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ON THE PRODUCT OF SEMIGROUP VALUED MEASURES

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Abstract. A product measure construction is examined for semigroup valued mappings. It is based on a version of Alexandroff theorem. The results are applied to the existence of the joint observable for some types of observables.

Introduction

The classical Alexandroff theorem states that an additive, regular non-negative real-valued mapping is σ-additive. As an application of the theorem the product of measures can be constructed. The method was examined for various types of real-valued mappings, vector-valued mappings [4, 5, 6, 7] and also for group-valued mappings [21, 22, 25, 26, 27].

In this paper the relation between regularity and continuity is investigated from a general point of view (considering semigroup-valued mappings).

Further a commutative lattice group $(G,+)$ with G^+ denoting the set of positive elements is considered together with a partial binary operation $\cdot : G^+ \times G^+ \to G^+$ such that $a \cdot (b+c) = a \cdot b + a \cdot c$ for every $a, b, c \in G^+$. For such groups a G-valued product measure theorem is proved.

In a quantum mechanics model a set $\mathcal{F} \subset \langle 0,1\rangle^\Omega$ is considered. Instead of a random variable from the Kolmogorov probability scheme, an observable is taken as a homomorphism $x\colon \mathcal{B}(R) \to \mathcal{F}\big(\mathcal{B}(R) = \text{Borel sets in reals } R\big)$. In this paper we study a more general situation (see [14]) substituing the unit interval $\langle 0,1\rangle$ by an interval in an ordered group G. Applying the general product measure theorem we prove the existence of the joint observable of two observables. Recall that the notion of the joint observable in quantum models corresponds to the notion of a random vector in the Kolmogorov theory.

In the first part of the paper a version of Alexandroff theorem will be proved for semigroup valued mappings. The second part will contain another version of Alexandroff theorem and a product measure theorem for group valued mappings. Finally the third part of the paper is devoted to the proof of the existence of the joint observable of two observables.

1. Semigroup valued Alexandroff theorem

We shall consider a commutative semigroup $(G,+)$, i.e., a set with a commutative and associative binary operation $+$.

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Theorem 1. *Let G be a commutative semigroup, $\mathcal{D}$ a semiring of subsets of a set, $s(\mathcal{D})$ the generated ring, $\lambda\colon \mathcal{D} \to G$ an additive mapping. Then there exists exactly one additive extension $\bar{\lambda}\colon s(\mathcal{D}) \to G$ of λ.*

Proof. Every set of $s(\mathcal{D})$ can be expressed in the form $\bigcup_{i=1}^{n} A_i$, where $A_i \in \mathcal{D}$ are pairwise disjoint. If $\bigcup_{j=1}^{m} C_j$ is another expression of the same set, then

$$A_i = \bigcup_{j=1}^{m} (A_i \cap C_j), \ C_j = \bigcup_{i=1}^{n} (A_i \cap C_j),$$

hence

$$\begin{aligned}\sum_{i=1}^{n} \lambda(A_i) &= \sum_{i=1}^{n}\sum_{j=1}^{m} \lambda(A_i \cap C_j) = \\ &= \sum_{j=1}^{m}\sum_{i=1}^{n} \lambda(A_i \cap C_j) = \sum_{j=1}^{m} \lambda(C_j).\end{aligned}$$

Therefore we can define $\bar{\lambda}\colon s(\mathcal{D}) \to G$ by the formula

$$\bar{\lambda}\Big(\bigcup_{i=1}^{n} A_i\Big) = \sum_{i=1}^{n} \lambda(A_i)$$

and evidently $\bar{\lambda}$ is an additive mapping. On the other hand, if κ is an arbitrary additive extension of λ to $s(\mathcal{D})$ and $A = \bigcup_{i=1}^{n} A_i$, $A_i \in \mathcal{D}$ $(i = 1, \dots, n)$, $A_i \cap A_j = \emptyset$ $(i \neq j)$, then

$$\kappa(A) = \sum_{i=1}^{n} \kappa(A_i) = \sum_{i=1}^{n} \lambda(A_i) = \bar{\lambda}(A),$$

hence $\kappa = \bar{\lambda}$.

