \{\alpha\} \bmod 1$.

If $\{\alpha\} > \frac{1}{2}$, then the copula $g(x,y)$ has the form

$$g(x,y) = \begin{cases} 0, & \text{if } (x,y) \in A, \\ x, & \text{if } (x,y) \in B, \\ \{\alpha\} - y, & \text{if } (x,y) \in C, \\ \{\alpha\} - y + x - (1 - \{\alpha\}), & \text{if } (x,y) \in D, \\ x - (1 - \{\alpha\}), & \text{if } (x,y) \in E, \\ y, & \text{if } (x,y) \in F \end{cases} \tag{558}$$

and from it follows

$$g(x,x)=\begin{cases}0, & \text{if } x\in[0,1-\{\alpha\}],\\ x-(1-\{\alpha\}), & \text{if } x\in[1-\{\alpha\},\{\alpha\}],\\ x-(1-\{\alpha\})+x-\{\alpha\}, & \text{if } x\in[\{\alpha\},1].\end{cases} \tag{559}$$

Then (559) and (579) implies (556), since

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}|\{n\alpha\}-\{(n+1)\alpha\}|=1-2\int_0^1 g(x,x)\mathrm{d}x=2\{\alpha\}(1-\{\alpha\}).$$

Similarly for $\{\alpha\}<\frac{1}{2}$. □

Note that simplify (557) to

$$\{(n+1)\alpha\}=\begin{cases}\{n\alpha\}+\{\alpha\}, & \text{if } \{n\alpha\}\in[0,1-\{\alpha\}),\\ \{n\alpha\}+\{\alpha\}-1, & \text{if } \{n\alpha\}\in[1-\{\alpha\},1)\end{cases}$$

we have

$$|\{(n+1)\alpha\}-\{n\alpha\}|=\begin{cases}\{\alpha\}, & \text{if } \{n\alpha\}\in[0,1-\{\alpha\}),\\ 1-\{\alpha\}, & \text{if } \{n\alpha\}\in[1-\{\alpha\},1)\end{cases} \tag{560}$$

and then for $F(x,y)=|x-y|$ we can use

$$\begin{aligned}\frac{1}{N}\sum_{n=1}^{N}F(\{n\alpha\},\{(n+1)\alpha\})&\to\int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y)\\ &=\int_0^{1-\{\alpha\}}F(x,x+\{\alpha\})\mathrm{d}x+\int_{1-\{\alpha\}}^1 F(x,x+\{\alpha\}-1)\mathrm{d}x\end{aligned}$$

which gives (556), again.
More generally: Let
   $x_n\in[0,1)$ be u.d.,
   $y_n=f(x_n)$, where $f:[0,1)\to[0,1)$ be continuous,
   $F(x,y)$ be continuous,
   $g(x,y)$ be d.f. of $(x_n,y_n)$. Then, simultaneously

$$\frac{1}{N}\sum_{n=1}^{N}F(x_n,y_n)\to\int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y)$$

$$\frac{1}{N}\sum_{n=1}^{N}F(x_n,f(x_n))\to\int_0^1 F(x,f(x))\mathrm{d}x.$$

**Example 90.** Similar method can be used to prove Steinerberger [151] result that

$$\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}(x_n-y_n)^2\le\frac{1}{3} \tag{561}$$

for every two u.d. sequences $x_n$ and $y_n$ in $[0,1)$.

Put $F(x,y)=(x-y)^2$. Then
$F(1,1)=0$;
$\mathrm{d}_yF(1,y)=-2(1-y)\mathrm{d}y$;
$\mathrm{d}_xF(x,1)=-2(1-x)\mathrm{d}x$;
$\mathrm{d}_x\mathrm{d}_yF(x,y)=-2\mathrm{d}x\mathrm{d}y$.
Applying (573) then for every copula $g(x,y)$ we have

$$\int_0^1\int_0^1(x-y)^2\mathrm{d}_x\mathrm{d}_yg(x,y)=\frac{2}{3}-2\int_0^1\int_0^1 g(x,y)\mathrm{d}x\mathrm{d}y.$$

Now, using lower bound in (575) and computing

$$\int_0^1\int_0^1\max(x+y-1,0)\mathrm{d}x\mathrm{d}y=\frac{1}{6}$$

then we have (561) in the form

$$\int_0^1\int_0^1(x-y)^2\mathrm{d}_x\mathrm{d}_yg(x,y)\le\frac{1}{3}$$

for an arbitrary copula $g(x,y)$.

**Theorem 212.** *For every copula $g(x,y)$ we have*

$$\int_0^1\int_0^1|x-y|\mathrm{d}_x\mathrm{d}_yg(x,y)\le\frac{1}{2}.$$

*Proof.* Input $F(x,y)=|x-y|$ to (541) and bearing in mind $F(1,1)=0$, $F(1,y)=1-y$, $F(x,1)=1-x$,
$\mathrm{d}_x\mathrm{d}_y|x-y|$
$=\mathrm{d}_x\mathrm{d}_y(y-x)=(y+\mathrm{d}y-(x+\mathrm{d}x))+(y-x)-(y-(x+\mathrm{d}x))-(y+\mathrm{d}y-x)=0$
for $y>x$, similarly for $y<x$ and for $y=x$, $\mathrm{d}x=\mathrm{d}y$

$\mathrm{d}_x\mathrm{d}_y|x-y|$
$= |x+\mathrm{d}x-(x+\mathrm{d}x)| + |x-x| - |(x+\mathrm{d}x)-x| - |x-(x+\mathrm{d}x)| = -2\mathrm{d}x$
we have

$$\int_0^1\int_0^1 |x-y|\mathrm{d}_x\mathrm{d}_y g(x,y) = 1-2\int_0^1 g(x,x)\mathrm{d}x. \tag{562}$$

Thus if $g_1(x,y) \le g_2(x,y)$ for $(x,y)\in[0,1]^2$, then

$$\int_0^1\int_0^1 |x-y|\mathrm{d}_x\mathrm{d}_y g_2(x,y) \le \int_0^1\int_0^1 |x-y|\mathrm{d}_x\mathrm{d}_y g_1(x,y).$$

The lower bond $\max(x+y-1,0)\le g(x,y)$ and

$$\int_0^1\int_0^1 |x-y|\mathrm{d}_x\mathrm{d}_y \max(x+y-1,0) = \frac{1}{2}$$

implies theorem. □

New, let again $x_n$ and $y_n$, $n=1,2,\dots$, be two u.d. sequences in the unit interval $[0,1)$. F. Pillichshammer and S. Steinerberger in [127] study the sequence $\frac{1}{N}\sum_{n=1}^N |x_n-y_n|$ and proved that

$$\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^N |x_n-y_n| \le \frac{1}{2}. \tag{563}$$

Putting $y_n = x_{n+1}$, $n=1,2,\dots$, they found a new necessary condition

$$\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^N |x_{n+1}-x_n| \le \frac{1}{2} \tag{564}$$

for u.d. of the sequence $x_n$. They also found

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} |x_{n+1}-x_n| = \frac{2(b-1)}{b^2}$$

for van der Corput sequence $x_n=\gamma_b(n)$ in the base $b$ and

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} |x_{n+1}-x_n| = 2\{\alpha\}(1-\{\alpha\})$$

for $x_n = n\alpha \bmod 1$, where $\alpha$ is irrational. Alternative proofs via d.f.s are in Examples 92 and 89.

## 8.6 Sequences $(x_n, y_n, z_n)$ where $(x_n, y_n)$, $(x_n, z_n)$, and $(y_n, z_n)$ are u.d.

**Theorem 213.** *Let $(x_n, y_n, z_n)$, $n = 1, 2, \dots,$ be a sequence in $[0,1)^3$ such that both the marginal sequences $(x_n, y_n)$, $n = 1, 2, \dots,$ $(x_n, z_n)$, $n = 1, 2, \dots$ and $(y_n, z_n)$, $n = 1, 2, \dots$ are u.d. in $[0,1)^2$. Then the sequence $(x_n, y_n, z_n)$, $n = 1, 2, \dots$ is u.d. if and only if one of the following conditions is satisfies:*

(i) $\displaystyle\lim_{N\to\infty} \frac{1}{N^2} \sum_{m,n=1}^{N} F_0((x_m, y_m, z_m), (x_n, y_n, z_n)) = 0;$

(ii) $$\lim_{N\to\infty} \frac{1}{N^6} \sum_{m,n,k,l=1}^{N} F_1((x_m, y_m, z_m), (x_n, y_n, z_n), (x_k, y_k, z_k), (x_{m'}, y_{m'}, z_{m'}), (x_{n'}, y_{n'}, z_{n'}), (x_{k'}, y_{k'}, z_{k'})) = 0;$$

(iii) $$\lim_{N\to\infty} \frac{1}{N^4} \sum_{m,n,k,l=1}^{N} F_1((x_m, y_m, z_m), (x_n, y_n, z_n), (x_{m'}, y_{m'}, z_{m'}), (x_{n'}, y_{n'}, z_{n'})) = 0.$$

*Here*

$$\begin{aligned}
&F_1((x_m, y_m, z_m), (x_n, y_n, z_n), (x_k, y_k, z_k), (x_{m'}, y_{m'}, z_{m'}), (x_{n'}, y_{n'}, z_{n'}),\\
&(x_{k'}, y_{k'}, z_{k'})) =\\
&= \Big((1 - \max(x_m, x_{m'}))(1 - \max(y_m, y_{m'}))(1 - \max(z_m, z_{m'}))\\
&+ (1 - \max(x_m, x_{m'}))(1 - \max(y_n, y_{n'}))(1 - \max(z_k, z_{k'}))\\
&- 2(1 - \max(x_m, x_{m'}))(1 - \max(y_m, y_{n'}))(1 - \max(z_m, z_{k'}))\Big);\\
&F_1((x_m, y_m, z_m), (x_n, y_n, z_n), (x_{m'}, y_{m'}, z_{m'}), (x_{n'}, y_{n'}, z_{n'})) =\\
&= \Big((1 - \max(x_m, x_n))(1 - \max(y_m, y_n))(1 - \max(z_m, z_n))\\
&+ (1 - \max(x_m, x_{m'}))(1 - \max(y_m, y_{m'}))(1 - \max(z_n, z_{n'}))\\
&- 2(1 - \max(x_m, x_{m'}))(1 - \max(y_m, y_{m'}))(1 - \max(z_m, z_{n'}))\Big);
\end{aligned}$$

*and $F_0((x_m, y_m, z_m), (x_n, y_n, z_n))$ is given in* (517).

*Proof.* Assume that the marginal sequences $(x_n, y_n)$, $(y_n, z_n)$ and $(x_n, z_n)$ of $(x_n, y_n, z_n)$ are u.d. Using the theory of $L^2$ discrepancy, every of the following zero-limits

$$\text{(i)} \lim_{N\to\infty} \int_0^1 \int_0^1 \int_0^1 (F_N(x,y,z) - xyz)^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z = 0,$$

$$\text{(ii)} \lim_{N\to\infty} \int_0^1 \int_0^1 \int_0^1 \left(F_N(x,y,z) - F_N^{(1)}(x)F_N^{(2)}(y)F_N^{(3)}(z)\right)^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z = 0,$$

$$\text{(iii)} \lim_{N\to\infty} \int_0^1 \int_0^1 \int_0^1 \left(F_N(x,y,z) - F_N(x,y)F_N^{(3)}(z)\right)^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z = 0,$$

implies the u.d. of the sequence $(x_n, y_n, z_n)$. Here (i) is the limit of the classical $L^2$ discrepancy characterizing u.d. of $(x_n, y_n, z_n)$ and limits (ii) and (iii) are of $L^2$ discrepancies of statistical independence. The (ii) implies that every d.f. $g(x,y,z) \in G((x_n, y_n, z_n))$ has the form $g(x,y,z) = g(x,1,1).g(1,y,1).g(1,1,z)(= x.y.z)$ and (iii) implies

$$g(x,y,z) = g(x,y,1).g(1,1,z)(= xy.z)$$

. $\square$

## 8.7 Method of d.f.s for $\frac{1}{N}\sum_{n=1}^{N} F(x_n, y_n)$

Let $F(x,y)$ be a continuous function defined on $[0,1]^2$ and $x_n$, $y_n$, $n = 1, 2, \dots$, be two sequences in $[0,1)$. In this section we study the limit points of the sequence of arithmetic means

$$\frac{1}{N}\sum_{n=1}^{N} F(x_n, y_n), \quad N = 1, 2, \dots. \tag{565}$$

To do this we use a theory of distribution functions (d.f.s). This theory we also use for computing integrals of the type

$$\int_0^1 f_1(\Phi(x)) f_2(\Psi(x)) dx \tag{566}$$

where $f_1, f_2$ are Riemann integrable functions and $\Phi, \Psi : [0,1] \to [0,1]$ are so called uniformly distribution preserving (u.d.p.) map (see Section 12.3). This problem was introduced by Steinerberger [151, p. 127].

Recapitulation:

Let $x_n$ and $y_n$, $n = 1, 2, \dots$, be an arbitrary two sequences in $[0, 1)$. Denote the step d.f.

$$F_N(x, y) = \frac{\#\{n \le N; (x_n, y_n) \in [0, x) \times [0, y)\}}{N},$$

and let $G((x_n, y_n))$ be the set of all possible limits $F_{N_k}(x, y) \to g(x, y)$, which hold for all continuity points $(x, y)$ of $g(x, y)$. These $g(x, y)$ are called d.f.s of the sequence $(x_n, y_n)$, $n = 1, 2, \dots$. If $G((x_n, y_n))$ is singleton, i.e., $G((x_n, y_n)) = \{g(x, y)\}$ then $g(x, y)$ is called asymptotic d.f. (a.d.f.) of $(x_n, y_n)$, $n = 1, 2, \dots$.

By Riemann-Stieltjes integration we have

$$\frac{1}{N} \sum_{n=1}^{N} F(x_n, y_n) = \int_0^1 \int_0^1 F(x, y) \mathrm{d}_x \mathrm{d}_y F_N(x, y) \tag{567}$$

a) If $F_{N_k}(x, y) \to g(x, y)$ as $k \to \infty$ then by the second Helly theorem the equation (567) implies

$$\frac{1}{N_k} \sum_{n=1}^{N_k} F(x_n, y_n) \to \int_0^1 \int_0^1 F(x, y) \mathrm{d}_x \mathrm{d}_y g(x, y) \quad \text{as } k \to \infty. \tag{568}$$

b) If $\frac{1}{N_k} \sum_{n=1}^{N_k} F(x_n, y_n) \to A$ as $k \to \infty$ then by the first Helly theorem there exists subsequence $N'_k$ of $N_k$ such that for some d.f. $g(x, y)$ of $(x_n, y_n)$ we have $F_{N'_k}(x, y) \to g(x, y)$.

Summary, the set of limit points of (565) has the form

$$\left\{ \int_0^1 \int_0^1 F(x, y) \mathrm{d}_x \mathrm{d}_y g(x, y); g(x, y) \in G((x_n, y_n)) \right\}. \tag{569}$$

In (569) we can apply

$$\begin{aligned}
\int_0^1 \int_0^1 F(x, y) \mathrm{d}_x \mathrm{d}_y g(x, y) &= F(1, 1) - \int_0^1 g(1, y) \mathrm{d}_y F(1, y) \\
- \int_0^1 g(x, 1) \mathrm{d}_x F(x, 1) &+ \int_0^1 \int_0^1 g(x, y) \mathrm{d}_x \mathrm{d}_y F(x, y).
\end{aligned} \tag{570}$$

It is proved by integration by parts in Riemann-Stieltjes integration in Theorem 202, or by induction in Theorem 216, p. 372.

Thus problem is to find extreme values of $\int_0^1 \int_0^1 F(x, y) \mathrm{d}_x \mathrm{d}_y g(x, y)$, where $g(x, y)$ is a copula.

## 8.8 Boundaries of $\frac{1}{N}\sum_{n=1}^{N} F(x_n, y_n)$

Let $x_n, y_n \in [0,1)$, $n = 1,2,\dots$, both u.d. sequences. In this case two-dimensional sequence $(x_n, y_n)$ need not be u.d. but every d.f. $g(x,y) \in G((x_n, y_n))$ satisfies

(i) $g(x,1) = x$ for $x \in [0,1]$ and

(ii) $g(1,y) = y$ for $y \in [0,1]$.

The d.f. $g(x,y)$ satisfying (i) and (ii) is called copula and its basic theory can be found in R.B. Nelsen [114]. Applying this theory we find (see also Theorem 217)

**Theorem 214.** *Let $F(x,y)$ be a continuous function defined on $[0,1]^2$. For differential of $F(x,y)$ assume that $\mathrm{d}_x\mathrm{d}_yF(x,y) > 0$ for every $(x,y) \in [0,1]^2$. Then for every two u.d. sequences $x_n, y_n \in [0,1)$, $n = 1,2,\dots$, we have*

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} F(x_n, y_n) \le \int_0^1 F(x,x)\mathrm{d}x, \tag{571}$$

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} F(x_n, y_n) \ge \int_0^1 F(x,1-x)\mathrm{d}x. \tag{572}$$

*Furthermore, for the sequence $(x_n, y_n)$, $n = 1,2,\dots$, we have*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} F(x_n, y_n) = \int_0^1 F(x,x)\mathrm{d}x$$

*if and only if $(x_n, y_n)$ has a.d.f. $g(x,y) = \min(x,y)$, and we have*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} F(x_n, y_n) = \int_0^1 F(x,1-x)\mathrm{d}x$$

*if and only if $g(x,y) = \max(x+y-1, 0)$.*

*If $\mathrm{d}_x\mathrm{d}_yF(x,y) < 0$ the right hand sides of (571) and (572) are exchanged.*

*Proof.* For a copula $g(x,y)$ the equation (570) has the form

$$\int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y) = F(1,1) - \int_0^1 y\mathrm{d}_yF(1,y)$$
$$- \int_0^1 x\mathrm{d}_xF(x,1) + \int_0^1\int_0^1 g(x,y)\mathrm{d}_x\mathrm{d}_yF(x,y). \tag{573}$$

Thus the set (569) of limit points of (565) coincide with

$$\left\{\int_0^1\int_0^1 g(x,y)\mathrm{d}_x\mathrm{d}_yF(x,y)\right\} \tag{574}$$

shifting by $F(1,1)-\int_0^1 y\mathrm{d}_yF(1,y)-\int_0^1 x\mathrm{d}_xF(x,1)$. Then we use Fréchet-Hoeffding bounds [114, p. 9]

$$\max(x+y-1,0)\le g(x,y)\le\min(x,y) \tag{575}$$

which holds for every $(x,y)\in[0,1]^2$ and for every copula $g(x,y)$. The assumption $\mathrm{d}_x\mathrm{d}_yF(x,y)>0$ implies

$$\begin{aligned}\int_0^1\int_0^1\max(x+y-1,0)\mathrm{d}_x\mathrm{d}_yF(x,y)&\le\int_0^1\int_0^1 g(x,y)\mathrm{d}_x\mathrm{d}_yF(x,y)\\&\le\int_0^1\int_0^1\min(x,y)\mathrm{d}_x\mathrm{d}_yF(x,y).\end{aligned}$$

Since every copula is continuous, then the left inequality is attained if and only if $g(x,y)=\max(x+y-1,0)$ and the right if and only if $g(x,y)=\min(x,y)$.

Directly by definition of a.d.f., for every u.d. sequence $x_n\in[0,1)$, it can be proved that

a) the sequence $(x_n,x_n)$, $n=1,2,\dots$, $x_n$ is not u.d., has the a.d.f. $g(x,y)=\min(x,y)$ and

b) the sequence $(x_n,1-x_n)$, $n=1,2,\dots$, has the a.d.f. $g(x,y)=\max(x+y-1,0)$. From it

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^N F(x_n,x_n)=\int_0^1 F(x,x)\mathrm{d}x=\int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y\min(x,y), \tag{576}$$

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^N F(x_n,1-x_n)&=\int_0^1 F(x,1-x)\mathrm{d}x\\&=\int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y\max(x+y-1,0).\end{aligned} \tag{577}$$

□

Once again in the following we prove the result (563) appeared in Pillichshammer and Steinerberger [127] repeating the proof of Theorem 212.

**Example 91.** Putting $F(x,y)=|x-y|$, we have $F(1,1)=0$, $F(1,x)=1-x$, $F(y,1)=1-y$, and computing, for $y>x$,

$$\mathrm{d}_x\mathrm{d}_y|y-x| = (y+\mathrm{d}y-(x+\mathrm{d}x)) = (y-x)-(y-(x+\mathrm{d}x))-(y+\mathrm{d}y-x) = 0,$$

and for $y=x$, $\mathrm{d}y=\mathrm{d}x$,

$$\mathrm{d}_x\mathrm{d}_y|y-x| = |x+\mathrm{d}x-(x+\mathrm{d}x)|+|x-x|-|(x+\mathrm{d}x)-x|-|x-(x+\mathrm{d}x)| = -2\mathrm{d}x$$

and then applying (573) we have

$$\int_0^1\int_0^1 |x-y|\mathrm{d}_x\mathrm{d}_y g(x,y) = \int_0^1 g(x,1)\mathrm{d}x + \int_0^1 g(1,y)\mathrm{d}y - 2\int_0^1 g(x,x)\mathrm{d}x. \tag{578}$$

Thus for a copula $g(x,y)$, $g(x,1)=x$, $g(1,y)=y$ we have

$$\int_0^1\int_0^1 |x-y|\mathrm{d}_x\mathrm{d}_y g(x,y) = 1-2\int_0^1 g(x,x)\mathrm{d}x. \tag{579}$$

Finally, the lower bound in (575) for copulas $g(x,y)$ give F. Pillichshammer and S. Steinerberger result [127] in the form

$$\int_0^1\int_0^1 |x-y|\mathrm{d}_x\mathrm{d}_y g(x,y) \le \int_0^1\int_0^1 |x-y|\mathrm{d}_x\mathrm{d}_y \max(x+y-1,0) = \frac{1}{2}. \tag{580}$$

## 8.9 D.f. $g(x,y)$ with given marginal $g(1,y)$ and $g(x,1)$

In the following theorem the sequences $x_n$ and $y_n$ are not be u.d., but with assigned a.d.f. $g_1(x)$ and $g_2(x)$, respectively.

**Theorem 215.** *Let $x_n\in[0,1)$ be a sequence with an a.d.f. $g_1(x)$ and $y_n\in[0,1)$ with an a.d.f. $g_2(x)$. Assume that $F(x,y)$ is a continuous function such that $\mathrm{d}_x\mathrm{d}_yF(x,y)\ge 0$ for every $(x,y)\in[0,1]^2$. Then we have*

$$\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^N F(x_n,y_n) \le \int_0^1 F(g_1^{-1}(x),g_2^{-1}(x))\mathrm{d}x, \tag{581}$$

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} F(x_n,y_n) \geq \int_0^1 F(g_1^{-1}(x), g_2^{-1}(1-x))\mathrm{d}x. \tag{582}$$

*Furthermore, for the sequence* $(x_n, y_n)$, $n = 1, 2, \dots,$ *we have*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} F(x_n,y_n) = \int_0^1 F(g_1^{-1}(x), g_2^{-1}(x))\mathrm{d}x \tag{583}$$

*if* $(x_n, y_n)$ *has the a.d.f.* $g(x,y) = \min(g_1(x), g_2(y))$ *and we have*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} F(x_n,y_n) = \int_0^1 F(g_1^{-1}(x), g_2^{-1}(1-x))\mathrm{d}x \tag{584}$$

*if* $g(x,y) = \max(g_1(x) + g_2(y) - 1, 0)$.

*If* $\mathrm{d}_x\mathrm{d}_y F(x,y) \leq 0$ *the right hand sides of* (581) *and* (582) *are exchanged.*

*Proof.* Let $g(x,y)$ be a d.f. of the sequence $(x_n, y_n)$, i.e. there exists a sequence $N_k \to \infty$ such that $F_{N_k}(x,y) \to g(x,y)$ and

$$\lim_{k\to\infty} \frac{1}{N_k}\sum_{n=1}^{N_k} F(x_n,y_n) \to \int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y).$$

Furthermore we have $g(x,1) = g_1(x)$ and $g(1,y) = g_2(y)$. Then by the Sklar's theorem for any such d.f. $g(x,y)$ there exists a copula $c(x,y)$ such that

$$g(x,y) = c(g_1(x), g_2(y))$$

for every $(x,y) \in [0,1]^2$. Applying the Fréchet-Hoeffding bounds (575) we find

$$\max(g_1(x) + g_2(y) - 1, 0) \leq g(x,y) \leq \min(g_1(x), g_2(y)). \tag{585}$$

Now, apply (573) we find (581) and (582).

Let $z_n$, $n = 1, 2, \dots,$ be u.d. sequence in $[0,1)$. Then the sequence

(i) $(g_1^{-1}(z_n), g_2^{-1}(z_n))$ has the a.d.f. $g(x,y) = \min(g_1(x), g_2(y))$, and

(ii) $(g_1^{-1}(z_n), g_2^{-1}(1-z_n))$ has the a.d.f. $g(x,y) = \max(g_1(x)+g_2(y)-1, 0)$.

By Helly theorem, for sequences (i) and (ii) we have (583) and (584)

$$\lim_{k\to\infty} \frac{1}{N_k}\sum_{n=1}^{N_k} F(g_1^{-1}(z_n), g_2^{-1}(z_n)) \to \int_0^1 F(g_1^{-1}(x), g_2^{-1}(x))\mathrm{d}x$$

$$= \int_0^1 \int_0^1 F(x,y) \mathrm{d}_x \mathrm{d}_y \min(g_1(x), g_2(y)),$$

$$\lim_{k\to\infty} \frac{1}{N_k} \sum_{n=1}^{N_k} F(g_1^{-1}(z_n), g_2^{-1}(1-z_n)) \to \int_0^1 F(g_1^{-1}(x), g_2^{-1}(1-x)) \mathrm{d}x$$

$$= \int_0^1 \int_0^1 F(x,y) \mathrm{d}_x \mathrm{d}_y \max(g_1(x) + g_2(y) - 1, 0).$$

Proof of (i) and (ii):

Assume that $g_1(x)$ and $g_2(x)$ are strictly increasing.
For (i) we note that $g_1^{-1}(z_n) < x \Leftrightarrow z_n < g_1(x)$ and $g_2^{-1}(z_n) < y \Leftrightarrow z_n < g_2(y)$ and thus $(g_1^{-1}(z_n), g_2^{-1}(z_n)) \in [0,x) \times [0,y) \Leftrightarrow z_n \in [0, \min(g_1(x), g_2(y)))$.
For (ii) we note that $g_1^{-1}(z_n) < x \Leftrightarrow z_n < g_1(x)$ and $g_2^{-1}(1-z_n) < y \Leftrightarrow 1 - z_n < g_2(y)$, $1 - z_n < g_2(y) \Leftrightarrow 1 - g_2(y) < z_n$ and thus $(g_1^{-1}(z_n), g_2^{-1}(1-z_n)) \in [0,x) \times [0,y) \Leftrightarrow z_n \in (1 - g_2(y), g_1(x))$.

On the contrary, if a d.f. $g(x)$ is constant on the interval $I = (\alpha, \beta)$ with value $c$ and to the left of $\alpha$ and to the right of $\beta$ d.f. $g(x)$ increases simultaneously, then we put in (i) and (ii)

$$g^{-1}(c) = \beta \tag{586}$$

because in this case the $g_1^{-1}(z_n) < x \Leftrightarrow z_n < g_1(x)$ also holds for $z_n = c$.

Finally, the uniqueness of extremal d.f. $g(x,y)$ follows from the existence of common point $(x,y)$ of continuity for any two d.f.s $g(x,y)$. □

Completing the above proof we prove (573) in the following discrete form (587) assuming $F_N(x,y) \to g(x,y)$.

**Theorem 216.** *Let $F(x,y)$ be a continuous function defined on $[0,1]^2$. Then for arbitrary $N$-terms sequence $(x_1,y_1),\dots,(x_N,y_N)$ in $[0,1)^2$ with the step d.f. $F_N(x,y,)$ we have*

$$\begin{aligned} N\int_0^1 \int_0^1 F(x,y) \mathrm{d}_x \mathrm{d}_y F_N(x,y) &= NF(1,1) - N\int_0^1 F_N(1,y) \mathrm{d}_y F(1,y) \\ - N\int_0^1 F_N(x,1) \mathrm{d}_x F(x,1) &+ N\int_0^1 \int_0^1 F_N(x,y) \mathrm{d}_x \mathrm{d}_y F(x,y). \end{aligned} \tag{587}$$

*Proof.* We employ induction. Assume that for $N$ the equation (587) holds and add in $(x_1,y_1),\dots,(x_N,y_N)$ a new point $(x_{N+1},y_{N+1})$ and exchange

$F_N(x,y)$ by $F_{N+1}(x,y)$. We have $(N+1)F(1,1) = NF(1,1) + F(1,1)$ and

$$\begin{aligned}(N+1)\int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y F_{N+1}(x,y) =& N\int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y F_N(x,y)\\ &+ F(x_{N+1},y_{N+1}),\end{aligned}$$

$$\begin{aligned}-(N+1)\int_0^1 F_{N+1}(x,1)\mathrm{d}_x F(x,1) =& -N\int_0^1 F_N(x,1)\mathrm{d}_x F(x,1)\\ &- 1.(F(1,1) - F(x_{N+1},1)),\end{aligned}$$

$$\begin{aligned}-(N+1)\int_0^1 F_{N+1}(1,y)\mathrm{d}_y F(1,y) =& -N\int_0^1 F_N(1,y)\mathrm{d}_y F(1,y)\\ &- 1.(F(1,1) - F(1,y_{N+1})),\end{aligned}$$

$$\begin{aligned}(N+1)\int_0^1\int_0^1 F_{N+1}(x,y)\mathrm{d}_x\mathrm{d}_y F(x,y) =& N\int_0^1\int_0^1 F_N(x,y)\mathrm{d}_x\mathrm{d}_y F(x,y)\\ +(F(1,1) + F(x_{N+1},y_{N+1}) &- F(x_{N+1},1) - F(1,y_{N+1})).\end{aligned}$$

Summing up

$$\begin{aligned}F(1,1) &- 1.(F(1,1) - F(x_{N+1},1)) - 1.(F(1,1) - F(1,y_{N+1}))\\ &+ (F(1,1) + F(x_{N+1},y_{N+1}) - F(x_{N+1},1) - F(1,y_{N+1}))\\ &= F(x_{N+1},y_{N+1})\end{aligned}$$

we have that (587) valid also for $N+1$. □

## 8.10 D.f. of $(x_n, x_{n+1}, \dots, x_{n+s-1})$ where $x_n$ is u.d.

### 8.10.1 Two-dimensional shifted van der Corput sequence

**Example 92.** For van der Corput sequence $x_n = \gamma_q(n)$, $n = 0, 1, \dots$, in the base $q$ we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}|x_{n+1} - x_n| = \frac{2(q-1)}{q^2}. \tag{588}$$

*Proof.* Every point $(\gamma_q(n), \gamma_q(n+1))$, $n = 0, 1, 2, \dots$, lie on the line segments

$$Y = X + \frac{1}{q}, \quad X \in \left[0, 1 - \frac{1}{q}\right], \tag{589}$$

$$Y = X - 1 + \frac{1}{q^i} + \frac{1}{q^{i+1}}, \quad X \in \left[1 - \frac{1}{q^i}, 1 - \frac{1}{q^{i+1}}\right], \quad i = 1, 2, \dots \tag{590}$$

*Proof.* Express an $n$ in the base $q$

$$n = n_k q^k + n_{k-1} q^{k-1} + \dots + n_1 q + n_0,$$

where $n_i < q$ and $n_k > 0$. We consider two following cases:

$1^0$. $n_0 < q - 1$,

$2^0$. $n_0 = q - 1$.

Let

$1^0.n_0 < q - 1$. Then

$n = n_k q^k + \dots + n_0$,

$n + 1 = n_k q^k + \dots + n_0 + 1$ and by (11)

$\gamma_q(n+1) - \gamma_q(n) = \frac{1}{q}$. In this case

$\gamma_q(n) = \frac{n_0}{q} + \dots + \frac{n_k}{q^{k+1}} \le \frac{q-2}{q} + \frac{q-1}{q^2} + \dots = \frac{q-1}{q}$.

Thus such $(\gamma_q(n), \gamma_q(n+1))$ lies on the line-segment (589).

Let

$2^0.n_0 = q - 1$. Then

$n = n_k q^k + \dots + n_{i+1} q^{i+1} + (q-1)q^i + (q-1)q^{i-1} + \dots + (q-1)$ and $n_{i+1} < q - 1$, where $i = 0, 1, 2, \dots$. Then

$n + 1 = n_k q^k + \dots + (n_{i+1} + 1)q^{i+1} + 0.q^i + 0.q^{i-1} + \dots + 0$. Thus

$\gamma_q(n) = \frac{q-1}{q} + \dots + \frac{q-1}{q^{i+1}} + \frac{n_{i+1}}{q^{i+2}} + \dots + \frac{n_k}{q^{k+1}}$,

$\gamma_q(n+1) = \frac{n_{i+1}+1}{q^{i+2}} + \dots + \frac{n_k}{q^{k+1}}$, and we have

$\gamma_q(n+1) - \gamma_q(n) = \frac{1}{q^{i+2}} - \frac{q-1}{q}(1 + \frac{1}{q} + \dots + \frac{1}{q^i}) = \frac{1}{q^{i+2}} - 1 + \frac{1}{q^{i+1}}$ and

$1 - \frac{1}{q^{i+1}} = \frac{q-1}{q} + \dots + \frac{q-1}{q^{i+1}} \le \gamma_q(n)$ and

$\gamma_q(n) \le \frac{q-1}{q} + \dots + \frac{q-1}{q^{i+1}} + \frac{q-2}{q^{i+2}} + \frac{q-1}{q^{i+3}} + \dots = 1 - \frac{1}{q^{i+2}}$. Thus such $(\gamma_q(n), \gamma_q(n+1))$ lies on the segment (590). Thus, for $1^0$, the sequence $(\gamma_q(n), \gamma_q(n+1))$ lies on the diagonal of the interval

$$I_X \times I_Y := \left[0, 1 - \frac{1}{q}\right] \times \left[\frac{1}{q}, 1\right] \tag{591}$$

and for $2^0$, the sequence $(\gamma_q(n), \gamma_q(n+1))$ lies on the diagonals of the intervals

$$I_X^{(i)} \times I_Y^{(i)} := \left[1 - \frac{1}{q^i}, 1 - \frac{1}{q^{i+1}}\right] \times \left[\frac{1}{q^{i+1}}, \frac{1}{q^i}\right], \quad i = 1, 2, \dots \tag{592}$$

These intervals are maximal with respect to inclusion. □

Adding the maps (589) and (590) we found so-called the von Neumann-Kakutani transformation $T : [0,1] \to [0,1]$, see the following Fig. Because $\gamma_q(n)$ is u.d., the sequence $(\gamma_q(n), \gamma_q(n+1))$ has a.d.f. copula $g(x,y)$ of the form

$$\begin{aligned} g(x,y) &= |\mathrm{Project}_X(([0,x] \times [0,y]) \cap \mathrm{graph}\ T)| \\ &= \min\left(\left|[0,x] \cap I_X\right|, \left|[0,y] \cap I_Y\right|\right) \\ &+ \sum_{i=1}^{\infty} \min\left(\left|[0,x] \cap I_X^{(i)}\right|, \left|[0,y] \cap I_Y^{(i)}\right|\right), \end{aligned} \tag{593}$$

where $\mathrm{Project}_X$ is the projection of a two dimensional set to the $X$-axis.

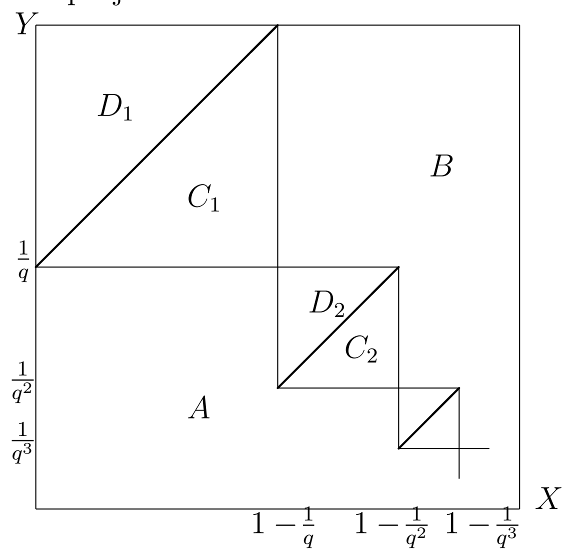



Figure: Line segments containing $(\gamma_q(n), \gamma_q(n+1)), n = 1, 2, \dots$ The graph of the von Neumann-Kakutani transformation $T$.

The sum (593) implies

$$g(x,y)=\begin{cases}0 & \text{if } (x,y)\in A,\\ 1-(1-y)-(1-x)=x+y-1 & \text{if } (x,y)\in B,\\ y-\frac{1}{q^i} & \text{if } (x,y)\in C_i,\\ x-1+\frac{1}{q^{i-1}} & \text{if } (x,y)\in D_i,\end{cases} \tag{594}$$

$i=1,2,\dots$ Thus

$$g(x,x)=\begin{cases}0, & \text{if } x\in\left[0,\frac{1}{q}\right],\\ x-\frac{1}{q}, & \text{if } x\in\left[\frac{1}{q},1-\frac{1}{q}\right],\\ 2x-1, & \text{if } x\in\left[1-\frac{1}{q},1\right]\end{cases} \tag{595}$$

and by (579)

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}|\gamma_q(n)-\gamma_q(n+1)|=1-2\int_0^1 g(x,x)\mathrm{d}x=\frac{2(q-1)}{q^2}.$$

□

**Example 93.** All terms of the sequence $(\gamma_q(n),\gamma_q(n+2)), n=1,2,...,$ lie in the line segments

$$Y=X+\frac{2}{q},\quad X\in\left[0,1-\frac{2}{q}\right),\ \text{or} \tag{596}$$

$$Y=X+\frac{1}{q}+\frac{1}{q^{i+1}}+\frac{1}{q^{i+2}}-1,\quad X\in\left[1-\frac{1}{q}-\frac{1}{q^{i+1}},1-\frac{1}{q}-\frac{1}{q^{i+2}}\right)\ \text{or} \tag{597}$$

$$Y=X+\frac{1}{q}+\frac{1}{q^{i+1}}+\frac{1}{q^{i+2}}-1,\quad X\in\left[1-\frac{1}{q^{i+1}},1-\frac{1}{q^{i+2}}\right) \tag{598}$$

for $i=0,1,\dots$.

This is the second iteration $T^2$ of the von Neumann-Kakutany transformation $T:[0,1]\to[0,1]$.

*Proof.* Express an $n$ in the base $q$

$$n=n_kq^k+n_{k-1}q^{k-1}+\cdots+n_1q+n_0, \tag{599}$$

where $n_i < q$ and $n_k > 0$. We consider three following cases:

$1^0$. $n_0 < q-2$,

$2^0$. $n_0 = q-2$,

$3^0$. $n_0 = q-1$.

Let

$1^0 . n_0 < q-2$. Then

$n = n_k q^k + \cdots + n_0$,

$n+2 = n_k q^k + \cdots + n_0 + 2$ and

$\gamma_q(n+2) - \gamma_q(n) = \frac{2}{q}$. In this case

$0 \leq \gamma_q(n) = \frac{n_0}{q} + \cdots + \frac{n_k}{q^{k+1}} < \frac{q-3}{q} + \frac{q-1}{q^2} + \cdots = \frac{q-2}{q}$, and

thus such $(\gamma_q(n), \gamma_q(n+2))$ lies on the line-segment (596).

Let

$2^0$. $n_0 = q-2$. Then

$n = n_k q^k + \cdots + n_{i+1} q^{i+1} + (q-1)q^i + (q-1)q^{i-1} + \cdots + (q-2)$ and $n_{i+1} < q-1$, then

$n+2 = n_k q^k + \cdots + (n_{i+1}+1)q^{i+1} + 0.q^i + 0.q^{i-1} + \cdots + 0$. Thus

$\gamma_q(n) = \frac{q-2}{q} + \cdots + \frac{q-1}{q^{i+1}} + \frac{n_{i+1}}{q^{i+2}} + \cdots + \frac{n_k}{q^{k+1}}$,

$\gamma_q(n+2) = \frac{n_{i+1}+1}{q^{i+2}} + \cdots + \frac{n_k}{q^{k+1}}$, and we have

$\gamma_q(n+2) - \gamma_q(n) = \frac{1}{q^{i+2}} + \frac{1}{q} - \frac{q-1}{q}(1 + \frac{1}{q} + \cdots + \frac{1}{q^i}) = \frac{1}{q^{i+2}} + \frac{1}{q} - 1 + \frac{1}{q^{i+1}}$.

Furthermore

$1 - \frac{1}{q^{i+1}} - \frac{1}{q} = \frac{q-2}{q} + \cdots + \frac{q-1}{q^{i+1}} \leq \gamma_q(n)$ and

$\gamma_q(n) \leq \frac{q-2}{q} + \cdots + \frac{q-1}{q^{i+1}} + \frac{q-2}{q^{i+2}} + \frac{q-1}{q^{i+3}} + \cdots = 1 - \frac{1}{q^{i+2}} - \frac{1}{q}$.

Thus in this case $(\gamma_q(n), \gamma_q(n+2))$ lies on the line-segment (597).

Let

$3^0$. $n_0 = q-1$. then

$n = n_k q^k + \cdots + n_{i+1} q^{i+1} + (q-1)q^i + (q-1)q^{i-1} + \cdots + (q-1)$ and $n_{i+1} < q-1$, then

$n+2 = n_k q^k + \cdots + (n_{i+1}+1)q^{i+1} + 0.q^i + 0.q^{i-1} + \cdots + 1$. Thus

$\gamma_q(n) = \frac{q-1}{q} + \cdots + \frac{q-1}{q^{i+1}} + \frac{n_{i+1}}{q^{i+2}} + \cdots + \frac{n_k}{q^{k+1}}$,

$\gamma_q(n+2) = \frac{1}{q} + \frac{n_{i+1}+1}{q^{i+2}} + \cdots + \frac{n_k}{q^{k+1}}$, and we have

$\gamma_q(n+2) - \gamma_q(n) = \frac{1}{q} + \frac{1}{q^{i+2}} - \frac{q-1}{q}(1 + \frac{1}{q} + \cdots + \frac{1}{q^i}) = \frac{1}{q^{i+2}} + \frac{1}{q} - 1 + \frac{1}{q^{i+1}}$.

Furthermore

$1 - \frac{1}{q^{i+1}} = \frac{q-1}{q} + \cdots + \frac{q-1}{q^{i+1}} \leq \gamma_q(n)$ and

$\gamma_q(n) \leq \frac{q-1}{q} + \cdots + \frac{q-1}{q^{i+1}} + \frac{q-2}{q^{i+2}} + \frac{q-1}{q^{i+3}} + \cdots = 1 - \frac{1}{q^{i+2}}$.

This gives (598).

