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FUZZY SET REPRESENTATIONS OF SOME QUANTUM STRUCTURES

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Abstract. Quantum structures are algebraic structures like quantum logics (orthomodular posets and lattices), orthoalgebras, effect algebras (D-posets) which model event structures of quantum mechanics. The elements of orthomodular posets can be represented by question observables having their spectrum in $\{0, 1\}$, so they are considered as two-valued physical quantities. We show that if such a structure possesses an order determining system of states (= probability measures), it can be represented by a system of fuzzy sets and conversely.

1. Introduction

Birkhoff and von Neumann [BiNe] used the $\{0, 1\}$-analogy to justify the interpretation of projections of a Hilbert space as the "experimental propositions" pertaining to a quantum physical system. In orthodox Hilbert space quantum mechanics, projection operators and self-adjoint operators represent events and observables, respectively. Since projections have the spectrum in $\{0, 1\}$, they are assumed as $\{0, 1\}$-physical quantities.

One of standard algebraic models of the event structure of quantum mechanics are orthomodular posets (or orthomodular lattices or quantum logics). This structure can appear in view of a variation of axioms presented originally by Mackey [Mac] (see also [PtPu]).

Let $\mathcal{B}(\mathbb{R})$ be the set of all Borel subsets of the real line $\mathbb{R}$. We suppose that we are given two abstract sets $\mathcal{O}$ (set of all observables) and $\mathcal{S}$ (set of all states) and a function $p : \mathcal{O} \times \mathcal{S} \times \mathcal{B}(\mathbb{R}) \to [0, 1]$. We interpret $p(A, m, E)$ as the probability that a measurement in the state m ($m \in \mathcal{S}$) of the observable A ($A \in \mathcal{O}$) will lead to a value in $E \in \mathcal{B}(\mathbb{R})$. We assume that p satisfies the following axioms:

(I) For any $A \in \mathcal{O}$ and any $m \in \mathcal{S}$,

$$P(A, m, \mathbb{R}) = 1.$$

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(II) For any $A \in \mathcal{O}$ and any $m \in \mathcal{S}$,

$$p(A, m, E \cup F) = p(A, m, E) + p(A, m, F)$$

whenever $E, F \in \mathcal{B}(\mathbb{R}), E \cap F = \emptyset$.

We define an equivalence relation $\sim_{\mathcal{S}}$ on $\mathcal{O} \times \mathcal{B}(\mathbb{R})$ via $(A, E) \sim_{\mathcal{S}} (B, F)$ iff $p(A, m, E) = p(B, m, F)$ for any $m \in \mathcal{S}$ and put

$$L = \{|A, E| : A \in \mathcal{O}, E \in \mathcal{B}(\mathbb{R})\},$$

where $|A, E| = \{(B, F) \in \mathcal{O} \times \mathcal{B}(\mathbb{R}) : (B, F) \sim_{\mathcal{S}} (A, E)\}$. The set L is said to be the event structure (or a propositional logic) and the elements of L are called the events. If $a = |A, E| \in L$ and $m \in \mathcal{S}$, we define two functions $\bar{a} : \mathcal{S} \to [0, 1]$ and $m : L \to [0, 1]$ via

$$\bar{a}(m) := p(A, m, E) =: m(a). \tag{1.1}$$

The set $\mathcal{S}$ defines on L a partial ordering $\le$ via $a \le b$, $a, b \in L$ iff $m(a) \le m(b)$ for any $m \in \mathcal{S}$ with the least and last elements $0 := |A, \emptyset|$ and $1 := |A, \mathbb{R}|$, and a unary operation $^{\perp} : L \to L$, called orthocomplementation, defined via $|A, E|^{\perp} := |A, \mathbb{R} \setminus E|$. Then

(OMi) $(a^{\perp})^{\perp} = a$ for any $a \in L$;

(OMii) if $a \le b$, then $b^{\perp} \le a^{\perp}$, $a, b \in L$.