Our version of the Alexandroff theorem will be formulated and proved for a type of semigroups. We shall say that a commutative ordered semigroup G with a neutral element 0 is weakly regular, if for any $a \in G$, $a > 0$ and any double sequence (a_n^i) such that $a_n^i \searrow 0$ $(i \to \infty,\ n = 1, 2, \dots)$ there are $i_1, i_2, \dots$ such that

$$a \le \sum_{j=1}^{n} a_j^{i_j}$$

for no n.

Recall that every commutative lattice ordered group is weakly regular according to [26].

The mapping $\lambda\colon \mathcal{R} \to G$, $\mathcal{R}$ being a ring acting in our version of the Alexandroff theorem will be supposed to be monotone, subadditive and compact. A mapping $\lambda\colon \mathcal{R} \to G$ is called to be monotone, if $A \subset B$, $A, B \in \mathcal{R}$ implies $\lambda(A) \le \lambda(B)$, it is called subadditive, if $\lambda(A \cup B) \le \lambda(A) + \lambda(B)$ for every $A, B \in \mathcal{R}$. A family $\mathcal{C}$ of subsets of a set is said to be compact, if it is closed under finite intersections and every decreasing sequence of non-empty sets of $\mathcal{C}$ has a non-empty intersection. A mapping $\lambda\colon \mathcal{R} \to G$ is said to be compact, if there is a compact family $\mathcal{C} \subset \mathcal{R}$ such that to every $E \in \mathcal{R}$ there are $C_n \in \mathcal{C}$ $(n = 1, 2, \dots)$ such that $C_n \subset C_{n+1} \subset E$ $(n = 1, 2, \dots)$ and $\lambda(E \setminus C_n) \searrow 0$.

Theorem 2. *Let G be a weakly regular, σ-complete, commutative semigroup. Let $\lambda\colon \mathcal{R} \to G$ be monotone, subadditive, compact and $\lambda(\emptyset) = 0$. Then λ is upper continuous in $\emptyset$, i.e., $(A_n) \subset \mathcal{R}$, $A_n \searrow \emptyset$ implies $\lambda(A_n) \searrow 0$.*

Proof. Let $(A_n) \subset \mathcal{R}$, $A_n \searrow \emptyset$. We want to prove that $\lambda(A_n) \searrow 0$. Let us prove it indirectly. Since G is σ-complete, there is $a > 0$ such that $\lambda(A_n) \geq a$ $(n = 1, 2, \dots)$. Since λ is compact, to any n there are $C_n^i \in \mathcal{C}$ $(i = 1, 2, \dots)$ such that $C_n^i \subset C_n^{i+1} \subset A_n$ $(i = 1, 2, \dots)$ and $\lambda(A_n \setminus C_n^i) \searrow 0$ $(i \to \infty)$. Put $a_n^i = \lambda(A_n \setminus C_n^i)$. Choose $i_1, i_2, \dots$ by the weak regularity of G. Now put

$$D_1 = C_1^{i_1},\ D_2 = C_1^{i_1} \cap C_2^{i_2}, \dots, D_n = C_1^{i_1} \cdots \cap C_n^{i_n}, \dots .$$

Then $D_n \in \mathcal{C}$, $D_n \supset D_{n+1}$ $(n = 1, 2, \dots)$. We prove that $D_n \neq \emptyset$ $(n = 1, 2, \dots)$. If $D_n = \emptyset$, then

$$\begin{aligned} a \leq \lambda(A_n) &\leq \lambda\Big(\Big(\bigcup_{j=1}^{n} (A_j \setminus C_j^{i_j})\Big) \cup \Big(\bigcap_{j=1}^{n} C_j^{i_j}\Big)\Big) \leq \\ &\leq \Big(\sum_{j=1}^{n} \lambda(A_j \setminus C_j^{i_j})\Big) + \lambda(D_n) = \sum_{j=1} a_j^{i_j} \end{aligned}$$

which is imposible.

Since $D_n \supset D_{n+1}$, $D_n \in \mathcal{C}$, $D_n \neq \emptyset$ $(n = 1, 2, \dots)$, we have $\bigcap_{n=1}^{\infty} D_n \neq \emptyset$. But $D_n \subset A_n$ $(n = 1, 2, \dots)$, hence $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$, which is a contradiction.