Summary, if the $n$ satisfies $1^0$, then $(\gamma_q(n), \gamma_q(n+2))$ is contained in the diagonal of

$$I_X \times I_Z := \left[0, 1-\frac{2}{q}\right] \times \left[\frac{2}{q}, 1\right], \tag{600}$$

for $2^0$ in the diagonal

$$I_X^{(i)} \times I_Z^{(i)} := \left[1-\frac{1}{q^i}, 1-\frac{1}{q^{i+1}}\right] \times \left[\frac{1}{q}+\frac{1}{q^{i+1}}, \frac{1}{q}+\frac{1}{q^i}\right], i = 1, 2, \ldots, \tag{601}$$

and for $3^0$ in the diagonals

$$J_X^{(j)} \times J_Z^{(j)} := \left[1-\frac{1}{q}-\frac{1}{q^j}, 1-\frac{1}{q}-\frac{1}{q^{j+1}}\right] \times \left[\frac{1}{q^{j+1}}, \frac{1}{q^j}\right], j = 1, 2, \ldots \tag{602}$$

□

The results (596), (597) and (598) gives the following graph of line segments

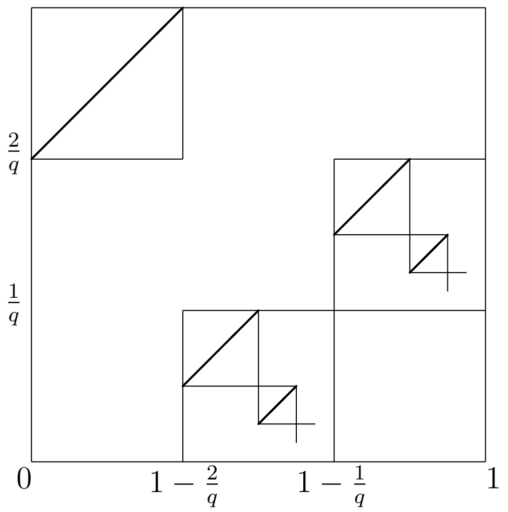
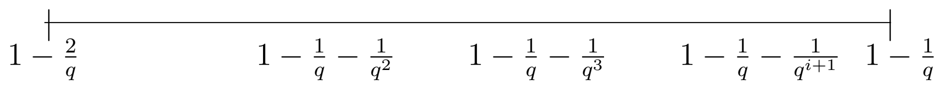
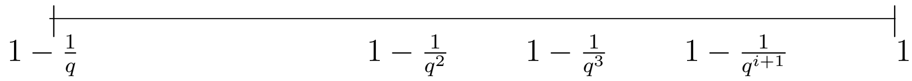



Figure: Straight lines containing $(\gamma_q(n), \gamma_q(n+2)), n = 01, 2, \ldots$ This is the second iteration $T^2$ of the von Neumann-Kakutany transformation.

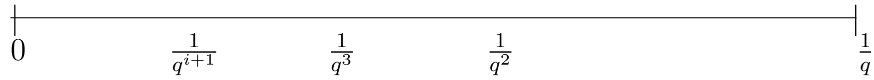
Here the interval $\left[1-\frac{2}{q}, 1-\frac{1}{q}\right)$ on $X$-axis is divided by



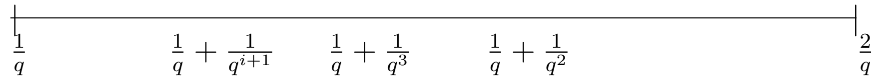
and the interval $\left[1-\frac{1}{q},1\right)$ is divided as




On $Y$-axes the interval $\left[0,\frac{1}{q}\right)$ is divided as



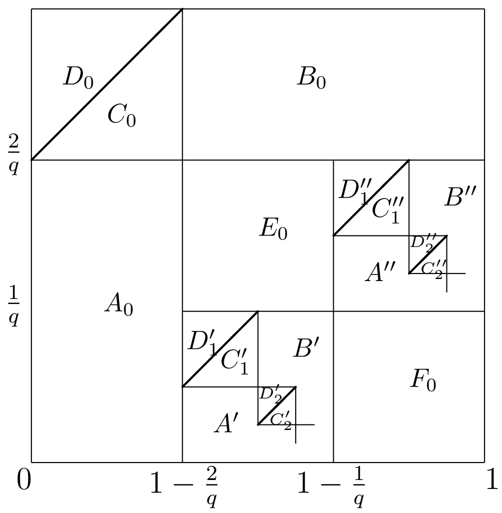

and the interval $\left[\frac{1}{q},\frac{2}{q}\right)$ is divided as




and in all above cases $i=1,2,\dots$ Note that for $q=2$, the interval $\left[0,1-\frac{2}{q}\right]\times\left[\frac{2}{q},1\right]$ is zero. Similarly as in (593) we have a.d.f. $g(x,y)$ of the sequence $(\gamma_q(n),\gamma_q(n+2))$ is

$$\begin{aligned} g(x,y) &= \min\left(|[0,x]\cap I_X|,|[0,y]\cap I_Y|\right) \\ &\quad+\sum_{i=1}^{\infty}\min\left(|[0,x]\cap I_X^{(i)}|,|[0,y]\cap I_Y^{(i)}|\right) \\ &\quad+\sum_{k=1}^{\infty}\min\left(|[0,x]\cap J_X^{(k)}|,|[0,y]\cap J_Y^{(k)}|,\right). \end{aligned} \tag{603}$$

Divide $[0,1]^2$ in the following figure



we have the following explicit d.f. $g(x,y)$ of the sequence $(\gamma_q(n), \gamma_q(n+2))$.

$$g(x,y) = \begin{cases} x & \text{if } (x,y) \in D_0, \\ y - \frac{2}{q} & \text{if } (x,y) \in C_0, \\ 0 & \text{if } (x,y) \in A_0, \\ y + x - 1 & \text{if } (x,y) \in B_0, \\ x - 1 + \frac{2}{q} & \text{if } (x,y) \in E_0, \\ y & \text{if } (x,y) \in F_0, \\ 0 & \text{if } (x,y) \in A', \\ x + y - 1 + \frac{1}{q} & \text{if } (x,y) \in B', \\ x - 1 + \frac{1}{q} + \frac{1}{q^i} & \text{if } (x,y) \in D'_i, \\ y - \frac{1}{q^{i+1}} & \text{if } (x,y) \in C'_i, \\ \frac{1}{q} & \text{if } (x,y) \in A'', \\ x + y - 1 & \text{if } (x,y) \in B'', \\ x - 1 + \frac{1}{q} + \frac{1}{q^i} & \text{if } (x,y) \in D''_i, \\ y - \frac{1}{q^{i+1}} & \text{if } (x,y) \in C''_i. \end{cases} \tag{604}$$

### 8.10.2 Three-dimensional shifted van der Corput sequence

Cf. [52]:

**Example 94.** Firstly we find all maximal 3-dimensional intervals $I$ containing points $(\gamma_q(n), \gamma_q(n+1), \gamma_q(n+2))$ in the form $I = (I_X, I_Y, I_Z)$, where $I_X, I_Y, I_Z$ are projection of $I$ to $X, Y, Z$, axes respectively. These intervals $I_X, I_Y, I_Z$ are constructed in Example 92 and 93 such that if $\gamma_q(n) \in I_X$ then $\gamma_q(n+1) \in I_Y$ and $\gamma_q(n+2) \in I_Z$. Thus for the lengths we have $|I_X| = |I_Y| = |I_Z|$. Combining intervals (591), (592), (600), (601), (602) of equal lengths by following Fig.

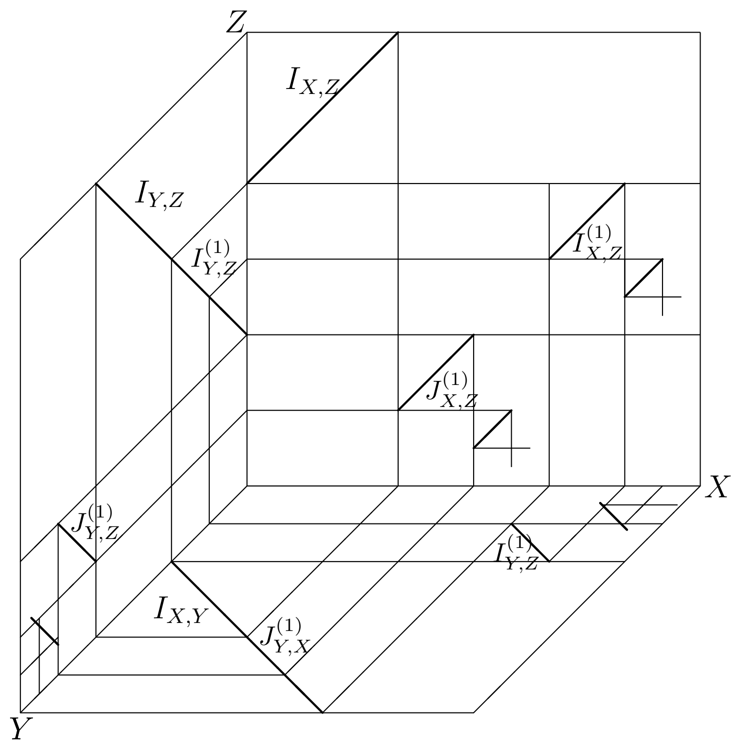



Figure: Mapping between intervals with equal lengths.

we find two following infinite sequences of intervals and an one separate interval:

$$I = \left(\left[0, 1-\frac{2}{q}\right], \left[\frac{1}{q}, 1-\frac{1}{q}\right], \left[\frac{2}{q}, 1\right]\right) \tag{605}$$

$$I^{(i)} = \left(\left[1-\frac{1}{q^i}, 1-\frac{1}{q^{i+1}}\right], \left[\frac{1}{q^{i+1}}, \frac{1}{q^i}\right], \left[\frac{1}{q}+\frac{1}{q^{i+1}}, \frac{1}{q}+\frac{1}{q^i}\right]\right), i = 1, 2, \ldots, \tag{606}$$

$$J^{(k)} = \left(\left[1-\frac{1}{q}-\frac{1}{q^k}, 1-\frac{1}{q}-\frac{1}{q^{k+1}}\right], \left[1-\frac{1}{q^k}, 1-\frac{1}{q^{k+1}}\right], \left[\frac{1}{q^{k+1}}, \frac{1}{q^k}\right]\right), k = 1, 2, \ldots, \tag{607}$$

All these interval are maximal with respect to inclusion. Similarly as in (593), let $D$ be a union of diagonals of (605), (606) and (607). Then

$$g(x,y,z) = |\mathrm{Project}_X([0,x]\times[0,y]\times[0,z]\cap D)| \tag{608}$$

and it can be rewritten as

$$\begin{aligned} g(x,y,z) = \min\big(&|[0,x]\cap I_X|, |[0,y]\cap I_Y|, |[0,z]\cap I_Z|\big) \\ &+ \sum_{i=1}^{\infty} \min\big(|[0,x]\cap I_X^{(i)}|, |[0,y]\cap I_Y^{(i)}|, |[0,z]\cap I_Z^{(i)}|\big) \\ &+ \sum_{k=1}^{\infty} \min\big(|[0,x]\cap J_X^{(k)}|, |[0,y]\cap J_Y^{(k)}|, |[0,z]\cap J_Z^{(k)}|\big). \end{aligned} \tag{609}$$

To calculate minimums in (609) we can use the following Fig. (here $q=3$):

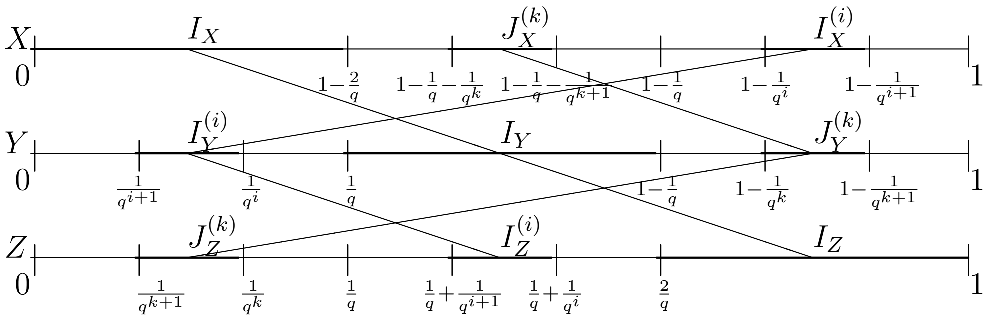



Figure: Projections of intervals $I, I^{(i)}, J^{(k)}$ on axes $X, Y, Z$.

As an example of application of (106) and Fig. 4 we compute $g(x,x,x)$ for $q\geq 3$ without the need to know of $g(x,y,z)$, [57]

$$g(x,x,x) = \begin{cases} 0 & \text{if } x\in\left[0,\frac{2}{q}\right], \\ x-\frac{2}{q} & \text{if } x\in\left[\frac{2}{q},1-\frac{1}{q}\right], \\ 3x-2 & \text{if } x\in\left[1-\frac{1}{q},1\right]. \end{cases} \tag{610}$$

[57] For $q=3$ the middle member in (610) is omitting.

*Proof.* **1.** Let $x \in [0, \frac{1}{q}]$.

Then $|[0,x] \cap I_Z| = 0$, $|[0,x] \cap I_Z^{(i)}| = 0$, $|[0,x] \cap J_Y^{(k)}| = 0$, consequently $g(x,x,x) = 0$.

**2.** Let $x \in [\frac{1}{q}, \frac{2}{q}]$.

Then $|[0,x] \cap I_Z| = 0$, $|[0,x] \cap J_Y^{(k)}| = 0$, $|[0,x] \cap I_X^{(i)}| = 0$, consequently $g(x,x,x) = 0$.

**3.** Let $x \in [\frac{2}{q}, 1 - \frac{1}{q}]$.

Then $|[0,x] \cap I_X^{(i)}| = 0$, $|[0,x] \cap J_Y^{(k)}| = 0$, consequently $g(x,x,x) = \min\left(1 - \frac{2}{q}, x - \frac{1}{q}, x - \frac{2}{q}\right) = x - \frac{2}{q}$.

**4.** Let $x \in [1 - \frac{1}{q}, 1]$.

Specify $x \in I_X^{(k_1)}$, $x \in J_Y^{(k_1)}$. Then $|[0,x] \cap I_X^{(k)}| = 0$, $|[0,x] \cap J_Y^{(k)}| = 0$ for $k > k_1$. Thus (106) implies

$$\begin{aligned}
g(x,x,x) &= \min\left(1 - \frac{2}{q}, 1 - \frac{1}{q} - \frac{1}{q}, x - \frac{2}{q}\right) \\
&+ \sum_{i=1}^{k_1} \min\left(|[0,x] \cap I_X^{(i)}|, |[0,y] \cap I_Y^{(i)}|, |[0,z] \cap I_Z^{(i)}|\right) \\
&+ \sum_{k=1}^{k_1} \min\left(|[0,x] \cap J_X^{(k)}|, |[0,y] \cap J_Y^{(k)}|, |[0,z] \cap J_Z^{(k)}|\right) \\
&= x - \frac{2}{q} + \sum_{i=1}^{k_1 - 1}\left(\frac{1}{q^i} - \frac{1}{q^{i+1}}\right) + x - 1 + \frac{1}{q^{k_1}} \\
&+ \sum_{k=1}^{k_1 - 1}\left(\frac{1}{q^k} - \frac{1}{q^{k+1}}\right) + x - 1 + \frac{1}{q^{k_1}} = 3x - 2.
\end{aligned}$$

□

For $q = 2$ we have

$$g(x,x,x) = \begin{cases} 0 & \text{if } x \in \left[0, \frac{1}{2}\right], \\ x - \frac{1}{2} & \text{if } x \in \left[\frac{1}{2}, \frac{3}{4}\right], \\ 3x - 2 & \text{if } x \in \left[\frac{3}{4}, 1\right]. \end{cases} \tag{611}$$

Explicit form of $g(x,y,z)$. Let $q \geq 3$ be an integer.
Motivated by the abode Fig. we decompose the unit interval $[0,1]$ on $X$, $Y$ and $Z$ axes in the following Fig. of intervals (here $q = 4$):

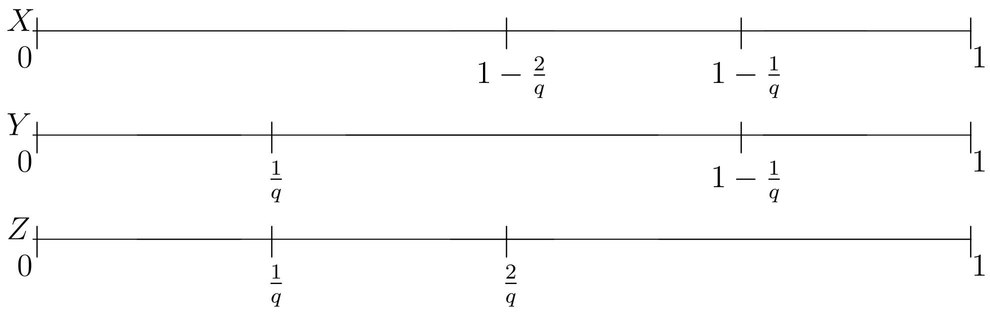


Figure: Divisions of the unit intervals.

In this decomposition, for $(x, y, z) \in [0,1]^3$, we have 27 possibilities. All cases can be found in [52]:

**Notes 44.** This is the most complicated d.f. in this book.

**1.** Let $x \in [0, 1-\frac{2}{q}], y \in [0, \frac{1}{q}], z \in [0, \frac{1}{q}]$. Then $g(x, y, z) = 0$.
**2.** Let $x \in [0, 1-\frac{2}{q}], y \in [0, \frac{1}{q}], z \in [\frac{1}{q}, \frac{2}{q}]$. Then $g(x, y, z) = 0$.
**3.** Let $x \in [0, 1-\frac{2}{q}], y \in [0, \frac{1}{q}], z \in [\frac{2}{q}, 1]$. Then $g(x, y, z) = 0$.
**4.** Let $x \in [0, 1-\frac{2}{q}], y \in [\frac{1}{q}, 1-\frac{1}{q}], z \in [0, \frac{1}{q}]$. Then $g(x, y, z) = 0$.
**5.** Let $x \in [0, 1-\frac{2}{q}], y \in [\frac{1}{q}, 1-\frac{1}{q}], z \in [\frac{1}{q}, \frac{2}{q}]$. Then $g(x, y, z) = 0$.
**6.** Let $x \in [0, 1-\frac{2}{q}], y \in [\frac{1}{q}, 1-\frac{1}{q}], z \in [\frac{2}{q}, 1]$. Then $g(x, y, z) = \min\left(x, y-\frac{1}{q}, z-\frac{2}{q}\right)$.
**7.** Let $x \in [0, 1-\frac{2}{q}], y \in [1-\frac{1}{q}, 1], z \in [0, \frac{1}{q}]$. Then $g(x, y, z) = 0$.
**8.** Let $x \in [0, 1-\frac{2}{q}], y \in [1-\frac{1}{q}, 1], z \in [\frac{1}{q}, \frac{2}{q}]$. Then $g(x, y, z) = 0$.
**9.** Let $x \in [0, 1-\frac{2}{q}], y \in [1-\frac{1}{q}, 1], z \in [\frac{2}{q}, 1]$. Then $g(x, y, z) = \min\left(x, z-\frac{2}{q}\right)$.
**10.** Let $x \in [1-\frac{2}{q}, 1-\frac{1}{q}], y \in [0, \frac{1}{q}], z \in [0, \frac{1}{q}]$. Then $g(x, y, z) = 0$.
**11.** Let $x \in [1-\frac{2}{q}, 1-\frac{1}{q}], y \in [0, \frac{1}{q}], z \in [\frac{1}{q}, \frac{2}{q}]$. Then $g(x, y, z) = 0$.
**12.** Let $x \in [1-\frac{2}{q}, 1-\frac{1}{q}], y \in [0, \frac{1}{q}], z \in [\frac{2}{q}, 1]$. Then $g(x, y, z) = 0$.
**13.** Let $x \in [1-\frac{2}{q}, 1-\frac{1}{q}], y \in [\frac{1}{q}, 1-\frac{1}{q}], z \in [0, \frac{1}{q}]$. Then

$g(x,y,z)=0.$
**14.** Let $x\in[1-\frac{2}{q},1-\frac{1}{q}], y\in[\frac{1}{q},1-\frac{1}{q}], z\in[\frac{1}{q},\frac{2}{q}]$. Then
$g(x,y,z)=0.$
**15.** Let $x\in[1-\frac{2}{q},1-\frac{1}{q}], y\in[\frac{1}{q},1-\frac{1}{q}], z\in[\frac{2}{q},1]$. Then
$g(x,y,z)=\min\left(y-\frac{1}{q},z-\frac{2}{q}\right)$.
**16.** Let $x\in[1-\frac{2}{q},1-\frac{1}{q}], y\in[1-\frac{1}{q},1], z\in[0,\frac{1}{q}]$.
Specify $x\in J_X^{(k_1)}$, $y\in J_Y^{(k_2)}$, $z\in J_Z^{(k_3)}$. Then

$$g(x,y,z)=\begin{cases}0, & \text{if } \min(k_1,k_2)<k_3,\\ \min\left(x-1+\frac{1}{q}+\frac{1}{q^{k_3}},y-1+\frac{1}{q^{k_3}},z-\frac{1}{q^{k_3+1}}\right), & \text{if } k_3=k_1=k_2,\\ \min\left(x-1+\frac{1}{q}+\frac{1}{q^{k_3}},z-\frac{1}{q^{k_3+1}}\right), & \text{if } k_3=k_1<k_2,\\ \min\left(y-1+\frac{1}{q^{k_3}},z-\frac{1}{q^{k_3+1}}\right), & \text{if } k_3=k_2<k_1,\\ z+x-1+\frac{1}{q}, & \text{if } k_3<k_1<k_2,\\ z+\min\left(x-1+\frac{1}{q},y-1\right), & \text{if } k_3<k_1=k_2,\\ z+y-1 & \text{if } k_3<k_2<k_1.\end{cases}$$

**17.** Let $x\in[1-\frac{2}{q},1-\frac{1}{q}], y\in[1-\frac{1}{q},1], z\in[\frac{1}{q},\frac{2}{q}]$.
Specify $x\in J_X^{(k_1)}$, $y\in J_Y^{(k_2)}$. Then

$$g(x,y,z)=\begin{cases}\min\left(x-1+\frac{2}{q},y-1+\frac{1}{q}\right), & \text{if } k_1=k_2,\\ x-1+\frac{2}{q}, & \text{if } k_1<k_2,\\ y-1+\frac{1}{q}, & \text{if } k_1>k_2.\end{cases}$$

**18.** Let $x\in[1-\frac{2}{q},1-\frac{1}{q}], y\in[1-\frac{1}{q},1], z\in[\frac{2}{q},1]$.
Specify $x\in J_X^{(k_1)}$, $y\in J_Y^{(k_2)}$. Then

$$g(x,y,z)=\begin{cases}\min\left(x-1+\frac{2}{q},y-1+\frac{1}{q}\right)+z-\frac{2}{q}, & \text{if } k_1=k_2,\\ x+z-1, & \text{if } k_1<k_2,\\ y+z-1-\frac{1}{q}, & \text{if } k_2<k_1.\end{cases}$$

**19.** Let $x\in[1-\frac{1}{q},1], y\in[0,\frac{1}{q}], z\in[0,\frac{1}{q}]$. Then
$g(x,y,z)=0.$
**20.** Let $x\in[1-\frac{1}{q},1], y\in[0,\frac{1}{q}], z\in[\frac{1}{q},\frac{2}{q}]$.
Specify $x\in I_X^{(i_1)}$, $y\in I_Y^{(i_2)}$, $z\in I_Z^{(i_3)}$. Then we have

$$g(x,y,z)=\begin{cases}0, & \text{if } i_1<\max(i_2,i_3),\\ \min\left(x-1+\frac{1}{q^{i_1}},y-\frac{1}{q^{i_1+1}},z-\frac{1}{q}-\frac{1}{q^{i_1+1}}\right), & \text{if } i_3=i_2=i_1,\\ \min\left(y,z-\frac{1}{q}\right)+x-1, & \text{if } i_3=i_2<i_1,\\ \min\left(x-1+\frac{1}{q^{i_1}},y-\frac{1}{q^{i_1+1}}\right), & \text{if } i_3<i_2=i_1,\\ \min\left(x-1+\frac{1}{q^{i_1}},z-\frac{1}{q}-\frac{1}{q^{i_1+1}}\right), & \text{if } i_2<i_3=i_1,\\ x+z-1-\frac{1}{q}, & \text{if } i_2<i_3<i_1,\\ x+y-1, & \text{if } i_3<i_2<i_1.\end{cases}$$

**21.** Let $x \in [1-\frac{1}{q},1], y \in [0,\frac{1}{q}], z \in [\frac{2}{q},1]$.
Specify $x \in I_X^{(i_1)}$, $y \in I_Y^{(i_2)}$. Then

$$g(x,y,z) = \begin{cases} 0, & \text{if } i_1 < i_2, \\ \min\left(x-1+\frac{1}{q^{i_1}}, y-\frac{1}{q^{i_1+1}}\right), & \text{if } i_1 = i_2, \\ x+y-1, & \text{if } i_2 < i_1. \end{cases}$$

**22.** Let $x \in [1-\frac{1}{q},1], y \in [\frac{1}{q},1-\frac{1}{q}], z \in [0,\frac{1}{q}]$. Then
$g(x,y,z) = 0$.
**23.** Let $x \in [1-\frac{1}{q},1], y \in [\frac{1}{q},1-\frac{1}{q}], z \in [\frac{1}{q},\frac{2}{q}]$.
Specify $x \in I_X^{(i_1)}$, $z \in I_Z^{(i_2)}$. Then

$$g(x,y,z) = \begin{cases} 0, & \text{if } i_1 < i_2, \\ \min\left(x-1+\frac{1}{q^{i_1}}, z-\frac{1}{q}-\frac{1}{q^{i_1+1}}\right), & \text{if } i_1 = i_2, \\ x+z-1-\frac{1}{q}, & \text{if } i_2 < i_1. \end{cases}$$

**24.** Let $x \in [1-\frac{1}{q},1], y \in [\frac{1}{q},1-\frac{1}{q}], z \in [\frac{2}{q},1]$. Then
$g(x,y,z) = \min\left(y-\frac{1}{q}, z-\frac{2}{q}\right) + x - 1 + \frac{1}{q}$.
**25.** Let $x \in [1-\frac{1}{q},1], y \in [1-\frac{1}{q},1], z \in [0,\frac{1}{q}]$.
Specify $y \in J_Y^{(k_1)}$, $z \in J_Z^{(k_2)}$. Then

$$g(x,y,z) = \begin{cases} 0, & \text{if } k_1 < k_2, \\ \min\left(y-1+\frac{1}{q^{k_1}}, z-\frac{1}{q^{k_1+1}}\right), & \text{if } k_1 = k_2, \\ y+z-1, & \text{if } k_2 < k_1. \end{cases}$$

**26.** Let $x \in [1-\frac{1}{q},1], y \in [1-\frac{1}{q},1], z \in [\frac{1}{q},\frac{2}{q}]$.
Specify $x \in I_X^{(i_1)}$, $y \in J_Y^{(k_1)}$ and $z \in I_Z^{(i_2)}$. Then

$$g(z,y,z) = \begin{cases} y-1+\frac{1}{q}, & \text{if } i_1 < i_2, \\ y-1+\frac{1}{q}+\min\left(x-1+\frac{1}{q^{i_1}}, z-\frac{1}{q}-\frac{1}{q^{i_1+1}}\right), & \text{if } i_1 = i_2, \\ x+y+z-2, & \text{if } i_2 < i_1. \end{cases}$$

**27.** Let $x \in [1-\frac{1}{q},1], y \in [1-\frac{1}{q},1], z \in [\frac{2}{q},1]$. Then
$g(z,y,z) = x+y+z-2$.

**Example 95.** The sequence $(\gamma_q(n), \gamma_q(n+3)), n = 1,2,\dots$ lies on the diagonals of the following intervals

$$\left[0, 1-\frac{3}{q}\right] \times \left[\frac{3}{q}, 1\right],$$
$$\left[1-\frac{2}{q}-\frac{1}{q^{i+1}}, 1-\frac{1}{q^{i+2}}-\frac{2}{q}\right] \times \left[\frac{1}{q^{i+2}}, \frac{1}{q^{i+1}}\right], i = 0,1,2,\dots,$$

$$\left[1-\frac{1}{q}-\frac{1}{q^{i+1}},1-\frac{1}{q^{i+2}}-\frac{1}{q}\right]\times\left[\frac{1}{q^{i+2}}+\frac{1}{q},\frac{1}{q^{i+1}}+\frac{1}{q}\right], i=0,1,2,\dots,$$
$$\left[1-\frac{1}{q^{i+1}},1-\frac{1}{q^{i+2}}\right]\times\left[\frac{1}{q^{i+2}}+\frac{2}{q},\frac{1}{q^{i+1}}+\frac{2}{q}\right], i=0,1,2,\dots \tag{612}$$

These intervals are maximal. This is the third iteration $T^3$ of the von Neumann-Kakutany transformation $T:[0,1]\to[0,1]$.

### 8.10.3 Four-dimensional shifted van der Corput sequence

Cf. [7]:

**Example 96.** The sequence $(\gamma_q(n),\gamma_q(n+1),\gamma_q(n+2),\gamma_q(n+3))$, $n=0,1,2,\dots,$ lies in the diagonals of the following 4-dimensional maximal intervals:

$$I=\left[0,1-\frac{3}{q}\right]\times\left[\frac{1}{q},1-\frac{2}{q}\right]\times\left[\frac{2}{q},1-\frac{1}{q}\right]\times\left[\frac{3}{q},1\right] \tag{613}$$
$$I^{(i)}=\left[1-\frac{1}{q^i},1-\frac{1}{q^{i+1}}\right]\times\left[\frac{1}{q^{i+1}},\frac{1}{q^i}\right]\times\left[\frac{1}{q}+\frac{1}{q^{i+1}},\frac{1}{q}+\frac{1}{q^i}\right]$$
$$\times\left[\frac{2}{q}+\frac{1}{q^{i+1}},\frac{2}{q}+\frac{1}{q^i}\right],\quad i=1,2,\dots \tag{614}$$
$$J^{(j)}=\left[1-\frac{2}{q}-\frac{1}{q^j},1-\frac{2}{q}-\frac{1}{q^{j+1}}\right]\times\left[1-\frac{1}{q}-\frac{1}{q^j},1-\frac{1}{q}-\frac{1}{q^{j+1}}\right]$$
$$\times\left[1-\frac{1}{q^j},1-\frac{1}{q^{j+1}}\right]\times\left[\frac{1}{q^{j+1}},\frac{1}{q^j}\right],\quad j=1,2,\dots \tag{615}$$
$$K^{(k)}=\left[1-\frac{1}{q}-\frac{1}{q^k},1-\frac{1}{q}-\frac{1}{q^{k+1}}\right]\times\left[1-\frac{1}{q^k},1-\frac{1}{q^{k+1}}\right]\times\left[\frac{1}{q^{k+1}},\frac{1}{q^k}\right]$$
$$\times\left[\frac{1}{q}+\frac{1}{q^{k+1}},\frac{1}{q}+\frac{1}{q^k}\right], k=1,2,\dots \tag{616}$$

Proof: We started with 2-dimensional intervals in $(X,Y)=(Y,Z)=(Z,U)$ axis, respectively, containing $(\gamma_q(n),\gamma_q(n+1))$, $n=0,1,2,\dots,$ in diagonals. By [52] they are:
$\left[0,1-\frac{1}{q}\right]\times\left[\frac{1}{q},1\right]$;
$\left[1-\frac{1}{q^i},1-\frac{1}{q^{i+1}}\right]\times\left[\frac{1}{q^{i+1}},\frac{1}{q^i}\right]$, $i=1,2,\dots;$

Adding intervals in $(X,U)$ axis containing $(\gamma_q(n),\gamma_q(n+3))$ in their diagonal, see (115), (116), (117) and (172):

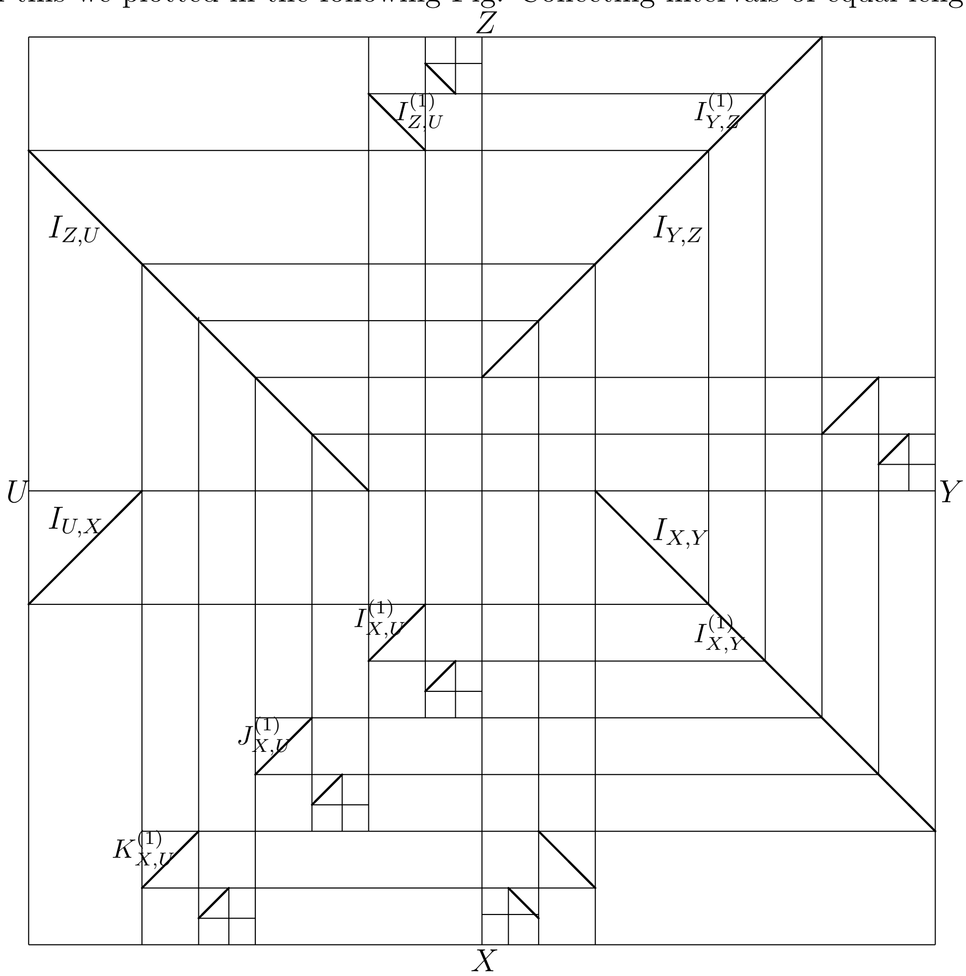
$\left[0,1-\frac{3}{q}\right]\times\left[\frac{3}{q},1\right]$;

$\left[1-\frac{2}{q}-\frac{1}{q^i},1-\frac{2}{q}-\frac{1}{q^{i+1}}\right]\times\left[\frac{1}{q^i},\frac{1}{q^{i+1}}\right]$, $i=1,2,\dots$;

$\left[1-\frac{1}{q}-\frac{1}{q^i},1-\frac{1}{q}-\frac{1}{q^{i+1}}\right]\times\left[\frac{1}{q}+\frac{1}{q^{i+1}},\frac{1}{q}+\frac{1}{q^i}\right]$, $i=1,2,\dots$;

$\left[1-\frac{1}{q^i},1-\frac{1}{q^{i+1}}\right]\times\left[\frac{2}{q}+\frac{1}{q^{i+1}},\frac{2}{q}+\frac{1}{q^i}\right]$, $i=1,2,\dots$;

All this we plotted in the following Fig. Collecting intervals of equal length



we find that maximal 4-dimensional intervals containing points

$$(\gamma_q(n), \gamma_q(n+1), \gamma_q(n+2), \gamma_q(n+3)), \quad n = 0, 1, 2, \ldots$$

Now, let $D$ be a union of diagonals of (613), (614), (615) and (616). Then

$$g(x, y, z, u) = |\mathrm{Project}_X([0, x] \times [0, y] \times [0, z] \times [0, u] \cap D)| \tag{617}$$

and it can be rewritten as

$$\begin{aligned}
&g(x, y, z, u) \\
&= \min\big(|[0, x] \cap I_X|, |[0, y] \cap I_Y|, |[0, z] \cap I_Z|, |[0, u] \cap I_U|\big) \\
&+ \sum_{i=1}^{\infty} \min\big(|[0, x] \cap I_X^{(i)}|, |[0, y] \cap I_Y^{(i)}|, |[0, z] \cap I_Z^{(i)}|, |[0, u] \cap I_U^{(i)}|\big) \\
&+ \sum_{j=1}^{\infty} \min\big(|[0, x] \cap J_X^{(j)}|, |[0, y] \cap J_Y^{(j)}|, |[0, z] \cap J_Z^{(j)}|, |[0, u] \cap J_U^{(j)}|\big) \\
&+ \sum_{k=1}^{\infty} \min\big(|[0, x] \cap K_X^{(k)}|, |[0, y] \cap K_Y^{(k)}|, |[0, z] \cap K_Z^{(k)}|, |[0, u] \cap K_U^{(k)}|\big)
\end{aligned} \tag{618}$$

Computing $g(x, y, z, u)$ is complicated. By using the following picture

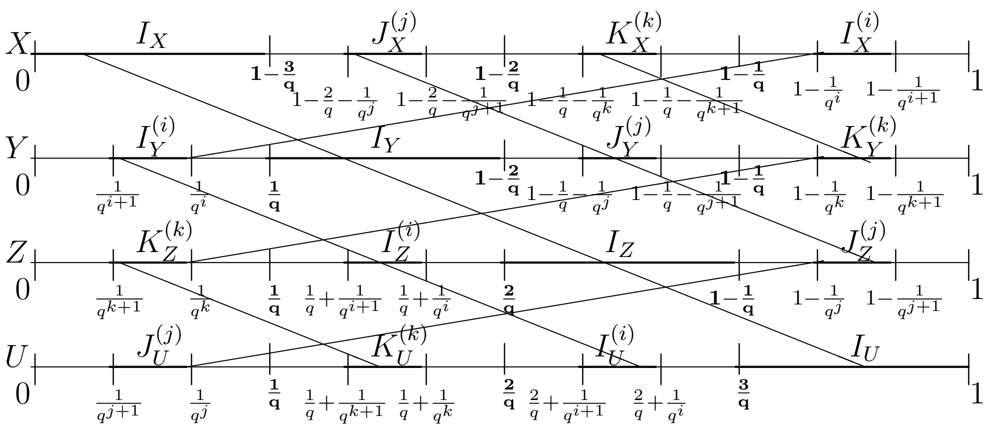



Figure: Projections of intervals $I, I^{(i)}, J^{(j)}, K^{(k)}$ on axes $X, Y, Z, U$.

it can be find

$$g(x,x,x,x)=\begin{cases}0, & \text{if } x\in\left[0,\frac{3}{q}\right],\\ x-\frac{3}{q}, & \text{if } x\in\left[\frac{3}{q},1-\frac{1}{q}\right],\\ 4x-3, & \text{if } x\in\left[1-\frac{1}{q},1\right]\end{cases} \tag{619}$$

for $q\geq 4$.

Application. We apply Weyl's limit relation

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}F(\gamma_q(n),\gamma_q(n+1),\gamma_q(n+2),\gamma_q(n+3))$$
$$=\int_0^1\int_0^1\int_0^1\int_0^1 F(x,y,z,u)\mathrm{d}_x\mathrm{d}_y\mathrm{d}_x\mathrm{d}_u g(x,y,z,u). \tag{620}$$

Assume that $F(x,y,z,u)$ is a continuous in $[0,1]^4$ and $g(x,y,z,u)$ is a d.f. Then

$$\int_0^1\int_0^1\int_0^1\int_0^1 F(x,y,z,u)\mathrm{d}_x\mathrm{d}_y\mathrm{d}_z\mathrm{d}_u g(x,y,z,u)=F(1,1,1,1)$$
$$-\int_0^1 g(x,1,1,1)\mathrm{d}_xF(x,1,1,1)-\int_0^1 g(1,y,1,1)\mathrm{d}_yF(1,y,1,1)$$
$$-\int_0^1 g(1,1,z,1)\mathrm{d}_zF(1,1,z,1)-\int_0^1 g(1,1,1,u)\mathrm{d}_uF(1,1,1,u)$$
$$+\int_0^1\int_0^1 g(x,y,1,1)\mathrm{d}_x\mathrm{d}_yF(x,y,1,1)+\int_0^1\int_0^1 g(x,1,z,1)\mathrm{d}_x\mathrm{d}_zF(x,1,z,1)$$
$$+\int_0^1\int_0^1 g(1,y,z,1)\mathrm{d}_y\mathrm{d}_zF(1,y,z,1)+\int_0^1\int_0^1 g(x,1,1,u)\mathrm{d}_x\mathrm{d}_uF(x,1,1,u)$$
$$+\int_0^1\int_0^1 g(1,y,1,u)\mathrm{d}_y\mathrm{d}_uF(1,y,1,u)+\int_0^1\int_0^1 g(1,1,z,u)\mathrm{d}_z\mathrm{d}_uF(1,1,z,u)$$
$$-\int_0^1\int_0^1\int_0^1 g(x,y,z,1)\mathrm{d}_x\mathrm{d}_y\mathrm{d}_zF(x,y,z,1)$$
$$-\int_0^1\int_0^1\int_0^1 g(1,y,z,u)\mathrm{d}_u\mathrm{d}_y\mathrm{d}_zF(1,y,z,u)$$
$$-\int_0^1\int_0^1\int_0^1 g(x,1,z,u)\mathrm{d}_x\mathrm{d}_z\mathrm{d}_uF(x,1,z,u)$$
$$-\int_0^1\int_0^1\int_0^1 g(x,y,1,u)\mathrm{d}_u\mathrm{d}_y\mathrm{d}_xF(x,y,1,u)$$

$$+\int_0^1\int_0^1\int_0^1\int_0^1 g(x,y,z,u)\mathrm{d}_x\mathrm{d}_y\mathrm{d}_z\mathrm{d}_u F(x,y,z,u).$$

Put $F(x,y,z,u)=\max(x,y,z,u)$. Then

$\mathrm{d}_xF(x,1,1,1)=\mathrm{d}_yF(1,y,1,1)=\mathrm{d}_zF(1,1,z,1)=\mathrm{d}_uF(1,1,1,u)=0,$

$\mathrm{d}_x\mathrm{d}_yF(x,y,1,1)=\mathrm{d}_x\mathrm{d}_zF(x,1,z,1)=\mathrm{d}_y\mathrm{d}_zF(1,y,z,1)=\mathrm{d}_x\mathrm{d}_uF(x,1,1,u)=$
$\mathrm{d}_y\mathrm{d}_uF(1,y,1,u)=\mathrm{d}_z\mathrm{d}_uF(1,1,z,u)=0,$

$\mathrm{d}_x\mathrm{d}_y\mathrm{d}_zF(x,y,z,1)=\mathrm{d}_x\mathrm{d}_y\mathrm{d}_uF(x,y,1,u)=\mathrm{d}_x\mathrm{d}_z\mathrm{d}_uF(x,1,z,u)$
$=\mathrm{d}_y\mathrm{d}_z\mathrm{d}_uF(1,y,z,u)=0.$
The differential $\mathrm{d}_x\mathrm{d}_y\mathrm{d}_z\mathrm{d}_uF(x,y,z,u)$ is non-zero if and only if

$$x=y=z=u$$

and in this case

$$\mathrm{d}_x\mathrm{d}_y\mathrm{d}_z\mathrm{d}_uF(x,y,z,u)=-\mathrm{d}x.$$

Then

$$\begin{aligned}&\int_0^1\int_0^1\int_0^1\int_0^1 F(x,y,z,u)\mathrm{d}_x\mathrm{d}_y\mathrm{d}_z\mathrm{d}_u g(x,y,z,u)\\&=1+\int_0^1\int_0^1\int_0^1\int_0^1 g(x,y,z,u)\mathrm{d}_x\mathrm{d}_y\mathrm{d}_z\mathrm{d}_uF(x,y,z,u)\\&=1-\int_0^1 g(x,x,x,x)\mathrm{d}x.\end{aligned}\tag{621}$$

For $q\geq 4$ and by (121) we have

$$\int_0^1 g(x,x,x,x)\mathrm{d}x=\int_{\frac{3}{q}}^{1-\frac{1}{q}}\left(x-\frac{3}{q}\right)\mathrm{d}x+\int_{1-\frac{1}{q}}^1(4x-3)\mathrm{d}x=\frac{1}{2}-\frac{3}{q}+\frac{6}{q^2}.$$

Also see [7].