We say that two elements $a, b \in L$ are orthogonal, and we write $a \perp b$, iff

$$\bar{a}(m) + \bar{b}(m) \le 1, \ m \in \mathcal{S}.$$

We now assume the following orthogonality axiom:

(III) If, for $a, b, c \in L$, $a \perp b, b \perp c, a \perp c$, then there exists $d \in L$ such that

$$\bar{d}(m) + \bar{a}(m) + \bar{b}(m) + \bar{c}(m) = 1, \ m \in \mathcal{S}.$$

From this axiom we obtain that

(OMiii) $a \vee a^{\perp} = 1$ for any $a \in L$.

Indeed, if there is an element $b \in L$ such that $a, a^{\perp} \le b$, then $a \perp a^{\perp}$, $a \perp b^{\perp}$, $a^{\perp} \le b^{\perp}$, and by (III), there is an element $d \in L$ such that

$$\bar{d}(m) + \bar{a}(m) + \overline{a^{\perp}}(m) + \overline{b^{\perp}}(m) = 1, \ m \in \mathcal{S},$$

which implies $\bar{d}(m) = \overline{b^{\perp}}(m) = 0$, so that $b = 1$. In addition, if $a, b, \in L$ and $a \perp b$, then

(iv) $a \vee b \in L$.

This can be proved as follows: Since $a \perp 0$, $b \perp 0$, $a \perp b$, there is an element $d^{\perp} \in L$ such that

$$\overline{d^{\perp}}(m) + \bar{a}(m) + \bar{b}(m) + \bar{0}(m) = 1. \tag{1.2}$$

Hence $a, b \leq d$, and we assert that $a \vee b = d$. Indeed, if $a, b, \leq c$, $c \in L$, then $a \perp b, a \perp c^\perp, b \perp c^\perp$ and by (III), there is an element $e \in L$ such that

$$\bar{e}(m) + \bar{a}(m) + \bar{b}(m) + \overline{c^\perp}(m) = 1,\ m \in \mathcal{S},$$

which yields together with (1.2)

$$\bar{a}(m) + \bar{b}(m) = \bar{d}(m) = \bar{c}(m) - \bar{e}(m) \leq \bar{c}(m),$$

so that $d \leq c$.

Moreover, the orthomodular law

(OMv) if $a \leq b$, $a, b \in L$, then $b = a \vee (b^\perp \vee a)^\perp = a \vee (b \wedge a^\perp)$

holds.

The structure L with a partial order $\leq$, with the least and last elements 0 and 1 and with a unary operation $^\perp : L \to L$ satisfying (OMi)–(OMv) is said to be an *orthomodular poset*. Our L possesses an ordering system of states $\mathcal{S}$.

As we have seen, any element $a \in L$ defines a mapping $\bar{a} : \mathcal{S} \to [0, 1]$ via $\bar{a}(m) = m(a)$, $m \in \mathcal{S}$. We see that a propositional system satisfying physically plausible axioms (I)–(III), which are a simple variation of Mackey's axiomatical system [Mac], entails its representation by [0, 1]-valued functions. Mączyński [Mac a-b] was the first who represented any orthomodular poset (orthomodular lattice, orthomodular σ-lattice, quantum logic) admitting an order determining system of states via functions having values in the interval [0, 1]. We recall that if we define a general orthomodular poset $L = (L, \leq, ^\perp, 0, 1)$ satisfying (OMi)–(OMv), there are examples of stateless orthomodular posets (see [PtPu]).

Today any function $f \in [0, 1]^X$, where $X \neq \emptyset$, is said to be a *fuzzy set*. Pykacz [Pyk a,b] in a series of articles realized that Mączyński's result gives a representation of orthomodular posets via a system of fuzzy sets, and he showed that any attempt to use Łukasiewicz connectives to obtain a quantum logic leads to Mączyński's result. Mesiar [Mes a] obtained a similar result using connectives generated by t-norms.