Corollary. *Let G be a σ-complete, commutative ℓ-group, $\mathcal{R}$ be a ring of subsets of a set, $\lambda\colon \mathcal{R} \to G$ a monotone, subadditive, compact mapping such that $\lambda(\emptyset) = 0$. Then λ is upper continuous in $\emptyset$.*

Proof. By [26] every σ-complete, commutative ℓ-group is weakly regular.

2. σ-ADDITIVE GROUP-VALUED MEASURES AND PRODUCT MEASURE THEOREM

Let $\mathcal{R}$ be a ring of subsets of a set. A function $\lambda\colon \mathcal{R} \to G$ is said to be σ-additive, if $\lambda(A) = \bigvee_{n=1}^{\infty} \big(\lambda(A_1) + \dots + \lambda(A_n)\big)$, whenever $A \in \mathcal{R}$, $A_n \in \mathcal{R}$ $(n = 1, 2, \dots)$ $A_i \cap A_j = \emptyset$ $(i \neq j)$, $A = \bigcup_{n=1}^{\infty} A_n$.

Theorem 3. *Let G be a commutative partially ordered group, $\mathcal{R}$ be a ring of subsets of a set, $\lambda\colon \mathcal{R} \to G$ be an additive mapping. Then λ is σ-additive if and only if λ is upper continuous in $\emptyset$.*

Proof. If $A = \bigcup_{n=1}^{\infty} A_n$ and A_n are pairwise disjoint, then $A \setminus \bigcup_{i=1}^{n} A_i \searrow 0$. On the other hand, if $B_n \searrow 0$, then $B_1 = \bigcup_{i=1}^{\infty} (B_i \setminus B_{i+1})$, $(B_i \setminus B_{i+1})_i$ being pairwise disjoint.

Theorem 4. *Let G be a commutative, σ-complete ℓ-group, $\lambda\colon \mathcal{R} \to G$ be an additive non-negative and compact mapping. Then λ is σ-additive.*

Proof. It follows by Theorem 3 and Corollary of Theorem 2.

Further let two rings $\mathcal{R}_1$, resp. $\mathcal{R}_2$ of subsets of X_1, resp. X_2 be given. By $\mathcal{R}_1 \times \mathcal{R}_2$ the family of all sets $A \times B$ $(A \in \mathcal{R}_1, B \in \mathcal{R}_2)$ will be denoted. A function $\lambda\colon \mathcal{R}_1 \times \mathcal{R}_2 \to G$ will be called partially additive, if

$$\lambda(A \cup B, C) = \lambda(A, C) + \lambda(B, C)$$
$$\lambda(D, E \cup F) = \lambda(D, E) + \lambda(D, F)$$

whenever

$$A, B, D \in \mathcal{R}_1, C, E, F \in \mathcal{R}_2,\ A \cap B = \emptyset,\ E \cap F = \emptyset\,.$$

Theorem 5. *Let G be a commutative semigroup, $\lambda\colon \mathcal{R}_1 \times \mathcal{R}_2 \to G$ be a partially additive mapping. Then λ is additive.*

Proof. Let $A \times \mathcal{B} \in \mathcal{R}_1 \times \mathcal{R}_2$, $A_i \times B_i \in \mathcal{R}_1 \times \mathcal{R}_2$ $(i = 1, 2, \dots, n)$, $(A_i \times B_i) \cap (A_j \times B_j) = \emptyset$ $(i \neq j)$ $A \times B = \bigcup_{i=1}^n (A_i \times B_i)$. We want to prove

$$\lambda(A \times B) = \sum_{i=1}^n \lambda(A_i \times B_i)\,.$$

First we construct $C_1, \dots, C_k \in \mathcal{R}_1$ pairwise disjoint and such that

$$A_i = \bigcup_{j=1}^{n_i} C_{i,j} \quad i = 1, 2, \dots, n\,.$$

Similarly construct $D_1, \dots, D_\ell \in \mathcal{R}_2$ pairwise disjoint and such that