### 8.10.4 $s$-th iteration of von Neumann-Kakutani transformation

In this part we study distribution of the sequence $(\gamma_q(n),\gamma_q(n+s))$, $n=0,1,2,\dots$, where $q$ is an integer, $q\geq s$. Let $n=n_kq^k+n_{k-1}q^{k-1}+\dots+n_1q+n_0$. We investigate the following cases:

1. $n_0 < q - s$,
2. $n_0 = q - s$,
3. $n_0 = q - s + 1$,
...
$l$. $n_0 = q - s + l - 2$,
...
$(2 + s - 1)$. $n_0 = q - 1$.

In the general case $l$, $n_0 = q - s + l - 2$, we have

$$n = n_k q^k + \cdots + n_{i+1} q^{i+1} + (q-1)q^i + (q-1)q^{i-1} + \cdots + (q-1)q + q - s + l - 2,$$

where $n_{i+1} < q - 1$ and $i = 0, 1, 2, \ldots$,

$$n + s = n_k q^k + \cdots + (n_{i+1} + 1)q^{i+1} + 0q^i + 0q^{i-1} + \cdots + 0q + l - 2\ ,$$

$$\gamma_q(n) = \frac{q - s + l - 2}{q} + \frac{q-1}{q^2} + \cdots + \frac{q-1}{q^i} + \frac{q-1}{q^{i+1}} + \frac{n_{i+1}}{q^{i+2}} + \cdots + \frac{n_k}{q^{k+1}}\ ,$$

$$\gamma_q(n+s) = \frac{l-2}{q} + \frac{0}{q^2} + \cdots + \frac{0}{q^i} + \frac{0}{q^{i+1}} + \frac{n_{i+1}+1}{q^{i+2}} + \cdots + \frac{n_k}{q^{k+1}}\ ,$$

$$\gamma_q(n+s) - \gamma_q(n) = \frac{s-1}{q} + \frac{1}{q^{i+2}} + \frac{1}{q^{i+1}} - 1\ ,$$

$$1 - \frac{s-l+1}{q} - \frac{1}{q^{i+1}} \le \frac{q-s+l-2}{q} + \frac{q-1}{q^2} + \cdots + \frac{q-1}{q^{i+1}} + \frac{0}{q^{i+2}} + \cdots + \frac{1}{q^k} \le \gamma_q(n)\ ,$$

$$\gamma_q(n) \le \frac{q-s+l-2}{q} + \frac{q-1}{q^2} + \cdots + \frac{q-1}{q^{i+1}} + \frac{q-2}{q^{i+2}} + \frac{q-1}{q^{i+3}} + \cdots = 1 - \frac{s-l+1}{q} - \frac{1}{q^{i+2}}.$$

Thus, if $n$ satisfies the case $l$, then the point $(\gamma_q(n), \gamma_q(n+s))$ lie in the line segment

$$\begin{aligned} &Z = X + \frac{s-1}{q} - 1 + \frac{1}{q^{i+1}} + \frac{1}{q^{i+2}}, \\ &X \in \left[1 - \frac{s-l+1}{q} - \frac{1}{q^{i+1}}, 1 - \frac{1}{q^{i+2}} - \frac{s-l+1}{q}\right] \end{aligned} \tag{622}$$

and in the diagonal of

$$\left[1 - \frac{s-l+1}{q} - \frac{1}{q^{i+1}}, 1 - \frac{1}{q^{i+2}} - \frac{s-l+1}{q}\right] \times \left[\frac{l-2}{q} + \frac{1}{q^{i+2}}, \frac{l-2}{q} + \frac{1}{q^{i+1}}\right], \tag{623}$$

where $i = 0, 1, 2, \ldots$. In the following we shall reduce $(i+1) \to i$ and (622) gives $s$th iteration of von Neumann-Kakutani transformation $T$

$$T^s(x) = \begin{cases} x + \frac{s}{q} & \text{if } x \in \left[0, 1 - \frac{s}{q}\right], \\ x + \frac{s-1}{q} - 1 + \frac{1}{q^i} + \frac{1}{q^{i+1}} & \text{if } x \in \left[1 - \frac{s-1}{q} - \frac{1}{q^i}, 1 - \frac{s-1}{q} - \frac{1}{q^{i+1}}\right] \cup \\ & \cup \left[1 - \frac{s-2}{q} - \frac{1}{q^i}, 1 - \frac{s-2}{q} - \frac{1}{q^{i+1}}\right] \cup \\ & \ldots\ldots \\ & \cup \left[1 - \frac{s-l+1}{q} - \frac{1}{q^i}, 1 - \frac{s-l-1}{q} - \frac{1}{q^{i+1}}\right] \cup \\ & \ldots\ldots \\ & \cup \left[1 - \frac{1}{q^i}, 1 - \frac{1}{q^{i+1}}\right], \text{ where} \\ & \quad i = 1, 2, \ldots \end{cases} \tag{624}$$

Also the points $(\gamma_q(n), \gamma_q(n+s))$, $n = 0, 1, 2, \ldots$, are contained in diagonals of

$I_0 = \left[0, 1 - \frac{s}{q}\right] \times \left[\frac{s}{q}, 1\right]$,
$I_1^{(i)} = \left[1 - \frac{s-1}{q} - \frac{1}{q^i}, 1 - \frac{s-1}{q} - \frac{1}{q^{i+1}}\right] \times \left[\frac{1}{q^{i+1}}, \frac{1}{q^i}\right], i = 1, 2, \ldots,$
$I_2^{(i)} = \left[1 - \frac{s-2}{q} - \frac{1}{q^i}, 1 - \frac{s-2}{q} - \frac{1}{q^{i+1}}\right] \times \left[\frac{1}{q} + \frac{1}{q^{i+1}}, \frac{1}{q} + \frac{1}{q^i}\right], i = 1, 2, \ldots,$
$I_3^{(i)} = \left[1 - \frac{s-3}{q} - \frac{1}{q^i}, 1 - \frac{s-3}{q} - \frac{1}{q^{i+1}}\right] \times \left[\frac{2}{q} + \frac{1}{q^{i+1}}, \frac{2}{q} + \frac{1}{q^i}\right], i = 1, 2, \ldots,$
$I_4^{(i)} = \left[1 - \frac{s-4}{q} - \frac{1}{q^i}, 1 - \frac{s-4}{q} - \frac{1}{q^{i+1}}\right] \times \left[\frac{3}{q} + \frac{1}{q^{i+1}}, \frac{3}{q} + \frac{1}{q^i}\right], i = 1, 2, \ldots,$
$\ldots$
$I_{l-1}^{(i)} = \left[1 - \frac{s-l+1}{q} - \frac{1}{q^i}, 1 - \frac{s-l+1}{q} - \frac{1}{q^{i+1}}\right] \times \left[\frac{l-2}{q} + \frac{1}{q^{i+1}}, \frac{l-2}{q} + \frac{1}{q^i}\right], i = 1, 2, \ldots,$
$\ldots$
$I_s^{(i)} = \left[1 - \frac{1}{q^i}, 1 - \frac{1}{q^{i+1}}\right] \times \left[\frac{s-1}{q} + \frac{1}{q^{i+1}}, \frac{s-1}{q} + \frac{1}{q^i}\right], i = 1, 2, \ldots$

# 9 Extremes of $\int_0^1 \int_0^1 F(x, y) \mathrm{d}_x \mathrm{d}_y g(x, y)$ over copulas $g(x, y)$

In this section we comments [53] and [14].
In the previous Section 8.8 we were looking for the extreme limit points of
(565) $\frac{1}{N} \sum_{n=1}^{N} F(x_n, y_n)$
which related to the extremes of
(569) $\int_0^1 \int_0^1 F(x, y) \mathrm{d}_x \mathrm{d}_y g(x, y)$.
But the problem of optimizing the integral $\int_0^1 \int_0^1 F(x, y) \mathrm{d}C(x, y)$ over copulas $C(x, y)$ belongs to the mass transportation problems, or the Monge-

Kantorovich transportation problem, see e.g. L. Ambrosio and N. Gigli (2013) [4] or S.T. Rachev and L. Rüschendorf, Mass Transportation Problems, Volume I: Theory (1998) [132]. In this section we give a criterion for the copula which maximizes (569). The minimum can be studied similarly.

First of all we repeated and reformulated Theorem 214 and 215 from Section 8.8 to the following Theorem 217 and 218, respectively.

**Theorem 217.** *Let $F(x, y)$ be a Riemann integrable function defined on $[0, 1]^2$. Assume that $\mathrm{d}_x\mathrm{d}_yF(x, y) > 0$ for $(x, y) \in (0, 1)^2$. Then*

$$\max_{g(x,y)\text{-copula}} \int_0^1 \int_0^1 F(x, y)\mathrm{d}_x\mathrm{d}_y g(x, y) = \int_0^1 F(x, x)\mathrm{d}x,$$
$$\min_{g(x,y)\text{-copula}} \int_0^1 \int_0^1 F(x, y)\mathrm{d}_x\mathrm{d}_y g(x, y) = \int_0^1 F(x, 1 - x)\mathrm{d}x,$$

*where the* max *is attained in $g(x, y) = \min(x, y)$ and* min *in $g(x, y) = \max(x + y - 1, 0)$, uniquely.*

**Theorem 218.** *Let us assume that $F(x, y)$ is a continuous function such that $\mathrm{d}_x\mathrm{d}_yF(x, y) > 0$ for every $(x, y) \in (0, 1)^2$. Then for the extremes of the integral $\int_0^1 \int_0^1 F(x, y)\mathrm{d}_x\mathrm{d}_y g(x, y)$ for $g(x, y)$ for which $g(x, 1) = g_1(x)$ and $g(1, y) = g_2(y)$ we have*

$$\max_{g(x,y)} \int_0^1 \int_0^1 F(x, y)\mathrm{d}_x\mathrm{d}_y g(x, y) = \int_0^1 F(g_1^{-1}(x), g_2^{-1}(x))\mathrm{d}x,$$
$$\min_{g(x,y)} \int_0^1 \int_0^1 F(x, y)\mathrm{d}_x\mathrm{d}_y g(x, y) = \int_0^1 F(g_1^{-1}(x), g_2^{-1}(1 - x))\mathrm{d}x,$$

*where the maximum is attained in $g(x, y) = \min(g_1(x), g_2(y))$ and minimum in $g(x, y) = \max(g_1(x) + g_2(y) - 1, 0)$, uniquely.*

## 9.1 Criterion for maximality

Now, applying Theorem 218 we find the following criterion of a maximality in [53]:

**Theorem 219.** *Let us assume that a copula $g(x, y)$ maximizes the integral $\int_0^1 \int_0^1 F(x, y)\mathrm{d}_x\mathrm{d}_y g(x, y)$ and let $[X_1, X_2] \times [Y_1, Y_2]$ be an interval in $[0, 1]^2$ such that the differential*

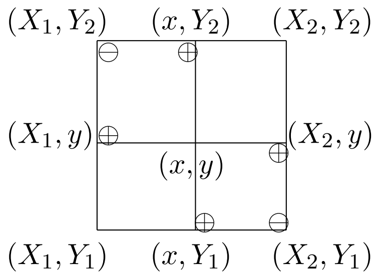
$$g(X_2, Y_2) + g(X_1, Y_1) - g(X_2, Y_1) - g(X_1, Y_2) > 0.$$

*If for every interior point* $(x,y)$ *of* $[X_1,X_2]\times[Y_1,Y_2]$ *the differential* $\mathrm{d}_x\mathrm{d}_y F(x,y)$ *has a constant sign, then:*

(i) *if* $\mathrm{d}_x\mathrm{d}_y F(x,y) > 0$*, then*

$$g(x,y) = \min(g(x,Y_2)+g(X_1,y)-g(X_1,Y_2), g(x,Y_1)+g(X_2,y)-g(X_2,Y_1)) \tag{625}$$

*Schematically,*

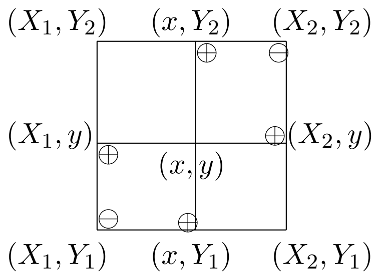



*Here, if in point* $(u,v)$ *we have add* $\oplus$ *or* $\ominus$ *then in* $g(x,y)$ *we add* $+g(u,v)$ *or* $-g(u,v)$*, respectively.*

(ii) *if* $\mathrm{d}_x\mathrm{d}_y F(x,y) < 0$*, then*

$$g(x,y) = \max(g(x,Y_2)+g(X_2,y)-g(X_2,Y_2), g(x,Y_1)+g(X_1,y)-g(X_1,Y_1)) \tag{626}$$

*for every* $(x,y)\in[X_1,X_2]\times[Y_1,Y_2]$*. Schematically*




*Proof.* $1^0$. We start with a linear map $[X_1,X_2]\times[Y_1,Y_2]\to[0,1]^2$

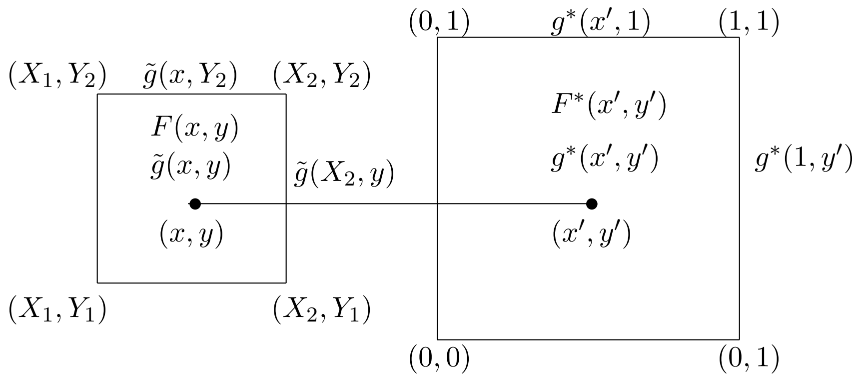



Figure: Linear map $(x,y) \to (x',y')$.

defined by

$$\frac{x - X_1}{X_2 - X_1} = x', \quad \frac{y - Y_1}{Y_2 - Y_1} = y' \tag{627}$$

with inverse

$$x = x'(X_2 - X_1) + X_1, \quad y = y'(Y_2 - Y_1) + Y_1. \tag{628}$$

Using this map $[X_1, X_2] \times [Y_1, Y_2] \to [0,1]^2$ to the $F(x,y)$ and $g(x,y)$ in $[X_1, X_2] \times [Y_1, Y_2] \to [0,1]^2$ we define the following related $F^*(x',y')$ and $g^*(x',y')$ in $[0,1]^2$:

$$F^*(x',y') = [F(x,y)]_{\substack{x=x'(X_2-X_1)+X_1, \\ y=y'(Y_2-Y_1)+Y_1}} = F(x'(X_2 - X_1) + X_1, y'(Y_2 - Y_1) + Y_1) \tag{629}$$

for $(x',y') \in [0,1]^2$. We have

$$\mathrm{d}_{x'}\mathrm{d}_{y'}F^*(x',y') = \mathrm{d}_x\mathrm{d}_y F(x,y)(X_2 - X_1)(Y_2 - Y_1).$$

So the differential of $F(x,y)$ has the same sign as the differential of $F^*(x',y')$.

For a definition of $g^*(x',y')$ we use auxiliary $\tilde{g}(x,y)$ defined by the following: Let us assume that from $(x_1,y_1)\ldots,(x_N,y_N)$ there are $M$-points in $[X_1,X_2] \times [Y_1,Y_2]$ with the local step d.f.

$$\tilde{F}_M(x,y) = \frac{1}{M}\#\{n \le N; (x_n,y_n) \in [0,x) \times [0,y)\}$$

and assume $\tilde{F}_M(x,y) \to \tilde{g}(x,y)$ a.e. on $[X_1,X_2] \times [Y_1,Y_2]$. Since

$$\tilde{F}_M(x,y) = \frac{F_N(x,y) + F_N(X_1,Y_1) - F_N(x,Y_1) - F_N(X_1,y)}{F_N(X_2,Y_2) + F_N(X_1,Y_1) - F_N(X_2,Y_1) - F_N(X_1,Y_2)},$$

we have

$$\tilde{g}(x,y) = \frac{g(x,y) + g(X_1,Y_1) - g(x,Y_1) - g(X_1,y)}{g(X_2,Y_2) + g(X_1,Y_1) - g(X_2,Y_1) - g(X_1,Y_2)}. \tag{630}$$

Now, we put

$$g^*(x',y') = [\tilde{g}(x,y)]_{\substack{x=x'(X_2-X_1)+X_1,\\ y=y'(Y_2-Y_1)+Y_1}} = \tilde{g}(x'(X_2-X_1)+X_1, y'(Y_2-Y_1)+Y_1) \tag{631}$$

for $x',y' \in [0,1]^2$. Since

$$\frac{1}{M}\sum_{n=1}^{M} F(x_n,y_n) = \frac{1}{M}\sum_{n=1}^{M} F^*(x'_n,y'_n), \tag{632}$$

$$\frac{1}{M}\sum_{n=1}^{M} F(x_n,y_n) \to \iint_{[X_1,X_2]\times[Y_1,Y_2]} F(x,y)\mathrm{d}_x\mathrm{d}_y\tilde{g}(x,y), \tag{633}$$

$$\frac{1}{M}\sum_{n=1}^{M} F^*(x'_n,y'_n) \to \int_0^1\int_0^1 F^*(x',y')\mathrm{d}_{x'}\mathrm{d}_{y'}g^*(x',y') \tag{634}$$

and the differential of $F^*(x',y')$ has a constant sign on the open unit square, it follows from the Theorem 218 that the integral (634) is maximal if and only if:

(i) $\mathrm{d}'_x\mathrm{d}'_yF^*(x',y') > 0$ implies $g^*(x',y') = \min(g^*(x',1), g^*(1,y'))$ and

(ii) $\mathrm{d}'_x\mathrm{d}'_yF^*(x',y') < 0$ implies $g^*(x',y') = \max(g^*(x',1)+g^*(1,y')-1,0)$.

From maps $g^*(x',1) \longleftrightarrow \tilde{g}(x,Y_2)$, $g^*(1,y') \longleftrightarrow \tilde{g}(X_2,y)$ and from (632), it follows that the integral (633) is maximal if and only if:

(i) $\mathrm{d}_x\mathrm{d}_yF(x,y) > 0$ implies $\tilde{g}(x,y) = \min(\tilde{g}(x,Y_2), \tilde{g}(X_2,y))$ and

(ii) $\mathrm{d}_x\mathrm{d}_yF(x,y) < 0$ implies $\tilde{g}(x,y) = \max(\tilde{g}(x,Y_2)+\tilde{g}(X_2,y)-1,0)$.

Here from (630) we have

$$\begin{aligned}\tilde{g}(x,Y_2) &= \frac{g(x,Y_2) + g(X_1,Y_1) - g(x,Y_1) - g(X_1,Y_2)}{g(X_2,Y_2) + g(X_1,Y_1) - g(X_2,Y_1) - g(X_1,Y_2)},\\ \tilde{g}(X_2,y) &= \frac{g(X_2,y) + g(X_1,Y_1) - g(X_2,Y_1) - g(X_1,y)}{g(X_2,Y_2) + g(X_1,Y_1) - g(X_2,Y_1) - g(X_1,Y_2)}.\end{aligned} \tag{635}$$

Using (635) in (i) and (ii) we find (625) and (626). □

For a fixed $F(x,y)$, the criterion in Theorem 219 can be fulfilling for several copulas, which form local extremes of $\int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_yg(x,y)$. For a global extreme of $\int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y)$ we need to choose $g(x,Y_i)$ and $g(X_i,y)$ optimally in Theorem 219, see the following theorem.

**Theorem 220.** *Let us divide* $[0,1]^2$ *to two parts* $[0,1]\times[Y,1]$ *and* $[0,1]\times[0,Y]$ *and define continuous* $F(x,y)$ *by a composition*

$$F(x,y)=\begin{cases} F_1(x,y), & \textit{if } (x,y)\in[0,1]\times[Y,1],\\ F_2(x,y), & \textit{if } (x,y)\in[0,1]\times[0,Y],\end{cases}$$

*where* $\mathrm{d}_x\mathrm{d}_y F_1(x,y)>0$ *for interior points of* $[0,1]\times[Y,1]$ *and* $\mathrm{d}_x\mathrm{d}_y F_1(x,y)<0$ *for interior points of* $[0,1]\times[0,Y]$*. Then we have*

$$\max_{g(x,y)\text{-}copula}\int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y)$$
$$=\max_{h(x)}\left(\int_0^1 F_1(x,x-h(x)+Y)(1-h'(x))\mathrm{d}x+\int_0^1 F_2(x,Y-h(x))h'(x)\mathrm{d}x\right), \tag{636}$$

*where the maximum in* (636) *is over nondecreasing* $h(x)$*,* $h(0)=0$*,* $h(1)=Y$ *and* $\max(x+Y-1,0)\le h(x)\le\min(x,Y)$*.*

*Proof.* As in the proof of Theorem 219 we start with two linear maps:

$$[0,1]\times[Y,1]\to[0,1]^2,\text{ where } x'=x,\quad y'=\frac{y-Y}{1-Y}, \tag{637}$$
$$[0,1]\times[0,Y]\to[0,1]^2,\text{ where } x'=x,\quad y'=\frac{y}{Y}. \tag{638}$$

Then, from (630) and from the properties $g(0,0)=g(0,y)=g(x,0)=0$, $g(1,1)=1$, $g(1,Y)=Y$ of copulas $g(x,y)$ we find

$$\tilde g_1(x,y)=\frac{g(x,y)-g(x,Y)}{1-Y},\quad \tilde g_2(x,y)=\frac{g(x,y)}{Y}. \tag{639}$$

Using transformation (637) and (638) we find

$$g_1^*(x',y')=[\tilde g_1(x,y)]_{\substack{x=x',\\ y=y'(1-Y)+Y}}=\frac{g(x',y'(1-Y)+Y)-g(x',Y)}{1-Y} \tag{640}$$
$$g_2^*(x',y')=[\tilde g_2(x,y)]_{\substack{x=x',\\ y=y'Y}}=\frac{g(x',Y)}{Y}\text{ see following Fig.} \tag{641}$$

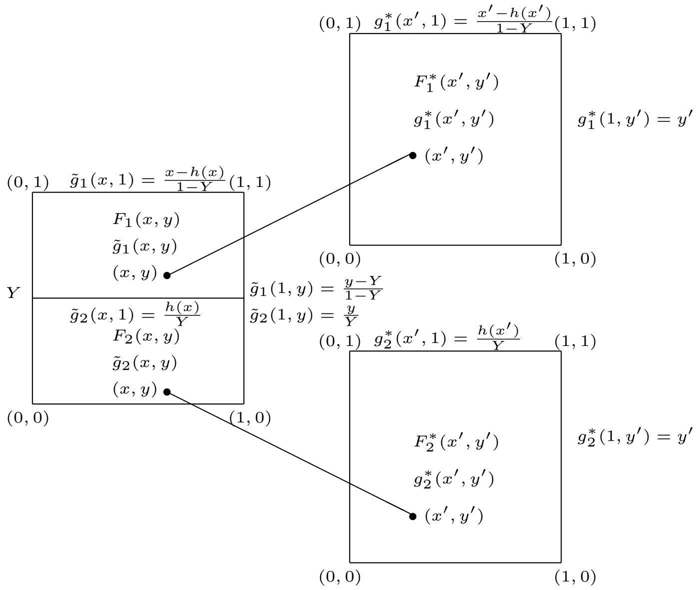



Figure: Linear maps $[0,1]\times[Y,1]\to[0,1]^2$ and $[0,1]\times[0,Y]\to[0,1]^2$.

Put $g(x,Y)=h(x)$. Since $g(x,y)$ is a copula and for every copula $\max(x+y-1,0)\le g(x,y)\le\min(x,y)$, then we have $h(0)=0$, $h(1)=Y$, $h(x)$ is nondecreasing and $\max(x+Y-1,0)\le h(x)\le\min(x,Y)$. Now, from (640) and (641) we find

$$g_1^*(x',1)=\frac{x'-h(x')}{1-Y},\quad g_1^*(1,y')=y',\tag{642}$$

$$g_2^*(x',1)=\frac{h(x')}{Y},\quad g_2^*(1,y')=y'.\tag{643}$$

Then by Theorem 218 the d.f. $g_1^*(x',y')$ maximize $\int_0^1\int_0^1 F_1^*(x',y')\mathrm{d}_{x'}\mathrm{d}_{y'}g(x',y')$ and $g_2^*(x',y')$ maximize $\int_0^1\int_0^1 F_2^*(x',y')\mathrm{d}_{x'}\mathrm{d}_{y'}g(x',y')$ for fixes (642) and (643) iff

$$g_1^*(x',y')=\min\left(\frac{x'-h(x')}{1-Y},y'\right),\tag{644}$$

$$g_2^*(x',y')=\max\left(\frac{h(x')}{Y}+y'-1,0\right).\tag{645}$$

Now, from (639) it follows

$$g(x,y)=\begin{cases}\tilde{g}_1(x,y)(1-Y)+h(x), & \text{if } (x,y)\in[0,1]\times[Y,1],\\ \tilde{g}_2(x,y)Y, & \text{if } (x,y)\in[0,1]\times[0,Y].\end{cases}\tag{646}$$

From backward transformations of (637) and (638) we find

$$\begin{aligned}\tilde{g}_1(x,y) &= [g_1^*(x',y')]_{\substack{x'=x,\\ y'=\frac{y-Y}{1-Y}}} = \min\left(\frac{x-h(x)}{1-Y},\frac{y-Y}{1-Y}\right)\\ \tilde{g}_2(x,y) &= [g_2^*(x',y')]_{\substack{x'=x,\\ y'=\frac{y}{Y}}} = \max\left(\frac{h(x)}{Y}+\frac{y}{Y}-1,0\right).\end{aligned} \tag{647}$$

By inserting (647) to (646) we find

$$g(x,y) = \begin{cases}\min(x-h(x),y-Y)+h(x), & \text{if } y\in[Y,1],\\ \max(h(x)+y-Y,0), & \text{if } y\in[0,Y]\end{cases} \tag{648}$$

and for every $x \in [0,1]$. It can be seen that $g(x,y)$ has a nonzero differential on the curves $y = x - h(x) + Y$ and $y = Y - h(x)$ only, where

$$\mathrm{d}_x\mathrm{d}_y g(x,y) = \begin{cases}(1-h'(x))\mathrm{d}x, & \text{if } y = x-h(x)+Y,\\ h'(x)\mathrm{d}x, & \text{if } y = Y-h(x).\end{cases} \tag{649}$$

Finally, (649) implies

$$\begin{aligned}\int_0^1\int_Y^1 F_1(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y) &= \int_0^1 F_1(x,x-h(x)+Y)(1-h'(x))\mathrm{d}x\\ \int_0^1\int_0^Y F_2(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y) &= \int_0^1 F_2(x,Y-h(x))h'(x)\mathrm{d}x.\end{aligned}$$

□

**Notes 45.** Rewrite (636) in the form $\int_0^1 G(x,h,h')\mathrm{d}x$. Then for searching optimal $h(x)$ we can apply Euler-Lagrange differential equation in variation method (see [149, p. 208]): If the integral $\int_0^1 G(x,h,h')\mathrm{d}x$ takes on $h_0(x)$ a local extreme over all continuous $h(x)$ with continuous derivative $h'(x)$, $h(0) = h_0(0) = 0$, $h(1) = h_0(1) = 1$, then $h_0(x)$ satisfies

$$\frac{\partial G}{\partial h} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial G}{\partial h'} = 0, \tag{650}$$

see the following example.

**Example 97.** Put $Y = \frac{1}{2}$ and define, for $x \in [0,1]$

$$F(x,y) = \begin{cases}F_1(x,y) = 2\left(y-\frac{1}{2}\right)x, & \text{if } y \in \left[\frac{1}{2},1\right],\\ F_2(x,y) = 2\left(\frac{1}{2}-y\right)x, & \text{if } y\in\left[0,\frac{1}{2}\right].\end{cases} \tag{651}$$

Then the integral in (636) has the form

$$\int_0^1 F_1(x, x - h(x) + Y)(1 - h'(x))\mathrm{d}x + \int_0^1 F_2(x, Y - h(x))h'(x)\mathrm{d}x$$
$$= \frac{1}{6} + 2\int_0^1 (xh(x) - h^2(x))\mathrm{d}x. \tag{652}$$

By Euler equation (650) we have

$$\frac{\partial(xh - h^2)}{\partial h} = x - 2h(x) = 0$$

which gives $h(x) = \frac{x}{2}$. Such $h(x)$ is nondecreasing, satisfies $\max(x+\frac{1}{2}-1, 0) \le h(x) \le \min(x, \frac{1}{2})$ and thus the searched global extreme is

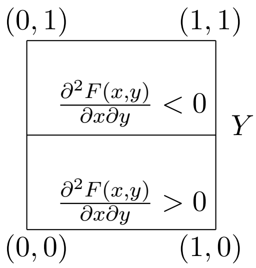
$$\max_{g(x,y)\text{-copula}} \int_0^1 \int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y) = \frac{1}{6} + 2\frac{1}{12} = \frac{1}{3}.$$

This maximum also holds omitting continuous derivative of $h$. It follows from the fact the point

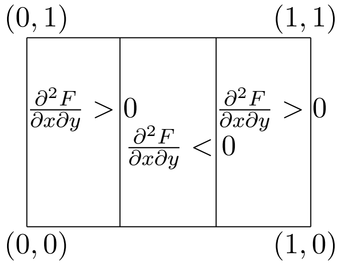
$$\left(\int_0^1 xh(x)\mathrm{d}x = \frac{1}{6}, \int_0^1 h^2(x)\mathrm{d}x = \frac{1}{12}\right)$$

lie on the boundary of the projection $X_3 \times X_2$ of the body $(X_1, X_2, X_3)$, where $X_1 = \int_0^1 g(x)\mathrm{d}x$, $X_2 = \int_0^1 xg(x)\mathrm{d}x$, $X_3 = \int_0^1 g^2(x)\mathrm{d}x$ and $g(x)$ run over all d.f.s. The straight line $X_2 - X_3 = \frac{1}{12}$ touches to this projection, see [159] and Section 4.10 in this book.

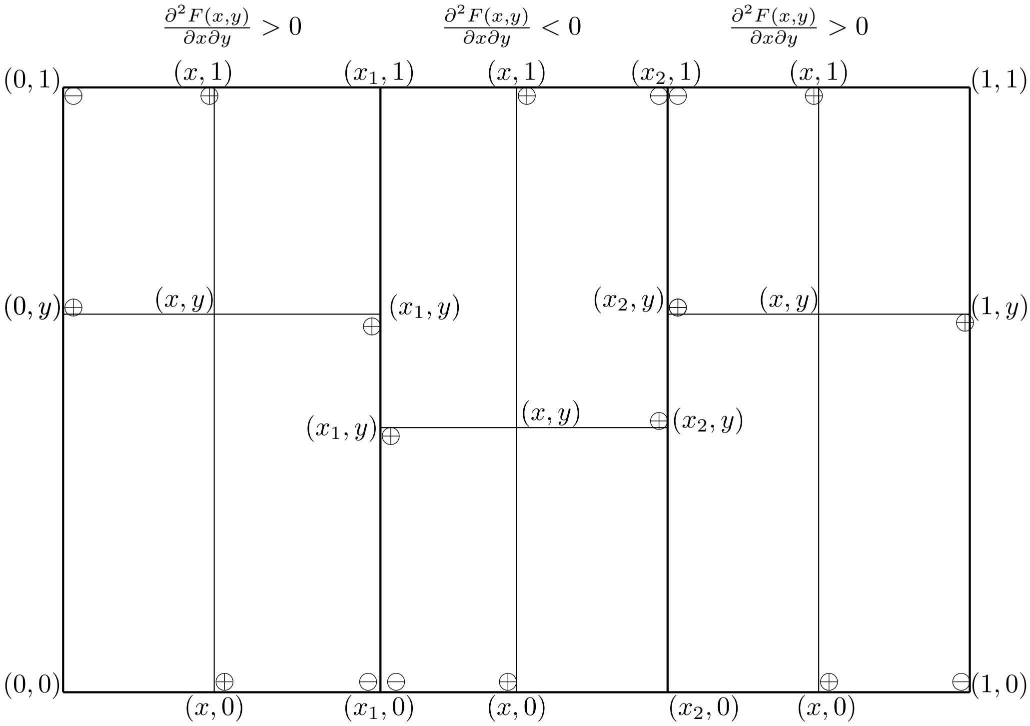
From example of the form




we go to (see [14])




In this case we divide unit square $[0,1]^2$ on $[0,x_1] \times [0,1]$, $[x_1,x_2] \times [0,1]$ and $[x_2,1] \times [0,1]$, see following Fig.




For computing a form of copula $g(x,y)$ which maximize $\int_0^1 \int_0^1 F(x,y) \mathrm{d}_x \mathrm{d}_y g(x,y)$ we use (625) if $x \in (0,x_1) \cup (x_2,1)$ and (626) if $x \in (x_1,x_2)$. Denote

$g(x_1,y) = h_1(y)$,

$g(x_2,y) = h_2(y)$.

If $x \in (0,x_1)$, then

$$g(x,y) = \min(g(0,y) + g(x,1) - g(0,1), g(x,0) + g(x_1,y) - g(x_1,0))$$

$$= \min(0 + x - 0, 0 + h_1(y) - 0) = \min(x, h_1(y)). \tag{653}$$

If $x \in (x_1, x_2)$, then

$$\begin{aligned} g(x,y) &= \max(g(x,1) + g(x_2,y) - g(x_2,1), g(x_1,y) + g(x,0) - g(x_1,0)) \\ &= \max(x + h_2(y) - x_2, h_1(y) + 0 - 0) = \max(x + h_2(y) - x_2, h_1(y)). \end{aligned} \tag{654}$$

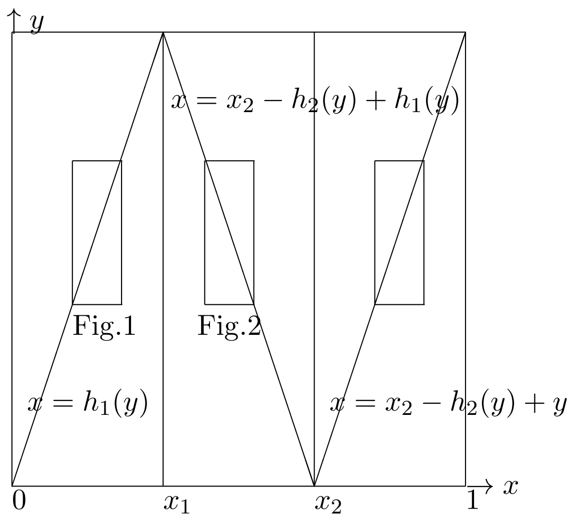
If $x \in (x_2, 1)$, then

$$\begin{aligned} g(x,y) &= \min(g(x_2,y) + g(x,1) - g(x_2,1), g(x,0) + g(1,y) - g(1,0)) \\ &= \min(h_2(y) + x - x_2, 0 + y - 0) = \min(x - x_2 + h_2(y), y). \end{aligned} \tag{655}$$

The differential $\mathrm{d}_x\mathrm{d}_y g(x,y)$ of the copula $g(x,y)$ defined by (653), (654) and (655) is nonzero only for points $(x,y)$ on the curves

$$\begin{aligned} &x = h_1(y), y \in [0,1], \\ &x = x_2 - h_2(y) + h_1(y), y \in [0,1], \\ &x = x_2 - h_2(y) + y, y \in [0,1]. \end{aligned} \tag{656}$$

Formally,




On the rectangle

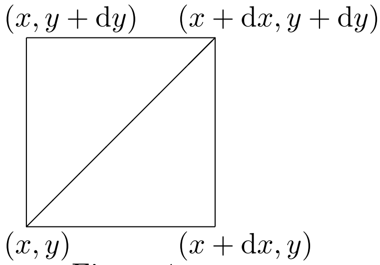


Figure 1.

the differential is defined as

$$\mathrm{d}_x\mathrm{d}_y g(x,y) = g(x,y) + g(x+\mathrm{d}x, y+\mathrm{d}y) - g(x, y+\mathrm{d}y) - g(x+\mathrm{d}x). \quad (657)$$

For the case $x \in (0, x_1)$
we have $x = h_1(y)$ and $g(x,y) = \min(x, h_1(y))$ and for the vertices of Fig. 1 we have

(i) $g(x,y) = x = h_1(y)$;
(ii) $g(x+\mathrm{d}x, y+\mathrm{d}y) = x+\mathrm{d}x = h_1(y+\mathrm{d}y) = h_1(y) + h_1'(y)\mathrm{d}y$;
(iii) $g(x, y+\mathrm{d}y) = \min(x, h_1(y+\mathrm{d}y) = x = h_1(y)$;
(iv) $g(x+\mathrm{d}x, y) = \min(x+\mathrm{d}x, h_1(y)) = h_1(y) = x$.

Thus by (657)

$$\mathrm{d}_x\mathrm{d}_y g(x,y) = h_1(y) + (h_1(y) + h_1'(y)\mathrm{d}y) - h_1(y) - h_1(y) = h_1'(y)\mathrm{d}y. \quad (658)$$

For the case $x \in (x_1, x_2)$
we have $x = x_2 - h_2(y) + h_1(y)$ and $g(x,y) = \max(x + h_2(y) - x_2, h_1(y))$ and the differential in the rectangle

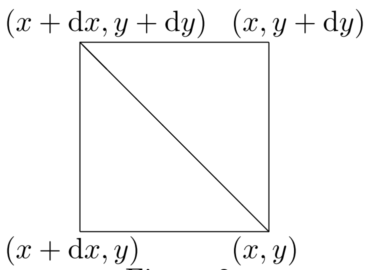


Figure 2.

is equal

$$\mathrm{d}_x\mathrm{d}_y g(x,y) = g(x+\mathrm{d}x, y) + g(x, y+\mathrm{d}y) - g(x+\mathrm{d}x, y+\mathrm{d}y) - g(x,y) \quad (659)$$

For the vertices of $g(x, y)$ in Fig. 2 we have

(i) $g(x, y) = \max(x + h_2(y) - x_2, h_1(y)) = x + h_2(y) - x_2 = h_1(y)$;

(ii) $g(x+\mathrm{d}x, y+\mathrm{d}y) = x+\mathrm{d}x+h_2(y+\mathrm{d}y)-x_2 = h_1(y+\mathrm{d}y) = h_1(y)+h_1'(y)$;

(iii) $g(x, y + \mathrm{d}y) = \max(x + h_2(y + \mathrm{d}y) - x_2, h_1(y + \mathrm{d}y))$
$= \max(h_1(y) + h_2'(y)\mathrm{d}y, h_1(y) + h_1'(y)\mathrm{d}y)$
$=h_1(y) + h_2'(y)\mathrm{d}y$
since for $(x, y)$ in the curve $x + h_2(y) - x_2 = h_1(y)$ we have
$\frac{\mathrm{d}x}{\mathrm{d}y} + h_2'(y) = h_1'(y)$ and $\frac{\mathrm{d}x}{\mathrm{d}y} < 0$.

(iv) $g(x + \mathrm{d}x, y) = \max(x + \mathrm{d}x + h_2(y) - x_2, h_1(y)) = \max(h_1(y) + \mathrm{d}x, h_1(y)) = h_1(y)$ since $\mathrm{d}x < 0$. Thus by (659) we have

$$\begin{aligned}\mathrm{d}_x\mathrm{d}_y g(x, y) &= h_1(y) + (h_1(y) + h_2'(y)\mathrm{d}y) - h_1(y) - h_1'(y) - h_1(y)\mathrm{d}y \\ &= (h_2'(y) - h_1'(y))\mathrm{d}y \end{aligned} \tag{660}$$

In the case $x \in (x_2, 1)$
we have $x = x_2 - h_2(y) + y$ and $g(x, y) = \min(x - x_2 + h_2(y), y)$ and we use Fig. 1 and the differential (732). In this case in vertices of Fig. 1 we have

(i) $g(x, y) = \min(x - x_2 + h_2(y), y) = x - x_2 + h_2(y) = y$;

(ii) $g(x + \mathrm{d}x, y + \mathrm{d}y) = \min(x + \mathrm{d}x - x_2 + h_2(y + \mathrm{d}y), y + \mathrm{d}y)$
$= \min(x + \mathrm{d}x - x_2 + h_2(y) + h_2'(y)\mathrm{d}y, y + \mathrm{d}y)$
$= \min(y + \mathrm{d}x + h_2'(y)\mathrm{d}y, y + \mathrm{d}y) = y + \mathrm{d}y$
since for $(x, y)$ in the curve $x - x_2 + h_2(y) = y$ we have $\mathrm{d}x + h_2'(y)\mathrm{d}y = \mathrm{d}y$.

(iii) $g(x, y + \mathrm{d}y) = \min(x - x_2 + h_2(y + \mathrm{d}y), y + \mathrm{d}y)$
$= \min(x - x_2 + h_2(y) + h_2'(y)\mathrm{d}y, y + \mathrm{d}y) = \min(y + h_2'(y)\mathrm{d}y, y + \mathrm{d}y)$
$= y + h_2'(y)\mathrm{d}y$
since $h_2'(y) \leq 1$, a proof follows.

(iv) $g(x + \mathrm{d}x, y) = \min(x + \mathrm{d}x - x_2 + h_2(y), y) = \min(y + \mathrm{d}x, y) = y$ since $\mathrm{d}x > 0$. By (732) we have

$$\begin{aligned}\mathrm{d}_x\mathrm{d}_y g(x, y) &= g(x, y) + g(x + \mathrm{d}x, y + \mathrm{d}y) - g(x, y + \mathrm{d}y) - g(x + \mathrm{d}x) \\ &= y + y + \mathrm{d}y - (y + h_2'(y)\mathrm{d}y) - y = (1 - h_2'(y))\mathrm{d}y. \end{aligned} \tag{661}$$

## 9.2 Summarization

[14]: Let

$$F(x, y) = \begin{cases} F_1(x, y) & \text{if } x \in (0, x_1), \frac{\partial^2 F_1(x,y)}{\partial x \partial y} > 0 \\ F_2(x, y) & \text{if } x \in (x_1, x_2), \frac{\partial^2 F_2(x,y)}{\partial x \partial y} < 0 \\ F_3(x, y) & \text{if } x \in (x_2, 1), \frac{\partial^2 F_3(x,y)}{\partial x \partial y} > 0. \end{cases} \tag{662}$$

By (653), (654) and (655) the copulas which maximize $\int_0^1 \int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y)$ has the form

$$g(x,y) = \begin{cases} \min(x, h_1(y)) & \text{if } x \in [0, x_1], \\ \max(x + h_2(y) - x_2, h_1(y)) & \text{if } x \in [x_1, x_2], \\ \min(x - x_2 + h_2(y), y) & \text{if } x \in [x_2, 1] \end{cases} \tag{663}$$

where $y \in [0,1]$ and $h_1(y)$ and $h_2(y)$ are variable functions which we need to calculate. By (658), (660) and (661) for nonzero differential $\mathrm{d}_x\mathrm{d}_y g(x,y)$ we have

$$\mathrm{d}_x\mathrm{d}_y g(x,y) = \begin{cases} h_1'(y)\mathrm{d}y & \text{if } x \in [0, x_1], x = h_1(y), \\ (h_2'(y) - h_1'(y))\mathrm{d}y & \text{if } x \in [x_1, x_2], x = x_2 - h_2(y) + h_1(y), \\ (1 - h_2'(y))\mathrm{d}y & \text{if } x \in [x_2, 1], x = x_2 - h_2(y) + y. \end{cases} \tag{664}$$

Now, we return to the integral (437).