In the present paper, we give characterizations of orthomodular posets, orthomodular posets, orthoalgebras, effect algebras and MV-algebras via a system of fuzzy sets giving an analogue of Mączyński's result. In addition, we give a negative answer to Mesiar's question [Mes b].

2. Fuzzy Set Representation of Effect Algebras

Recently there appeared a new axiomatical system of quantum structures, D-posets, introduced by Kôpka and Chovanec [KoCh]. This most general structure possesses orthomodular posets and lattices as well as orthoalgebras. It combines both algebraic and fuzzy ideas. An equivalent form is an effect algebra.

An *effect algebra* is a non-empty set L with two particular elements 0, 1 ($0 \neq 1$), and with a partial binary operation $\oplus : L \times L \to L$ such that for all $a, b, c \in L$ we have

(EAi) if $a \oplus b \in L$, then $b \oplus a \in L$ and $a \oplus b = b \oplus a$ (commutativity);

(EAii) if $b \oplus c \in L$ and $a \oplus (b \oplus c) \in L$, then $a \oplus b \in L$ and $(a \oplus b) \oplus c \in L$, and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$ (associativity);

(EAiii) for any $a \in L$ there is a unique $b \in L$ such that $a \oplus b$ is defined, and $a \oplus b = 1$ (orthocomplementation);

(EAiv) if $1 \oplus a$ is defined, then $a = 0$ (zero-one law).

Let a and b be two elements of an effect algebra L. We say that (i) a is *orthogonal* to b and write $a \perp b$ iff $a \oplus b$ is defined in L; (ii) a is *less than or equal to* b and write $a \le b$ iff there exists an element $c \in L$ such that $a \perp c$ and $a \oplus c = b$ (in this case we also write $b \ge a$); (iii) b is the *orthocomplement* of a iff b is a (unique) element of L such that $b \perp a$ and $a \oplus b = 1$ and it is written as $a^\perp$.

We shall write $L = (L, \oplus, 0, 1,)$ for an effect algebra.

It is well-known that any Boolean algebra, orthomodular lattice, and orthomodular poset can be organized into an effect algebra supposing that $a \oplus b$ is defined iff $a \perp b$ and then $a \oplus b := a \vee b$.

An *atom* of L is a non-zero element $a \in L$ such that if $b \le a$ for $b \in L$, then either $b = a$ or $b = 0$.

Let L be an effect algebra. Let $F = \{a_1, \ldots, a_n\}$ be a finite sequence in L. Recursively we define for $n \ge 3$

$$a_1 \oplus \cdots \oplus a_n := (a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n, \tag{2.1}$$

supposing that $a_1 \oplus \cdots \oplus a_{n-1}$ and $(a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n$ exist in L. From the associativity of $\oplus$ in effect algebras we conclude that (2.1) is correctly defined. By definition we put $a_1 \oplus \cdots \oplus a_n = a_1$ if $n = 1$, and $a_1 \oplus \cdots \oplus a_n = 0$ if $n = 0$. Then for any permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$ and any k with $1 \le k \le n$ we have

$$a_1 \oplus \cdots \oplus a_n = a_{i_1} \oplus \cdots \oplus a_{i_n}, \tag{2.2}$$

$$a_1 \oplus \cdots \oplus a_n = (a_1 \oplus \cdots \oplus a_k) \oplus (a_{k+1} \oplus \cdots \oplus a_n). \tag{2.3}$$

We say that a finite sequence $F = \{a_1, \ldots, a_n\}$ in L is $\bigoplus$-*orthogonal* if $a_1 \oplus \cdots \oplus a_n$ exists in L. In this case we say that F has a $\bigoplus$-*sum*, $\bigoplus_{i=1}^n a_i$, defined via

$$\bigoplus_{i=1}^n a_i = a_1 \oplus \cdots \oplus a_n. \tag{2.4}$$

It is clear that two elements a and b of L are orthogonal, i.e. $a \perp b$, iff $\{a, b\}$ is $\bigoplus$-orthogonal.