$$B_i = \bigcup_{p=1}^{m_i} D_{i,p} \quad i = 1, 2, \dots, n\,.$$

($C_{i,j}$, $D_{i,p}$ denoting C_j, D_p belonging to A_i, B_i, respectively.) Then

$$\begin{aligned}
\lambda(A \times B) &= \lambda\Big(\Big(\bigcup_{j=1}^k C_j\Big) \times B\Big) = \sum_{j=1}^k \lambda(C_j \times B) = \sum_{j=1}^k \lambda\Big(C_j \times \bigcup_{p=1}^\ell D_p\Big) = \\
&= \sum_{j=1}^k \sum_{p=1}^\ell \lambda(C_j \times D_p) = \sum_{i=1}^n \sum_{j=1}^{n_i} \sum_{p=1}^{m_i} \lambda(C_{i,j} \times D_{i,p}) = \\
&= \sum_{i=1}^n \lambda\Big(\bigcup_{j=1}^{n_i} C_{i,j} \times \bigcup_{p=1}^{m_i} D_{i,p}\Big) = \sum_{\ell=1}^k \lambda(A_i \times B_i)\,.
\end{aligned}$$

Now we shall formulate our G–valued product measure theorem. It works under the assumption of weak σ-distributivity [12], [16], [17], [18], [28]. A σ-complete comutative ℓ-group G is said to be weakly σ-distributive, if for every bounded sequence $(a_{ij}) \subset G$ such that $a_{ij} \searrow 0$ $(j \to \infty, i = 1, 2, \dots)$ it is$\bigwedge_{\varphi \in N^N} \bigvee_{i=1}^\infty a_{i\varphi(i)} = 0$.

Theorem 6. *Let G be a commutative, weakly σ-distributive ℓ-group with a partial commutative binary operation $\cdot\colon G^+ \times G^+ \to G^+$ satisfying the distributive law: $a\cdot(b+c) = a\cdot b + a\cdot c$ for every $a,b,c \in G^+ = \{x\in G;\ x\ge 0\}$.*

If $\mu\colon \mathcal{B}(R) \to G$, $\nu\colon \mathcal{B}(R)\to G$ are positive measures (i.e., σ-additive, non-negative and having 0 in $\emptyset$), then there exists exactly one measure $\bar\lambda\colon \mathcal{B}(R^2)\to G$ such that $\bar\lambda(A\times B) = \mu(A)\cdot\nu(B)$ for every $A\in\mathcal{B}(R)$, $B\in\mathcal{B}(R)$.

Proof. Define first on $\mathcal{D} = \{\langle a,b)\times\langle c,d);\ a,b,c,d\in R\}$ the function $\lambda\colon \mathcal{D}\to G$ by the formula

$$\lambda(A\times B) = \mu(A)\cdot\nu(B)\,.$$

Evidently λ is partially additive, hence additive by Theorem 5. It can be uniquelly extended to an additive mapping $\bar\lambda\colon s(\mathcal{D})\to G$ defined by

$$\bar\lambda\left(\bigcup_{i=1}^{n}(A_i\times B_i)\right) = \sum_{i=1}^{n}\mu(A_i)\nu(B_i)\,,$$

$(A_i\times B_i)\cap(A_j\times B_j) = \emptyset$ $(i\ne j)$. The function $\bar\lambda$ is compact with respect to $\mathcal{D}_0 = \{C\subset R^2,\ C \text{ compact}\}$. The function $\bar\lambda$ is σ-additive by Theorem 4, hence $\bar\lambda\colon s(\mathcal{D})\to G$ is a measure. By [22] $\bar\lambda$ can be uniquely extended to a measure on the generated σ-algebra $\sigma\big(s(\mathcal{D})\big) = \mathcal{B}(R^2)$.

In the group-valued case we are able to prove a variant of the Alexandroff theorem for a semiring $\mathcal{D}$ instead of a ring $\mathcal{R}$.

Lemma 1. *Let $\mathcal{D}$ be a semiring of subsets of a set, G be a commutative ℓ-group, $\lambda\colon\mathcal{D}\to G$ be a non-negative, additive mapping. Then the following properties hold:*

(i) $\lambda(\emptyset) = 0$.