**Theorem 221.** *Denote*

$$\begin{aligned} &G(y, h_1, h_2, h_1', h_2') \\ &= F_1(h_1(y), y)h_1'(y) + F_2(x_2 - h_2(y) + h_1(y), y)(h_2'(y) - h_1'(y)) \\ &+ F_3(x_2 - h_2(y) + y, y)(1 - h_2'(y)). \end{aligned} \tag{665}$$

*Then*

$$\max_{g(x,y)-\ copula} \int_0^1 \int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y) = \max_{h_1,h_2} \int_0^1 G(y, h_1, h_2, h_1', h_2')\mathrm{d}y, \tag{666}$$

*where $h_1, h_2$ give a copula in* (663). *If the solutions* $(h_1, h_2)$ *of* (666) *not give a copula in* (663), *then we have an upper bound*

$$\max_{g(x,y)-copula} \int_0^1 \int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y) \le \int_0^1 G(y, h_1, h_2, h_1', h_2')\mathrm{d}y. \tag{667}$$

*Proof.* The two-dimensional Stieltjes integral $\int_0^1 \int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y)$ over $F(x,y)$ in (662) and over $g(x,y)$ defined by (663) with differential (664) and $\mathrm{d}_x\mathrm{d}_y g(x,y) = 0$ others, is calculated as the following one-dimensional integral

$$\int_0^1 \int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y) = \int_0^1 F_1(h_1(y), y)h_1'(y)\mathrm{d}y \tag{668}$$

$$+ \int_0^1 F_2(x_2 - h_2(y) + h_1(y), y)(h_2'(y) - h_1'(y))\mathrm{d}y \tag{669}$$

$$+ \int_0^1 F_3(x_2 - h_2(y) + y, y)(1 - h_2'(y))\mathrm{d}y. \tag{670}$$

□

To compute extremes in $\int_0^1 G(y, h_1, h_2, h_1', h_2')\mathrm{d}y$ we can apply calculus of variation (cf [191, p. 33]): If $(h_1, h_2)$ extremize the integral $\int_0^1 G(y, h_1, h_2, h_1', h_2')\mathrm{d}y$ then $(h_1, h_2)$ must be satisfied the following system of Euler-Lagrange differential equations

$$\frac{\partial G}{\partial h_1} - \frac{\mathrm{d}}{\mathrm{d}y}\frac{\partial G}{\partial h_1'} = 0, \tag{671}$$

$$\frac{\partial G}{\partial h_2} - \frac{\mathrm{d}}{\mathrm{d}y}\frac{\partial G}{\partial h_2'} = 0. \tag{672}$$

The solution $(h_1, h_2)$ maximize $\int_0^1 G(y, h_1, h_2, h_1', h_2')\mathrm{d}y$ if

$$\frac{\partial^2 G}{\partial h_1' \partial h_1'} \leq 0, \quad \begin{vmatrix} \frac{\partial^2 G}{\partial h_1' \partial h_1'} & \frac{\partial^2 G}{\partial h_1' \partial h_2'} \\ \frac{\partial^2 G}{\partial h_2' \partial h_1'} & \frac{\partial^2 G}{\partial h_2' \partial h_2'} \end{vmatrix} \leq 0. \tag{673}$$

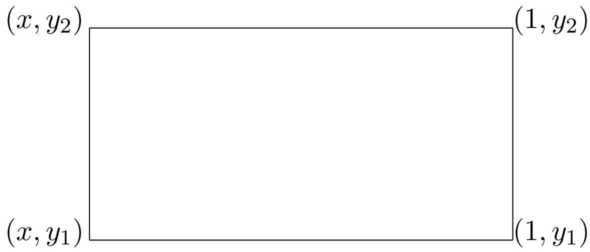
**Theorem 222.** *The function $g(x, y)$ defined by* (663) *is a copula if and only if*

(i) $h_1(y)$ *and* $h_2(y)$ *are increasing;*
(ii) $h_1(0) = 0$, $h_2(0) = 0$*;*
(iii) $h_1(1) = x_1$, $h_2(1) = x_2$*;*
(iv) $0 \leq h_1(y) \leq h_2(y) \leq y$*;*
(v) $0 \leq h_1'(y) \leq h_2'(y) \leq 1$*;*

*Proof.* $1^0$. Necessary. Let $g(x, y)$ be a copula and $h_1(y) = g(x_1, y)$ and $h_2(y) = g(x_2, y)$. The properties (i)-(iii) are clear.

Now we prove (v).

For an arbitrary copula $g(x, y)$ the differential $g(x, y)$ on the rectangle




is

$$g(x, y_1) + g(1, y_2) - g(x, y_2) - g(1, y_1) = g(x, y_1) + y_2 - g(x, y_2) - y_1 \geq 0 \quad (674)$$

which gives

$y_2 - y_1 \geq g(x, y_2) - g(x, y_1)$, which gives

$g'(x, y) \leq 1$ a.e. and which implies

$h'(y) \leq 1$ a.e. for $h(y) = g(x, y)$ (see also [114, p. 11, Th. 2.2.7.] ). Furthermore

$h_1'(y) \leq h_2'(y)$ because for $(x, y)$ on the curve $x + h_2(y) - x_2 = h_1(y)$, $x \in [x_1, x_2]$ we have

$\frac{\mathrm{d}x}{\mathrm{d}y} + h_2'(y) = h_1'(y)$ and $\frac{\mathrm{d}x}{\mathrm{d}y} < 0$. This is the end of proof (v).

Proof of (iv).

The differential of $g(x, y)$ on the rectangle in Fig.


is

$$g(x, 0) + g(1, y_1) - g(x, y_1) - g(1, 0) = 0 + y_1 - g(x, y_1) - 0 \geq 0, \quad (675)$$

thus $y_1 \geq g(x, y_1)$. This is the end of proof (iv).

$2^0$. Sufficiency. From (v) and (664) follows that the differential $\mathrm{d}_x \mathrm{d}_y g(x, y)$ of (663) is nonnegative for every $(x, y) \in [0, 1]^2$. Also by (663) $g(x, 0) =$

$g(0,y)=0$. Thus $g(x,y)$ is a distribution function. Now, for copula we need to show $g(x,1)=x$ and $g(1,y)=y$ for every $(x,y)\in[0,1]^2$. Indeed we have

$$g(x,1)=\begin{cases}\min(x,h_1(1))=\min(x,x_1)=x & \text{if } x\in[0,x_1],\\ \max(x+h_2(1)-x_2,h_1(1))=\max(x,x_1)=x & \text{if } x\in[x_1,x_2],\\ \min(x-x_2+h_2(1),1)=\min(x,1)=x & \text{if } x\in[x_2,1].\end{cases}$$

For $x=1$ we need

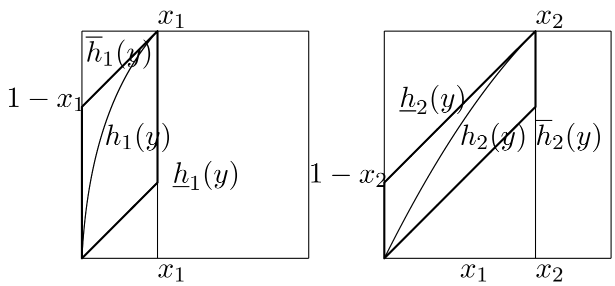
$$g(1,y)=\min(1-x_2+h_2(y),y)=y. \tag{676}$$

Since $h_2(1)=x_2$, then (676) is equivalent

$$1-y\geq h_2(1)-h_2(y). \tag{677}$$

It holds, since $h_2(1)-h_2(y)=(1-y)h_2'(y^*)$ and the derivative satisfies (v). $\square$

Theorem 222 implies the following boundaries



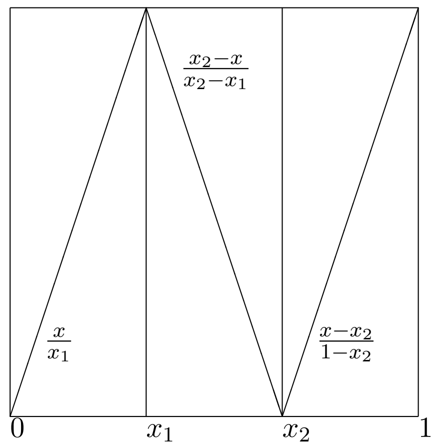

$$\overline{h}_1(y)=\begin{cases}y & \text{if } y\in[0,x_1],\\ x_1 & \text{if } y\in[x_1,1],\end{cases}\quad \underline{h}_1(y)=\begin{cases}0 & \text{if } y\in[0,1-x_1],\\ y-(1-x_1) & \text{if } y\in[1-x_1,1],\end{cases}$$

$$\overline{h}_2(y)=\begin{cases}y & \text{if } y\in[0,x_2],\\ x_1 & \text{if } y\in[x_2,1],\end{cases}\quad \underline{h}_2(y)=\begin{cases}0 & \text{if } y\in[0,1-x_2],\\ y-(1-x_1) & \text{if } y\in[1-x_2,1],\end{cases}$$

where

$$\underline{h}_1(y)\leq h_1(y)\leq\overline{h}_1(y),\quad \underline{h}_2(y)\leq h_2(y)\leq\overline{h}_2(y). \tag{678}$$

**Example 98.** In (662) define $F(x,y) = f(x).y$, where $f(x)$ is given by Fig.




and denote

$F_1(x,y) = \frac{x}{x_1}y$, and the integrand in (668) has the form
$F_1(h_1(y),y)h_1'(y) = \frac{h_1}{x_1}yh_1'$.
$F_2(x,y) = \frac{x_2-x}{x_2-x_1}y$, and the integrand in (669) has the form
$F_2(x_2-h_2(y)+h_1(y),y)(h_2'(y)-h_1'(y)) = \frac{h_2-h_1}{x_2-x_1}y(h_2'-h_1')$.
$F_3(x,y) = \frac{x-x_2}{1-x_2}y$, and the integrand in (670) has the form
$F_3(x_2-h_2(y)+y,y)(1-h_2'(y)) = \frac{y-h_2}{1-x_2}y(1-h_2')$.
Summing up this, then $G$ in (665) has the form

$$G = \frac{h_1}{x_1}yh_1' + \frac{h_2-h_1}{x_2-x_1}y(h_2'-h_1') + \frac{y-h_2}{1-x_2}y(1-h_2'), \tag{679}$$

$$\frac{\partial G}{\partial h_1} - \frac{\mathrm{d}}{\mathrm{d}y}\frac{\partial G}{\partial h_1'} = \frac{h_1'}{x_1}y - \frac{h_2'-h_1'}{x_2-x_1}y - \frac{h_1}{x_1} + \frac{h_2-h_1}{x_2-x_1} = 0, \tag{680}$$

$$\frac{\partial G}{\partial h_2} - \frac{\mathrm{d}}{\mathrm{d}y}\frac{\partial G}{\partial h_2'} = \frac{h_2'-h_1'}{x_2-x_1}y - \frac{1-h_2'}{1-x_2}y - \frac{h_2-h_1}{x_2-x_1} + \frac{y-h_2}{1-x_2} + \frac{y}{1-x_2} = 0. \tag{681}$$

Now, $((680)+(681)).(\frac{1-x_2}{y})$ gives

$$h_1'\left(\frac{1-x_2}{x_1}\right) + h_2' = \frac{h_1}{y}\left(\frac{1-x_2}{x_1}\right) + \frac{h_2}{y} - 1, \tag{682}$$

and $(680).(\frac{x_2-x_1}{y})$ gives

$$h_1'\left(\frac{x_2-x_1}{x_1}+1\right) - h_2' = \frac{h_1}{y}\left(\frac{x_2-x_1}{x_1}+1\right) - \frac{h_2}{y}, \tag{683}$$

Summing up (682) and (683) and by some simple operation we find

$$h_1' = \frac{h_1}{y} - x_1. \tag{684}$$

Insert (684) to (683) we find

$$h_2' = \frac{h_2}{y} - x_2. \tag{685}$$

The general solution $h(y)$ of the differential equation

$$h' = \frac{h}{y} - x \tag{686}$$

has the form

$$h(y) = c.y - xy.\log y. \tag{687}$$

Assuming boundary assumptions $h(1) = x$ and $h(0) = 0$, then $h(y) = xy(1 - \log y)$ and $h'(y) = x(-\log y)$. Thus

$$h_1(y) = x_1 y(1 - \log y), \quad h_2(y) = x_2 y(1 - \log y) \tag{688}$$

not satisfy the condition (v) in Theorem 222. Here $G$ in (679) imputing (688) has the form

$$G = y^2 \log y\left(\frac{x_2}{1-x_2}\right) + y^2(\log y)^2\left(\frac{x_2}{1-x_2}\right) + y^2$$

which gives

$$\int_0^1 G\mathrm{d}y = \frac{x_2}{1-x_2}\left(-\frac{1}{27}\right) + \frac{1}{3}. \tag{689}$$

In other case imputing $h_1(y) = x_1 y$ and $h_2(y) = x_2 y$ in $G$ in (679) gives $G = y^2$ and $\int_0^1 G\mathrm{d}y = \frac{1}{3}$. Thus (688) not maximize $\int_0^1 G\mathrm{d}y$.

Note that for $F(x, y) = f(x).y$, where $f(x)$ is uniformly distributed preserving map (u.d.p.) [58] we have

$$\max_{x_n, y_n \text{ are u.d.}} \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(x_n)y_n = \max_{g(x,y)-\text{copula}} \int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y), \tag{690}$$

[58]Definition: For every u.d sequence $x_n$ the mapping sequence $f(x_n)$ is also u.d., see Section 12.3.

$$\max_{x_n-\text{u.d},\Phi-\text{u.d.p.}} \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(x_n)\Phi(x_n) = \max_{\Phi-\text{u.d.p.}} \int_0^1\int_0^1 f(x)\Phi(x)\mathrm{d}x \quad (691)$$

Then (691) is small that (690). By [53, p. 123, Corollary 3] for u.d.p. $f(x)$ we have

$$\max_{\Phi-\text{u.d.p.}} \int_0^1\int_0^1 f(x)\Phi(x)\mathrm{d}x = \frac{1}{3}.$$

Then $\int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y) \geq \frac{1}{3}$.

## 9.3 Extremes of $\int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y)$ attained at shuffles of $M$

**Definition 14.** Let $I_i$, $i=1,2,\dots,n$ be a decomposition of the unit interval $[0,1]$, let $\pi$ be a permutation of $(1,2,\dots,n)$, and let $T:[0,1]\to[0,1]$ be an one-to-one map whose graph $T$ is formed by diagonals or anti-diagonals of squares $I_i\times I_{\pi(i)}$, $i=1,2,\dots,n$. Then the copula $C(x,y)$ defined by

$$C(x,y) = |\mathrm{Project}_x(([0,x)\times[0,y))\cap T)| \quad (692)$$

is called the shuffle of $M$.

Note that if $x_n$, $n=1,2,\dots,$ is an u.d. sequence, then two-dimensional sequence $(x_n,T(x_n))$ has a.d.f $C(x,y)$ and thus for every continuous $F(x,y)$ we have

$$\int_0^1\int_0^1 F(x,y)\mathrm{d}C(x,y) = \int_0^1 F(x,T(x)\mathrm{d}x. \quad (693)$$

For example (594), (604), (558) are shuffles of $M$.

M. Hofer and M.R. Iacò [75] was proved:

**Theorem 223.** *Let $(a_{i,j})$, $i,j=1,2,\dots,n$ be a real-valued $n\times n$ matrix. Let $I_{i,j}=\left[\frac{i-1}{n},\frac{i}{n}\right]\times\left[\frac{j-1}{n},\frac{j}{n}\right]$, $i,j=1,2,\dots,n$ and let the piecewise constant function $F(x,y)$ be defined as*

$$F(x,y) = a_{i,j} \text{ if } (x,y)\in I_{i,j}, i,j=1,2,\dots,n.$$

*Then*

$$\max_{g(x,y)\text{-copula}} \int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y) = \frac{1}{n}\sum_{i=1}^{n} a_{i,\pi^*(i)}. \quad (694)$$

*Here $\pi^*(i)$ maximize $\sum_{i=1}^n a_{i,\pi(i)}$, where $\pi$ is a permutation of $(1,2,\dots,n)$. The maximum in (694) is attained at $g(x,y) = C(x,y)$, where $C(x,y)$ is the shuffle of $M$ whose graph $T$ is formed by diagonals or anti-diagonals in $I_{i,\pi^*(i)}$, $i = 1,2,\dots,n$.*

*Proof.* a) Let $C_k(x,y)$, $k = 1,2,\dots,n! = N$ be all copulas defined by shuffles of $M$ and $t_k \geq 0$, $\sum_{k=1}^N t_k = 1$. Then $C(x,y) = \sum_{k=1}^N t_k C_k(x,y)$ is a copula and satisfies $\int_{[0,1]^2} f(x,y)\mathrm{d}C(x,y) \leq \frac{1}{n}\sum_{i=1}^n a_{i,\pi^*(i)}$.

b) To every copula $C(x,y)$ we add the matrix

$C(i,j) = n\int_{I_{i,j}} 1.\mathrm{d}C(x,y)$. Then $C(i,j)$ is a doubly-stochastic and $C_k(i,j)$ is a permutation matrix. By Birkhoff theorem [22] (see [104]) the set of doubly-stochastic matrices is identical with the convex hull of the set of of permutation matrices. Thus there exists $t_k \geq 0$, $\sum_{k=1}^N t_k = 1$ such that

$C(i,j) = \sum_{k=1}^N t_k C_k(i,j)$ for every $i,j$,

$\frac{1}{n}a_{i,j}C(i,j) = \sum_{k=1}^N t_k \frac{1}{n} a_{i,j} C_k(i,j)$, and thus

$\int_{[0,1]^2} f(x,y)\mathrm{d}C(x,y) = \sum_{k=1}^N t_k \int_{[0,1]^2} f(x,y)\mathrm{d}C_k(x,y) \leq \frac{1}{n}\sum_{i=1}^n a_{i,\pi^*(i)}$.

□

Applying Theorem 223 M. Hofer and M.R. Iacò [75] approximate extremes of $\int_0^1\int_0^1 F(x,y)\mathrm{d}g(x,y)$ which respect to copulas $g(x,y)$ by the following.

**Theorem 224.** *For continuous $F(x,y)$ on $[0,1]^2$ define piecewise constant functions $F_1(x,y)$, $F_2(x,y)$ as*

$$F_1(x,y) = \min_{(u,v)\in I_{i,j}} F(u,v) \text{ if } (x,y) \in I_{i,j}, i,j = 1,2,\dots,n,$$

$$F_2(x,y) = \max_{(u,v)\in I_{i,j}} F(u,v) \text{ if } (x,y) \in I_{i,j}, i,j = 1,2,\dots,n, \text{ where}$$

$$I_{i,j} = \left[\frac{i-1}{n}, \frac{i}{n}\right] \times \left[\frac{j-1}{n}, \frac{j}{n}\right].$$

*Let $C_0(x,y), C_1(x,y), C_2(x,y)$ be copulas such that*

*$C_1(x,y)$ maximize $\int_0^1\int_0^1 F_1(x,y)\mathrm{d}g(x,y)$,*

*$C_2(x,y)$ maximize $\int_0^1\int_0^1 F_2(x,y)\mathrm{d}g(x,y)$ and*

*$C_0(x,y)$ maximize $\int_0^1\int_0^1 F(x,y)\mathrm{d}g(x,y)$*

*over all copulas $g(x,y)$. Then*

$$\int_0^1\int_0^1 F(x,y)\mathrm{d}C_0(x,y)$$

$$= \lim_{n\to\infty} \int_0^1 \int_0^1 F_1(x,y)\mathrm{d}C_1(x,y) = \lim_{n\to\infty} \int_0^1 \int_0^1 F_2(x,y)\mathrm{d}C_2(x,y).$$

*Proof.* Directly from definition we have

$$\int_0^1 \int_0^1 F_1(x,y)\mathrm{d}C_1(x,y) \le \int_0^1 \int_0^1 F(x,y)\mathrm{d}C_0(x,y)$$
$$\le \int_0^1 \int_0^1 F_2(x,y)\mathrm{d}C_2(x,y).$$

From continuity of $F(x,y)$ follows that

$$F_2(x,y) \le F_1(x,y) + \varepsilon$$

for all sufficiently large $n$, for every $(x,y) \in [0,1]^2$. Then

$$\int_0^1 \int_0^1 F_2(x,y)\mathrm{d}C_2(x,y) \le \int_0^1 \int_0^1 (F_1(x,y) + \varepsilon)\mathrm{d}C_2(x,y)$$
$$\le \int_0^1 \int_0^1 F_1(x,y)\mathrm{d}C_2(x,y) + \varepsilon \le \int_0^1 \int_0^1 F_1(x,y)\mathrm{d}C_1(x,y) + \varepsilon.$$

□

## 9.4 Extremes of $\int_0^1 \int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y)$ for $F(x,y)$ of the form $F(x,y) = \Phi(x+y)$

Uckelmann (1997) [182] was proved: Let

$F(x,y) = \Phi(x+y)$ for $(x,y) \in [0,1]^2$;

For $0 < k_1 < k_2 < 2$ let $\Phi(x)$ be a twice differentiable function such that $\Phi(x)$ is strictly convex on $[0,k_1] \cup [k_2,2]$ and concave on $[k_1,k_2]$, i.e.

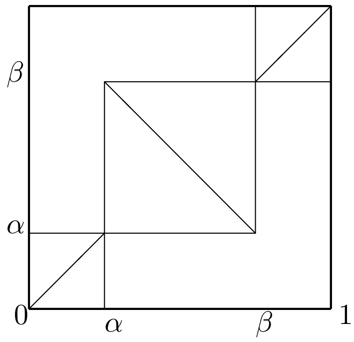
$\Phi''(x) > 0$ for $x \in [0,k_1) \cup (k_2,2]$,

$\Phi''(x) > 0$ for $x \in [0,k_1) \cup (k_2,2]$.

If $\alpha$ and $\beta$ are the solutions of

$$\Phi(2\alpha) - \Phi(\alpha+\beta) + (\beta-\alpha)\Phi'(\alpha+\beta) = 0,$$
$$\Phi(2\beta) - \Phi(\alpha+\beta) + (\alpha-\beta)\Phi'(\alpha+\beta) = 0$$

such that $0 < \alpha < \beta < 1$, then the optimal copula $C(x,y)$ is the shuffle of $M$




with the support

$$\Gamma(x) = \begin{cases} x & \text{for } x \in [0, \alpha] \cup [\beta, 1], \\ \alpha + \beta - x & \text{for } x \in (\alpha, \beta). \end{cases}$$

Then

$$\max \int_0^1 \int_0^1 F(x, y) dC(x, y) = \int_0^{\alpha} \Phi(2x) dx + (\beta - \alpha)\Phi(0) + \int_{\beta}^1 \Phi(2x) dx.$$

## 9.5 Extremes of $\int_0^1 f_1(\Phi(x)) f_2(\Psi(x)) \mathrm{d}x$

The problem of this part was introduced by S. Steinerberger [151]. Motivation is a study of extremes of (565) for $F(x, y) = f_1(x) f_2(x)$ and for u.d. sequences $x_n = \Phi(z_n)$ and $y_n = \Psi(z_n)$, where $\Phi(x)$ and $\Psi(x)$ are u.d.p. functions and $z_n$ is a u.d. sequence. His solution is given in the following Theorem 225, where given proof does not contain a theory of d.f.s.

**Definition 15.** Let $f : [0, 1] \to \mathbb{R}$ be a Lebesgue measurable function, we see it as a random variable, and $g(x) = |f^{-1}([0, x))|$ be its d.f. Put
$f^*(x) = g^{-1}(x)$.
Here, if $g(x)$ is constantly equal to $c$ on interval $(\alpha, \beta)$ (maximal with respect to inclusion) then we put $g^{-1}(c) = \beta$ In the following we shall write such $g(x)$ as $G_f(x)$. [59]

**Notes 46.** In Example 36 we have notion $g(x) = x_f(x)$ and in the following Theorems 226, 227, 229 and 232 we used

$$G_f(x) = |f^{-1}([0, x))| \tag{695}$$

[59] the same value as in the end of the proof of Theorem 215.

**Theorem 225.** *Let $f_1, f_2$ be the Riemann integrable functions on $[0,1]$. Let $\Phi(x), \Psi(x)$ be arbitrary u.d.p. transformations. Then*

$$\int_0^1 f_1^*(x) f_2^*(1-x)\mathrm{d}x \le \int_0^1 f_1(\Phi(x)) f_2(\Psi(x))\mathrm{d}x \le \int_0^1 f_1^*(x) f_2^*(x)\mathrm{d}x \quad (696)$$

*and these bounds are the best possible. Also, every number within the bounds is attained by some u.d.p. $\Phi(x), \Psi(x)$.*

In his proof Steinerberger used the Hardy-Littlewood inequality [Hardy, Littlewood and Pólya [1934, Th. 378]:

$$\int_0^1 f_1(x) f_2(x)\mathrm{d}x \le \int_0^1 f_1^*(x) f_2^*(x)\mathrm{d}x. \quad (697)$$

## 9.6 Extremes of $\int_0^1 f(x)\Phi(x)\mathrm{d}x$

Now, we consider $F(x,y)$ in the form $F(x,y) = f(x)\cdot y$ and study the limit points of $\frac{1}{N}\sum_{n=1}^N f(x_n)\cdot y_n$, where $x_n$ is u.d. sequence and u.d. sequence $y_n$ is given by $y_n = \Phi(x_n)$, where $\Phi(x)$ is a u.d.p. This problem is equivalent to calculate

$$\max_{\Phi(x)-\text{u.d.p.}} \int_0^1 f(x)\Phi(x)\mathrm{d}x, \quad \min_{\Phi(x)-\text{u.d.p.}} \int_0^1 f(x)\Phi(x)\mathrm{d}x. \quad (698)$$

Theorem 225 implies

**Theorem 226.** *For every Riemann integrable $f(x)$ we have*

$$\max_{\Phi(x)-u.d.p.} \int_0^1 f(x)\Phi(x)\mathrm{d}x = \int_0^1 f(x) G_f(f(x))\mathrm{d}x, \quad (699)$$

$$\min_{\Phi(x)-u.d.p.} \int_0^1 f(x)\Phi(x)\mathrm{d}x = \int_0^1 f(x)(1 - G_f(f(x))\mathrm{d}x. \quad (700)$$

*Proof.*

$$\max_{\Phi(x)-\text{u.d.p.}} \int_0^1 f(x)\Phi(x)\mathrm{d}x = \int_0^1 f^*(x)x\mathrm{d}x = \int_0^1 G_f^{-1}(x)x\mathrm{d}x$$
$$= \int_0^1 G_f^{-1}(G_f(f(x)))G_f(f(x))\mathrm{d}x = \int_0^1 f(x)\Psi(x)\mathrm{d}x. \quad (701)$$

Analogously,

$$\min_{\Phi(x)-\text{u.d.p.}} \int_0^1 f(x)\Phi(x)\mathrm{d}x = \int_0^1 f^*(x)(1-x)\mathrm{d}x = \int_0^1 G_f^{-1}(x)(1-x)\mathrm{d}x$$
$$= \int_0^1 f(x)(1-\Psi(x))\mathrm{d}x. \tag{702}$$

Here $\Psi(x) = G_f(f(x))$ is an u.d.p. map. It follows from that $G_f(f(x_n)$ is u.d. for an arbitrary u.d. sequence $x_n$, $n = 1, 2, \dots$, see [171, 2.3.7, pp. 2–28]. $\square$

In the following part we will find an explicit form of u.d.p. $\Psi(x) = G_f(f(x))$ for piecewise linear function $f : [0,1] \to [0,1]$.

### 9.6.1 Piecewise linear $f(x)$

**Definition 16.** A function $f : [0,1] \to [0,1]$ is *piecewise linear* (p.l.) if there exists a system of ordinate intervals $J_j$, $j = 1, 2, \dots, k$ which are disjoint and fulfil the unit interval $[0,1]$. For every $J_j$ there exists a related system of abscissa intervals $I_{j,i}$, $i = 1, 2, \dots, l_j$, such that $f(x)/I_{j,i}$ is the increasing or decreasing diagonal of $I_{j,i} \times J_j$.

So every p.l. function can be defined by the sets $J_j$ and $I_{j,i}$ and a sign of derivative on each $I_{j,i} \times J_j$. For example

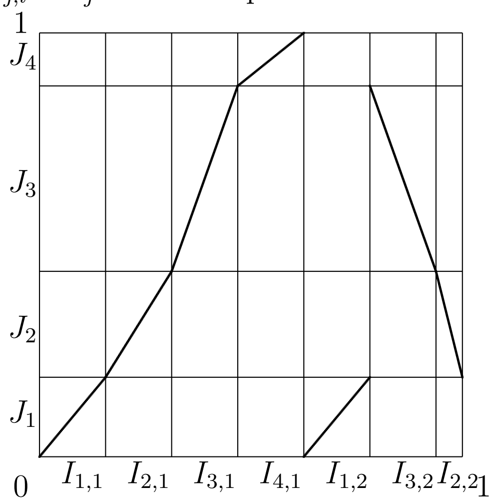



Figure: A piecewise linear function.

In J. Fialová [51] is proved:

**Theorem 227.** *Let $f : [0,1] \to [0,1]$ be a p.l. function with ordinate decomposition $J_j$, $j = 1, 2, \ldots, k$, and abscissa decomposition $I_{j,i}$, $i = 1, 2, \ldots, l_j$. Define a p.l. function $\Psi(x)$ in the same abscissa decomposition $I_{j,i}$ but in a new ordinate decomposition $J'_j$, with the lengths*

$$|J'_j| = \sum_{i=1}^{l_j} |I_{j,i}|, \quad j = 1, 2, \ldots, k, \tag{703}$$

*and with the same ordering as $J_j$, $j = 1, 2, \ldots, k$. Put the graph of $\Psi(x)/I_{j,i}$ on $I_{j,i} \times J'_j$ as the increasing diagonal $\nearrow$ or decreasing $\searrow$ if and only if $f(x)/I_{j,i}$ is the $\nearrow$ or $\searrow$ diagonal. Note that if $f(x)$ is a constant on the interval $I_{j,i}$, then $J_i$ is a point, and the graph $\Psi(x)/I_{j,i}$ can be defined arbitrary, either increasing or decreasing in $I_{j,i} \times J'_j$.*

*Then $\Psi(x)$ is the u.d.p. map and*

$$\int_0^1 f(x)\Psi(x)\mathrm{d}x = \max_{\Phi(x)\text{-}u.d.p.} \int_0^1 f(x)\Phi(x)\mathrm{d}x.$$

For example

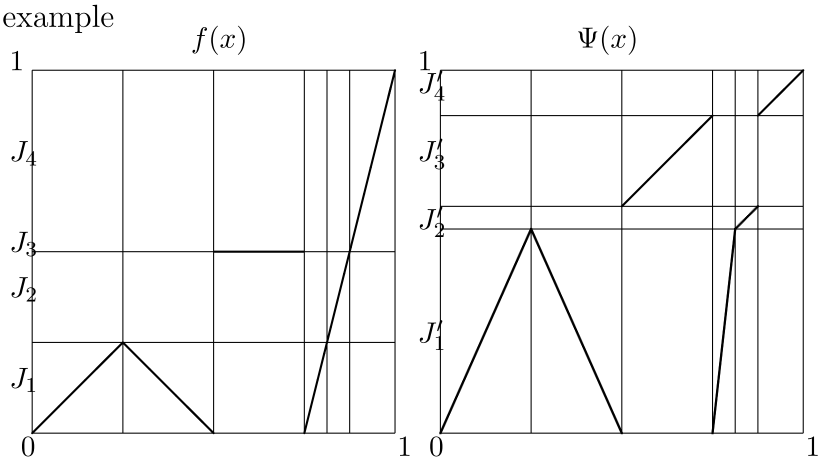



Figure: The best u.d.p. approximation.

*Proof.* By Theorem 226 it is sufficed to prove that $\Psi(x)$ defined in Theorem 227 is equal to $\Psi(x) = G_f(f(x))$. Without loos of generality, we transform the p.l. function $f(x)$ in the following way: the intervals $I_{j,i}$ are rearranged such that, for the same $j$ the intervals $I_{j,i}$ are fused into $I_j$ and these are ordered according to $j$ from the left to the right. This can be done also backwards, so the following picture shows that the function $G_f(f(x))$ is the same as the function $\Psi(x)$. Note that for such reorganized $f(x)$ the function $G_f(x)$ is still the same one.

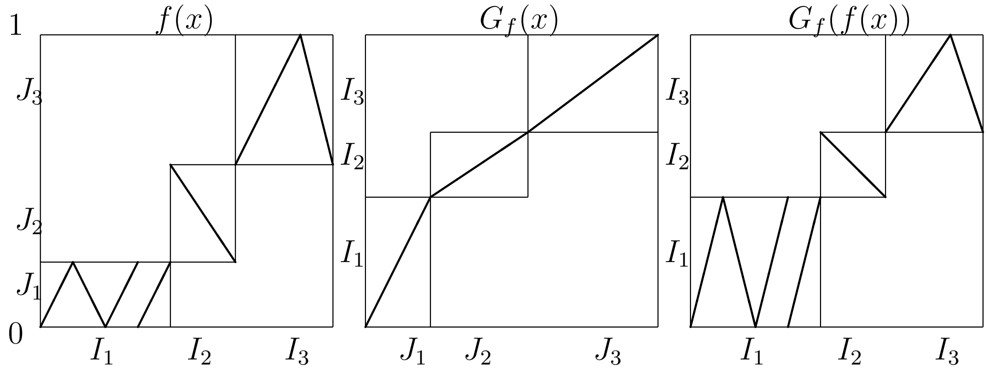



Figure: The equality between $\Psi(x)$ and $G_f(f(x))$.

□

In the originally proof of Theorem 227 presented in the Strobl UDT2010 conference, J. Fialová uses the following basic property of an extreme.

**Theorem 228.** *For an arbitrary Riemann integrable $f : [0,1] \to \mathbb{R}$ we have*

$$\int_0^1 f(x)\Psi(x)\mathrm{d}x = \max_{\Phi(x)\text{-}u.d.p.} \int_0^1 f(x)\Phi(x)\mathrm{d}x$$

*if and only if*

$$\int_0^1 (f(x) - \Psi(x))^2\mathrm{d}x = \min_{\Phi(x)\text{-}u.d.p.} \int_0^1 (f(x) - \Phi(x))^2\mathrm{d}x.$$

*We will call such u.d.p. $\Psi(x)$ as the best u.d.p. approximation of $f(x)$.*

*Proof.* Let $\Psi_1(x), \Psi_2(x)$ be two u.d.p. functions, and $f(x)$ is given. Then

$$\int_0^1 (f(x) - \Psi_1(x))^2\mathrm{d}x < \int_0^1 (f(x) - \Psi_2(x))^2\mathrm{d}x \Leftrightarrow$$
$$\int_0^1 f(x)\Psi_1(x)\mathrm{d}x > \int_0^1 f(x)\Psi_2(x)\mathrm{d}x.$$

Here the following property of u.d.p. functions is used

$$\int_0^1 \Psi_1^2(x)\mathrm{d}x = \int_0^1 \Psi_2^2(x)\mathrm{d}x = \int_0^1 x^2\mathrm{d}x = \frac{1}{3}.$$

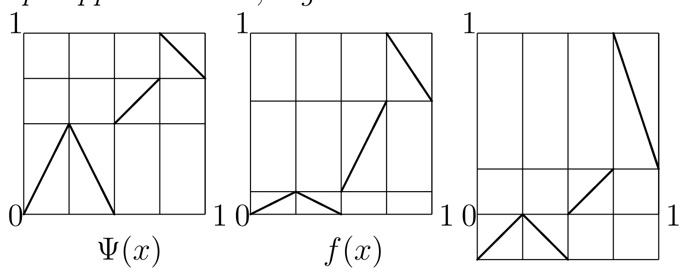
$\square$

By using Theorem 228, for some special cases, it can be proved that $\Psi(x)$ is the best approximation of p.l. function $f(x)$. But the proof of Theorem 123 via Steinerberger's Theorem 225 is to simple. Here we add some consequences of Theorem 228, 227 and 226:

**Theorem 229.** *If $f(x)$ is u.d.p. function, then*

$$\max_{\Phi(x)\text{-}u.d.p.} \int_0^1 f(x)\Phi(x)\mathrm{d}x = \int_0^1 f^2(x)\mathrm{d}x. \tag{704}$$

**Theorem 230** ([51])**.** *The best u.d.p. approximation of the function $f(x)$ is independent of the lengths of the ordinate intervals $J_j$. Thus, for the given u.d.p. function $\Psi(x)$ there are infinity many functions $f(x)$ for which $\Psi(x)$ is the best u.d.p. approximation, e.g.*




$\Psi(x)$ $f(x)$

**Theorem 231** ([51])**.** *For u.d.p. function $\Psi(x)$ in Theorem 227, $\Psi(x)/I_{j,i}$ can be expressed as*

$$\Psi(x) = \frac{|J_j'|}{|J_j|}f(x) + t_{j-1}' - t_{j-1}\frac{|J_j'|}{|J_j|},$$

*where $|J_j'| = \sum_{i=1}^{l_j}|I_{j,i}|$, and $t_j'$ is given recurrently as $t_0' = 0$ and $t_j' = t_{j-1}' + |J_j'|$. So the integral $\int_0^1 f(x)\Psi(x)\mathrm{d}x$ can be calculated directly from $J_j = (t_{j-1}, t_j)$ and $J_j' = (t_{j-1}', t_j')$ as*

$$\int_0^1 f(x)\Psi(x)\mathrm{d}x = \sum_{j=1}^{k} |J_j'|\left(t_{j-1}t_{j-1}' + \frac{1}{2}t_{j-1}|J_j'| + \frac{1}{2}t_{j-1}'|J_j| + \frac{1}{3}|J_j||J_j'|\right). \tag{705}$$

**Theorem 232.** *Let $f : [0,1] \to [0,1]$ be a Riemann-integrable function. Then*

$$\max_{\Phi(x)-u.d.p.} \int_0^1 f(x)\Phi(x)\mathrm{d}x = \int_0^1 x g_f(x)\mathrm{d}G_f(x) = \frac{1}{2}\left(1 - \int_0^1 G_f^2(x)\mathrm{d}x\right). \tag{706}$$

*Proof.* Here as in Definition 15, $G_f(x) = |f^{-1}[0,x)|$ for $x \in [0,1]$. For every u.d. sequence $x_n$, $n = 1, 2, \dots$, the sequence $f(x_n)$ has a.d.f. $G_f(x)$ and then $G_f(f(x_n))$ is u.d. sequence. By Helly theorem we have two alternatives

$$\frac{1}{N}\sum_{n=1}^{N} f(x_n)G_f(f(x_n)) \to \int_0^1 xG_f(x)\mathrm{d}G_f(x) \tag{707}$$

$$\to \int_0^1 f(x)G_f(f(x))\mathrm{d}x. \tag{708}$$

In (707) we use integration by parts and (708) is in (699). □

## 9.7 Computation of $\int_0^1 f_1(\Phi(x))f_2(\Psi(x))\mathrm{d}x$

**Theorem 233.** *Let $f_1(x)$ and $f_2(x)$ be two functions defined on $[0,1]$ having continuous derivatives $f_1'(x)$ and $f_2'(x)$. Let $\Phi(x)$ and $\Psi(x)$ be two u.d.p maps on $[0,1]$ and put for $(x,y) \in [0,1]^2$*

$$g(x,y) = |\Phi^{-1}([0,x)) \cap \Psi^{-1}([0,y))|, \tag{709}$$

*where $|X|$ is the Lebesgue measure of the set $X$. Then*

$$\begin{aligned}\int_0^1 f_1(\Phi(x))f_2(\Psi(x))\mathrm{d}x =& f_1(1)f_2(1) - f(1)\int_0^1 yf_2'(y)\mathrm{d}y - f_2(1)\int_0^1 xf_1'(x)\mathrm{d}x \\ &+ \int_0^1\int_0^1 g(x,y)f_1'(x)f_2'(y)\mathrm{d}x\mathrm{d}y. \end{aligned}\tag{710}$$

*Proof.* Let $x_n$, $n = 1, 2, \dots$, be a u.d. sequence in $[0,1)$ and let $g(x,y)$ be a d.f. of the two dimensional sequence $(\Phi(x_n), \Psi(x_n))$, $n = 1, 2, \dots$. Then by Weyl and by Helly theorem the limit of the following arithmetic means can be computed by two methods

$$\frac{1}{N}\sum_{n=1}^{N} f_1(\Phi(x_n))f_2(\Psi(x_n)) \to \int_0^1 f_1(\Phi(x))f_2(\Psi(x))\mathrm{d}x, \tag{711}$$

$$\to \int_0^1 \int_0^1 f_1(x)f_2(x)\mathrm{d}_x\mathrm{d}_y g(x,y). \tag{712}$$

Since

$$(\Phi(x_n) < x) \wedge (\Psi(x_n) < y) \Leftrightarrow (x_n \in \Phi^{-1}([0,x))) \wedge (x_n \in \Psi^{-1}([0,y)))$$

and $x_n$ is u.d. then the sequence $(\Phi(x_n), \Psi(x_n))$ has d.f

$$\frac{\#\{n \le N; x_n \in \Phi^{-1}([0,x)) \cap \Psi^{-1}([0,y))\}}{N} \to |\Phi^{-1}([0,x)) \cap \Psi^{-1}([0,y))|$$

which is furthermore a.d.f.

Now, to computing (712) we apply (573) for $F(x,y) = f(x)g(y)$. □

**Notes 47.** $1^0$. Steinerberger [151] generalized (566) to extremes of

$$\int_0^1 f_1(\Phi_1(x))f_2(\Phi_2(x))\dots f_s(\Phi_s(x))\mathrm{d}x \tag{713}$$

for Riemann integrable $f_1,\dots,f_s$ and u.d.p. maps $\Phi_1,\dots,\Phi_s$. He proved, e.g., the following partial results:

a) $\max_{\Phi_1,\dots,\Phi_s} \int_0^1 f_1(\Phi_1(x))f_2(\Phi_2(x))\dots f_s(\Phi_s(x))\mathrm{d}x$
$\le \left(\prod_{i=1}^n \int_0^1 |f_i(x)|^s \,\mathrm{d}x\right)^{\frac{1}{s}}$.

b) $\min_{\Phi_1,\dots,\Phi_n} \int_0^1 \Phi_1(x)\Phi_2(x)\dots\Phi_s(x)dx \ge \frac{1}{e^s}$.

c) $\min_{\Phi_1,\dots,\Phi_s} \int_0^1 \Phi_1(x)\Phi_2(x)\dots\Phi_s(x)\mathrm{d}x \le e^{\frac{1}{6s}} \frac{s}{s-2}\frac{4}{\pi}\frac{1}{e^s}$.

Using d.f.s theory, (713) can be reformulated: Let $x_n$, $n = 1,2,\dots$, be a u.d. sequence in $[0,1)$ and $g(t_1,\dots,t_s)$ be an a.d.f. of the $s$-dimensional sequence $(\Phi_1(x_n),\dots,\Phi_s(x_n))$, $n = 1,2,\dots$ We have

$$g(t_1,\dots,t_s) = |\Phi_1^{-1}([0,t_1)) \cap \dots \cap \Phi_s^{-1}([0,t_s))|,$$

$g(1\dots,t_i,1\dots,1) = t_i$ for $i = 1,\dots,s$, i.e., it is a copula and

$$\int_0^1 f_1(\Phi_1(x))\dots f_s(\Phi_s(x))\mathrm{d}x = \int_{[0,1]^s} f_1(t_1)\dots f_s(t_s)\mathrm{d}g(t_1,\dots,t_s).$$

Thus, we arrive to the **open problem:** Find extreme values of $\int_{[0,1]^s} F(\mathbf{x})\mathrm{d}g(\mathbf{x})$, where $g(\mathbf{x})$ is a copula.

$2^0$. Steinerberger [152] generalized (563) to give bounds for the asymptotic behavior of

$$\frac{1}{N}\sum_{n=1}^N ||\mathbf{x}_n - \mathbf{y}_n||, \tag{714}$$

where $\mathbf{x}_n, \mathbf{y}_n$ are u.d. sequences in a bounded Jordan measurable domain $\Omega$. E.g., for $s$-dimensional ball he found the sharp inequality $\frac{1}{N}\sum_{n=1}^N ||\mathbf{x}_n - \mathbf{y}_n|| \le \frac{2s}{s+1}$.