(ii) *If $A\subset B$, $A,B\in\mathcal{D}$, then $\lambda(A)\le\lambda(B)$.*

(iii) *If $A,B,C\in\mathcal{D}$, $B\cup C\subset A$, then $\lambda(B)+\lambda(C)\le\lambda(A)+\lambda(B\cap C)$.*

Proof. Since $\lambda(\emptyset) = \lambda(\emptyset\cup\emptyset) = \lambda(\emptyset)+\lambda(\emptyset)$, we obtain $\lambda(\emptyset)=0$. If $A\subset B$, $A,B\in\mathcal{D}$, then there are $A_1,\dots,A_n\in\mathcal{D}$ pairwise disjoint, $A_i\subset B\setminus A$ such that $B = A\cup A_1\cup\dots\cup A_n$.
Therefore

$$\lambda(B) = \lambda(A)+\lambda(A_1)+\dots+\lambda(A_n)\ge\lambda(A)\,.$$

Finally, if $A,B,C\in\mathcal{D}$, $B\cup C\subset A$, then we construct $A_1,\dots,A_n\subset A\setminus(B\cap C)$ pairwise disjoint such that $A = (B\cap C)\cup\bigcup_{i=1}^n A_i$, $B_1,\dots,B_m\subset A\setminus B$ such that $A = B\cup\bigcup_{i=1}^m B_i$ and $C_1,\dots,C_p\subset A\setminus C$ such that $A = C\cup\bigcup_{i=1}^p C_i$. Using all intersections $A_i\cap B_j\cap C_k$ we obtain a finite collection $D_1,D_2,\dots,D_\ell$ of pairwise disjoint sets of $\mathcal{D}$ such that $D_i\subset A\setminus(B\cap C)$ and

$$A\setminus(B\cap C) = \bigcup_{i=1}^{\ell}D_i,\ \ B\setminus B\cap C = \bigcup_{i=1}^{q}D_{n_i},\ \ C\setminus B\cap C = \bigcup_{i=1}^{r}D_{p_i}\,.$$

Then

$$\lambda(B)+\lambda(C)=\lambda(B\cap C)+\sum_{i=1}^{q}\lambda(D_{n_i})+\lambda(B\cap C)+\sum_{i=1}^{r}\lambda(D_{p_i})\le$$
$$\le\lambda(B\cap C)+\sum_{i=1}^{\ell}\lambda(D_i)=\lambda(B\cap C)+\lambda(A).$$

Lemma 2. *Let $\mathcal{D}$ be a family of sets closed under intersections. Let $\lambda(\emptyset)=0$ and $\lambda(E)\le\lambda(F)$, whenever $E,F\in\mathcal{D}$, $E\subset F$ and*

$$\lambda(B)+\lambda(C)\le\lambda(B\cap C)+\lambda(A),$$

whenever $A,B,C\in\mathcal{D}$, $B\cup C\subset A$. Let $A_1,\dots,A_n\in\mathcal{D}$, $C_1,\dots,C_n\in\mathcal{D}$, $A_1\supset A_2\supset\dots\supset A_n$, $C_i\subset A_i$ $(i=1,\dots,n)$. Then

$$\sum_{i=1}^{n}\lambda(C_i)\le\sum_{i=1}^{n-1}\lambda(A_i)+\lambda(C_1\cap C_2\cap\dots\cap C_n).$$

Proof. We shall prove it by induction. The first step is a consequence of the first assumption. Let the assertion hold for some n and let $C_1,\dots,C_n$, $C_{n+1}\in\mathcal{D}$, $A_1\supset A_2\supset\dots\supset A_n\supset A_{n+1}$, $C_i\subset A_i$ $(i=1,\dots,n+1)$. Then

$$\sum_{i=1}^{n+1}\lambda(C_i)=\sum_{i=1}^{n}\lambda(C_i)+\lambda(C_{n+1})\le$$
$$\le\sum_{i=1}^{n-1}\lambda(A_i)+\lambda(C_1\cap\dots\cap C_n)+\lambda(C_{n+1}).$$