## 9.8 Extremes of $\int_0^1\int_0^1\int_0^1 F(x,y,z)\mathrm{d}_x\mathrm{d}_y\mathrm{d}_z g(x,y,z)$

The result (570) can be extend for $s=3$ (cf. Theorem 204)

**Theorem 234.** *Assume that $F(x,y,z)$ is a continuous in $[0,1]^3$ and $g(x,y,z)$ is a d.f. Then*

$$\begin{aligned}
&\int_0^1\int_0^1\int_0^1 F(x,y,z)\mathrm{d}_x\mathrm{d}_y\mathrm{d}_z g(x,y,z)\\
&= F(1,1,1,) - \int_0^1 g(1,1,z)\mathrm{d}_z F(1,1,z) - \int_0^1 g(1,y,1)\mathrm{d}_y F(1,y,1)\\
&- \int_0^1 g(x,1,1)\mathrm{d}_x F(x,1,1) + \int_0^1\int_0^1 g(1,y,z)\mathrm{d}_y\mathrm{d}_z F(1,y,z)\\
&+ \int_0^1\int_0^1 g(x,1,z)\mathrm{d}_x\mathrm{d}_z F(x,1,z) + \int_0^1\int_0^1 g(x,y,1)\mathrm{d}_x\mathrm{d}_y F(x,y,1)\\
&- \int_0^1\int_0^1\int_0^1 g(x,y,z)\mathrm{d}_x\mathrm{d}_y\mathrm{d}_z F(x,y,z). \qquad (715)
\end{aligned}$$

This gives the following generalization of Theorem 214:

**Theorem 235.** *Let $x_n, y_n, z_n$ are u.d. sequences in $[0,1)$ such that $(x_n,y_n)$, $(x_n,z_n)$, $(y_n,z_n)$ are u.d. in $[0,1)^2$. If $\mathrm{d}_x\mathrm{d}_y\mathrm{d}_z F(x,y,z) < 0$ in $(0,1)^3$, then*

$$\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1} F(x_n,y_n,z_n) \le \int_0^1\int_0^1\int_0^1 F(x,y,z)\mathrm{d}_x\mathrm{d}_y\mathrm{d}_z\min(xy,xz,yz). \qquad (716)$$

# 10 Solution of $\int_0^1\int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y) = 0$ in d.f.s $g(x)$

These investigation is motivated by Theorem 126 and mostly published in [164]. Also extend Section 4.4. Again, for a given continuous $F:[0,1]^2\to\mathbb{R}$, let $G(F)$ denote the set of all d.f.s $g(x)$ which solve the moment problem

$$\int_0^1\int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y) = 0. \qquad (717)$$

## 10.1 Examples of calculations of $\int_0^1\int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y)$

**Example 99.** Following [158, p. 618] let us denote

$$F_{\tilde{g}}(x,y)=\int_0^1\tilde{g}^2(t)\mathrm{d}t-\int_x^1\tilde{g}(t)\mathrm{d}t-\int_y^1\tilde{g}(t)\mathrm{d}t+1-\max(x,y), \tag{718}$$

or alternatively

$$=\int_0^1\tilde{g}^2(t)\mathrm{d}t+\int_0^x\tilde{g}(t)\mathrm{d}t+\int_0^y\tilde{g}(t)\mathrm{d}t-2\int_0^1\tilde{g}(t)\mathrm{d}t$$
$$+1-\frac{x+y+|x-y|}{2}. \tag{719}$$

From the relation (see (30))

$$\int_0^1(g(x)-\tilde{g}(x))^2\mathrm{d}x=\int_0^1\int_0^1 F_{\tilde{g}}(x,y)\mathrm{d}g(x)\mathrm{d}g(y), \tag{720}$$

we see that the moment problem (717) with $F(x,y)=F_{\tilde{g}}(x,y)$ has the unique solution $g(x)=\tilde{g}(x)$. Applying (720) to $g(x)=(g_1(x)+g_2(x))/2$ also holds

$$\int_0^1\int_0^1 F_{\tilde{g}}(x,y)\mathrm{d}g_1(x)\mathrm{d}g_2(y)=\int_0^1(g_1(x)-\tilde{g}(x))(g_2(x)-\tilde{g}(x))\mathrm{d}x,$$

The function $F_{\tilde{g}}$ for $\tilde{g}(x)=x$ will be denoted by $F_0$ and

$$\begin{aligned}F_0(x,y)\quad&=\frac{1}{3}+\frac{x^2+y^2}{2}-\max(x,y)\\&=\frac{1}{3}+\frac{x^2+y^2}{2}-\frac{x+y}{2}-\frac{|x-y|}{2}\\&=\frac{1}{4}+xy-\frac{x+y}{2}+\frac{1}{2\pi^2}\sum_{h=1}^{\infty}\frac{1}{h^2}\cos 2\pi h(x-y)\\&=\left(x-\frac{1}{2}\right)\left(y-\frac{1}{2}\right)+\int_0^1\left(\{x+t\}-\frac{1}{2}\right)\left(\{y+t\}-\frac{1}{2}\right)\mathrm{d}t,\end{aligned}$$

where the final equation can be found in [92, p. 144, Th. 5.2]. This function $F_0(x,y)$ is inspired by the classical $L^2$ discrepancy $\int_0^1(F_N(x)-x)^2\mathrm{d}x$ since

$$\int_0^1(F_N(x)-x)^2\mathrm{d}x=\frac{1}{N^2}\sum_{m,n=1}^{N}F_0(x_m,x_n).$$

**Example 100.** In connection with a diaphony (cf. [158, p. 620]) we introduce

$$\begin{aligned} F_{\tilde{g}}^{(1)}(x,y) \quad &= \int_0^1 \tilde{g}^2(t)\mathrm{d}t - \left(\int_0^1 \tilde{g}(t)\mathrm{d}t\right)^2 - (x+y)\int_0^1 \tilde{g}(t)\mathrm{d}t \\ &+ \int_0^x \tilde{g}(t)\mathrm{d}t + \int_0^y \tilde{g}(t)\mathrm{d}t + \min(x,y) - xy. \end{aligned}$$

Since

$$\begin{aligned} \int_0^1\int_0^1 F_{\tilde{g}}^{(1)}(x,y)\mathrm{d}g(x)\mathrm{d}g(y) &= \iint\limits_{0\le x\le y\le 1} ((\tilde{g}(y)-\tilde{g}(x))-(g(y)-g(x)))^2\mathrm{d}x\mathrm{d}y \\ &= \int_0^1 (\tilde{g}(x)-g(x))^2\mathrm{d}x - \left(\int_0^1 (\tilde{g}(x)-g(x))\mathrm{d}x\right)^2, \end{aligned}$$

the moment problem (717) with $F = F_{\tilde{g}}^{(1)}$ is equivalent to $\tilde{g}(x)-g(x)$ =constant in every point $x$ of continuity and thus has a unique solution $g(x) = \tilde{g}(x)$ if, e.g. $\tilde{g}(x)$ is continuous at 0 and 1.

**Example 101.** For a continuous distribution function $\tilde{g}$, the moment problem

$$\int_0^1\int_0^1 F_0(\tilde{g}(x),\tilde{g}(y))\mathrm{d}g(x)\mathrm{d}g(y) = 0$$

has the unique solution $g(x) = \tilde{g}(x)$. This is a consequence of [92, Exerc. 7.19, p. 68] which states that if $\tilde{g}$ is a continuous limit law of the sequence $x_n$ in $[0,1]$, then the sequence $\tilde{g}(x_n)$ is u.d. in $[0,1]$.

**Example 102.** For a d.f. $g(x)$ and a piecewise continuous $f : [0,1] \to [0,1]$ we denote by $g_f$ the d.f. (see Section 4.8.1)

$$g_f(x) = \int_{f^{-1}([0,x))} 1.\mathrm{d}g(x).$$

In [158, p. 628] we have proved, for two piecewise continuous $f, g$,

$$\int_0^1 (g_f(x) - g_h(x))^2\mathrm{d}x = \int_0^1\int_0^1 F_{f,h}(x,y)\mathrm{d}g(x)\mathrm{d}g(y)$$

where

$$F_{f,h}(x,y) = \max(f(x),h(y)) + \max(f(y),h(x)) - \max(f(x),f(y))$$

$$- \max(h(x), h(y)). \tag{721}$$

Thus, the moment problem (717) with $F = F_{f,h}$ is equivalent to the functional equation $g_f(x) = g_h(x)$ a.e. on $[0,1]$. For $f(x) = \{1/x\}$ and $h(x) = x$ and applying the Gauss-Kuzmin theorem (see Theorem 178), the functional equation $g_f = g$ has unique solution $g(x) = \log(1+x)/\log 2$ on the set $G'$ of all distribution functions having continuous first derivative. Thus, for $F = F_{f,h}$, the moment problem (717) has a unique solution on $G'$. Here

$$\begin{aligned} F_{f,h}(x,y) &= \max(\{1/x\}, y) + \max(\{1/y\}, x) - \max(x,y) \\ &\quad - \max(\{1/x\}, \{1/y\}) \\ &= \frac{1}{2}\big(|\{1/x\} - y| + |\{1/y\} - x| - |x-y| - |\{1/x\} - \{1/y\}|\big) \end{aligned} \tag{722}$$

and

$$g_f(x) = \sum_{n=1}^{\infty} \left( g\left(\frac{1}{n}\right) - g\left(\frac{1}{n+x}\right)\right).$$

**Example 103.** Combining examples 100 and 101 we can define

$$\iint\limits_{0\le x\le y\le 1} \big((g_f(y) - g_f(x)) - (g_h(y) - g_h(x))\big)^2 \mathrm{d}x\mathrm{d}y$$

$$= \int_0^1 \int_0^1 F_{f,h}^{(1)}(x,y)\mathrm{d}g(x)\mathrm{d}g(y),$$

where

$$F_{f,h}^{(1)}(x,y) = F_{f,h}(x,y) - (f(x) - h(x))(f(y) - h(y)).$$

Moment problems (717) with $F = F_{f,h}$ and $F = F_{f,h}^{(1)}$ have the same solutions assuming some additional conditions on $g$, $f$ and $h$.

**Example 104.** Since

$$\int_0^1 (g_0(x) - g_f(x))^2 \mathrm{d}x = \int_0^1 \int_0^1 F_{g_0}(f(x), f(y))\mathrm{d}g(x)\mathrm{d}g(y)$$

the moment problem (717) with $F(x,y) = F_{g_0}(f(x), f(y))$ is equivalent to the functional equation $g_0(x) = g_f(x)$ a.e.

**Example 105.** Define $F^*_{g_1,g_2}(x,y)$ as $F_1(x,y)$ in Theorem 63, i.e.

$$\begin{aligned}\int_0^1\int_0^1 F^*_{g_1,g_2}(x,y)\mathrm{d}g(x)\mathrm{d}g(y) &= \int_0^1 (g(x)-g_2(x))^2\mathrm{d}x\int_0^1 (g_2(x)-g_1(x))^2\mathrm{d}x\\ &- \left(\int_0^1 (g(x)-g_2(x))(g_2(x)-g_1(x))\mathrm{d}x\right)^2.\end{aligned}$$

Then

$$\int_0^1\int_0^1 F^*_{g_1,g_2}(x,y)\mathrm{d}g(x)\mathrm{d}g(y)=0 \iff \exists(t\in\mathbb{R})g(x)=tg_1(x)+(1-t)g_2(x) \text{ a.e.}$$

If we assume that given $g_1, g_2$ have derivatives $g_1', g_2'$, then the parameter $t$ must satisfy

$$tg_1'(x)+(1-t)g_2'(x)\geq 0 \iff t(g_1'(x)-g_2'(x))\geq -g_2'(x).$$

Putting

$$X_1=\{x\in[0,1]; g_1'(x)-g_2'(x)>0\},\quad X_2=\{x\in[0,1]; g_1'(x)-g_2'(x)<0\},$$

we can express

$$G(F^*_{g_1,g_2})=\left\{tg_1(x)+(1-t)g_2(x); t\in\left[\sup_{x\in X_1}\frac{-g_2'(x)}{g_1'(x)-g_2'(x)},\inf_{x\in X_2}\frac{-g_2'(x)}{g_1'(x)-g_2'(x)}\right]\right\}.$$

Here we have the explicit form

$$F^*_{g_1,g_2}(x,y)=F_{g_2}(x,y)\int_0^1 (g_2(x)-g_1(x))^2\mathrm{d}x - F_{g_1,g_2}(x)F_{g_1,g_2}(y),$$

where $F_{g_2}(x,y)$ is defined by Example 99 and

$$F_{g_1,g_2}(x)=\int_0^x (g_2(t)-g_1(t))\mathrm{d}t-\int_0^1 (1-g_2(x))(g_2(x)-g_1(x))\mathrm{d}x.$$

**Example 106.** We repeat Theorem 23: For every polynomial $\psi(y)=a(x)y^2+b(x)y+c(x)$ with continuous coefficients the one-dimensional integral $\int_0^1\psi(g(x))\mathrm{d}x$ can be express as the double integral

$$\int_0^1\psi(g(x))\mathrm{d}x=\int_0^1\int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y)$$

for every d.f. $g$, where

$$F(x,y)=\int_{\max(x,y)}^1 a(t)\mathrm{d}t+\frac{1}{2}\int_x^1 b(t)\mathrm{d}t+\frac{1}{2}\int_y^1 b(t)\mathrm{d}t+\int_0^1 c(t)\mathrm{d}t.$$

**Notes 48.** Assume that
$F(x,y) = \sum_{i=1}^{k} F_i(x,y)$, and
$\int_0^1 \int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y) = 0$ has a unique solution $g(x) = g_0(x)$.
Denote $\int_0^1 \int_0^1 F_i(x,y)\mathrm{d}g_0(x)\mathrm{d}g_0(y) = \alpha_i, \quad i = 1,2,\dots,k$. Then the system of equations

$$\int_0^1 \int_0^1 F_i(x,y)\mathrm{d}g(x)\mathrm{d}g(y) = \alpha_i, \quad i = 1,2,\dots,k.$$

has a unique solution $g(x) = g_0(x)$.

## 10.2 Theorems of calculations of $\int_0^1 \int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y)$

Here we apply Riemann-Stieltjes integration and as usually

$$\mathrm{d}_x F(x,y) = F(x+\mathrm{d}x,y) - F(x,y), \quad \mathrm{d}_x F(x,1) = \left[\mathrm{d}_x F(x,y)\right]_{y=1},$$

$$\mathrm{d}_x \mathrm{d}_y F(x,y) = F(x+\mathrm{d}x, y+\mathrm{d}y) - F(x+\mathrm{d}x,y) - F(x,y+\mathrm{d}y) + F(x,y)$$

and

$$\mathrm{d}_x F(x,y) = F'_x(x,y)\mathrm{d}x, \quad \mathrm{d}_x \mathrm{d}_y F(x,y) = F''_{xy}(x,y)\mathrm{d}x\mathrm{d}y$$

provided that related partial derivatives $F'_x = \frac{\partial F(x,y)}{\partial x}$ and $F''_{xy} = \frac{\partial^2 F(x,y)}{\partial x \partial y}$ are exist. For an introduction we mention the following simple properties.

**Theorem 236.** *Assume that $F(x,y) = 0$ for $(x,y) \in \{(x_i,x_j); i,j = 1,2,\dots,m\}$. Then any step d.f. $g(x)$ having jumps only in $x_i$, $i = 1,2,\dots,m$ is a solution of* (717)*.*

**Theorem 237.** *If $F(x,y)$ and $H(x,y)$ are continuous symmetric and for every d.f. $g$*

$$\int_0^1 \int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y) = \int_0^1 \int_0^1 H(x,y)\mathrm{d}g(x)\mathrm{d}g(y),$$

*then $F(x,y) = H(x,y)$ for all $(x,y) \in [0,1]^2$.*

*Proof.* Indeed, $F(\alpha,\alpha) = H(\alpha,\alpha)$ follows from the assumed integral equation with $g(x) = c_\alpha(x)$ and $F(\alpha,\beta) = H(\alpha,\beta)$ we find by using $g(x) = tc_\alpha(x) + (1-t)c_\beta(x)$. $\square$

**Theorem 238.** *Let $F : [0,1]^2 \to \mathbb{R}$ be a continuous and symmetric function. For every d.f.s $g(x)$, $\tilde{g}(x)$ we have*

$$\int_0^1\int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y) = 0 \Longleftrightarrow \int_0^1\int_0^1 F(x,y)\mathrm{d}\tilde{g}(x)\mathrm{d}\tilde{g}(y)$$
$$= \int_0^1 (g(x)-\tilde{g}(x))\left(2\mathrm{d}_x F(x,1) - \int_0^1 (g(y)+\tilde{g}(y))\mathrm{d}_y\mathrm{d}_x F(x,y)\right).$$

*Proof.* Using integration by parts, for any two d.f.s $g, \tilde{g}$ we have

$$\int_0^1\int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y) - \int_0^1\int_0^1 F(x,y)\mathrm{d}\tilde{g}(x)\mathrm{d}\tilde{g}(y)$$
$$= \int_0^1\int_0^1 F(x,y)\mathrm{d}(g(x)-\tilde{g}(x))\mathrm{d}(g(y)+\tilde{g}(y))$$
$$= -\int_0^1\left(\int_0^1 (g(x)-\tilde{g}(x))\mathrm{d}_x F(x,y)\right)\mathrm{d}(g(y)+\tilde{g}(y))$$
$$= -\left[\int_0^1 (g(x)-\tilde{g}(x))\mathrm{d}_x F(x,y)\cdot(g(y)+\tilde{g}(y))\right]_{y=0}^{y=1}$$
$$+ \int_0^1 (g(y)+\tilde{g}(y))\mathrm{d}_y\left(\int_0^1 (g(x)-\tilde{g}(x))\mathrm{d}_x F(x,y)\right).$$

This implies

$$\int_0^1\int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y) - \int_0^1\int_0^1 F(x,y)\mathrm{d}\tilde{g}(x)\mathrm{d}\tilde{g}(y)$$
$$= \int_0^1 (g(x)-\tilde{g}(x))\left(-2\mathrm{d}_x F(x,1) + \int_0^1 (g(y)+\tilde{g}(y))\mathrm{d}_y\mathrm{d}_x F(x,y)\right) \quad (723)$$
$$= \int_0^1\int_0^1 (g(x)-\tilde{g}(x))(g(y)-\tilde{g}(y))\mathrm{d}_y\mathrm{d}_x F(x,y)$$
$$+ 2\int_0^1 (g(x)-\tilde{g}(x))\left(-\mathrm{d}_x F(x,1) + \int_0^1 \tilde{g}(y)\mathrm{d}_y\mathrm{d}_x F(x,y)\right). \quad (724)$$

and (723) implies theorem. □

**Example 107.** Especially, putting $\tilde{g}(x) = c_0(x)$, we have

$$\int_0^1\int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y) = 0 \Longleftrightarrow$$

$$F(0,0) = \int_0^1 (g(x)-1)\left(2\mathrm{d}_x F(x,1) - \int_0^1 (g(y)+1)\mathrm{d}_y \mathrm{d}_x F(x,y)\right). \quad (725)$$

For $F(x,y) = F_0(x,y)$, where $F_0(x,y) = \frac{1}{3} + \frac{x^2+y^2}{2} - \max(x,y)$ is defined in Example 99, we have (725) in the form

$$g(x) = x \Longleftrightarrow \frac{1}{3} = \int_0^1 g(x)(2x - g(x))\mathrm{d}x.$$

Here $\mathrm{d}_y \mathrm{d}_x F_0(x,y) = 0$ if $x \neq y$ and for $x = y$

$$\mathrm{d}_y \mathrm{d}_x F_0(x,y) = F_0(x,y) + F_0(x+\mathrm{d}x, y+\mathrm{d}y) - F_0(x, y+\mathrm{d}y) - F_0(x+\mathrm{d}x, y) = \mathrm{d}x = \mathrm{d}y.$$

Then

$$\int_0^1 \int_0^1 (g(x)-1)(g(y)+1)\mathrm{d}_y \mathrm{d}_x F_0(x,y) = \int_0^1 (g(x)-1)(g(x)+1)\mathrm{d}x.$$

A direct proof of (725) can be found by using theory [159] of the moment problem 4.10.

**Example 108.** For control, if $F(x,y) = xy$, then (725) has the form

$$\begin{aligned}
&\int_0^1 \int_0^1 xy \mathrm{d}g(x)\mathrm{d}g(y) = 0 \Longleftrightarrow \\
&0 = \int_0^1 (g(x)-1)\left(2\mathrm{d}x - \int_0^1 (g(y)+1)\mathrm{d}y\mathrm{d}x\right) \\
&= \int_0^1 (g(x)-1)\left(2 - \int_0^1 (g(y)+1)\mathrm{d}y\right)\mathrm{d}x \\
&= \int_0^1 (g(x)-1)\mathrm{d}x \left(1 - \int_0^1 g(y)\mathrm{d}y\right) = -\left(\int_0^1 (g(x)-1)\mathrm{d}x\right)^2
\end{aligned}$$

Thus $\int_0^1 x\mathrm{d}g(x) = 0 \Longleftrightarrow \int_0^1 g(x)\mathrm{d}x = 1$ which is trivial.

**Example 109.** For $F(x,y) = x+y$, $g(x)$, $\tilde{g}(x)$ the (723) gives

$$\int_0^1 \int_0^1 (x+y)\mathrm{d}x^2\mathrm{d}y^2 - \int_0^1 \int_0^1 (x+y)\mathrm{d}x\mathrm{d}y = -2\int_0^1 (x^2-x)\mathrm{d}x.$$

This holds since $8(1/3)(1/2) - 2(1/2) = 1/3$.

**Theorem 239.** *Assume that continuous $f : [0,1] \to [0,1]$ has finitely many inverse functions $f_1^{-1}, \dots, f_m^{-1}$ such that, for every $i = 1, 2, \dots, m$ we have $f_i^{-1} : [0,1] \to [\alpha_i, \beta_i]$. Then*

$$\int_0^1 (g_f(x) - g_{1_f}(x))^2 \mathrm{d}x = \int_0^1 (g(x) - g_1(x)) f'(x)(g_f(f(x)) - g_{1_f}(f(x)))\mathrm{d}x \tag{726}$$

*Proof.* We give a simple proof for the special $f(x) = 2x \bmod 1$. Firstly split the integral (726) on two parts $\int_0^{1/2}$ and $\int_{1/2}^1$. Substitute

$y = f(x)$
$\frac{y}{2} = f_1^{-1}(y) = x$
into $\int_0^{1/2}$ we find

$$\begin{aligned}&\int_0^{1/2} (g(x) - g_1(x)) f'(x)(g_f(f(x)) - g_{1_f}(f(x)))\mathrm{d}x \\ &= \int_0^1 (g(f_1^{-1}(y)) - g_1(f_1^{-1}(y)))2(g_f(y) - g_{1_f}(y))\frac{\mathrm{d}y}{2}.\end{aligned} \tag{727}$$

Substitute

$y = f(x)$
$\frac{y+1}{2} = f_2^{-1}(y) = x$
into $\int_{1/2}^1$ we find

$$\begin{aligned}&\int_{1/2}^{1} (g(x) - g_1(x)) f'(x)(g_f(f(x)) - g_{1_f}(f(x)))\mathrm{d}x \\ &= \int_0^1 (g(f_2^{-1}(y)) - g_1(f_2^{-1}(y)))2(g_f(y) - g_{1_f}(y))\frac{\mathrm{d}y}{2}.\end{aligned} \tag{728}$$

Summing up (727) and (728) then we find (726). □

**Example 110.** Once again (726): Put

$g_1(x) = x$, and $g(x)$ is an arbitrary d.f.,

$$f(x) = \begin{cases} 2x, & x \in [0, 1/2]; \\ 2 - 2x, & x \in [1/2, 1]; \end{cases}$$

$f_1^{-1}(x) = x/2$ and $f_2^{-1}(x) = 1 - (x/2)$;

$g_f(x) = g(f_1^{-1}(x)) + 1 - g(f_2^{-1}(x))$;
$g_{1f}(x) = x$;

$$f'(x) = \begin{cases} 2, & x \in [0,1/2]; \\ -2, & x \in [1/2,1]; \end{cases}$$

$$f_1^{-1}(f(x)) = \begin{cases} x, & x \in [0,1/2]; \\ 1-x, & x \in [1/2,1]; \end{cases}$$

$$f_2^{-1}(f(x)) = \begin{cases} 1-x, & x \in [0,1/2]; \\ x, & x \in [1/2,1]; \end{cases}$$

Then we have

$$g_f((f(x)) = \begin{cases} g(x) + 1 - g(1-x), & x \in [0,1/2]; \\ g(1-x) + 1 - g(x), & x \in [1/2,1]; \end{cases}$$

$$g_{1f}((f(x)) = \begin{cases} 2x, & x \in [0,1/2]; \\ 2-2x, & x \in [1/2,1]; \end{cases}$$

The (726) has the form

$$\begin{aligned} &\int_0^1 (g(x/2) + 1 - g(1-x/2) - x)^2 \mathrm{d}x \\ &= \int_0^{1/2} (g(x) - x) 2 (g(x) + 1 - g(1-x) - 2x) \mathrm{d}x \\ &+ \int_{1/2}^1 (g(x) - x)(-2)(g(1-x) + 1 - g(x) - (2-2x)) \mathrm{d}x \end{aligned}$$

which holds for every d.f. $g(x)$.

• In the following we denote the set of jumps of $F'_x(x,y)$ as

$$H(F) = \{(x,y) \in [0,1]^2; \mathrm{d}_y F'_x(x,y) \neq 0 \text{ as } \mathrm{d}y \to 0\}. \tag{729}$$

Furthermore assume that $H(F)$ can be expressed as

$$H(F) = \{(x, u_i(x)); i = 1,2,\dots,k, x \in [0,1]\}, \tag{730}$$

and we denote a jump of partial derivative $F'_x$ at the point $(x, u_i(x))$ as $v_i(x)$, i.e.

$$v_i(x) = F'_x(x, u_i(x)+0) - F'_x(x, u_i(x)-0). \tag{731}$$

**Theorem 240.** *Let $F(x,y)$ be a continuous, symmetric and $F''_{xy}=0$ a.e. and such that the set $H(F)$ of jumps of $F'_x(x,y)$ satisfies* (730) *and* (731)*. Furthermore assume that $u_i(x)$ and $v_i(x)$, for $i=1,2,\dots,k$, are continuous. Then for every solution $g_1$ of the moment problem* (717) *we have*

$$\begin{aligned}&\int_0^1\int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y)\\&=\int_0^1(g(x)-g_1(x))\left(-2F'_x(x,1)+\sum_{i=1}^k\big(g(u_i(x))+g_1(u_i(x))\big)v_i(x)\right)\mathrm{d}x.\end{aligned}\tag{732}$$

*for any d.f. $g$.*

*Proof.* Directly from (723) and by using Riemann-Stieltjes integration. □

**Theorem 241.** *From assumption of Theorem* 240 *follows*

$$\int_0^1 g_1(x)\bigg(\sum_{i=1}^k g(u_i(x))v_i(x)\bigg)\mathrm{d}x=\int_0^1 g(x)\bigg(\sum_{i=1}^k g_1(u_i(x))v_i(x)\bigg)\mathrm{d}x \tag{733}$$

*for every d.f. $g(x)$.*

*Proof.* From symmetry of $F$ we see that partial derivative $F'_y(x,y)$ has jumps in $(y,u_i(y))$ of size $v_i(y)$. Thus

$$\begin{aligned}&\int_0^1\int_0^1(g(x)-g_1(x))(g(y)+g_1(y))\mathrm{d}_y\mathrm{d}_xF(x,y)\\&=\int_0^1(g(x)-g_1(x))\left(\sum_{i=1}^k\big(g(u_i(x))+g_1(u_i(x))\big)v_i(x)\right)\mathrm{d}x\\&=\int_0^1(g(x)+g_1(x))\left(\sum_{i=1}^k\big(g(u_i(x))-g_1(u_i(x))\big)v_i(x)\right)\mathrm{d}x.\end{aligned}$$

□

## 10.3 Copositive $F(x,y)$

A symmetric matrix $\mathbf{A}$ is called *copositive* if the quadratic form $\mathbf{x}\mathbf{A}\mathbf{x}^T$ is nonnegative for all nonnegative values of the variables $(x_1,x_2,\dots,x_n)=\mathbf{x}$. This generalization of positive definiteness was first discussed by T. S. Motzkin [108]. All criteria for copositivity of real symmetric matrices $\mathbf{A}$ of order $n$ are inductive, e.g. (cf. K. P. Hadeler [67, Th. 2]): A real symmetric matrix $\mathbf{A}$ of order $n$ is copositive if and only if

(i) every principal submatrix of order $n-1$ is copositive,

(ii) det $\mathbf{A} \geq 0$ or the matrix of signed cofactors adj $\mathbf{A}$ contains a negative element.

In the following we give an extension of copositivity to $F(x,y)$ for which the moment problem (717) can be transformed to some simple integral equation.

**Definition 17.** A symmetric continuous $F(x,y)$ defined on $[0,1]^2$ is called *copositive* if

$$\int_0^1 \int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y) \geq 0$$

for all d.f.s $g : [0,1] \to [0,1]$.

**Theorem 242.** (I) *A copositive $F(x,y)$ need not have $F(x,y) \geq 0$ for every $(x,y) \in [0,1]^2$ but for every $(x,y) \in [0,1]^2$ it must be $F(x,x) \geq 0$, $F(y,y) \geq 0$ and $F(x,x)F(y,y) \geq F^2(x,y)$ or $F(x,y) \geq 0$.*

(II) *A real continuous symmetric $F(x,y)$ is copositive if and only if for every finite sequence $x_1, \dots, x_N$ in $[0,1]$ the matrix $\mathbf{A} = (F(x_m,x_n))_{1\leq m,n\leq N}$ is copositive.*

(III) *If $F_1(x,y)$ is copositive and $F(x,y) \geq F_1(x,y)$ on $[0,1]^2$, then $F(x,y)$ is copositive. If $F_1(x,y)$ and $F_2(x,y)$ are copositive, then $F_1(x,y)+F_2(x,y)$ and $\max(F_1(x,y),F_2(x,y))$ are also copositive.*

(IV) *Copositivity of $F(x,y)$ implies copositivity of $F(f(x),f(y))$ for every continuous $f : [0,1] \to [0,1]$.*

*Proof.* (I) The known criterion of copositivity for symmetric matrix $\mathbf{A} = (a_{ij})$ of rank 2 states: The matrix $\mathbf{A}$ is copositive if and only if $a_{11} \geq 0$, $a_{22} \geq 0$ and $a_{11}a_{22} - a_{12}^2 \geq 0$ or $a_{12} \geq 0$.

(II) Sufficiency follows from the Helly theorem.

(III) Clearly.

(IV) It follows from

$\int_0^1 \int_0^1 F(f(x),f(y))\mathrm{d}g(x)\mathrm{d}g(y) = \int_0^1 \int_0^1 F(x,y)\mathrm{d}g_f(x)\mathrm{d}g_f(y)$. □

**Theorem 243.** (I) *If* $F(x,y) \geq \max\big(F_{g_1}(x,y), F_{g_2}(x,y)\big)$, *for some two different distribution functions* $g_1$, $g_2$, *then the moment problem* (717) *has no solution.*

(II) *If continuous symmetric* $F(x,y)$ *is not copositive together with* $-F(x,y)$, *then the moment problem* (717) *has infinitely many solutions* $g$.

*Proof.* (I) $G(F) \subset G(F_{g_1}) \cap G(F_{g_2}) = \emptyset$.

(II) Assume that, for given distribution functions $g_1$ and $g_2$, we have

$$\int_0^1\int_0^1 F(x,y)\mathrm{d}g_1(x)\mathrm{d}g_1(y) > 0 \text{ and } \int_0^1\int_0^1 F(x,y)\mathrm{d}g_2(x)\mathrm{d}g_2(y) < 0.$$

For these $g_1, g_2$ we can find (in the interval $[0,1]$) finite sequences $x_n$ and $y_n$, $n = 1, 2, \dots, N$ with step d.f.s.

$$F_N^{(1)}(x) = \frac{1}{N}\sum_{n=1}^N c_{[0,x)}(x_n), F_N^{(2)}(x) = \frac{1}{N}\sum_{n=1}^N c_{[0,x)}(y_n),$$

such that

$$\int_0^1\int_0^1 F(x,y)\mathrm{d}F_N^{(1)}(x)\mathrm{d}F_N^{(1)}(y) = \frac{1}{N^2}\sum_{m,n=1}^N F(x_m, x_n) > 0,$$

$$\int_0^1\int_0^1 F(x,y)\mathrm{d}F_N^{(2)}(x)\mathrm{d}F_N^{(2)}(y) = \frac{1}{N^2}\sum_{m,n=1}^N F(y_m, y_n) < 0.$$

From continuity of $F(x,y)$ it follows the existence of a sequence $z_n$, $n = 1, 2, \dots, N$ in $[0,1]$ for which its step distribution function

$$F_N^{(3)}(x) = \frac{1}{N}\sum_{n=1}^N c_{[0,x)}(z_n)$$

solves (717). This can be realized for every sufficiently large $N$. Moreover, for any given $t_1, t_2, \dots, t_k$ in $[0,1]$ and for sufficiently large $N \geq N(k)$ we can find such $z_n$ with the initial $k$-terms segment $t_1, t_2, \dots, t_k$. Furthermore, if $F_{N_1}^{(3)}(x) = F_{N_2}^{(3)}(x)$, then either $(N_1, N_2) = d > 1$, or $F_{N_1}^{(3)}(x) = F_{N_2}^{(3)}(x) = c_\alpha(x)$. Then the infinitely many solutions of (717) can be realized by using $t_1 \neq t_2$ and an infinite sequence $N_1 < N_2 < \dots$ of pairwise coprime terms. $\square$

**Theorem 244.** *Let $F(x,y)$ be a copositive function having continuous $F'_x(x,1)$ a.e. and let $g_1(x)$ be a strictly increasing solution of the moment problem* (717). *Then for every strictly increasing d.f. $g(x)$ we have*

$$\int_0^1\int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y)=0 \Longleftrightarrow F'_x(x,1)=\int_0^1 g(y)\mathrm{d}_yF'_x(x,y)\ \textit{a.e. on } [0,1], \tag{734}$$

*Proof.* Let $g_1(x)$ be a solution of (717). Assume that $x_0$ is a common point of continuity of $g_1(x)$ and $F'_x(x,1)$. Furthermore, assume that $g_1(x)$ is strictly increasing on the interval $I_\varepsilon=[x_0-\varepsilon,x_0+\varepsilon]$. Then there exists $z(x)$ defined on $[0,1]$ such that

(i) $z(x)=0$ for $x\in[0,1]-I_\varepsilon$,

(ii) $z(x)\neq 0$ for $x\in I_\varepsilon$,

(iii) $g(x)=g_1(x)+z(x)$ is nondecreasing.

Then by (724) we have

$$\begin{aligned}\int_0^1\int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y)&=\iint\limits_{I_\varepsilon^2} z(x)z(y)\mathrm{d}_x\mathrm{d}_yF(x,y)\\&\quad+2\int_{I_\varepsilon} z(x)\left(-\mathrm{d}_xF(x,1)+\int_0^1 g_1(y)\mathrm{d}_y\mathrm{d}_xF(x,y)\right).\end{aligned} \tag{735}$$

Assuming $(-\mathrm{d}_xF(x,1)+\int_0^1 g_1(y)\mathrm{d}_y\mathrm{d}_xF(x,y))\neq 0$, the sign of the double integral $\int_0^1\int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y)$ is dependent on sign of $z(x)$ which is a contradiction and thus we have the
$\left(-\mathrm{d}_xF(x,1)+\int_0^1 g_1(y)\mathrm{d}_y\mathrm{d}_xF(x,y)\right)=0$ a.e. and thus we have
$\left(-\mathrm{d}_xF(x,1)+\int_0^1 g(y)\mathrm{d}_y\mathrm{d}_xF(x,y)\right)=0$ a.e. for every strictly increasing solution $g(x)$ of $\int_0^1\int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y)=0$.

Thus we have $\Rightarrow$ implication in (734). Opposite implication $\Leftarrow$ in (734) follows from (736). □

**Theorem 245.** *The equation* (723) *can be reduced as*

$$\begin{aligned}&\int_0^1\int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y)\\&=\int_0^1 (g(x)-g_1(x))\left(-F'_x(x,1)+\int_0^1 g(y)\mathrm{d}_yF'_x(x,y)\right)\mathrm{d}x\end{aligned} \tag{736}$$

$$= \int_0^1 (g(x) - g_1(x)) \left( -F'_x(x,1) + \sum_{i=1}^{k} g(u_i(x))v_i(x) \right) \mathrm{d}x \tag{737}$$

$$= \int_0^1 \int_0^1 (g(x) - g_1(x))(g(y) - g_1(y)) \mathrm{d}_x \mathrm{d}_y F(x,y) \tag{738}$$

*where $F(x,y)$ is copositive, symmetric and continuous and $g_1(x)$ is a strictly increasing solution of the moment problem* (717)*.*

*Proof.* The equation (736) follows from (723) and (734). The (738) follows from (724) and (734). □

Difference of (738) for two increasing solutions $g_1(x)$ and $g_2(x)$ of (717) gives

$$0 = \int_0^1 (g_2(x) - g_1(x)) \left( -F'_x(x,1) + \sum_{i=1}^{k} g(u_i(x))v_i(x) \right) \mathrm{d}x. \tag{739}$$

Since (739) holds for arbitrary d.f. $g(x)$ then

$$0 = \int_0^1 (g_2(x) - g_1(x)) F'_x(x,1) \mathrm{d}x \tag{740}$$

$$0 = \int_0^1 (g_2(x) - g_1(x)) \left( \sum_{i=1}^{k} g(u_i(x))v_i(x) \right) \mathrm{d}x, \tag{741}$$

for arbitrary d.f. $g(x)$.

**Theorem 246.** *Let $g_1(x)$ be a strictly increasing solution of* (717) *with copositive $F(x,y)$. Then*

$$0 = \int_0^1 F(x,y) \mathrm{d}g_1(y) \tag{742}$$

*for every $x \in [0,1]$.*

*Proof.* Instead of $g(x)$ with $(g(x) + g_1(x))/2$ in the formula (738):

$$\int_0^1 \int_0^1 F(x,y) \mathrm{d}\left( \frac{g(x) + g_1(x)}{2} \right) \mathrm{d}\left( \frac{g(y) + g_1(y)}{2} \right)$$

$$\int_0^1 \left( \frac{g(x) + g_1(x)}{2} - \frac{g_0(x) + g_0(x)}{2} \right) \times$$

$$\times \Bigg( - F'_x(x,1) + \frac{1}{2}\sum_{i=1}^{k} g(u_i(x))v_i(x) + \frac{1}{2}\sum_{i=1}^{k} g_1(u_i(x))v_i(x)\Bigg)\mathrm{d}x.$$

Then

$$\begin{aligned}
&\int_0^1\int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g_1(y)\\
&= \frac{1}{2}\int_0^1 (g(x)-g_0(x))\left(-F'_x(x,1)+\sum_{i=1}^{k} g_1(u_i(x))v_i(x)\right)\mathrm{d}x\\
&+ \frac{1}{2}\int_0^1 (g_1(x)-g_0(x))\left(-F'_x(x,1)+\sum_{i=1}^{k} g_1(u_i(x))v_i(x)\right)\mathrm{d}x.
\end{aligned}$$

Assuming that $g_1(x)$ is an increasing solution of (717), then bearing in mind (734) and (739)

$$\begin{aligned}
&0 = -F'_x(x,1)+\sum_{i=1}^{k} g_1(u_i(x))v_i(x),\\
&0 = \int_0^1 (g_1(x)-g_0(x))\left(-F'_x(x,1)+\sum_{i=1}^{k} g_1(u_i(x))v_i(x)\right)\mathrm{d}x
\end{aligned}$$

and we find

$$\int_0^1\int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g_1(y) = 0 \tag{743}$$

for any d.f. $g(x)$. Putting $g(x)=c_\alpha(x)$ we have $0=\int_0^1 F(\alpha,y)\mathrm{d}g_1(y)$ then

$$\int_0^1 F(x,y)\mathrm{d}g_1(y) = 0 \tag{744}$$

for a.e. $x\in[0,1]$ and for every increasing solution $g_1(x)$ of (717), i.e. $\int_0^1\int_0^1 F(x,y)\mathrm{d}g_1(x)\mathrm{d}g_1(y)=0$. □

**Example 111.** We have

$$\int_0^1 \left(\frac{1}{3}+\frac{x^2+y^2}{2}-\max(x,y)\right)\mathrm{d}y = 0$$

and

$$\int_0^1 \left(\frac{1}{3}+\frac{x^2+y^2}{2}-\max(x,y)\right)\mathrm{d}x\mathrm{d}y = \int_0^1 (g(x)-x)^2\mathrm{d}x.$$

In this case $g_1(x)=x$.

**Theorem 247.** (I) *If $g_1(x)$ and $g_2(x)$ are two strictly increasing solutions of the moment problem* (717) *with copositive $F(x,y)$, then their convex linear combination $tg_1(x)+(1-t)g_2(x)$, $t\in[0,1]$ also solved* (717).

(II) *Let $g_1(x)$ be a strictly increasing solution of* (717). *with continuous symmetric and copositive $F(x,y)$. Then $F(1,1)=\int_0^1 g_1(x)F'_x(x,1)\mathrm{d}x$.*

(III) *If* (717) *with copositive $F(x,y)$ has a strictly increasing solution $g_1(x)$, then $F''_{xy}(x,x)\geq 0$ for any common point $x\in[0,1]$ of continuity of $F''_{xy}(x,x)$ and $g_1(x)$.*

*Proof.* (I) It follows directly from (734)).

(II) Putting $g(x)=c_1(x)$ in (736) we find the desired result.

(III) Directly from (735). □

For some copositive $F(x,y)$ the moment problem can be transform to functional equation

## 10.4 Copositive $F(x,y)$ having $F''_{xy}=0$

**Theorem 248.** *Let $F(x,y)$ be copositive and $F''_{xy}=0$ a.e. such that the set $H(F)$ of jumps of $F'_x(x,y)$ satisfies* (730) *and* (731) *and moreover $u_i(x)$ and $v_i(x)$, for $i=1,2,\dots,k$, are continuous. Finally, let $g_1(x)$ be a strictly increasing solution of* (717). *Then for every strictly increasing d.f. $g(x)$ we have*

$$\int_0^1\int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y)=0 \Longleftrightarrow F'_x(x,1)=\sum_{i=1}^k g(u_i(x))v_i(x)\ \textit{a.e. on } [0,1]. \tag{745}$$

*Proof.* Immediately follows from Theorem 244. □

To Theorem 248 we can add that for any distribution function $g(x)$ we have

$$\begin{aligned}
&\int_0^1\int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y)\\
&=\int_0^1 (g(x)-g_1(x))\left(\sum_{i=1}^k (g(u_i(x))-g_1(u_i(x)))v_i(x)\right)\mathrm{d}x \qquad (746)\\
&=\int_0^1 (g(x)-g_1(x))\left(-F'_x(x,1)+\sum_{i=1}^k g(u_i(x))v_i(x)\right)\mathrm{d}x. \qquad (747)
\end{aligned}$$

Let $g_1(x)$ and $g_2(x)$ be two strictly increasing solutions of (437). Then for every two distribution functions $g(x)$ and $\tilde{g}(x)$ we have

$$0 = \int_0^1 (g_2(x) - g_1(x)) \left( \sum_{i=1}^{k} (\tilde{g}(u_i(x)) - g(u_i(x)))v_i(x) \right) \mathrm{d}x. \qquad (748)$$

## 10.5 The matrix form of $\int_0^1 \int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y)$

In the following we shall transform the moment problem (717) to a matrix form. To do this we cover $H(F)$,

$$H(F) \subset \bigcup_{i,j=1}^{M} \{(x_i(t), x_j(t)); t \in [\alpha, \beta)\} \qquad (749)$$

where the sets

$$\{x_1(t); t \in [\alpha, \beta)\}, \dots, \{x_M(t); t \in [\alpha, \beta)\} \qquad (750)$$

are pairwise disjoint. For this $F$ and $x_i(t)$ we define an associated matrix $\mathbf{A}(t)$ as follows,

$$\left(\mathbf{A}(t)\right)_{i,j} = \frac{1}{2}\left(\mathrm{d}_y F'_x(x_i(t), x_j(t))|x'_i(t)| + \mathrm{d}_y F'_x(x_j(t), x_i(t))|x'_j(t)|\right). \qquad (751)$$

Finally we denote by

$$\mathbf{g}(t) = (g(x_1(t)), g(x_2(t)), \dots, g(x_M(t)))$$

the vector associated with $g : [0,1] \to [0,1]$.

**Theorem 249.** *Let $F(x,y)$ be continuous, symmetric, copositive and $F''_{xy} = 0$ a.e. such that the set $H(F)$ of jumps of $F'_x(x,y)$ is covered by* (749) *with pairwise disjoint sets* (750)*. Assume that the derivatives $x'_i(t)$, $i = 1, 2, \dots, M$, are continuous and let $\mathbf{A}(t)$ denote the associated matrix defined by* (751)*. Finally, let $g_1$ be a strictly increasing solution of the moment problem* (717)*. Then we have*

$$\int_0^1 \int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y) = \int_\alpha^\beta (\mathbf{g}(t) - \mathbf{g}_1(t))\mathbf{A}(t)(\mathbf{g}(t) - \mathbf{g}_1(t))^T \mathrm{d}t \qquad (752)$$

*for all distribution functions $g : [0,1] \to [0,1]$. If $\det \mathbf{A}(t) \neq 0$ a.e., then the moment problem* (717) *has the unique solution $g_1$.*

*Proof.* Let $g_1$ be a strictly increasing solution of (717). Suppose that the set of points $(x_i(t), x_j(t))$, $t \in [\alpha, \beta)$, covers (with disjoint sets (750)) $H(F)$ and denote $\{x_i(t); t \in [\alpha, \beta)\} = [\alpha_i, \beta_i)$. Then

$$\int_{\alpha_i}^{\beta_i} (g(x) - g_1(x))(g(u_s(x) - g_1(u_s(x))v_s(x)\mathrm{d}x$$
$$= \int_{\alpha}^{\beta} \big(g(x_i(t)) - g_1(x_i(t))\big)\big(g(x_j(t)) - g_1(x_j(t))\big)v_s(x_i(t))|x_i'(t)|\mathrm{d}t,$$

where $u_s(x_i(t)) = x_j(t)$ and $v_s(x_i(t)) = \mathrm{d}_y F_x'(x_i(t), x_j(t))$. If we sum the last integral over $i$ and $j$, and according to (738), we obtain

$$\int_0^1 \int_0^1 F(x, y)\mathrm{d}g(x)\mathrm{d}g(y) =$$
$$\int_{\alpha}^{\beta} \sum_{i,j=1}^{M} \big(g(x_i(t)) - g_1(x_i(t))\big)\big(g(x_j(t)) - g_1(x_j(t))\big)\mathrm{d}_y F_x'(x_i(t), x_j(t))|x_i'(t)|\mathrm{d}t,$$

and the desired formula (752)) follows immediately. □

## 10.6 Examples of the matrix forms

**Example 112.** For continuous $f(x)$ the function $F(x, y) = F_{g_0}(f(x), f(y))$ described in example (345) is continuous, symmetric and copositive on the unit square $[0, 1]^2$. The first partial derivative of $F(x, y)$ use

$$\frac{\partial}{\partial x}(-\max(f(x), f(y))) = \begin{cases} -f'(x), & \text{if } f(x) > f(y), \\ 0, & \text{if } f(x) < f(y) \end{cases} \tag{753}$$

is piecewise independent of the variable $y$ and the set $H(F)$ of points $(x, y)$ in which $F_x'(x, y)$ has a jump has a form

$$H(F) = \{(x, y) \in [0, 1]^2; f(x) = f(y)\}. \tag{754}$$

In all cases the absolute value of a jump in $(x, y) \in H(F)$ is $|v_i(x)| = |f'(x)|$. Assuming that $g_0(x) = g_{1_f}(x)$ for some strictly increasing d.f. $g_1(x)$ and applying (746) we find

$$\int_0^1 (g_0(x) - g_f(x))^2 \mathrm{d}x = \int_0^1 (g(x) - g_1(x)) \left( \sum_{i=1}^{k} (g(u_i(x)) - g_1(u_i(x)))v_i(x) \right) \mathrm{d}x$$

for every distribution function $g(x)$. In the following, for special $f$, we shall find this formula and also matrix formula directly.

Assume that continuous $f:[0,1]\to[0,1]$ has finitely many inverse functions $f_1^{-1},\dots,f_m^{-1}$ such that, for every $i=1,2,\dots,m$ we have $f_i^{-1}:[0,1]\to[\alpha_i,\beta_i]$, $f_i^{-1}(f(x))=x$ for $x\in[\alpha_i,\beta_i]$, $[0,1]=\cup_{i=1}^m[\alpha_i,\beta_i]$ and $\beta_i\le\alpha_{i+1}$. Furthermore, denote $f_i^{-1}(f(t))=x_i(t)$ for $t\in[\alpha_1,\beta_1]$. In this case

$$g_f(x)=\sum_{i=1}^m \varepsilon_i g(f_i^{-1}(x))+\varepsilon$$

where $\varepsilon_i=\operatorname{sign}(f_i^{-1}(x))'$ and $\varepsilon=(1-\operatorname{sign}(f_m^{-1}(x))'(-1)^m)/2$. Finally, assume that $g_0(x)=g_{1_f}(x)$ a.e. for some (not necessarily strictly increasing) distribution function $g_1(x)$.

Integration by change of variables gives

$$\begin{aligned}
&\int_0^1(g_f(x)-g_{1_f}(x))^2\mathrm{d}x=\int_0^1(g(x)-g_1(x))f'(x)(g_f(f(x))-g_{1_f}(f(x)))\mathrm{d}x\\
&=\sum_{i=1}^m\int_{\alpha_i}^{\beta_i}(g(x)-g_1(x))f'(x)\left(\sum_{j=1}^m(g(f_j^{-1}(f(x)))-g_1(f_j^{-1}(f(x))))\varepsilon_j\right)\mathrm{d}x\\
&=\sum_{i,j=1}^m\int_{\alpha_1}^{\beta_1}(g(x_i(t))-g_1(x_i(t)))f'(x_i(t))\varepsilon_j|x_i'(t)|(g(x_j(t))-g_1(x_j(t)))\mathrm{d}t,
\end{aligned}$$

where in the final step we substitute $x=x_i(t)$ and apply

$$f_j^{-1}f(x_i(t))=f_j^{-1}ff_i^{-1}f(t)=x_j(t).$$

Directly, the associated $m\times m$ matrix $\mathbf{A}(t)=(a_{ij}(t))$ has the terms

$$a_{ij}(t)=\frac{f'(x_i(t))\varepsilon_j|x_i'(t)|+f'(x_j(t))\varepsilon_i|x_j'(t)|}{2}$$

and for every distribution functions $g(x)$ and $g_1(x)$ we have

$$\int_0^1(g_f(x)-g_{1_f}(x))^2\mathrm{d}x=\int_{\alpha_1}^{\beta_1}(\mathbf{g}(t)-\mathbf{g}_1(t))\mathbf{A}(t)(\mathbf{g}(t)-\mathbf{g}_1(t))^T\mathrm{d}t$$

where $\mathbf{g}(t)=(g(x_1(t)),\dots,g(x_m(t)))$ and similarly $\mathbf{g}_1(t)=(g_1(x_1(t)),\dots,g_1(x_m(t)))$ for $t\in[\alpha_1,\beta_1]$.

**Example 113.** Now denote

$$f(x) = \begin{cases} 2x, & x \in [0, 1/2]; \\ 2 - 2x, & x \in [1/2, 1], \end{cases}$$

$$h(x) = \begin{cases} 3x, & x \in [0, 1/3]; \\ 2 - 3x, & x \in [1/3, 2/3]; \\ 3x - 2, & x \in [2/3, 1]. \end{cases}$$

and define $F(x, y)$ by

$$\int_0^1 (x - g_f(x))^2 \mathrm{d}x + \int_0^1 (x - g_h(x))^2 \mathrm{d}x = \int_0^1 \int_0^1 F(x, y) \mathrm{d}g(x) \mathrm{d}g(y).$$

Using definition of $F_0(x, y)$ in Example 99 we have $F(x, y) = F_0(f(x), f(y)) + F_0(h(x), h(y))$. Thus

$$\begin{aligned} F(x, y) = &\frac{1}{3} + \frac{f^2(x) + f^2(y)}{2} - \max(f(x), f(y)) + \\ &\frac{1}{3} + \frac{h^2(x) + h^2(y)}{2} - \max(h(x), h(y)). \end{aligned}$$

The set $H(F)$ is equal to

$$H(F) = \{(x, y) \in [0, 1]^2; f(x) = f(y)\} \cup \{(x, y) \in [0, 1]^2; h(x) = h(y)\}$$

and has the form

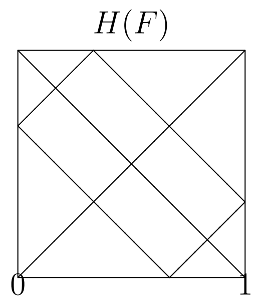



Fig. 3.

It can be covered by $(x_i(t), x_j(t))$, where

$$x_1(t) = t, \; x_2(t) = 2/6 - t, \; x_3(t) = 2/6 + t,$$

$$x_4(t) = 4/6 - t,\ x_5(t) = 4/6 + t,\ x_6(t) = 1 - t \text{ pre } t \in [0, 1/6].$$

This gives

$$\mathbf{A} = \begin{pmatrix} 5 & 0 & 0 & -3 & 3 & -2 \\ 0 & 5 & -3 & 0 & -2 & 3 \\ 0 & -3 & 5 & -2 & 0 & -3 \\ -3 & 0 & -2 & 5 & -3 & 0 \\ 3 & -2 & 0 & -3 & 5 & 0 \\ -2 & 3 & -3 & 0 & 0 & 5 \end{pmatrix},$$

The matrix $\mathbf{A}$ can be decomposed as $\mathbf{A} = \mathbf{B}^T\mathbf{D}\mathbf{B}$, where

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 & -3/5 & 3/5 & -2/5 \\ 0 & 1 & -3/5 & 0 & -2/5 & 3/5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & -1/10 & 1/10 & 0 & 0 \\ 0 & 0 & -1/10 & -1/10 & 1/5 & 1/5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 260 & 0 & 0 \\ 0 & 0 & 0 & 0 & 60 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

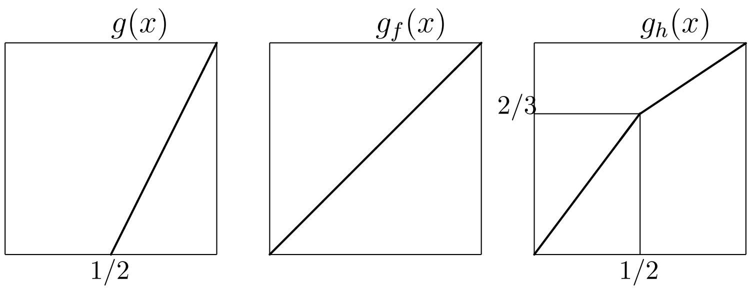
and applying Theorem 249 to $g_1(x) = x$ we have

$$\int_0^1 (x - g_f(x))^2 \mathrm{d}x + \int_0^1 (x - g_h(x))^2 \mathrm{d}x =$$

$$\int_0^{1/6} \Big[ 5\big((g(x_1) - x_1) - 3/5(g(x_4) - x_4) + 3/5(g(x_5) - x_5) - 2/5(g(x_6) - x_6)\big)^2$$
$$+ 5\big((g(x_2) - x_2) - 3/5(g(x_3) - x_3) - 2/5(g(x_5) - x_5) + 3/5(g(x_6) - x_6)\big)^2$$
$$+ 260\big(-1/10(g(x_3) - x_3) + 1/10(g(x_4) - x_4)\big)^2$$
$$+ 60\big(-1/10(g(x_3) - x_3) - 1/10(g(x_4) - x_4) + 1/5(g(x_5) - x_5)$$
$$+ 1/5(g(x_6) - x_6)\big)^2 \Big] \mathrm{d}t \tag{755}$$

for every d.f. $g : [0, 1] \to [0, 1]$.

**Example 114.** To testing (755) we putt $g(x)$ and compute $g_f(x)$ and $g_h(x)$ by the following Fig.



Then
$$\int_0^1 (x-g_f(x))^2 \mathrm{d}x + \int_0^1 (x-g_h(x))^2 \mathrm{d}x = \frac{1}{2^2.3^3}$$
which is the same as (755). Note that $\int_0^1 (x-g_f(x))^2\mathrm{d}x = 0$.

**Example 115.** In the following we consider three functions
$$g_0(x) = \frac{2}{\pi}\arcsin(\sqrt{x}),\ \ f(x) = 4x(1-x),\ \ h(x) = 9x - 24x^2 + 16x^3,$$
on the unit interval $[0,1]$. Let us study the function $F(x,y)$ defined by
$$\int_0^1 (g_0(x)-g_f(x))^2\mathrm{d}x + \int_0^1 (g_0(x)-g_h(x))^2\mathrm{d}x = \int_0^1\int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y).$$
Here we have
$$g_f(x) = g(f_1^{-1}(x)) + 1 - g(f_2^{-1}(x)),$$
$$g_h(x) = g(h_1^{-1}(x)) + g(h_3^{-1}(x)) - g(h_2^{-1}(x)),$$
where $f_1^{-1}(x) = (1-\sqrt{1-x})/2$ and $f_2^{-1}(x) = (1+\sqrt{1-x})/2$ are the inverse functions to $f$, and $h_1^{-1}(x)$, $h_2^{-1}(x)$ and $h_3^{-1}(x)$ denote the inverse functions to $h$ (these functions do not admit simple explicit forms). The explicit expression for $F(x,y)$ can be determined according to
$$F(x,y) = F_{g_0}(f(x),f(y)) + F_{g_0}(h(x),h(y)),$$
where $F_g(x,y)$ is defined in Example 99 as (718) or (719). It is readily seen that
$$\mathrm{d}_yF'_x(x,y) = \mathrm{d}_y\left(-\frac{\partial}{\partial x}\max(f(x),f(y))\right) + \mathrm{d}_y\left(-\frac{\partial}{\partial x}\max(h(x),h(y))\right),$$
and
$$H(F) = \{(x,y)\in[0,1]^2; f(x)=f(y)\}\cup\{(x,y)\in[0,1]^2; h(x)=h(y)\}$$

Let us consider two roots
$$\begin{aligned} y_1(x) &= \frac{3-2x}{4} - \frac{\sqrt{3x(1-x)}}{2}, \\ y_2(x) &= \frac{3-2x}{4} + \frac{\sqrt{3x(1-x)}}{2}, \end{aligned}$$

of the equation $h(x) = h(y)$. The set $H(F)$ can be covered by the set of points $(x_i(t), x_j(t))$, $i, j = 1, \ldots, 6$, where

$$\begin{aligned} x_1(t) &= t, \\ x_2(t) &= 1 - y_2(t), \\ x_3(t) &= 1 - y_1(t), \\ x_4(t) &= y_1(t), \\ x_5(t) &= y_2(t), \\ x_6(t) &= 1 - t, \end{aligned}$$

and $t \in [0, (2 - \sqrt{3})/4]$.

The matrix $\big(\mathrm{d}_y F'_x(x_i, x_j)\big)_{i,j=1}^{6}$ has the form

$$\begin{pmatrix} f'(x_1)+h'(x_1) & 0 & 0 & h'(x_4) & h'(x_5) & f'(x_6) \\ 0 & f'(x_2)+h'(x_2) & h'(x_3) & 0 & f'(x_5) & h'(x_6) \\ 0 & -h'(x_2) & f'(x_3)-h'(x_3) & f'(x_4) & 0 & -h'(x_6) \\ -h'(x_1) & 0 & -f'(x_3) & -f'(x_4)-h'(x_4) & -h'(x_5) & 0 \\ h'(x_1) & -f'(x_2) & 0 & h'(x_4) & -f'(x_5)+h'(x_5) & 0 \\ -f'(x_1) & h'(x_2) & h'(x_3) & 0 & 0 & -f'(x_6)+h'(x_6) \end{pmatrix},$$

For our further aims we need the diagonal matrix

$$\mathrm{diag}\ (|x'_1(t)|, \ldots, |x'_6(t)|) =$$

$$\mathrm{diag}\ (1, a(t) - 1/2, a(t) + 1/2, a(t) + 1/2, a(t) - 1/2, 1),$$

where

$$a(t) = \frac{3(1-2t)}{4\sqrt{3t(1-t)}}.$$

Putting

$$\mathbf{C}(t) = \big(\mathrm{d}_y F'_x(x_i, x_j)\big)_{i,j=1}^{6} \cdot \mathrm{diag}\ (|x'_1(t)|, \ldots, |x'_6(t)|),$$

then we have

$$\mathbf{A}(t) = \frac{1}{2}\big(\mathbf{C}(t) + \mathbf{C}^T(t)\big)$$

Computing the determinant of $\mathbf{A}(t)$ we find $\det \mathbf{A}(t) = 0$ which does not give uniqueness for solutions of (717). For the expression (752)

$$\int_0^1 \int_0^1 F(x, y)\mathrm{d}g(x)\mathrm{d}g(y)$$

$$= \int_0^1 (g_0(x) - g_f(x))^2 \mathrm{d}x + \int_0^1 (g_0(x) - g_h(x))^2 \mathrm{d}x$$
$$= \int_0^{(2-\sqrt{3})/4} (\mathbf{g}(t) - \mathbf{g}_0(t))\mathbf{A}(t)(\mathbf{g}(t) - \mathbf{g}_0(t))^T \mathrm{d}t$$

we need to prove that $g_0(x)$ is a solution of $g_f(x) = g_h(x)$.

*Proof.* Denote $f_1(x) = 2x \bmod 1$ and $h_1(x) = 3x \bmod 1$ and $s(x) = \sin^2 2\pi x$. Clearly $f \circ s = s \circ f_1$, $h \circ s = s \circ h_1$. Thus for every distribution function $g$ we have

$$(g_s)_f = g_{f\circ s} = g_{s\circ f_1} = (g_{f_1})_s$$
$$(g_s)_h = g_{h\circ s} = g_{s\circ h_1} = (g_{h_1})_s.$$

For $g(x) = x$ we have $g_0 = g_s$ and $g_{f_1} = g_{h_1}$ which gives the proof. □

Discrete solutions of $g_f = g_h$ are

$$c_0(x), \quad c_{\frac{5-\sqrt{5}}{8}}(x), \quad c_{\frac{5+\sqrt{5}}{8}}(x).$$

Other solutions can be found by using solutions of $g_{f_1} = g_{h_1}$ in Section 6.1.10 or [162].

**Notes 49.** Note that the above example is motivated by the fact that any distribution function $g(x)$ of the sequence

$$\sin^2(2\pi(3/2)^n), \ n = 1, 2, \dots$$

satisfies $g_f = g_h$, since all distribution functions of $(3/2)^n \bmod 1$ satisfy $g_{f_1} = g_{h_1}$, cf. Section 3.8 (XV) or [162].

# 11 Miscellaneous examples of d.f.s

In [171] the following sequences involving trigonometric and logarithmic functions can be found:

## 11.1 Trigonometric sequences

**Example 116.** [171, 3.12.1., p. 3–49]. Let $1, \omega_1, \omega_2$ be linearly independent over the rational numbers. Then the sequence

$$(\cos 2\pi n\omega_1, \cos 2\pi n\omega_2)$$

has the a.d.f.

$$\begin{aligned} g(x,y) =& 4\left(\frac{1}{4} - g_1(x)\right)\left(\frac{1}{4} - g_1(y)\right) + 2\left(\frac{1}{4} - g_1(x)\right)\big(1 - 2g_2(y)\big) \\ &+ 2\big(1 - 2g_2(x)\big)\left(\frac{1}{4} - g_1(y)\right) + \big(1 - 2g_2(x)\big)\big(1 - 2g_2(y)\big), \end{aligned}$$

where

$$g_1(x) = \frac{1}{2\pi}\arccos x \quad \text{and} \quad g_2(x) = \frac{1}{2\pi}\arccos(x-1).$$

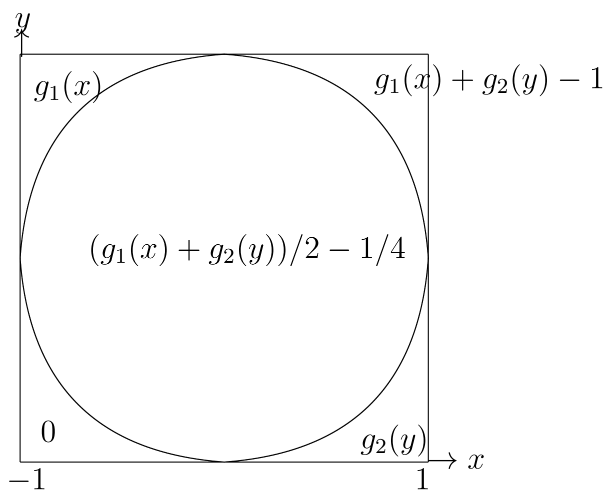
This was proved by R.F. Tichy [178].

**Example 117.** Denote

$x_n$ is u.d. sequence in $[0,1)$,
$y_n = \sin 2\pi x_n$,
$z_n = \cos 2\pi x_n$.

For a.d.f.s we have:

(i) $y_n$ has a.d.f. $g_1(x) = \frac{1}{2\pi}(\pi + 2\arcsin x)$ in $[-1,1]$.
(ii) $z_n$ has a.d.f. $g_2(x) = \frac{1}{2\pi}(2\pi - 2\arccos x)$ in $[-1,1]$.
(iii) $(y_n, z_n)$ is u.d. on the unit circle and has a.d.f. $g(x,y)$ in $[-1,1]^2$ by the Fig.



Now, put

(iv) $x_n = \log(\frac{3}{2})^n \bmod 1 = n\log(\frac{3}{2}) \bmod 1$.

Then from

$2\left(\frac{3}{2}\right)^{n+1} = 3\left(\frac{3}{2}\right)^n$ we have

$\log 2 + (n+1)\log(\frac{3}{2}) = \log 3 + n\log(\frac{3}{2})$. Applying $\sin 2\pi x$ we find

$$\sin(2\pi\log 2).z_{n+1} + \cos(2\pi\log 2)y_{n+1} = \sin(2\pi\log 3).z_n + \cos(2\pi\log 3)y_n.$$

Thus the sequences $w_{n+1}^{(1)}$ and $w_n^{(2)}$ of inner product

$w_{n+1}^{(1)} = \sin(2\pi\log 2).z_{n+1} + \cos(2\pi\log 2)y_{n+1} = \mathbf{e}^{(1)}\cdot(y_{n+1}, z_{n+1})$,

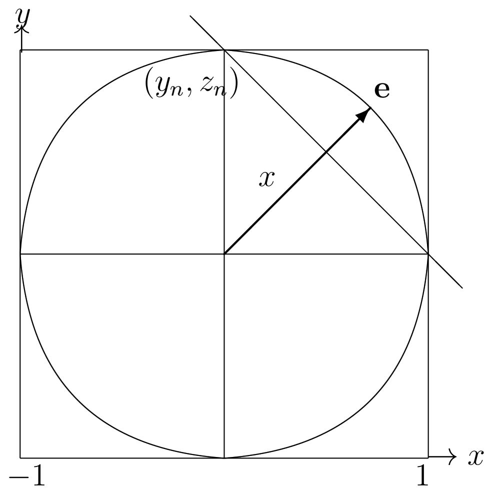
$w_n^{(2)} = \sin(2\pi\log 3).z_n + \cos(2\pi\log 3)y_n = \mathbf{e}^{(2)}\cdot(y_{n+1}, z_{n+1})$

have the same a.d.f.

$$g(x) = \lim_{N\to\infty}\frac{1}{N}\#\{n \le N; w_n^{(1)} < x\} = \lim_{N\to\infty}\frac{1}{N}\#\{n \le N; w_n^{(2)} < x\}. \quad (756)$$

By the following Fig. we see that (756) is not only property of $x_n = \log(\frac{3}{2})^n \bmod 1$ but for arbitrary u.d. $x_n$, for arbitrary unit vector $\mathbf{e}$, and $(y_n, z_n) = (\sin 2\pi x_n, \cos\pi x_n)$ we have the same

$g(x) = \lim_{N\to\infty}\frac{1}{N}\#\{n \le N; \mathbf{e}\cdot(y_n, z_n) < x\}$,




because for a fixed $x$ the length of the family of arcs is constant, independent on $\mathbf{x}$.

## 11.2 Logarithmic sequences

**Example 118.** Let $g(x,y)$ be a d.f. of the sequence $(\{\log n\},\{\log(n+1)\})$, $n=1,2,\dots$. All such d.f.s. can be computed by following:

We have $\{\log n\}\in[0,x)\Leftrightarrow n\in\cup_{k=0}^{\infty}(e^k,e^{k+x})$ and put $N=e^{K+u}$. Then

$$\begin{aligned}F_N(x,y)=&\frac{\{n\le N;(\{\log n\},\{\log(n+1)\})\in[0,x)\times[0,y)\}}{N}\\=&\frac{\sum_{k=0}^{K-1}\left(e^{k+\min(x,y)}-e^k+O(1)\right)}{N}+\frac{e^{K+\min(x,y,u)}-e^K}{N}.\end{aligned}$$

Now, if $N$ letting the sequence of integers such that $\{\log N\}\to u$, then $F_N(x,y)\to g_u(x,y)$, where

$$g_u(x,y)=\frac{e^{\min(x,y)}-1}{e-1}.\frac{1}{e^u}+\frac{e^{\min(x,y,u)}-1}{e^u} \tag{757}$$

By the formula (578) we compute

$$\begin{aligned}\int_0^1\int_0^1|x-y|\mathrm{d}_x\mathrm{d}_y g_u(x,y)=&\int_0^1\left(\frac{e^{\min(x,1)}-1}{e-1}\cdot\frac{1}{e^u}+\frac{e^{\min(x,1,u)}-1}{e^u}\right)\mathrm{d}x\\&+\int_0^1\left(\frac{e^{\min(1,y)}-1}{e-1}\cdot\frac{1}{e^u}+\frac{e^{\min(1,y,u)}-1}{e^u}\right)\mathrm{d}x\\&-2\int_0^1\left(\frac{e^{x}-1}{e-1}\cdot\frac{1}{e^u}+\frac{e^{\min(x,u)}-1}{e^u}\right)\mathrm{d}x=0\end{aligned}$$

which implies

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}|\{\log(n+1)\}-\{\log n\}|=0. \tag{758}$$

This result also follows direct from the fact, that for *almost all* $n$ we have $[\log n]=[\log(n+1)]$ and so $|\{\log(n+1)\}-\{\log n\}|=|\log(n+1)-\log n|=\log(n+1)-\log n$. From this follows

$$\begin{aligned}&\frac{1}{N}\sum_{n=1}^{N}|\{\log(n+1)\}-\{\log n\}|=\frac{1}{N}\sum_{n=1}^{N}(\log(n+1)-\log n)+o(N)\\&=\frac{1}{N}\log(N+1)+o(N)\to 0.\end{aligned}$$

## 11.3 Sequence involving $n\alpha$ mod 1

**Example 119.** (J. Fialová [51].) Let $\alpha$ be an irrational number smaller than $\frac{1}{2}$ and

$$x_n = \alpha, \alpha, 2\alpha, 2\alpha, \{3\alpha\}, \{3\alpha\}, \{4\alpha\}, ...$$
$$y_n = \alpha, 2\alpha, \{3\alpha\}, \{4\alpha\}, \{5\alpha\}, \{6\alpha\}, ...$$

Both sequences are u.d. mod1 and of the two dimensional sequence $(x_n, y_n)$ we can take out two subsequences with different a.d.f.s. First is the subsequence $(\{n\alpha\}, \{2n\alpha\})$ and the second is $(\{n\alpha\}, \{(2n-1)\alpha\})$:

$$g_1(x,y) = \begin{cases} \min\left(x, \frac{y}{2}\right) & \text{if } x \le \frac{1}{2}, \\ \frac{y}{2} + \min\left(x - \frac{1}{2}, \frac{y}{2}\right) & \text{if } x > \frac{1}{2} \end{cases}$$

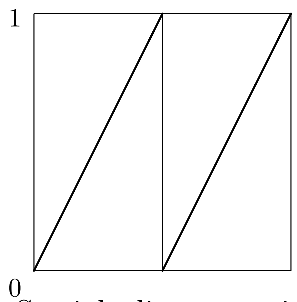



Figure: Straight lines containing $(\{n\alpha\}, \{2n\alpha\})$

$$g_2(x,y) = \begin{cases} 0 & \text{if } (x,y) \in A, \\ x & \text{if } (x,y) \in B_1, \\ \frac{y-(1-\alpha)}{2} & \text{if } (x,y) \in B_2, \\ x - \frac{\alpha}{2} & \text{if } (x,y) \in C_1, \\ \frac{y}{2} & \text{if } (x,y) \in C_2, \\ \frac{y}{2} + x - \frac{1}{2} & \text{if } (x,y) \in D, \\ x - \frac{1+\alpha}{2} & \text{if } (x,y) \in E_1, \\ y & \text{if } (x,y) \in E_2, \\ x + y - 1 & \text{if } (x,y) \in F \end{cases}$$

where

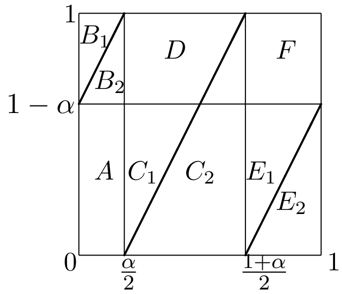



Figure: Straight lines containing $(\{n\alpha\}, \{(2n-1)\alpha\})$

Then the sequence $(x_n, y_n)$, $n = 1, 2, \dots$, has the a.d.f.

$$g(x, y) = \frac{1}{2} g_1(x, y) + \frac{1}{2} g_2(x, y).$$

In the following example we shall find d.f.s of the given two-dimensional sequences $(\{2x_n\}, \{3x_n\})$.

**Example 120** ([51])**.** Let $x_n$, $n = 1, 2, \dots$ be u.d. sequence in $[0, 1)$ and consider two u.d.p. maps $\Phi(x) = 2x \bmod 1$ and $\Psi(x) = 3x \bmod 1$. [60] Then we can compute a.d.f. $g(x, y)$ of the sequence $(\{2x_n\}, \{3x_n\})$ applying the formula (115) $g(x, y) = |\Phi^{-1}([0, x)) \cap \Psi^{-1}([0, y))|$ and by the following graphs

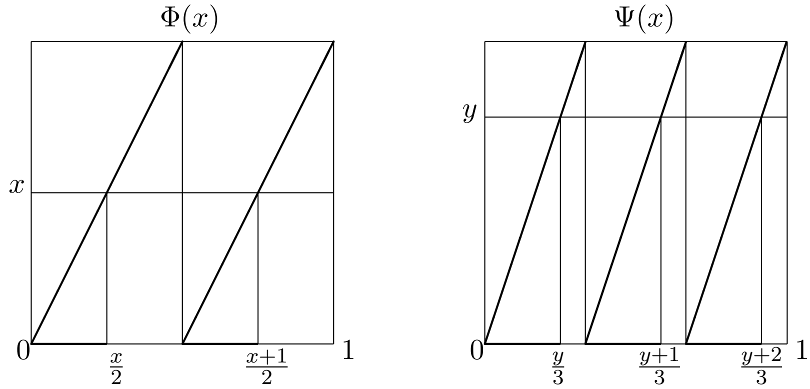



Figure 9: Intervals $\Phi^{-1}([0, x))$ and $\Psi^{-1}([0, y))$.

[60] Before referred to as $f(x)$ and $h(x)$.

$$g(x,y)=\begin{cases}\min\left(\frac{x}{2},\frac{y}{3}\right), & \text{if } x\in A,\\ \frac{x-1}{2}+\min\left(\frac{x+1}{2},\frac{y+1}{3}\right), & \text{if } x\in B,\\ \frac{y-2}{3}+\min\left(\frac{x+1}{2},\frac{y+2}{3}\right), & \text{if } x\in C,\\ \frac{x}{2}+\frac{y-1}{3}+\min\left(\frac{x}{2},\frac{y}{3}\right), & \text{if } x\in D,\\ \frac{2y-1}{3}+\min\left(\frac{x}{2},\frac{y+1}{3}\right), & \text{if } x\in E,\\ \frac{x-1}{2}+\frac{2y-2}{3}+\min\left(\frac{x+1}{2},\frac{y+2}{3}\right), & \text{if } x\in F,\end{cases}$$

where

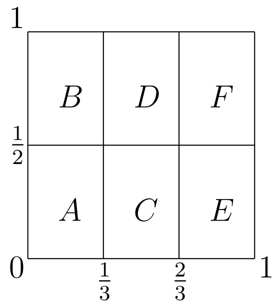



Figure 10: Areas $A,\dots,E$ of $(x,y)$.

## 11.4 Sequence $(\{n\alpha\},\{(n+1)\alpha\},\{(n+2)\alpha\})$

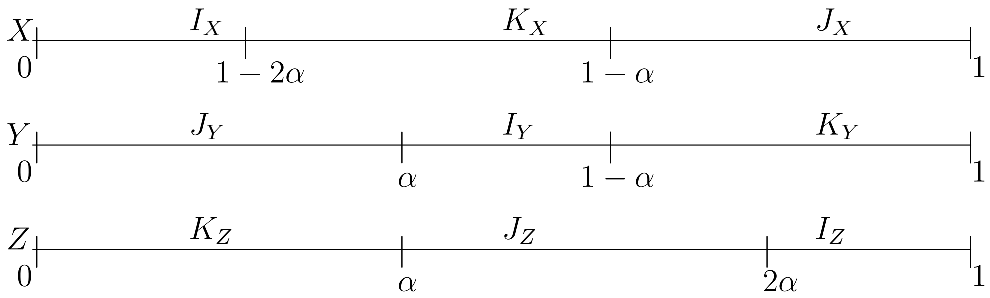
**Example 121.** Let $\alpha$ be irrational and $1/3<\alpha<1/2$. Then the points

$$(\{n\alpha\},\{(n+1)\alpha\},\{(n+2)\alpha\}), n=0,1,2,\dots \tag{759}$$

lie on diagonals
$I=[0,1-2\alpha]\times[\alpha,1-\alpha]\times[2\alpha,1]=(I_X,I_Y,Y_Z)$,
$J=[1-\alpha,1]\times[0,\alpha]\times[\alpha,2\alpha]=(J_X,J_Y,J_Z)$,
$K=[1-2\alpha,1-\alpha]\times[1,1-\alpha]\times[0,\alpha]=(K_X,K_Y,K_Z)$
Let $T$ be a sum of diagonals $I,J,K$ and let $g(x,y,z)$ be a.d.f. of the sequence (759). Then

$$\begin{aligned}g(x,y,z)&=\text{Proj }_X\big([0,x]\times[0,y]\times[0,z]\cap T\big)\\ &=\min\big([0,x]\cap I_X,[0,y]\cap I_Y,[0,z]\cap I_Z\big)\\ &+\min\big([0,x]\cap J_X,[0,y]\cap J_Y,[0,z]\cap J_Z\big)\\ &+\min\big([0,x]\cap K_X,[0,y]\cap K_Y,[0,z]\cap K_Z\big)\end{aligned}$$

To calculate minimums we can use the following Fig.:




For example

$$g(x,x,x) = \begin{cases} 0 & \text{if } x \in [0, 1-\alpha], \\ 2(x-(1-\alpha)) & \text{if } x \in [1-\alpha, 2\alpha], \\ 2(x-(1-\alpha)) + x - 2\alpha & \text{if } x \in [2\alpha, 1]. \end{cases} \tag{760}$$

## 11.5 Sequences of the type $f(x_n)$

In the next example we shall listed some simple d.f. of a sequence $(x_n, y_n)$, $n = 1, 2, \dots$, $x_n, y_n \in [0, 1)$.

**Example 122.** (i) If $(x_n, y_n)$ has a.d.f. $g(x, y)$ then the sequence $(x_n, 1-y_n)$, $n = 1, 2, \dots$, has a.d.f $\tilde{g}(x, y) = g(x, 1) - g(x, 1-y)$ because

$$x_n < x \wedge 1 - y_n < y \Longleftrightarrow x_n < x \wedge 1 - y < y_n.$$

(ii) For u.d. sequence $x_n \in [0, 1)$, the sequence $(x_n, x_n)$ has a.d.f. $g(x, y) = \min(x, y)$ and the sequence $(x_n, 1-x_n)$ has a.d.f. $\tilde{g}(x, y) = g(x, 1) - g(x, 1-y) = x - \min(x, 1-y) = \max(x + y - 1, 0)$. Thus we find upper and lower copulas.

(iii) Let $x_n$ be u.d. sequence in $[0, 1)$ and $\Psi : [0, 1] \to [0, 1]$ be u.d.p. Then the sequence $(x_n, \Psi(x_n))$ has as an a.d.f. the following copula

$$g(x, y) = |[0, x) \cap \Psi^{-1}([0, y))|,$$

because

$$x_n < x \wedge \Psi(x_n) < y \Longleftrightarrow x_n \in [0, x) \wedge x_n \in \Psi^{-1}([0, y)).$$

(iv) The a.d.f. in (iii) (see (709)) can also be extend to non-u.d.p. functions: If $x_n \in [0, 1)$ is u.d. sequence and $f_i : [0, 1] \to [0, 1]$, $i = 1, 2, \dots, s$ are

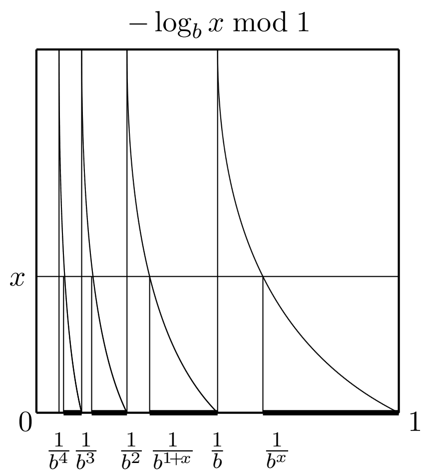
Riemann's integrable, then the s-dimensional sequence $(f_1(x_n), f_2(x_n), \dots, f_s(x_n))$ has a.d.f.

$$g(y_1, y_2, \dots, y_s) = |f_1^{-1}([0, y_1)) \cap f_2^{-1}([0, y_2)) \cap \dots \cap f_s^{-1}([0, y_s))|.$$

(iv) ([51]) For u.d. sequence $x_n \in (0,1)$ the sequence $(x_n, \{\log x_n\})$ has a.d.f

$$g(x,y) = \begin{cases} \frac{e^y-1}{e-1} & \text{if } x \in (e^{y-1}, 1), \\ \frac{e^y-1}{e(e-1)} + x - \frac{1}{e} & \text{if } x \in (\frac{1}{e}, e^{y-1}), \\ \frac{e^y-1}{e(e-1)}, & \text{if } x \in (e^{y-2}, \frac{1}{e}), \\ \frac{e^y-1}{e^2(e-1)} + x - \frac{1}{e^2} & \text{if } x \in (\frac{1}{e^2}, e^{y-2}), \\ \dots \end{cases} \tag{761}$$

It follows from intervals $f_i^{-1}([0,x))$, $i = 0, 1, 2, \dots$




(v) Every u.d. sequence $x_n \in [0,1)$, $n = 1, 2, \dots$, is statistically independent with the sequence $\{\log n\}$, $n = 1, 2, \dots$. Then every d.f. $g(x,y)$ of the two-dimensional sequence $(x_n, \{\log n\})$ has the form $g(x,y) = x g_u(y)$, where

$$g_u(y) = \frac{e^y - 1}{e - 1}\frac{1}{e^u} + \frac{e^{\min(y,u)} - 1}{e^u}, \tag{762}$$

$u \in [0,1]$ all d.f. of $\{\log n\}$, $n = 1, 2, \dots$. For a proof see Theorem 91.

(vi) For u.d. sequence $x_n$ in $[0,1)$ the sequence $(x_n, \{2x_n\})$ has a.d.f

$$g(x,y) = \begin{cases} \min\left(x, \frac{y}{2}\right) & \text{if } x \le \frac{1}{2}, \\ \frac{y}{2} + \min\left(x - \frac{1}{2}, \frac{y}{2}\right) & \text{if } x > \frac{1}{2} \end{cases} \tag{763}$$

**Example 123.** Let $x_n$, $n = 1, 2, \dots$, be u.d. sequence in $[0,1)$.

For $f : [0,1] \to [0,1]$ denote $g(x) = |f^{-1}([0,x))|$.

Then the sequence $f(x_n)$ has a.d.f. $g(x)$.

Assume that the sequence $(x_n, f(x_n))$ has a.d.f. $g(x,y)$.