Since $(C_1\cap\dots\cap C_n)\cup C_{n+1}\subset A_n\cup A_{n+1}=A_n$, we obtain

$$\sum_{i=1}^{n+1}\lambda(C_i)\le\sum_{i=1}^{n-1}\lambda(A_i)+\lambda(C_1\cap\dots\cap C_n\cap C_{n+1})+\lambda(A_n)=$$
$$=\sum_{i=1}^{n}\lambda(A_i)+\lambda(C_1,\cap\dots\cap C_{n+1}).$$

Theorem 7. *Let G be a σ-complete, commutative ℓ-group. Let $\mathcal{D}$ be a family of subsets of a set, closed under intersections and containing $\emptyset$, $\lambda\colon\mathcal{D}\to G$ be a mapping satisfying the following conditions:*

(i) $\lambda(\emptyset)=0$.

(ii) *If $A,B\in\mathcal{D}$, $A\subset B$, then $\lambda(A)\le\lambda(B)$.*

(iii) *If $A,B,C\in\mathcal{D}$, $B\cup C\subset A$, then $\lambda(B)+\lambda(C)\le\lambda(B\cap C)+\lambda(A)$.*

(iv) *There is a compact family $\mathcal{C}$ such that to every $E\in\mathcal{D}$ there exists a sequence $(C_n)\subset\mathcal{C}$ such that $C_n\subset C_{n+1}\subset E$ $(n=1,2,\dots)$ and $\lambda(C_n)\nearrow\lambda(E)$.*

Then λ is upper continuous in $\emptyset$, i.e. $(A_n) \subset \mathcal{D}$, $A_n \searrow \emptyset$ implies $\lambda(A_n) \searrow 0$.

Proof. Let $(A_n) \subset \mathcal{D}$, $A_n \searrow \emptyset$. We want to prove that $\lambda(A_n) \searrow 0$. We shall show it indirectly. Since G is σ-complete, there is $a > 0$ such that $\lambda(A_n) \geq a$ $(n = 1, 2, \dots)$. By the condition (iv) to any n there are $C_n^i \in \mathcal{C}$ $(i = 1, 2, \dots)$ such that $C_n^i \subset C_n^{i+1} \subset A_n$ $(i = 1, 2, \dots)$ and $\lambda(A_n) - \lambda(C_n^i) \searrow 0$ $(i \to \infty)$. Put $a_n^i = \lambda(A_n) - \lambda(C_n^i)$. Choose $i_1, i_2, \dots$ by the weak regularity of G ([25]) and put $D_n = \bigcap_{j=1}^n C_j^{i_j}$. Then $D_n \in \mathcal{C}$, $D_n \supset D_{n+1}$ $(n = 1, 2, \dots)$ and by Lemma 2

$$\sum_{j=1}^{n} \lambda(C_j^{i_j}) \leq \sum_{i=1}^{n-1} \lambda(A_i) + \lambda(D_n).$$

We want to prove that $D_n \neq \emptyset$ $(n = 1, 2, \dots)$.

If $D_n = \emptyset$ for some n, then

$$a \leq \lambda(A_n) \leq \sum_{i=1}^{n} \lambda(A_i) - \sum_{j=1}^{n} \lambda(C_j^{i_j}) = \sum_{j=1}^{n} a_j^{i_j}$$

what is in a contradiction with the weak regularity of G.

Since $D_n \supset D_{n+1}$, $D_n \in \mathcal{C}$, $D_n \neq \emptyset$ $(n = 1, 2, \dots)$, we have $\bigcap_{n=1}^{\infty} D_n \neq \emptyset$. But $D_n \subset A_n$ $(n = 1, 2, \dots)$, hence $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$, which is a contradiction.

3. Joint observables for some types of MV algebras

If instead of a σ-algebra (in the Kolmogorov probability theory) we consider a set $\mathcal{F} \subset \langle 0, 1\rangle^{\Omega}$ of fuzzy subsets of a set Ω, we obtain a new model for the quantum theory [1], [2], [3], [8], [9], [10], [11], [13], [14], [15], [17], [18], [19], [20], [23], [24]. The basic idea is in the using of the operation $f \oplus g = \min(f + g, 1)$. This algebraic structure is an example of MV algebra. On the other hand, every MV algebra can be represented by a set $\langle 0, u\rangle^{\Omega}$, where $\langle 0, u\rangle$ is an interval in an ℓ-group G. So we are able to construct a convenient theory for a special case of MV algebras (with weakly σ-distributive G having a distributive multiplication). Recall that a theory for observables for weakly σ-distributive ℓ-groups G was constructed in [14].