By Helly theorem

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) f(x_n) = \begin{cases} \int_0^1 x^2 \mathrm{d}g(x), \\ \int_0^1 f^2(x) \mathrm{d}x, \\ \int_0^1 \int_0^1 f(x) y \mathrm{d}_x \mathrm{d}_y g(x,y), \\ f(1) - \int_0^1 x f'(x) \mathrm{d}x - \int_0^1 g(y) f(1) \mathrm{d}y \\ + \int_0^1 \int_0^1 g(x,y) f'(x) \mathrm{d}x \mathrm{d}y. \end{cases}$$

## 11.6 Sequence of a scalar product

**Example 124.** See [165]:

Define

$$g_s(t) = |\{(\mathbf{b}, \mathbf{x}) \in [0,1]^{2s}; \mathbf{b} \cdot \mathbf{x} < t\}|, \quad t \in [0,s].$$

For $s = 1$ we have

$$g_1(t) = t - t\log t, \quad t \in [0,1]$$

with the density $g_1'(t) = -\log t$. The $g_s(t)$ is an a.d.f. of the sequence

$$\mathbf{b}_n \cdot \mathbf{x}_n = \sum_{i=1}^{s} b_{i,n} x_{i,n}, \quad n = 1, 2, \dots,$$

where $\mathbf{b}_n = (b_{1,n}, \dots, b_{s,n})$ and $\mathbf{x}_n = (x_{1,n}, \dots, x_{s,n})$ are statistically independent and u.d. in $[0,1]^s$. Since the $s$-dimensional sequence

$$(b_{1,n} x_{1,n}, \dots, b_{s,n} x_{s,n})$$

also has statistically independent coordinates, it has a.d.f. $g(\mathbf{t})$, $\mathbf{t} = (t_1, \dots, t_s)$, of the form

$$g(\mathbf{t}) = (t_1 - t_1 \log t_1) \dots (t_s - t_s \log t_s)$$

which gives

$$g_s(t) = (-1)^s \int_{\substack{t_1+\dots+t_s<t \\ 0\le t_1\le 1,\dots,0\le t_s\le 1}} 1.\log t_1 \dots \log t_s \mathrm{d}t_1 \dots \mathrm{d}t_s.$$

In particular, for any decomposition $S_1 \cup S_2 = \{1, \dots, s\}$, the coordinates of the sequence

$$\left(\sum_{i\in S_1} b_{i,n}x_{i,n}, \sum_{i\in S_2} b_{i,n}x_{i,n}\right)$$

are also statistically independent. Thus

$$g_s(t) = \int_{x+y<t} 1.\mathrm{d}g_j(x)\mathrm{d}g_{s-j}(y)$$

and the Fig. 1

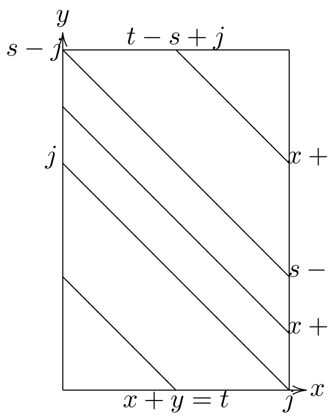



Fig. 1

implies that

$$g_s(t) = \begin{cases} \int_0^t \mathrm{d}g_j(x) \int_0^{t-x} \mathrm{d}g_{s-j}(y) & \text{for } t \in [0, j], \\ \int_0^j \mathrm{d}g_j(x) \int_0^{t-x} \mathrm{d}g_{s-j}(y) & \text{for } t \in [j, s-j], \\ \int_0^{t-s+j} \mathrm{d}g_j(x) + \int_{t-s+j}^j \mathrm{d}g_j(x) \int_0^{t-x} \mathrm{d}g_{s-j}(y) & \text{for } t \in [s-j, s], \end{cases}$$

for $j \le s - j$. Its densities are

$$g'_s(t) = \begin{cases} \int_0^t g'_j(x)g'_{s-j}(t-x)\mathrm{d}x & \text{for } t \in [0, j], \\ \int_0^j g'_j(x)g'_{s-j}(t-x)\mathrm{d}x & \text{for } t \in [j, s-j], \\ \int_{t-s+j}^j g'_j(x)g'_{s-j}(t-x)\mathrm{d}x & \text{for } t \in [s-j, s]. \end{cases} \tag{764}$$

Applying (764) we find

**Theorem 250.** *For $t \in [0,1]$,*

1. $g_2(t) = \frac{t^2}{2}\left((\log t)^2 - 3\log t + \frac{7}{2} - \frac{1}{6}\pi^2\right)$,

2. $g_3(t) = \frac{t^3}{27}\Bigg( - \frac{9}{2}(\log t)^3 + \frac{99}{4}(\log t)^2 + \left(-\frac{255}{4} + \frac{9}{4}\pi^2\right)\log t + \frac{575}{8} - \frac{33}{8}\pi^2 -$
   $9\zeta(3)\Bigg)$,

*where $\zeta(s)$ is the classical Riemann's zeta function.*

Applying substitution $x_i = \frac{t_i}{t}, \quad i = 1,2,\dots,s$, for $t \in (0,1)$, L. Habsieger (Bordeaux) found (personal communication) that

$$\begin{aligned} g_s(t) &= (-1)^s t^s \int_{\substack{x_1+\dots+x_s<1\\ 0\le x_1<1,\dots,0\le x_s<1}} (\log t + \log x_1)\dots(\log t + \log x_s)\mathrm{d}x_1\dots\mathrm{d}x_s \\ &= (-1)^s t^s \sum_{j=0}^{s} \binom{s}{j} (\log t)^{s-j}\tilde{g}_j, \end{aligned}$$

and then using substitution $x_1 + \dots + x_j = 1 - y_1 \dots y_j$, he found

$$\begin{aligned} \tilde{g}_j &= \int_{\substack{x_1+\dots+x_s<1\\ 0\le x_1<1,\dots,0\le x_s<1}} \log x_1 \dots \log x_j \mathrm{d}x_1\dots\mathrm{d}x_j \\ &= \frac{1}{(s-j)!}\int_{[0,1]^j}\prod_{i=1}^{j}(\log y_1 + \dots + \log y_{j-1} + \log(1-y_j))y_1^{s-1}\dots y_j^{s-j}\mathrm{d}y_1\dots\mathrm{d}y_j. \end{aligned}$$

He also observed that $\tilde{g}_j$ is a composition of integrals

$$\begin{aligned} &\int_0^1 (\log x)^m x^n \mathrm{d}x = \frac{(-1)^m m!}{(n+1)^{m+1}}, \\ &\int_0^1 (\log x)^m x^n \log(1-x)\mathrm{d}x = (-1)^{m+1}m!\sum_{k=1}^{\infty}\frac{1}{k(k+n+1)^{m+1}} \\ &= a_0 + a_1\zeta(2) + \dots + a_m\zeta(m+1) \text{ for some } a_i \in \mathbb{Q}. \end{aligned}$$

**Notes 50.** The explicit form of $g_s(t)$, for $t \in [1,s]$, is open even for $s = 2,3$.

## 11.7 $\beta$-van der Corput sequence

In [98] is proved: Let
• $\beta$ be a PV-number;
• $\phi_\beta(n)$ be the Monna map, and $\phi_\beta(n)$, $n = 0, 1, 2, \dots$ is $\beta$-van der Corput sequence; [61]
• $T : [0, 1) \to [0, 1)$ be von Neumann-Kakutani map defined by $\phi_\beta(n)$, i.e. $T^n(0) = \phi_\beta(n)$;
• $n_1, \dots, n_s$ be non-negative integers.
• $k_n$, $n = 1, 2, \dots$ be Hartman uniformly distributed and $L^p$-good universal for a $p \in [1, \infty]$.

Then the sequence

$$(\phi_\beta(k_n + n_1), \dots, \phi_\beta(k_n + n_s)), n = 1, 2, \dots \tag{765}$$

has an a.d.f in $[0, 1)^s$. Moreover,
• $([0, 1], T)$ is uniquely ergodic and

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^{k_n} x) = \int_0^1 f(y) \mathrm{d}\mu(y)$$

for all $x \in [0, 1]$ and all continuous $f : [0, 1] \to [0, 1]$.
• Using $\gamma(t) = (T^{n_1}(t), \dots), T^{n_s}(t))$ and $\Gamma = \{\gamma(t); t \in [0, 1]\}$,

then the a.d.f $\overline{\mu}$ of (765) as a measure $\overline{\mu}$ of Jordan measurable $B \subset [0, 1]^s$ has the form

$$\overline{\mu}(B) = \mu(\mathrm{proj}_1(B \cap \Gamma)). \tag{766}$$

## 11.8 Jager's example

**Example 125.** H. Jager [78] proved: Let $(x_n, y_n)$, $n = 1, 2, \dots$, be a two-dimensional sequence in $[0, 1)^2$ having a.d.f.

$$g(x, y) = \int_0^x \int_0^y f(u, v) \mathrm{d}u\mathrm{d}v,$$

with density

$$f(u, v) = \begin{cases} \frac{1}{\log 2} \frac{1}{\sqrt{1-4uv}} & \text{if } u + v < 1 \\ 0 & \text{others.} \end{cases}$$

[61] $\beta$-adic van der Corput sequence was introduced by G. Barat and P.J. Grabner (1996) [15].

Then the sequences

1. $(x_n + y_n)$ mod 1 has a.d.f.

$$g_1(x) = \frac{1}{2\log 2}((1+x)\log(1+x) + (1-x)\log(1-x)) \text{ if } x \in [0,1].$$

2. $|x_n - y_n|$ has a.d.f.

$$g_2(x) = \frac{1}{\log 2}\big(\tfrac{1}{2}x\pi - 2x\arctan x + \log(1+x^2)\big) \text{ if } x \in [0,1].$$

3. $x_n y_n$ has a.d.f.

$$g_3(x) = \begin{cases} 0 & \text{if } 1/4 < x \le 1, \\ \sqrt{1-4x} - \frac{1}{\log 2}\sqrt{1-4x}\log(1+\sqrt{1-4x}) & \\ -\frac{1}{2\log 2}(1-\sqrt{1-4x})\log x & \text{if } 0 \le x \le 1/4. \end{cases}$$

4. $x_n$ has a.d.f.

$$g_4(x) = \begin{cases} \frac{x}{\log 2} & \text{if } 0 \le x \le 1/2, \\ \frac{1}{2}(1 - x + \log 2x) & \text{if } 1/2 \le x \le 1. \end{cases}$$

# 12 Different themes

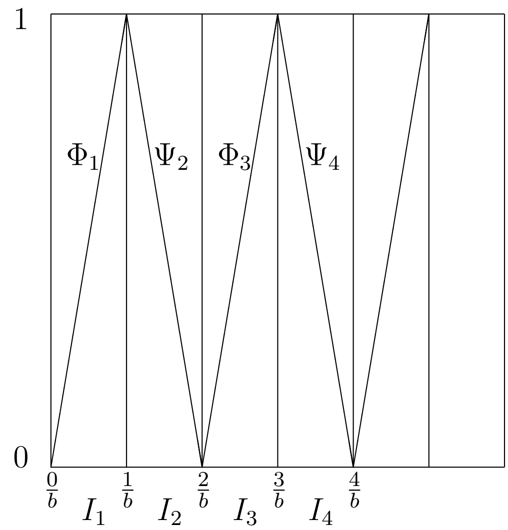
## 12.1 The mean square worst-case error

**Example 126.** In [8] we have defined:
• $x = \frac{x_0}{b} + \frac{x_1}{b^2} + \dots$ is a $b$-adic representation of $x \in [0,1)$, and
• $x = \frac{x_0}{b} + \frac{x_1}{b^2} + \dots$ is a $b$-adic representation of $x \in [0,1)$, and
• $\sigma = \frac{\sigma_0}{b} + \frac{\sigma_1}{b^2} + \dots$, then
• $x \oplus \sigma = \frac{x_0 + \sigma_0 \pmod b}{b} + \frac{x_1 + \sigma_1 \pmod b}{b^2} + \dots$
• $\sigma_i$, $i = 0, 1, \dots$, is a u.d. sequence in $[0,1)$,
• $g_{m,n}(x,y)$ is the asymptotic distribution function (a.d.f.) of the sequence $(x_m \oplus \sigma_i, x_n \oplus \sigma_i)$, $i = 0, 1, 2, \dots$,
• $H$ is Hilbert space with reproducing kernel $K(x,y)$,
• Let $K(x,y) = \max(x,y)$.

The one-dimensional tent map $\Phi(x) = 1 - |2x - 1|$ we extend to the map $\Phi(x)$: Putting $I_i = \left[\frac{i-1}{b}, \frac{i}{b}\right)$, $i = 1, 2, \ldots, b$, we define

$$\Phi(x) = \begin{cases} \Phi_i(x) = bx - (i-1) & \text{if } x \in I_i, 2|i, \\ \Psi_i(x) = -bx + i & \text{if } x \in I_i, 2 \nmid i, \end{cases} \tag{767}$$

with a graph in the following Fig.




In [8] we express the mean square worst-case error as

$$\int_0^1 \sup_{\substack{f \in H \\ ||f|| \leq 1}} \left| \frac{1}{N} \sum_{n=0}^{N-1} f\big(\Phi(x_n \oplus \sigma)\big) - \int_0^1 f(x) \mathrm{d}x \right|^2 \mathrm{d}\sigma =$$

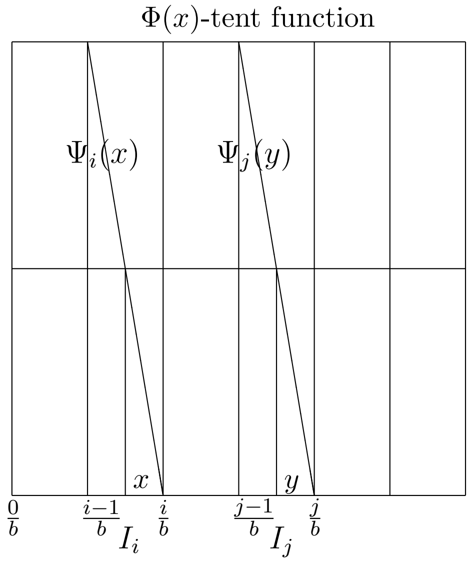
$$= \int_0^1 \int_0^1 \left( \frac{1}{N^2} \sum_{m,n=0}^{N-1} g_{m,n}(x, y) - xy \right) \mathrm{d}_x \mathrm{d}_y K\big(\Phi(x), \Phi(y)\big). \tag{768}$$

Putting $K\big(\Phi(x), \Phi(y)\big) = F(x, y)$ and $g_{m,n}(x, y) = g(x, y)$ to compute (768) we need computed the Riemann-Stiltjes integral

$$\int_0^1 \int_0^1 g(x, y) \mathrm{d}_x \mathrm{d}_y F(x, y) = \sum_{i,j=1}^{b} \iint_{I_i \times I_j} g(x, y) \mathrm{d}_x \mathrm{d}_y F(x, y). \tag{769}$$

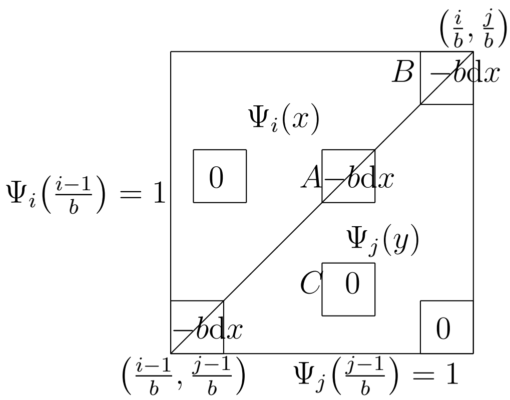
In the following we describe a method of expression of (769). We distinguish four cases:

$1^0$. $i, j$-are even.




$$\Psi_i(x) = \Psi_j(y) \Leftrightarrow b\left(\frac{i}{b} - x\right) = b\left(\frac{j-1}{b} - y\right) \Leftrightarrow y = x + \frac{j-i}{b}$$

and $\Psi_i(x) = \Psi_j(y) > \Psi_j(y + \mathrm{d}y)$ we have $F(x, y)$ on $I_i \times I_j$ by the following figure




where the differential $\square F(x, y)$ on the squares $A, B, C$ can be computed by definition:

$AF(x, y) = (\Psi_i(x_k) = \Psi_j(y_l)) + (\Psi_i(x_{k+1}) = \Psi_j(y_{l+1})) - \Psi_i(x_k) - \Psi_j(y_l)$
$= \Psi_i(x_{k+1}) - \Psi_i(x_k) = -b(x_{k+1} - x_k),$
$BF(x, y) = (\Psi_i(x_k) = \Psi_j(y_l)) + (\Psi_i(x_{k+1}) = \Psi_j(y_{l+1}) = 0) - \Psi_i(x_k) - \Psi_j(y_l) =$

$$= \Psi_i(x_{k+1}) - \Psi_i(x_k) = -b(x_{k+1} - x_k),$$
$$CF(x,y) = \Psi_j(y_l) + \Psi_j(y_{l+1}) - \Psi_j(y_l) - \Psi_j(y_{l+1}) = 0.$$

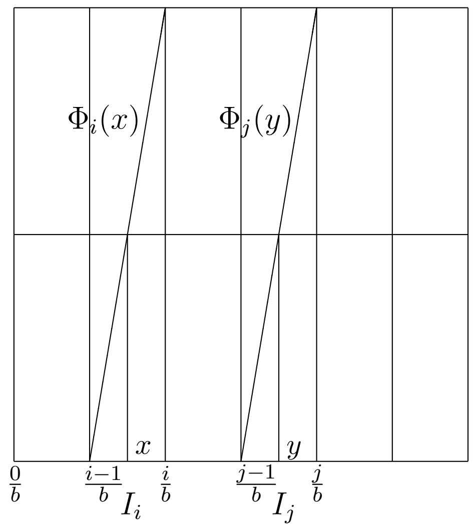
From it

$$\iint_{I_i \times I_j} g(x,y)\mathrm{d}_x\mathrm{d}_y F(x,y) = \int_{\frac{i-1}{b}}^{\frac{i}{b}} g\left(x, x + \frac{j-i}{b}\right)(-b)\mathrm{d}x \tag{770}$$

Similarly

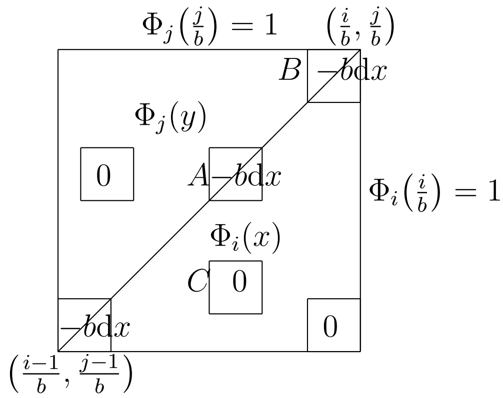
$2^0$. $i, j$-are odd.

$\Phi(x) =$ tent u.d.p. map




$$\Phi(x) = \Phi(y) \Leftrightarrow b\left(x - \frac{i-1}{b}\right) = b\left(y - \frac{j-1}{b}\right) \Leftrightarrow y = x + \frac{j-i}{b}$$

and $\Phi_i(x) = \Phi_j(y) < \Phi_j(y + \mathrm{d}y)$ we have $F(x,y)$ on $I_i \times I_j$ by the following figure



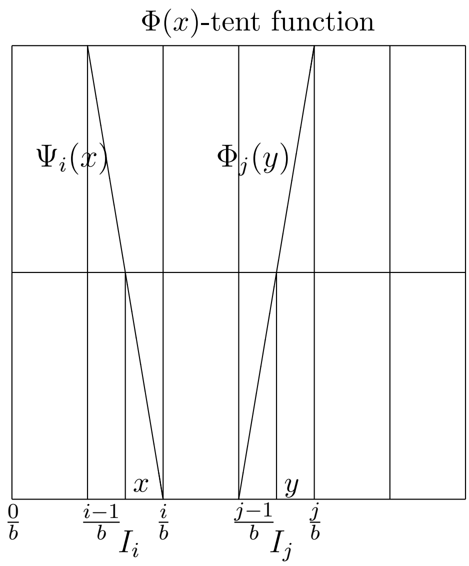
where the differential $\square F(x,y)$ on the squares $A, B, C$ can be computed by definition:

$$AF(x,y) = (\Phi_i(x_k) = \Phi_j(y_l)) + (\Phi_i(x_{k+1}) = \Phi_j(y_{l+1})) - \Phi_j(y_{l+1}) - \Phi_i(x_{k+1})$$
$$= \Phi_i(x_k) - \Phi_i(x_{k+1}) = -b(x_{k+1} - x_k),$$
$$BF(x,y) = (\Phi_i(x_k) = \Phi_j(y_l)) + 1 - 1 - 1 = \Phi(x_k) - \Phi(1) = -b(1 - x_k),$$
$$CF(x,y) = \Phi_i(x_k) + \Phi_i(x_{k+1}) - \Phi_i(x_k) - \Phi_i(x_{k+1}) = 0.$$

Similarly for others squares in $I_i \times I_j$. Then for Riemann-Stiltjes integral

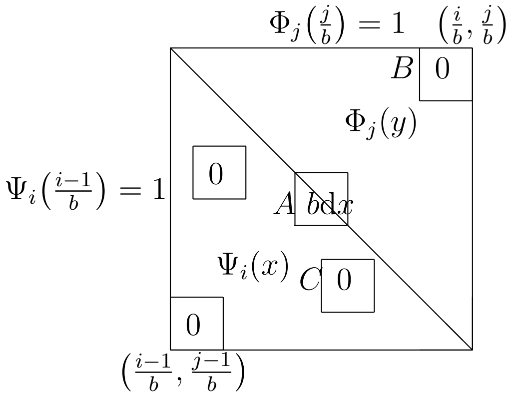
$$\iint_{I_i \times I_j} g(x,y) \mathrm{d}_x \mathrm{d}_y F(x,y) = \int_{\frac{i-1}{b}}^{\frac{i}{b}} g\left(x, x + \frac{j-i}{b}\right)(-b)\mathrm{d}x \tag{771}$$

$3^0$. $i$-even and $j$-odd.




$$\Psi(x) = \Psi(y) \Leftrightarrow x - \frac{i-1}{b} = \frac{j}{b} - y \Leftrightarrow y = -x + \frac{i+j-1}{b}$$

and $\Psi_i(x) = \Phi_j(y) < \Phi_j(y + \mathrm{d}y)$ we have $F(x,y)$ on $I_i \times I_j$ by the following figure




where the differential $\square F(x, y)$ on the squares $A, B, C$ can be computed by definition:

$$
\begin{aligned}
&AF(x, y) = \Psi_i(x_k) + \Phi_j(y_{l+1}) - (\Psi_i(x_k) = \Phi_j(y_{l+1})) - (\Psi_i(x_{k+1}) = \Phi_j(y_l)) \\
&= \Psi_i(x_k) - \Psi_i(x_{k+1}) = b(x_{k+1} - x_k), \\
&BF(x, y) = \Phi_j(y_l) + 1 - 1 - \Phi_j(y_l) = 0, \\
&CF(x, y) = \Psi_i(x_k) + \Psi_i(x_{k+1}) - \Psi_i(x_k) - \Psi_i(x_{k+1}) = 0.
\end{aligned}
$$

From it

$$
\iint_{I_i \times I_j} g(x, y) \mathrm{d}_x \mathrm{d}_y F(x, y) = \int_{\frac{i-1}{b}}^{\frac{i}{b}} g\left(x, -x + \frac{i + j - 1}{b}\right) b\mathrm{d}x. \tag{772}
$$

Similarly

$4^0$. $i$-odd and $j$-even.

$$
\iint_{I_i \times I_j} g(x, y) \mathrm{d}_x \mathrm{d}_y F(x, y) = \int_{\frac{i-1}{b}}^{\frac{i}{b}} g\left(x, -x + \frac{i + j - 1}{b}\right) b\mathrm{d}x. \tag{773}
$$

## 12.2 Infinite set of equations

U.d. of $q$-adic van der Corput sequence $x_n$, $n = 0, 1, 2, \ldots$, can be proved by using shift transformation (see also Section 8.10.4) [62] $x_{n+1} = T(x_n)$. In the base $q \geq 2$ we put $\gamma_i = \frac{1}{q^{i+1}}$, $i = 0, 1, 2, \ldots$ and

$$
\begin{aligned}
n &= n_0 + n_1 q + \cdots + n_k q^k, \\
x_n &= n_0 \gamma_0 + n_1 \gamma_1 + \cdots + n_k \gamma_k.
\end{aligned}
$$

[62] called von Neumann-Kakutani transformation

If $i$ is the first index [63] such that $n_{i+1} < q-1$, then

$$\begin{aligned} n &= (q-1) + (q-1)q + \cdots + (q-1)q^i + n_{i+1}q^{i+1} + \cdots + n_k q^k, \\ n+1 &= 0 + 0q + \cdots + 0q^i + (n_{i+1}+1)q^{i+1} + \cdots + n_k q^k, \\ x_n &= (q-1)(\gamma_0 + \gamma_1 + \cdots + \gamma_i) + \gamma_{i+1}n_{i+1} + \cdots + \gamma_k n_k, \\ x_{n+1} &= 0(\gamma_0 + \gamma_1 + \cdots + \gamma_i) + \gamma_{i+1}(n_{i+1}+1) + \cdots + \gamma_k n_k. \end{aligned}$$

From it

$$x_{n+1} = x_n - (q-1)(\gamma_0 + \gamma_1 + \cdots + \gamma_i) + \gamma_{i+1}$$

and thus every point $(x_n, x_{n+1})$ mod 1 lies on the straight lines

$$Y = X - (q-1)(\gamma_0 + \gamma_1 + \cdots + \gamma_i) + \gamma_{i+1}, i = 0, 1, 2, \ldots \text{and } Y = X + \gamma_0. \quad (774)$$

Using above expression of $x_n$ and $x_{n+1}$ for $i = 0, 1, 2, \ldots$ we have

$$(q-1)(\gamma_0 + \gamma_1 + \cdots + \gamma_i) \leq X < (q-1)(\gamma_0 + \gamma_1 + \gamma_2 + \ldots) - \gamma_{i+1}, \quad (775)$$

$$\gamma_{i+1} \leq Y < (q-1)(\gamma_{i+1} + \gamma_{i+2} + \gamma_{i+3} + \ldots). \quad (776)$$

If $i$ do not exists [64], then

$$0 \leq X < (q-1)(\gamma_0 + \gamma_1 + \gamma_2 + \ldots) \quad (777)$$

$$\gamma_0 \leq Y < \gamma_0 + (q-1)(\gamma_0 + \gamma_1 + \gamma_2 + \ldots). \quad (778)$$

Again putting $\gamma_i = \frac{1}{q^{i+1}}$, then the graph of $T : [0,1] \to [0,1]$ is

[63] If for $n$ such index $i$ do not exists, then $n_0 < q-1$, $x_n = n_0\gamma_0 + n_1\gamma_1 + \cdots + n_k\gamma_k$, $x_{n+1} = (n_0+1)\gamma_0 + n_1\gamma_1 + \cdots + n_k\gamma_k$, and then $x_{n+1} = x_n + \gamma_0$.

[64] If $q = 2$ and $i$ do tot exists, then $n_0 = 0$ and thus $0 \leq X < (\gamma_1 + \gamma_2 + \gamma_3 + \ldots)$ and $\gamma_0 \leq Y < (\gamma_0 + \gamma_1 + \gamma_2 + \gamma_3 + \ldots)$

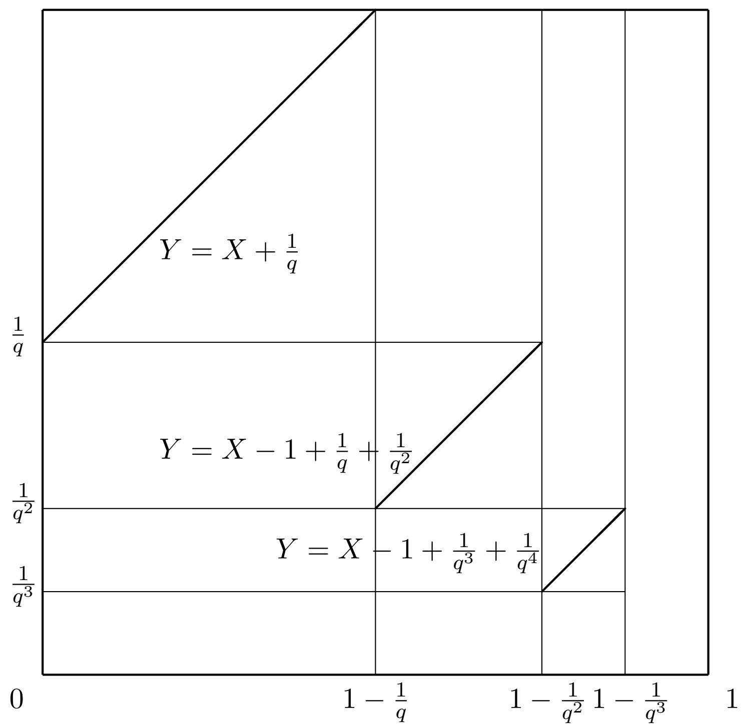



Figure 1: Graph of von Neumann-Kakutani $T$

Assume that $x_n$, $n = 0, 1, 2, \ldots$, has a d.f. $g(x)$. We compute $|T^{-1}([0, x))|$.

From Fig. 1 we see

$$g(x)=\begin{cases} g\left(x-\frac{1}{q}\right)+1-g\left(1-\frac{1}{q}\right) & \text{if } x\in\left(\frac{1}{q},1\right], \\ g\left(x+1-\frac{1}{q}-\frac{1}{q^2}\right)-g\left(1-\frac{1}{q}\right)+1-g\left(1-\frac{1}{q^2}\right) & \text{if } x\in\left(\frac{1}{q^2},\frac{1}{q}\right], \\ g\left(x+1-\frac{1}{q^2}-\frac{1}{q^3}\right)-g\left(1-\frac{1}{q^2}\right)+1-g\left(1-\frac{1}{q^3}\right) & \text{if } x\in\left(\frac{1}{q^3},\frac{1}{q^2}\right], \\ g\left(x+1-\frac{1}{q^3}-\frac{1}{q^4}\right)-g\left(1-\frac{1}{q^3}\right)+1-g\left(1-\frac{1}{q^4}\right) & \text{if } x\in\left(\frac{1}{q^4},\frac{1}{q^3}\right], \\ & \dots \\ g\left(x+1-\frac{1}{q^{i+1}}-\frac{1}{q^i}\right)-g\left(1-\frac{1}{q^i}\right)+1-g\left(1-\frac{1}{q^{i+1}}\right) & \text{if } x\in\left(\frac{1}{q^{i+1}},\frac{1}{q^i}\right], \\ & \dots \end{cases} \tag{779}$$

The u.d. of $x_n$ can be proved that only $g(x)=x$ satisfies the functional equations (779).

*Proof.* Let $I_1, I_2$, $|I_1|=|I_2|$ be intervals in $[0,1]$ and call the graph $g(x)/I_1$ similar to the graph $g(x)/I_2$ if it we can identify with sliding, i.e. if

$$g(x)/I_1 = g(x)/I_2 + \text{ constant.} \tag{780}$$

We denote this $I_1 \longleftrightarrow I_2$. The equations (779) imply that the graphs of $g(x)$ are similar on the pairs

$$\left[\frac{1}{q},1\right] \longleftrightarrow \left[0,1-\frac{1}{q}\right]; \tag{781}$$

$$\left[\frac{1}{q^2},\frac{1}{q}\right] \longleftrightarrow \left[1-\frac{1}{q},1-\frac{1}{q^2}\right]; \tag{782}$$

$$\left[\frac{1}{q^3},\frac{1}{q^2}\right] \longleftrightarrow \left[1-\frac{1}{q^2},1-\frac{1}{q^3}\right]; \tag{783}$$

$$\left[\frac{1}{q^4},\frac{1}{q^3}\right] \longleftrightarrow \left[1-\frac{1}{q^3},1-\frac{1}{q^4}\right]; \tag{784}$$

$$\dots$$

$$\left[\frac{1}{q^{i+1}},\frac{1}{q^i}\right] \longleftrightarrow \left[1-\frac{1}{q^i},1-\frac{1}{q^{i+1}}\right]; \tag{785}$$

For simplification we shall writing an increasing graph of $g(x)$ as linear.

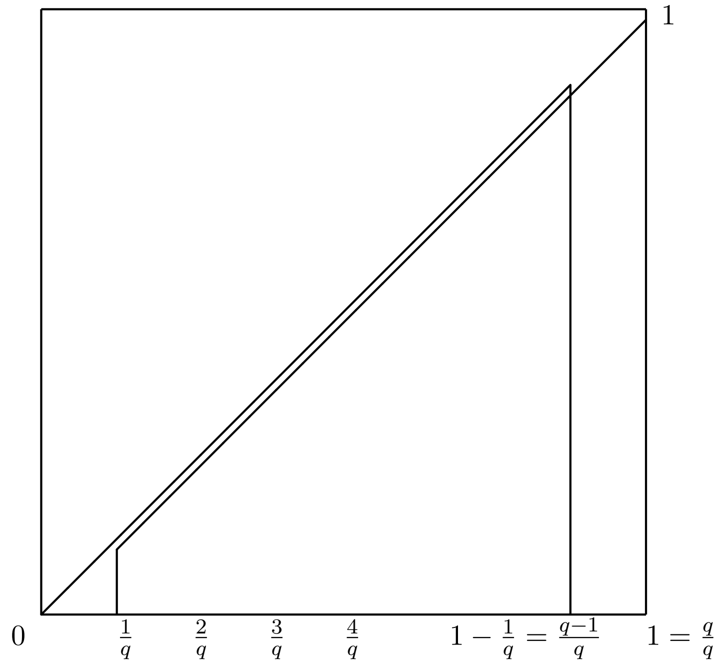



Figure 3: Similar graphs in Step 1.

**Step 1.** We starting with (781)
$\left[\frac{1}{q},1\right] \longleftrightarrow \left[0,1-\frac{1}{q}\right]$.
Shifted the graph $g(x)$ from $\left[\frac{1}{q},1\right]$ to $\left[0,1-\frac{1}{q}\right]$ by Fig. 3 we see that the graph of $g(x)$ is also similar on

$$\left[0,\frac{1}{q}\right] \longleftrightarrow \left[\frac{1}{q},\frac{2}{q}\right] \longleftrightarrow,\ldots,\longleftrightarrow \left[\frac{q-1}{q},\frac{q}{q}\right]. \tag{786}$$

Then $g\left(\frac{k}{q}\right) = kg\left(\frac{1}{q}\right)$ and $qg\left(\frac{1}{q}\right) = 1$. Thus

$$g\left(\frac{k}{q}\right) = \frac{k}{q},\ \text{for } k = 0,1,\ldots,q. \tag{787}$$

**Step 2.** We starting with (782)

$\left[\frac{1}{q^2}, \frac{1}{q}\right] \longleftrightarrow \left[1-\frac{1}{q}, 1-\frac{1}{q^2}\right]$.
Since by (786) we have
$\left[0, \frac{1}{q}\right] \longleftrightarrow \left[1-\frac{1}{q}, 1\right]$ then we can shift $\left[1-\frac{1}{q}, 1-\frac{1}{q^2}\right]$ to 0 and we have
$\left[1-\frac{1}{q}, 1-\frac{1}{q^2}\right] \longleftrightarrow \left[0, \frac{1}{q}-\frac{1}{q^2}\right]$.

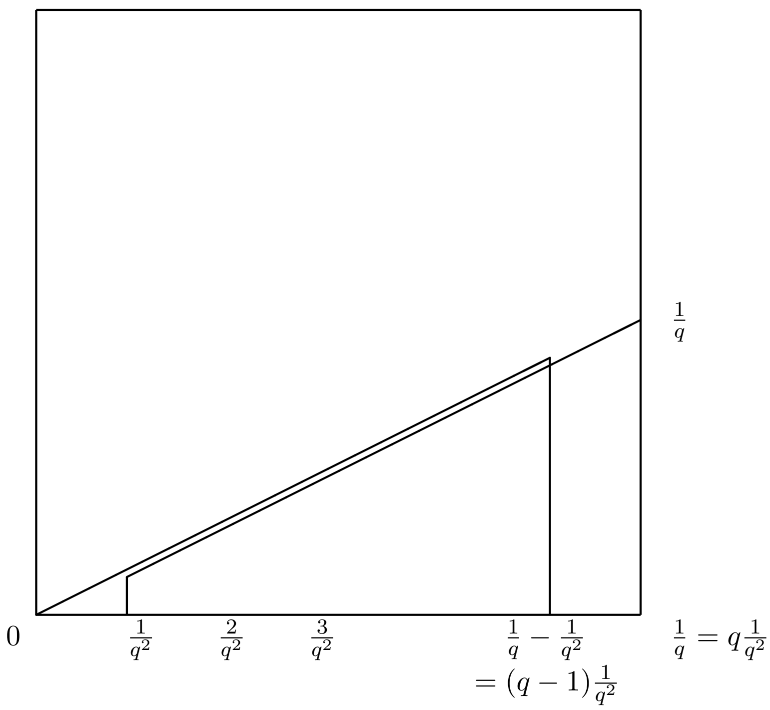



Figure 4: Similar graphs in Step 2.

By Fig. 4 we see that the graph of $g(x)$ is also similar on

$$\left[0, \frac{1}{q^2}\right] \longleftrightarrow \left[\frac{1}{q^2}, \frac{2}{q^2}\right] \longleftrightarrow, \ldots, \longleftrightarrow \left[(q-1)\frac{1}{q^2}, q\frac{1}{q^2}\right]. \tag{788}$$

Then $g\left(\frac{k}{q^2}\right) = kg\left(\frac{1}{q^2}\right)$ and $qg\left(\frac{1}{q^2}\right) = g\left(\frac{q}{q^2}\right) = g\left(\frac{1}{q}\right) = \frac{1}{q}$. Thus

$$g\left(\frac{k}{q^2}\right) = \frac{k}{q^2}, \text{ for } k = 0, 1, \ldots, q. \tag{789}$$

**Step 3.** We starting with (783)

$\left[\frac{1}{q^3}, \frac{1}{q^2}\right] \longleftrightarrow \left[1-\frac{1}{q^2}, 1-\frac{1}{q^3}\right]$; Since

$\left[0, \frac{1}{q}\right] \longleftrightarrow \left[1-\frac{1}{q}, 1\right]$ and

$\left[\frac{1}{q^2}, \frac{1}{q}\right] \longleftrightarrow \left[1-\frac{1}{q}, 1-\frac{1}{q^2}\right]$, then

$\left[0, \frac{1}{q^2}\right] \longleftrightarrow [1-\frac{1}{q^2}, 1]$ and then we can shift $\left[1-\frac{1}{q^2}, 1-\frac{1}{q^3}\right]$ to 0 and we have

$\left[1-\frac{1}{q^2}, 1-\frac{1}{q^3}\right] \longleftrightarrow \left[0, \frac{1}{q^2}-\frac{1}{q^3}\right]$.

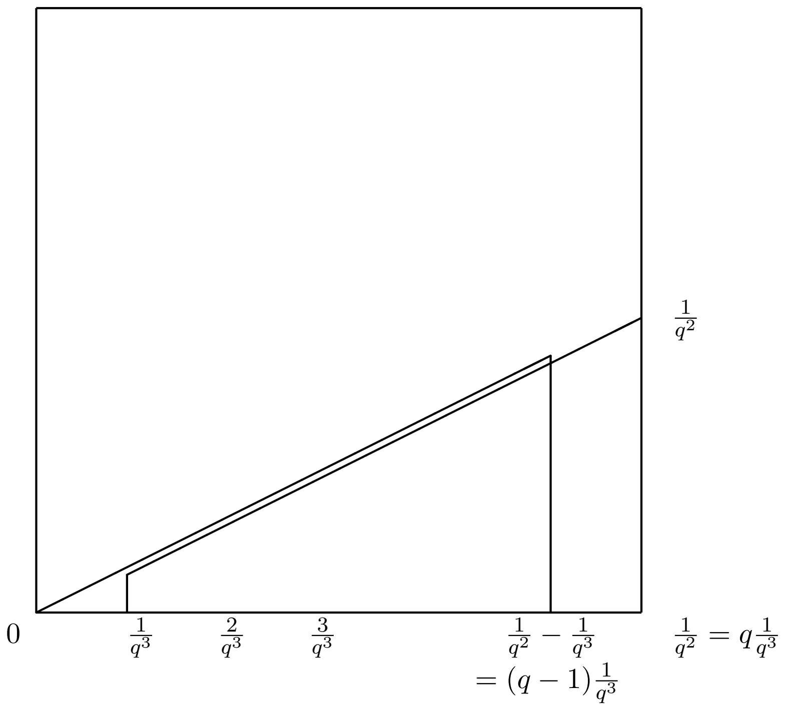



Figure 5: Similar graphs in Step 3.

By Fig. 5 we see that the graph of $g(x)$ is also similar on

$$\left[0, \frac{1}{q^3}\right] \longleftrightarrow \left[\frac{1}{q^3}, \frac{2}{q^3}\right] \longleftrightarrow, \ldots, \longleftrightarrow \left[(q-1)\frac{1}{q^3}, q\frac{1}{q^3}\right]. \tag{790}$$

Then $g\left(\frac{k}{q^3}\right) = kg\left(\frac{1}{q^3}\right)$ and $qg\left(\frac{1}{q^3}\right) = g\left(\frac{q}{q^3}\right) = g\left(\frac{1}{q^2}\right) = \frac{1}{q^2}$. Thus

$$g\left(\frac{k}{q^3}\right) = \frac{k}{q^3}, \text{ for } k = 0, 1, \dots, q. \tag{791}$$

Summary, for every $i = 1, 2, \dots$ and every $k = 0, 1, \dots, q^i$ we have

$$g\left(\frac{k}{q^i}\right) = \frac{k}{q^i}, \text{ for } k = 0, 1, \dots, q^i. \tag{792}$$

This gives $g(x) = x$ for $x \in [0, 1]$. $\square$

## 12.3 Uniform distribution preserving map

The map $u : [0, 1] \to [0, 1]$ is called *uniform distribution preserving* (abbreviated u.d.p.) if for any u.d. sequence $x_n$, $n = 1, 2, \dots$, in $[0, 1]$ the sequence $u(x_n)$ is also u.d. In this u.d.p. theory we register the following properties:

A Riemann integrable function $u : [0, 1] \to [0, 1]$ is a u.d.p. transformation if and only if one of the following conditions is satisfied:

(i) $\int_0^1 h(x)\mathrm{d}x = \int_0^1 h(u(x))\mathrm{d}x$ for every continuous $h : [0, 1] \to \mathbb{R}$.

(ii) $\int_0^1 (u(x))^k \mathrm{d}x = \frac{1}{k+1}$ for every $k = 1, 2, \dots$.

(iii) $\int_0^1 e^{2\pi i k u(x)}\mathrm{d}x = 0$ for every $k = \pm 1, \pm 2, \dots$.

(iv) There exists an increasing sequence of positive integers $N_k$ and an $N_k$–almost u.d. sequence $x_n$ for which the sequence $u(x_n)$ is also $N_k$–almost u.d.

(v) There exists an almost u.d. sequence $x_n$ in $[0, 1)$ such that the sequence $u(x_n) - x_n$ converges to a finite limit.

(vi) There exists at least one $x \in [0, 1]$ of which orbit $x, u(x), u(u(x)), \dots$ is almost u.d.

(vii) $u$ is measurable in the Jordan sense and $|u^{-1}(I)| = |I|$ for every subinterval $I \subset [0, 1]$.

(viii) $\int_0^1 u(x)\mathrm{d}x = \int_0^1 x\mathrm{d}x = \frac{1}{2}$,
$\int_0^1 (u(x))^2\mathrm{d}x = \int_0^1 x^2\mathrm{d}x = \frac{1}{3}$,
$\int_0^1\int_0^1 |u(x) - u(y)|\mathrm{d}x\mathrm{d}y = \int_0^1\int_0^1 |x - y|\mathrm{d}x\mathrm{d}y = \frac{1}{3}$.

From the other properties of u.d.p. transformations let us mention:

(ix) Let $u_1, u_2$ be u.d.p. transformations and $\alpha$ a real number. Then $u_1(u_2(x))$, $1 - u_1(x)$ and $u_1(x) + \alpha \bmod 1$ are again u.d.p. transformations.

(x) Let $u_n$ be a sequence of u.d.p. transformations uniformly converging to $u$. Then $u$ is u.d.p.

(xi) Let $u : [0, 1] \to [0, 1]$ be piecewise differentiable. Then $u$ is u.d.p. if and only if $\sum_{x\in u^{-1}(y)} \frac{1}{|u'(x)|} = 1$ for all but a finite number of points $y \in [0, 1]$.

(xii) A piecewise linear transformation $u : [0, 1] \to [0, 1]$ is u.d.p. if and only if $|J_j| = |I_{j,1}| + \cdots + |J_{j,n_j}|$ for every $J_j = (y_{j-1}, y_j)$, where $0 = y_0 < y_1 < \cdots < y_m = 1$ is the sequence of ordinates of the ends of line segment components of the graph of $f$ and $u^{-1}(J_j) = I_{j,1} \cup \cdots \cup J_{j,n_j}$.

(xiii) U.d.p. function $f : [0, 1] \to [0, 1]$ is equal $f(x) = x$ or $f(x) = 1 - x$, if hold some of the following properties:

$f$ is monotone;

$f$ has a derivative in every point of the interval $(0, 1)$;

$f$ has a Darboux property;

$f$ is continuous and either $f(x) \leq x$ for every $x \in [0, 1]$, or $f(x) \geq 1-x$ for every $x \in [0, 1]$.

The problem to find all continuous u.d.p. is formulated in Ja.-I. Rivkid (1973) [138]. The results (i)-(vii), (ix)-(xiii) are proved in Š. Porubský, T. Šalát and O. Strauch (1988) [130]. The criterion (viii) and (xiii) are given in O. Strauch (1999, p. 116, 67) [163]. R.F. Tichy and R. Winkler (1991) [180] gave a generalization for compact metric spaces. Some related results can be found in: M. Paštéka (1987) [125], Y. Sun (1993) [175], and (1995) [176], P. Schatte (1993) [141], S.H. Molnár (1994) [107] and J. Schmeling and R. Winkler (1995) [144].

### 12.3.1 Multidimensional u.d.p. map

$\Phi : [0,1]^s \to [0,1]^s$ is called uniformly distribution preserving (u.d.p.) map if for every uniformly distributed (u.d.) sequence $\mathbf{x}_n$, $n = 1, 2, \dots$, the image $\Phi(\mathbf{x}_n)$ is again u.d. For one-dimensional case basic properties of u.d.p. maps can be found in [166] and [171, 2.5.1].

For example, if $\Phi(x)$, $\Psi(x)$ are u.d.p. transformation and $\alpha$ a real number, then $\Psi(\Phi(x))$, $1 - \Phi(x)$, $\Phi(x) + \alpha \bmod 1$ are also u.d.p

For multi-dimensional case we have only known:

(i) $\Phi(\mathbf{x}) = \mathbf{x} \oplus \boldsymbol{\sigma}$;

(ii) $\Phi(\mathbf{x}) = (\Phi_1(x_1), \dots, \Phi_s(x_s))$, where $\Phi_n(x)$ are one-dimensional u.d.p. maps, especially

(iii) $\Phi(\mathbf{x}) = \mathbf{b}^{\boldsymbol{\alpha}}\mathbf{x} \bmod 1 = (b_1^{\alpha_1}x_1, \dots, b_s^{\alpha_s}x_s) \bmod 1$;

(iv) $\Phi(\mathbf{x}) = \mathbf{x} + \boldsymbol{\sigma} \bmod 1 = (x_1 + \sigma_1, \dots, x_s + \sigma_s) \bmod 1$;

(v) $\Phi(\mathbf{x}) = (A\mathbf{x})^T \bmod 1$, where $A$ is an $s \times s$ nonsingular integer matrix, cf. S. Steinerberger [151, Th.2];

(vi) $\Phi(\mathbf{x}) = \pi(\mathbf{x})$, where $\pi(\mathbf{x}) = (x_{\pi(1)}, \dots, x_{\pi(n)})$ is a permutation.

We have the following main criterion

**Theorem 251.** *A map $\Phi(\mathbf{x})$ is u.d.p. if and only if for every continuous $f : [0,1]^s \to \mathbb{R}$ we have*

$$\int_{[0,1]^s} f(\Phi(\mathbf{x}))\mathrm{d}\mathbf{x} = \int_{[0,1]^s} f(\mathbf{x})\mathrm{d}\mathbf{x}. \tag{793}$$

In V. Baláž, J. Fialová, V. Grozdanov, S. Stoilova and O. Strauch [9] is used a map $\Psi(\mathbf{x}, \boldsymbol{\sigma}) : [0,1]^{2s} \to [0,1]^s$ which is u.d.p. which respect to $\mathbf{x}$ and $\boldsymbol{\sigma}$, simultaneously. We know such maps only in the form $\Phi(\mathbf{x} \oplus \boldsymbol{\sigma})$ and $\Phi(\mathbf{x} + \boldsymbol{\sigma} \bmod 1)$, where $\Phi : [0,1]^s \to [0,1]^s$ is an arbitrary u.d.p. map.

Between u.d.p. maps and copulas we have

**Theorem 252.** *Let $f_i : [0,1] \to [0,1]$, $i = 1, 2, \dots s$ are u.d.p. maps, then*

$$g(y_1, y_2, \dots, y_s) = |f_1^{-1}([0, y_1)) \cap f_2^{-1}([0, y_2)) \cap \dots \cap f_s^{-1}([0, y_s))|. \tag{794}$$

*is s-dimensional copula.*

## 12.4 Discrepancies of $|x_m - x_n|$

In [156] for a quantitatively version of Theorem 104 is proved

$$12(D_N^{(2)}(x_n))^2 \le D_{N^2}^{(2)}(|x_m - x_n|) \le 12 D_N^{(2)}(x_n), \tag{795}$$

where in the left hand side of (795) we assume that $x_1, x_2, \dots, x_N$ is the same as $1-x_1, 1-x_2, \dots, 1-x_N$. Also is proved $D_{N^2}^{(2)}(|x_m - x_n|) \ge \frac{1}{12}(D_N^*)^6$. Here we prove, for the star discrepancy $D_{N^2}^*(|x_m - x_n|)$, that $ND_{N^2}^*(|x_m - x_n|) \nrightarrow 0$ for every u.d. $x_n$ in $[0,1)$, i.e., the sequence $|x_m - x_n|$, $m,n = 1,2,\dots$ has no good distribution.

*Proof.* Let $g(x) = 2x - x^2$ and $D_{N^2}^* = \sup_{x\in[0,1]} \left|\frac{A([0,x);N^2;|x_m-x_n|)}{N^2} - g(x)\right|$, where

$$A([0,x);N^2;|x_m - x_n|) = \#\{1 \le m,n \le N; |x_m - x_n| \in [0,x)\}.$$

Then

$$\frac{\#\{1 \le m,n \le N; |x_m - x_n| \in [0,x)\}}{N^2} = g(x) + O(D_{N^2}^*)$$

and

$$0 \le \frac{\#\{1 \le m \ne n \le N; |x_m - x_n| \in [0,x)\}}{N} = N(g(x) - 1) + O(ND_{N^2}^*).$$

Replace $x$ by $\frac{x}{N}$ we have

$$0 \le Ng\left(\frac{x}{N}\right) - N + O(ND_{N^2}^*) = 2x - \frac{x^2}{N} - N + O(ND_{N^2}^*)$$

which implies $ND_{N^2}^* \nrightarrow 0$. □

**Notes 51.** In [170] is given a method for computing discrepancy of $|x_j - y_j|$.

**Theorem 253.** *For an arbitrary $N = 1,2,\dots$ and for any two finite sequences $x_1,\dots,x_N$ and $y_1,\dots,y_N$ in $[0,1]$,*

$$D_N^*(|x_j - y_j|) \le 4\sqrt{D_N^*((x_j,y_j))}, \tag{796}$$

*where $D_N^*(|x_j - y_j|)$ is the star discrepancy of $|x_1 - y_1|, \dots, |x_N - y_N|$ with respect to d.f. $g(x) = 2x - x^2$, i.e.*

$$D_N^*(|x_j - y_j|) = \sup_{x\in[0,1]} \left|\frac{\#\{j \le N; |x_j - y_j| \in [0,x)\}}{N} - g(x)\right|.$$

*and the classical star discrepancy is*

$$D_N^*((x_j,y_j)) = \sup_{x,y\in[0,1]} \left|\frac{\#\{j \le N; (x_j,y_j) \in [0,x \times [0,y)\}}{N} - xy\right|.$$

*Proof.* Put $z_j = |x_j - y_j|$, let $c_I(x)$ be the indicator function of the interval $I$, $g(x) = 2x - x^2$, and define the following auxiliary functions: For any $\varepsilon > 0$ and $x_0 \in (0,1)$, let

$$f_1(x) = \begin{cases} 1 & \text{for } x \in [0, x_0), \\ 1 - (x - x_0)\frac{1}{\varepsilon} & \text{for } x \in [x_0, \min(1, x_0 + \varepsilon)), \\ 0 & \text{for } x \in [\min(1, x_0 + \varepsilon), 1]; \end{cases}$$

$$f_2(x) = \begin{cases} 1 & \text{for } x \in [0, \max(0, x_0 - \varepsilon)), \\ 1 - (x - (x_0 - \varepsilon))\frac{1}{\varepsilon} & \text{for } x \in [\max(0, x_0 - \varepsilon), x_0), \\ 0 & \text{for } x \in [x_0, 1]. \end{cases}$$

If $\frac{1}{N}\sum_{j=1}^{N} c_{[0,x_0)}(z_j) \geq \int_0^{x_0} 1.\mathrm{d}g(x)$, then

$$\begin{aligned} \frac{1}{N}\sum_{j=1}^{N} c_{[0,x_0)}(z_j) - \int_0^{x_0} 1.\mathrm{d}g(x) \quad &\leq \quad \left| \frac{1}{N}\sum_{j=1}^{N} f_1(z_j) - \int_0^1 f_1(x)\mathrm{d}g(x) \right| + \\ &+ \quad \int_{x_0}^{\min(1,x_0+\varepsilon)} f_1(x)\mathrm{d}g(x). \end{aligned}$$

If $\frac{1}{N}\sum_{j=1}^{N} c_{[0,x_0)}(z_j) \leq \int_0^{x_0} 1.\mathrm{d}g(x)$, then

$$\begin{aligned} \int_0^{x_0} 1.\mathrm{d}g(x) - \frac{1}{N}\sum_{j=1}^{N} c_{[0,x_0)}(z_j) \quad &\leq \quad \left| \frac{1}{N}\sum_{j=1}^{N} f_2(z_j) - \int_0^1 f_2(x)\mathrm{d}g(x) \right| + \\ &+ \quad \int_{\max(0,x_0-\varepsilon)}^{x_0} (1 - f_2(x))\mathrm{d}g(x). \end{aligned}$$

Clearly,

$$\int_{x_0}^{\min(1,x_0+\varepsilon)} f_1(x)\mathrm{d}g(x) \leq \varepsilon \text{ and } \int_{\max(0,x_0-\varepsilon)}^{x_0} (1 - f_2(x))\mathrm{d}g(x) \leq \varepsilon.$$

Putting $F_i(x,y) = f_i(|x-y|)$, $i = 1,2$, and since

$$\frac{1}{N}\sum_{j=1}^{N} f_i(z_j) - \int_0^1 f_i(x)\mathrm{d}g(x) = \frac{1}{N}\sum_{j=1}^{N} F_i(x_j, y_j) - \int_0^1\int_0^1 F_i(x,y)\mathrm{d}x\mathrm{d}y,$$

applying the Koksma-Hlawka inequality (see [92, p. 1–63]), we find

$$\left| \frac{1}{N}\sum_{j=1}^{N} c_{[0,x_0)}(z_j) - \int_0^{x_0} 1.\mathrm{d}g(x) \right| \leq \max(V(F_1(x,y)), V(F_2(x,y))).D_N^*((x_j, y_j)) + \varepsilon,$$

for any $N = 1, 2, \ldots$, $x_0 \in [0,1]$ and $\varepsilon > 0$. Now, we compute the Hardy-Krause variation of $F_i(x,y)$, $i = 1,2$. Directly from graphs of $F_i(x,1)$ we find the one-dimensional Vitali's variations

$$V^{(1)}(F_1(x,1)) = \begin{cases} 1 & \text{for } x_0 + \varepsilon \leq 1, \\ \frac{1-x_0}{\varepsilon} & \text{for } x_0 + \varepsilon > 1; \end{cases}$$

$$V^{(1)}(F_2(x,1)) = \begin{cases} 1 & \text{for } x_0-\varepsilon \geq 0, \\ 1-\frac{\varepsilon - x_0}{\varepsilon} & \text{for } x_0-\varepsilon < 0; \end{cases}$$

and the same hold for $F_i(1,y)$. For two-dimensional Vitali's variation of $F_i(x,y)$, let $P_0$ be a partition of $[0,1]^2$ defined by a cartesian product of the one-dimensional partition

$$0 = x'_0 < x'_1 < \cdots < x'_s = 1 \tag{797}$$

of $[0,1]$. We consider the following refinement: Let $t > 0$ be a sufficiently small real number and assume that $\frac{x_0}{t}$ and $\frac{x_0+\varepsilon}{t}$ are integers. Thus $\frac{x_0}{\varepsilon}$ must be rational. To the partitions (797) we added all points

$$t < 2t < \cdots < t\left[\frac{1}{t}\right]$$

and we call $P$ the cartesian product of the resulting partition of $[0,1]$. Thus, almost all rectangles (except at most $s^2$ terms) $[x'_i, x'_{i+1}] \times [y'_j, y'_{j+1}]$ in $P$ can be divided into the following subsets:

$P_1$ contains rectangles lying above the line $y = x + x_0 + \varepsilon$;
$P_2$ contains rectangles having diagonals on the line $y = x + x_0 + \varepsilon$;
$P_3$ contains rectangles lying between lines $y = x + x_0 + \varepsilon$ and $y = x + x_0$;
$P_4$ contains rectangles having diagonals on the line $y = x + x_0$;
$P_5$ contains rectangles lying between lines $y = x + x_0$ and $y = x$;
$P_6$ contains rectangles having diagonals on the line $y = x$;
$P_7$ contains rectangles lying between lines $y = x$ and $y = x - x_0$;
$P_8$ contains rectangles having diagonals on the line $y = x - x_0$;
$P_9$ contains rectangles lying between lines $y = x - x_0$ and $y = x - x_0 - \varepsilon$;
$P_{10}$ contains rectangles having diagonals on the line $y = x - x_0 - \varepsilon$;
$P_{11}$ contains rectangles lying under the line $y = x - x_0 - \varepsilon$.

We denote by $V_l^{(2)}(F_k)$ the supremum of the sum of

$$\left|F_k(x'_{i+1}, y'_{j+1}) - F_k(x'_i, y'_{j+1}) - F_k(x'_{i+1}, y'_j) + F_k(x'_i, y'_j)\right|$$

as $t \to 0$ and over the rectangles in $P_l$. Directly by computation it follows:
(I) For $x_0 + \varepsilon \leq 1$ we have nonzero $V_l^{(2)}(F_1)$ only for $l = 2, 4, 8, 10$ and
$V_2^{(2)} = V_{10}^{(2)} = \frac{1-x_0-\varepsilon}{\varepsilon}$, $V_4^{(2)} = V_8^{(2)} = \frac{1-x_0}{\varepsilon}$.
(II) For $x_0 + \varepsilon > 1$ we have nonzero $V_l^{(2)}(F_1)$ only for $l = 4, 8$ and
$V_4^{(2)} = V_8^{(2)} = \frac{1-x_0}{\varepsilon}$.
(III) For $x_0 - \varepsilon \geq 0$ and $V_l^{(2)}(F_2)$ we put $x_0 := x_0 - \varepsilon$ in (I).
(IV) For $x_0 - \varepsilon < 0$ we have nonzero $V_l^{(2)}(F_2)$ only for $l = 4, 6, 8$ and
$V_4^{(2)} = V_8^{(2)} = \frac{1-x_0}{\varepsilon}$, $V_6^{(2)} = \frac{2}{\varepsilon}$.

Thus for the Hardy-Krause variation

$$V(F_k(x,y)) = V^{(1)}(F_k(x,y)) + V^{(1)}(F_k(x,y)) + V^{(2)}(F_k(x,y))$$

we have

$$V(F_1(x,y)) = \begin{cases} 2 + 2\frac{1-x_0-\varepsilon}{\varepsilon} + 2\frac{1-x_0}{\varepsilon} = 4\frac{1-x_0}{\varepsilon} & \text{for } x_0+\varepsilon \leq 1, \\ 2\frac{1-x_0}{\varepsilon} + 2\frac{1-x_0}{\varepsilon} = 4\frac{1-x_0}{\varepsilon} & \text{for } x_0+\varepsilon > 1, \end{cases}$$

$$V(F_2(x,y)) = \begin{cases} 2 + 2\frac{1-(x_0-\varepsilon)-\varepsilon}{\varepsilon} + 2\frac{1-(x_0-\varepsilon)}{\varepsilon} = 4\frac{1-(x_0-\varepsilon)}{\varepsilon} & \text{for } x_0 - \varepsilon \geq 0, \\ 2\left(1 - \frac{\varepsilon - x_0}{\varepsilon}\right) + 2\frac{1-x_0}{\varepsilon} + \frac{2}{\varepsilon} = \frac{4}{\varepsilon} & \text{for } x_0 - \varepsilon < 0. \end{cases}$$

This implies

$$D_N^*(|x_j - y_j|) \leq \frac{4}{\varepsilon} D_N^*((x_j, y_j)) + \varepsilon$$

for any $\varepsilon > 0$, since the rationality of $\frac{x_0}{\varepsilon}$ can be omitted. To find (796) we put

$$\varepsilon = 2\sqrt{D_N^*((x_j, y_j))}.$$

□

In [170] is also proved

**Theorem 254.** *Let $(x_1, y_1), \dots, (x_N, y_N)$ be a sequence in $[0,1)^2$ invariant to $(x,y) \to (y,x)$ and $(x,y) \to (1-x, 1-y)$, i.e. for any $i \leq N$ there exists $j_1, j_2 \leq N$ such that $(x_{j_1}, y_{j_1}) = (y_i, x_i)$ and $(x_{j_2}, y_{j_2}) = (1-x_i, 1-y_i)$. Then*

$$D_N^*(|x_j - y_j|) \leq 3D_N((x_j, \{y_j - x_j\})) + D_N((x_j, y_j)).$$

# 13 Integral formulas

Here we repeat the following list of integral formulas used in this book, previously appeared from [163, pp. 132–135], and [155], [156], [159], [158], [161], [164].

## 13.1 Integral over $|x-y|$

(I) For every d.f. $g$, $\widetilde{g}$, $g_1$, $g_2$, $g_3$, and $g_4$ we have:

1. $$\int_0^1 \int_0^1 -\frac{|x-y|}{2} \mathrm{d}\,(g_1(x) - g_2(x)) \mathrm{d}\,(g_3(y) - g_4(y)) =$$
$$= \int_0^1 (g_1(x) - g_2(x))(g_3(x) - g_4(x)) \mathrm{d}x,$$

2. consequently (cf. [155, p. 130] [65]
$$\int_0^1 \int_0^1 -\frac{|x-y|}{2} \mathrm{d}\,(g(x) - \widetilde{g}(x)) \mathrm{d}\,(g(y) - \widetilde{g}(y)) = \int_0^1 (g(x) - \widetilde{g}(x))^2 \mathrm{d}x$$
and thus (see (33))

[65] The multidimensional integrals of the type $\int\int |\mathbf{x} - \mathbf{y}|^\alpha \mathrm{d}g(\mathbf{x})\mathrm{d}g(\mathbf{y})$ were studied many authors, see R. Alexander and K.B. Stolarsky [3] and R. Alexander[2] and others.

3.

$$\int_0^1 (g(x)-\tilde{g}(x))^2 \mathrm{d}x = \int_0^1 \int_0^1 |x-y| \mathrm{d}g(x)\mathrm{d}\tilde{g}(y)$$
$$- \frac{1}{2}\int_0^1 \int_0^1 |x-y|\mathrm{d}g(x)\mathrm{d}g(y) - \frac{1}{2}\int_0^1 \int_0^1 |x-y|\mathrm{d}\tilde{g}(x)\mathrm{d}\tilde{g}(y).$$

Similarly

4.

$$\int_0^1 \int_0^1 |x-y|\mathrm{d}g(x)\mathrm{d}\widetilde{g}(y) = \int_0^1 g(x)\mathrm{d}x + \int_0^1 \widetilde{g}(x)\mathrm{d}x - 2\int_0^1 g(x)\widetilde{g}(x)\mathrm{d}x,$$

or in a special case (cf. [159, p. 178])

5.

$$\begin{aligned}\int_0^1 \int_0^1 |x-y|\mathrm{d}g(x)\mathrm{d}g(y) &= 2\left(\int_0^1 g(x)\mathrm{d}x - \int_0^1 g^2(x)\mathrm{d}x\right)\\ &= 2\int_0^1 \left(\int_0^x g(t)\mathrm{d}t\right)\mathrm{d}g(x). \qquad (798)\end{aligned}$$

From (798) we have

6.

$$\sum_{m,n=1}^{N} |x_m - x_n| = 2N^2\left(\int_0^1 (F_N(x) - F_N^2(x))\mathrm{d}x\right).$$

In other words

7.

$$\int_0^1 \int_0^1 (1-\max(x,y))\mathrm{d}g(x)\mathrm{d}g(y) = \int_0^1 g^2(x)\mathrm{d}x,$$
$$\int_0^1 \int_0^1 (1-\max(x,y))\mathrm{d}g(x)\mathrm{d}\tilde{g}(y) = \int_0^1 g(x)\tilde{g}(x)\mathrm{d}x.$$

The multidimensional case (524)

8.
$$\int_{\mathbf{0}}^{\mathbf{1}}(g_1(\mathbf{x})-g_2(\mathbf{x}))^2\mathrm{d}\mathbf{x}$$
$$=\int_{\mathbf{0}}^{\mathbf{1}}\int_{\mathbf{0}}^{\mathbf{1}}(\mathbf{1}-\max(\mathbf{x},\mathbf{y}))\mathrm{d}(g_1(\mathbf{x})-g_2(\mathbf{x}))\mathrm{d}(g_1(\mathbf{y})-g_2(\mathbf{y})).$$

In the case of restricted integral range ($0\leq\alpha\leq1$) we have

9.
$$\int_0^\alpha\int_0^\alpha|x-y|\mathrm{d}g(x)\mathrm{d}g(y)=2\left(g(\alpha)\int_0^\alpha g(x)\mathrm{d}x-\int_0^\alpha g^2(x)\mathrm{d}x\right).$$

For $0\leq\alpha\leq\beta\leq1$ we have

10.
$$\int_0^1\int_0^1|x\alpha-y\beta|\mathrm{d}g(x)\mathrm{d}g(y)=$$
$$=2\beta\int_0^{\alpha/\beta}g(x)\mathrm{d}x+(\beta-\alpha)\left(1-\int_0^1 g(x)\mathrm{d}x\right)-$$
$$-2\alpha\int_0^1 g(x)g\left(\frac{x\alpha}{\beta}\right)\mathrm{d}x-\alpha\beta\left(\int_0^1 g(x)\mathrm{d}x\right)^2+$$
$$+\alpha\beta\int_0^1 g(x)\mathrm{d}x\int_0^{\alpha/\beta}g(x)\mathrm{d}x.$$

In [156, p. 251] is proved that

11.
$$\int_0^1\int_0^1|x-y|^k\mathrm{d}\left(g(x)-x\right)\mathrm{d}\left(g(y)-y\right)$$
$$=\begin{cases}0, & \text{if } k=0,\\ -2\int_0^1(g(x)-x)^2\mathrm{d}x, & \text{if } k=1,\\ -k(k-1)\int_0^1\int_0^1(g(x)-x)(g(y)-y)|x-y|^{k-2}\mathrm{d}x\mathrm{d}y, & \text{if } k\geq2,\end{cases}$$

and that
$$\int_0^1\int_0^1|x-y|^k\mathrm{d}\left(g(x)-x\right)\mathrm{d}y=-\int_0^1(g(x)-x)(x^k-(1-x)^k)\mathrm{d}x.$$

(II) If $f:[0,1]\to[0,1]$ and $H:[0,1]^2\to\mathbb{R}$ are continuous functions then for

$$g_f(x)=\int_{f^{-1}([0,x))}1.\mathrm{d}g(u),$$

we have the following integral transforms

$$\int_0^1\int_0^1 H(x,y)\mathrm{d}g_f(x)\mathrm{d}g_f(y)=\int_0^1\int_0^1 H(f(x),f(y))\mathrm{d}g(x)\mathrm{d}g(y).$$

If $f:[0,1]^2\to[0,1]$ is continuous and $g_f(x)=\int_{f^{-1}([0,x))}1\mathrm{d}g(u)\mathrm{d}g(v)$ then [66]

$$\int_0^1\int_0^1 H(x,y)\mathrm{d}g_f(x)\mathrm{d}g_f(y)=$$
$$=\int_0^1\int_0^1\int_0^1\int_0^1 H(f(x,y),f(u,v))\mathrm{d}g(x)\mathrm{d}g(y)\mathrm{d}g(u)\mathrm{d}g(v)$$

and in the special case

$$\int_0^1 h(f(x))\mathrm{d}g(x)=\int_0^1 h(x)\mathrm{d}g_f(x).$$

## 13.2 Generalized $L^2$ discrepancies

(III) If

$$F_{\widetilde{g}}(x,y)=\int_0^1\widetilde{g}^2(t)\mathrm{d}t-\int_x^1\widetilde{g}(t)\mathrm{d}t-\int_y^1\widetilde{g}(t)\mathrm{d}t+1-\max(x,y),$$

then (cf. [158, 618] and Example 99)

$$\int_0^1(g(x)-\widetilde{g}(x))^2\mathrm{d}x=\int_0^1\int_0^1 F_{\widetilde{g}}(x,y)\mathrm{d}g(x)\mathrm{d}g(y)$$

or more generally

$$\int_0^1(g_1(x)-\widetilde{g}(x))(g_2(x)-\widetilde{g}(x))\mathrm{d}x=\int_0^1\int_0^1 F_{\widetilde{g}}(x,y)\mathrm{d}g_1(x)\mathrm{d}g_2(y).$$

[66] In the probability it is called *transmission of integrals.*

(For the proof compute $\int_0^1\int_0^1 F_{\widetilde{g}}(x,y)\mathrm{d}\left(g_1(x)+g_2(y)\right)\mathrm{d}\left(g_1(y)+g_2(y)\right)$.)

$$\int_0^1 (g_f(x)-\widetilde{g}_f(x))^2\mathrm{d}x = \int_0^1\int_0^1 F_{\widetilde{g}_f}(f(x),f(y))\mathrm{d}g(x)\mathrm{d}g(y).$$

(IV) If

$$\begin{aligned} &F_{f,h}(x,y)= \\ &=\max(f(x),h(y))+\max(f(y),h(x))-\max(f(x),f(y))-\max(h(x),h(y))= \\ &\qquad=\frac{1}{2}\left(|f(x)-h(y)|+|f(y)-h(x)|-|f(x)-f(y)|-|h(x)-h(y)|\right), \end{aligned}$$

then (cf. [158, 628] for application see Theorem 111 and Example 102)

$$\int_0^1 (g_f(x)-g_h(x))^2\mathrm{d}x = \int_0^1\int_0^1 F_{f,h}(x,y)\mathrm{d}g(x)\mathrm{d}g(y).$$

There follows from the above that

$$\int_0^1\int_0^1 F_{f,h}(x,y)\mathrm{d}g(x)\mathrm{d}\widetilde{g}(y) = \int_0^1 \big(g_f(x)-g_h(x)\big)\big(\widetilde{g}_f(x)-\widetilde{g}_h(x)\big)\mathrm{d}x$$

and

$$\int_0^1 g_f^2(x)\mathrm{d}x = \int_0^1\int_0^1 \big(1-\max(f(x),f(y))\big)\mathrm{d}g(x)\mathrm{d}g(y).$$

In [164, 427] is proved that (see also Example 103)

$$\begin{aligned} &\iint_{0\le x\le y\le 1}\Big(\big(g_f(y)-g_f(x)\big)-\big(g_h(y)-g_h(x)\big)\Big)^2\mathrm{d}x\mathrm{d}y= \\ &\qquad\qquad=\int_0^1\int_0^1 F_{f,h}^{(1)}(x,y)\mathrm{d}g(x)\mathrm{d}g(y), \end{aligned}$$

where

$$F_{f,h}^{(1)}(x,y)=F_{f,h}(x,y)-\big(f(x)-h(x)\big)\big(f(y)-h(y)\big).$$

This follows from the fact that the integral on the right-hand side is equal to

$$\int_0^1 \left(g_f(x)-g_h(x)\right)^2\mathrm{d}x - \left(\int_0^1 \left(g_f(x)-g_h(x)\right)\mathrm{d}x\right)^2$$

and $\int_0^1 g_f(x)\mathrm{d}x = 1-\int_0^1 f(x)\mathrm{d}g(x)$ gives

$$\left(\int_0^1 \left(g_f(x)-g_h(x)\right)\mathrm{d}x\right)^2 = \int_0^1\int_0^1 \left(f(x)-h(x)\right)\left(f(y)-h(y)\right)\mathrm{d}g(x)\mathrm{d}g(y).$$

(V)

1. If $g_1$ is a strictly increasing solution of $g = g_f$, then

$$\int_0^1 \big(g(x) - g_f(x)\big)^2 \mathrm{d}x =$$
$$= \int_0^1 \big(g(x) - g_1(x)\big)\big(g(x) - g_f(x) + f'(x)\big(g_f(f(x)) - g(f(x))\big)\big)\mathrm{d}x,$$

and it is also true that

2.
$$\int_0^1 \big(g(x) - g_f(x)\big)^2 \mathrm{d}x = \int_0^1 \int_0^1 F_{g_f}(x,y)\mathrm{d}g(x)\mathrm{d}g(y).$$

(VI) Let $\psi(y) = a(x)y^2 + b(x)y + c(x)$ be a polynomial in the variable $y$, where $a(x)$, $b(x)$ and $c(x)$ are integrable functions in $[0,1]$ and put

$$F(x,y) = \int_{\max(x,y)}^1 a(t)\mathrm{d}t + \frac{1}{2}\int_x^1 b(t)\mathrm{d}t + \frac{1}{2}\int_y^1 b(t)\mathrm{d}t + \int_0^1 c(t)\mathrm{d}t.$$

Then (cf. [161, (1997) p. 219, Lemma 5] see also Theorem 23 in this book)

$$\int_0^1 \psi(g(x))\mathrm{d}x = \int_0^1 \int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y)$$

for every d.f. $g(x)$.
(VII) Given a finite sequence $x_1, x_2, \dots, x_N$ in $[0,1)$, a d.f. $g(x)$, and a continuous $f : [0,1] \to \mathbb{R}$, let $F_N(x) = \frac{A([0,x);N;x_n)}{N}$. Then

$$\frac{1}{N}\sum_{n=1}^N f(x_n) - \int_0^1 f(x)\mathrm{d}g(x) = -\int_0^1 (F_N(x) - g(x)\mathrm{d}f(x)$$

which implies

$$\sum_{n=1}^N f(x_n) = N\left(\int_0^1 f(x)\mathrm{d}g(x) - \int_0^1 (F_N(x) - g(x))\mathrm{d}f(x)\right).$$

(VIII) If $F(x,y)$ defined on $[0,1]^2$ is continuous and symmetric, then we have

$$\frac{1}{N^2}\sum_{m,n=1}^N F(x_m,x_n) - \int_0^1 \int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y)$$

$$= -2\int_0^1 (F_N(x)-g(x))\mathrm{d}_xF(x,1)$$
$$+\int_0^1\int_0^1 (F_N(x)-g(x))(F_N(y)+g(y))\mathrm{d}_y\mathrm{d}_xF(x,y).$$

(IX) Define

$$\tilde{g}(x)=\int_{g^{-1}([0,x))} 1.\mathrm{d}x = x_g.$$

Then

$$\int_0^1 x\mathrm{d}\tilde{g}(x)=\int_0^1 g(x)\mathrm{d}x, \int_0^1 x^2\mathrm{d}\tilde{g}(x)=\int_0^1 g^2(x)\mathrm{d}x,$$
$$\int_0^1\int_0^1 |x-y|\mathrm{d}\tilde{g}(x)\mathrm{d}\tilde{g}(y)=\int_0^1\int_0^1 |g(x)-g(y)|\mathrm{d}x\mathrm{d}y=$$
$$=4\int_0^1 xg(x)\mathrm{d}x-2\int_0^1 g(x)\mathrm{d}x.$$

## 13.3 Euclidean $L^2$ discrepancies

(X) In Section 7.6 we have used multidimensional integration by parts in:

1. $\int_0^1\int_0^1\int_0^1\int_0^1 F(x,y,u,v)\mathrm{d}_x\mathrm{d}_yg(x,y)\mathrm{d}_u\mathrm{d}_vg(u,v)$ (Theorem 203);
2. $\int_0^1\int_0^1\int_0^1 F(x,y,z)\mathrm{d}_x\mathrm{d}_y\mathrm{d}_zg(x,y,z)$ (Theorem 204);
3. $\int_0^1\int_0^1\int_0^1\int_0^1 F(x,y,z,u)\mathrm{d}_x\mathrm{d}_y\mathrm{d}_z\mathrm{d}_ug(x,y,z,u)$ (Theorem 205).

(XI) In Section 7.3 we have computed the following $L^2$ discrepancies:

1. $\int_0^1(F_N^{(1)}(x)-x)^2\mathrm{d}x$ - the $L^2$-discrepancy of $x_1,\dots,x_N$;
2. $\int_0^1\int_0^1(F_N(x,y)-xy)^2\mathrm{d}x\mathrm{d}y$ - the $L^2$ discrepancy of $(x_1,y_1),\dots,(x_N,y_N)$;
3. $\int_0^1\int_0^1\int_0^1(F_N(x,y,z)-xyz)^2\mathrm{d}x\mathrm{d}y\mathrm{d}z$ - the $L^2$ discrepancy of $(x_1,y_1,z_1),\dots$ $,(x_N,y_N,z_N)$;
4. $\int_0^1\int_0^1(F_N(x,y)-F_N^{(1)}(x)F_N^{(2)}(y))^2\mathrm{d}x\mathrm{d}y$ - the $L^2$ discrepancy of statistical independence of $x_1,\dots,x_N$ and $y_1,\dots,y_N$.

5. $\int_0^1\int_0^1\int_0^1(F_N(x,y,z)-F_N^{(1)}(x)F_N^{(2)}(y)F_N^{(3)}(z))^2\mathrm{d}x\mathrm{d}y\mathrm{d}z$ - the $L^2$ discrepancy of statistical independence of $x_1,\dots,x_N$, $y_1,\dots,y_N$, and $z_1,\dots,z_N$.

6. $\int_0^1\int_0^1\int_0^1(F_N(x,y,z)-F_N(x,y)F_N^{(3)}(z))^2\mathrm{d}x\mathrm{d}y\mathrm{d}z$ - the $L^2$ discrepancy of statistical independence of $(x_1,y_1)\dots,(x_N,y_N)$ and $z_1,\dots,z_N$.

(XII) In Section 10 we have used:

1. Assume that continuous $f:[0,1]\to[0,1]$ has finitely many inverse functions $f_1^{-1},\dots,f_m^{-1}$ such that, for every $i=1,2,\dots,m$ we have $f_i^{-1}:[0,1]\to[\alpha_i,\beta_i]$. If $g_1$ is a strictly increasing and $f'(x)$ pairwise exists, then (cf. [164, p. 437, Th. 4])

$$\int_0^1\big(g_f(x)-g_{1f}(x)\big)^2\mathrm{d}x=\int_0^1\big(g(x)-g_1(x)\big)f'(x)\big(g_f(f(x)-g_{1f}(f(x))\big)\mathrm{d}x.$$

In the following we listed only integral results without conditions:

2. (see (723)) $\int_0^1\int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y)-\int_0^1\int_0^1 F(x,y)\mathrm{d}\tilde{g}(x)\mathrm{d}\tilde{g}(y)$

$=\int_0^1(g(x)-\tilde{g}(x))\left(-2\mathrm{d}_xF(x,1)+\int_0^1(g(y)+\tilde{g}(y))\mathrm{d}_y\mathrm{d}_xF(x,y)\right).$

3. (see (732)) $\int_0^1\int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y)=\int_0^1(g(x)-g_1(x))$

$\left(-2F'_x(x,1)+\sum_{i=1}^k\big(g(u_i(x))+g_1(u_i(x))\big)v_i(x)\right)\mathrm{d}x.$

4. (see (733))

$\int_0^1 g_1(x)\left(\sum_{i=1}^k g(u_i(x))v_i(x)\right)\mathrm{d}x=\int_0^1 g(x)\left(\sum_{i=1}^k g_1(u_i(x))v_i(x)\right)\mathrm{d}x.$

5. (see (734))

$\int_0^1\int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y)=0\Longleftrightarrow F'_x(x,1)=\int_0^1 g(y)\mathrm{d}_yF'_x(x,y).$

6. (see (736)) $\int_0^1\int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y)$

$=\int_0^1(g(x)-g_1(x))\left(-F'_x(x,1)+\int_0^1 g(y)\mathrm{d}_yF'_x(x,y)\right)\mathrm{d}x.$

7. (see (737)) $\int_0^1\int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y)$

$= \int_0^1 (g(x) - g_1(x)) \left( -F'_x(x,1) + \sum_{i=1}^k g(u_i(x))v_i(x) \right) \mathrm{d}x.$

8. (see (738)) $\int_0^1 \int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y)$

   $= \int_0^1 \int_0^1 (g(x) - g_1(x))(g(y) - g_1(y))\mathrm{d}_x\mathrm{d}_y F(x,y).$

9. (see (740)) $\int_0^1 (g_2(x) - g_1(x))F'_x(x,1)\mathrm{d}x = 0.$

10. (see (741)) $\int_0^1 (g_2(x) - g_1(x)) \left( \sum_{i=1}^k g(u_i(x))v_i(x) \right) \mathrm{d}x = 0.$

11. (see (742)) $\int_0^1 F(x,y)\mathrm{d}g_1(y) = 0.$

12. (see (752)) $\int_0^1 \int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y)$

    $= \int_\alpha^\beta (\mathbf{g}(t) - \mathbf{g}_1(t))\mathbf{A}(t)(\mathbf{g}(t) - \mathbf{g}_1(t))^T \mathrm{d}t.$

(XIII) Put $R_N(x) = (F_N(x) - x)$. Then

$$\begin{aligned}
&\frac{1}{N^4} \sum_{m,n,r,s} f(x_m, x_n, x_r, x_s) - \int_0^1 \int_0^1 \int_0^1 \int_0^1 f(x,u,u,v)\mathrm{d}x\mathrm{d}y\mathrm{d}u\mathrm{d}v \\
&= \int_0^1 \int_0^1 \int_0^1 \int_0^1 f(x,u,u,v)\mathrm{d}R_N(x)\mathrm{d}R_N(y)\mathrm{d}R_N(u)\mathrm{d}R_N(v) \\
&+ 4 \int_0^1 \int_0^1 \int_0^1 \int_0^1 f(x,u,u,v)\mathrm{d}R_N(x)\mathrm{d}R_N(y)\mathrm{d}R_N(u)\mathrm{d}v \\
&+ 2 \int_0^1 \int_0^1 \int_0^1 \int_0^1 f(x,u,u,v)\mathrm{d}R_N(x)\mathrm{d}R_N(y)\mathrm{d}u\mathrm{d}v \\
&+ 4 \int_0^1 \int_0^1 \int_0^1 \int_0^1 f(x,u,u,v)\mathrm{d}R_N(x)\mathrm{d}y\mathrm{d}R_N(u)\mathrm{d}v \\
&+ 4 \int_0^1 \int_0^1 \int_0^1 \int_0^1 f(x,u,u,v)\mathrm{d}R_N(x)\mathrm{d}y\mathrm{d}u\mathrm{d}v,
\end{aligned}$$

(XIV) Suppose we are given a sequence $q_1, q_2, \dots$ of positive integers. Consider the sequence $x_n = \{q_n\alpha\}$ and set

$$R_N(x,y,\alpha) = A([x,y);N;x_n) - N(y-x).$$

Then [160]

$$\int_0^1 \int_0^1 R_N^2(0, y, \alpha)\mathrm{d}y\mathrm{d}\alpha = \frac{1}{12} \sum_{m,n=1}^{N} \frac{(q_m, q_n)^2}{q_m q_n} + \frac{1}{12} \sum_{\substack{m,n=1 \\ q_m=q_n}}^{N} 1.$$

For positive integers $a$ and $b$ we also have

$$\int_0^1 \Big(\min\big(\{ay\}, \{by\}\big) - \{ay\}\{by\}\Big)\mathrm{d}y = \begin{cases} \frac{1}{12}, & \text{for } a \neq b, \\ \frac{1}{6}, & \text{for } a = b, \end{cases}$$

and by J. Franel (1924)

$$\int_0^1 \left(\{ax\} - \frac{1}{2}\right)\left(\{bx\} - \frac{1}{2}\right) = \frac{1}{12}\frac{(a,b)^2}{ab}.$$

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**U**


# 16 Symbols and abbreviations

| | |
|---|---|
| $\mathbb{N}_0 = \mathbb{Z}_0^+$ | the set of non–negative integers |
| $\mathbb{N} = \mathbb{Z}^+$ | the set of positive integers |
| $\mathbb{Z}$ | the set of integers |
| $\mathbb{Q}$ | the set of rational numbers |
| $\mathbb{R}$ | the set of real numbers |
| $\mathbb{C}$ | the set of complex numbers |
| $\|X\|$ | Lebesgue's measure of the set $X$ |
| $\{x\}$ | fractional part of $x$ |
| $[x]$ | integer part of $x$ |
| $x \bmod 1 = \{x\}$ | |
| $\|\|x\|\| = \min(\{x\}, 1 - \{x\})$ | norm of $x$ |
| $x_n$ | sequence in $n = 1, 2, \dots$ or $n$-th term |
| a.d.f. | asymptotic distribution function |
| d.f. | distribution function |

| | |
|---|---|
| $g_f(x)$ | $\int_{f^{-1}([0,x))} 1.\mathrm{d}g(x)$ |
| $G_f(x)$ | $\lvert f^{-1}([0,x))\rvert$, p. 415 |
| u.d. | uniformly distributed, p. 35 |
| u.m. | uniformly maldistributed, p. 56 |
| u.m.s. | u.m. in the strict sense |
| u.q. | uniformly quick |
| B.L. | Benford's Law |
| $g_0(x)$ | d.f. of $\frac{\varphi(n)}{n}$, p. 257 |
| $c_\alpha(x)$ | one-step d.f. with the step 1 in $\alpha$ |
| $h_\beta(x)$ | constant d.f. $h_\beta(x) = \beta$ for $x \in (0,1)$ |
| $c_X(x)$ | characteristic function of $X$ |
| $F_N(x)$ | the step d.f. of $x_1, x_2, \dots, x_N$, p. 9 |
| $G(x_n)$ | the set of all d.f.s of the sequence $x_n$, p. 16 |
| $G(F)$ | the set of all d.f.s $g(x)$ satisfying $\int_0^1\int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y) = 0$, p. 16 |
| $\Omega(x_n)$ | see p. 11 |
| Graph $g$ | see p. 11 |
| $\mathrm{Project}_x(A)$ | project $A$ to $x$-axes |
| $\mathbf{A}(t)$ | matrix associated to $\int_0^1\int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y)$, p. 440 |
| $\underline{g}(x)$ and $\overline{g}(x)$ | lower and upper d.f.s of $x_n$, p. 11 |
| $\underline{g}_H(x)$ and $\overline{g}_H(x)$ | lower and upper d.f.s of $H$, p. 11 |
| $\underline{d}(A)$ and $\overline{d}(A)$ | lower and upper asymptotic density of $A$, p. 11 |
| $d(A)$ | asymptotic density of $A$, p. 11 |
| $R(A)$ | $\{a/b; a, b, \in A\}$ |
| $R(A)^l$ | the set of all limit points of $R(A)$ |
| $R(A)^d$ | the set of all accumulation points of $R(A)$ |
| $A([0,x); N; x_n)$ | counting function, p. 9 |
| $D_N$, $D_N^*$, and $D_N^{(2)}$ | extremal, star and $L^2$ discrepancies of $x_n$, p. 36 |
| $D_N^{(2)}(x_n, H)$ | $L^2$ discrepancy of $x_n$ with respect to $H$, p. 84 |
| $d(g_1, g_2)$ | metric in $G(x_n)$, p. 16 |
| $\varphi(n)$ | Euler function |
| $\pi(n)$ | number of all primes $\le n$ |
| $\Omega(n)$ | the total number of prime factors of $n$ |
| $p_n$ | the $n$th prime |
| $p\mid n$ | $p$ divided $n$ |
| $\mu(n)$ | Möbius function |
| $\mathrm{ord}_p(n) = \alpha$ | maximal $\alpha$ such that $p^\alpha \mid n$ |

| | |
|---|---|
| $\gamma_q(n)$ | van der Corput sequence |
| $h(n) = \min(\alpha_1, \dots, \alpha_k)$ | where $n = {p_1}^{\alpha_1} \dots {p_k}^{\alpha_k}$ p. 51 |
| $H(n) = \max(\alpha_1, \dots, \alpha_k)$ | p. 51 |
| $(a, b)$ | greatest common divisor of integer $a$ and $b$ |
| $\alpha = [a_0; a_1, a_2, \dots]$ | continued fraction expansion of $\alpha$ with partial quotients $a_0, a_1, \dots,$ |
| $\frac{p_n}{q_n} = [a_0; a_1, a_2, \dots, a_n]$ | the $n$th convergent of a continued fraction of $\alpha$ |
| $\lvert\lvert\mathbf{x}\rvert\rvert_\infty = \max_{1\le i\le s} \lvert x_i\rvert$ | norm of the vector $\mathbf{x} = (x_1, \dots, x_s)$ |
| $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{s} x_i y_i$ | the inner product of $\mathbf{x} = (x_1, \dots, x_s)$ and $\mathbf{y} = (y_1, \dots, y_s)$ |
| $\varrho(n) = \beta(n) + \imath\gamma(n)$ | root of dzeta function p. 19 |

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