Let G be a commutative ℓ-group, $u \in G$, $u > 0$, $\mathcal{F} = \langle 0, u\rangle^{\Omega}$. An observable is a mapping $x\colon \mathcal{B}(R) \to \mathcal{F}$ satisfying the following conditions:

(a) $x(R) = u_{\Omega}$.

(b) If $A, B \in \mathcal{B}(R)$, $A \cap B = \emptyset$, then $x(A \cup B) = x(A) + x(B)$.

(c) If $A_n \in \mathcal{B}(R)$ $(n = 1, 2, \dots)$, $A_n \nearrow A$, then $x(A_n) \nearrow x(A)$.

If there is given a commutative binary operation $\cdot$ on $\langle 0, u\rangle$, we can define the joint observable of two observables $x, y\colon \mathcal{B}(R) \to \mathcal{F}$ as a mapping $h\colon \mathcal{B}(R^2) \to \mathcal{F}$ satisfying the following conditions:

(i) $h(R^2) = u_{\Omega}$.

(ii) If $A, B \in \mathcal{B}(R^2)$, $A \cap B = \emptyset$, then $h(A \cup B) = h(A) + h(B)$.

(iii) If $A_n \in \mathcal{B}(R^2)$ $(n = 1, 2, \dots)$, $A_n \nearrow A$, then $h(A_n) \nearrow h(A)$.

(iv) If $A, B \in \mathcal{B}(R)$, then

$$h(A \times B) = x(A) \cdot y(B).$$

Theorem 8. *Let G be a commutative, weakly σ-distributive ℓ-group with a partial commutative binary operation $\cdot\colon G^+ \times G^+ \to G^+$ satisfying the distributive law. Let $a, b,\ c \in G^+$, $a \le b$ imply $a \cdot c \le b \cdot c$. Let $u \in G$, $u > 0$ be such an element that $u \cdot u = u$. Let $x, y\colon \mathcal{B}(R) \to \langle 0, u\rangle^\Omega$ be observables. Then there exists the joint observable of x, y.*

Proof. For fixed $\omega \in \Omega$, $A, B \in \mathcal{B}(R)$, put $\mu_\omega(A) = x(A)(\omega)$, $\nu_\omega(B) = y(B)(\omega)$. Evidently μ_ω and ν_ω are measures. Let $\bar\lambda$ be the product measure constructed in Theorem 6. For $C \in \mathcal{B}(R^2)$, $\omega \in \Omega$ put

$$h(C)(\omega) = \bar\lambda(C)\,.$$

Then $h\colon \mathcal{B}(R^2) \to \mathcal{F}$. We shall prove that h has the properties (i)–(iv). First

$$h(R^2)(\omega) = \bar\lambda(R^2) = \lambda(R \times R) = \mu_\omega(R) \cdot \nu_\omega(R) = u \cdot u = u$$

for every $\omega \in \Omega$. Therefore $h(R^2) = u_\Omega$. If $C, D \in \mathcal{B}(R^2)$, $C \cap D = \emptyset$ and $\omega \in \Omega$, then

$$\begin{aligned} h(C \cup D)(\omega) &= \bar\lambda(C \cup D) = \bar\lambda(C) + \bar\lambda(D) = \\ &= h(C)(\omega) + h(D)(\omega) \end{aligned}$$

for every $\omega \in \Omega$, hence $h(C \cup D) = h(C) + h(D)$. Similarly the property (iii) can be proved. Finally, for any $A, B \in \mathcal{B}(R)$ and any $\omega \in \Omega$

$$h(A \times B)(\omega) = \mu_\omega(A) \cdot \nu_\omega(B) = x(A)(\omega) \cdot y(B)(\omega)\,,$$

hence $h(A \times B) = x(A) \cdot y(B)$.

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