An arbitrary system $G = \{a_i\}_{i \in I}$ of not necessarily different elements of L is $\bigoplus$-*orthogonal* iff, for every finite subset F of I, the system $\{a_i\}_{i \in F}$ is $\bigoplus$-orthogonal. If $G = \{a_i\}_{i \in I}$ is $\bigoplus$-orthogonal, so is any $\{a_i\}_{i \in J}$ for any $J \subseteq I$. An $\bigoplus$-orthogonal system $G = \{a_i\}_{i \in I}$ of L has a $\bigoplus$-*sum* in L, written as $\bigoplus_{i \in I} a_i$, iff in L there exists the join

$$\bigoplus_{i \in J} a_i := \bigvee_J \bigoplus_{i \in J} a_i, \tag{2.5}$$

where J runs over all finite subsets in I. In this case, we also write $\bigoplus G := \bigoplus_{i \in I} a_i$.

It is evident that if $G = \{a_1, \ldots, a_n\}$ is $\bigoplus$-orthogonal, then the $\bigoplus$-sums defined by (2.4) and (2.5) coincide.

We say that an effect algebra L is a *σ-effect algebra* (*complete effect algebra*) if $\bigoplus_{i\in I} a_i$ belongs to L for any countable (arbitrary) system $\{a_i : i \in I\}$ of $\bigoplus$-orthogonal elements from L.

Two prototypes of effect algebras are the following two examples.

Example 2.1. The set $\mathcal{E}(H)$ of all Hermitian operators A on H such that $O \le A \le I$, where I is the identity operator on H, is an effect algebra (which is not an orthoalgebra, for definition see bellow); a partial ordering $\le$ is defined via $A \le B$ iff $(Ax, x) \le (Bx, x)$, $x \in H$, and $C = A \oplus B$, $C \in \mathcal{E}(H)$, iff $(Ax, x) + (Bx, x) = (Cx, x)$, $x \in H$. A system $\{A_i : i \in I\}$ from $\mathcal{E}(H)$ is $\bigoplus$-orthogonal iff $\sum_{i\in I} A_i \in \mathcal{E}(H)$ (where the convergence in the summation is e.g. weak or strong) and then $\bigoplus_{i\in I} A_i = \sum_{i\in I} A_i$. $\mathcal{E}(H)$ is a complete effect algebra which is not a lattice.

Example 2.2. Let the closed interval $[0, 1]$ be ordered by the natural way, and, for two numbers $a, b \in [0, 1]$, we define $a \oplus b$ iff $a + b \le 1$ and we put then $a \oplus b = a + b$. Then $[0, 1]$ is a totally ordered distributive lattice in that any $\bigoplus$-orthogonal system has the sum in it. We recall that $\{a_s\}$ is $\bigoplus$-orthogonal iff $\{a_s\}$ is summable and $\sum_s a_s \le 1$; then $\bigoplus_s a_s = \sum_s a_s$.

A real valued mapping m on an orthoalgebra L is said to be a *state* if (i) $m(1) = 1$, and (ii) $m(a \oplus b) = m(a) + m(b)$, $a, b \in L$. It is clear that $m(0) = 0$.

If for a state $m : L \to [0, 1]$ we have

$$m(\bigoplus_{i\in I} a_i) = \sum_{i\in I} m(a_i), \tag{2.6}$$

whenever $\bigoplus_{i\in I} a_i$ exists in L, m is said to be a *σ-additive* or *completely additive state* if (2.6) holds for any countable or any index set I, respectively.

A non-void system of states $\mathcal{S}$ on L is said to be *order determining* if, for $a, b \in L$, $a \le b$ iff $m(a) \le m(b)$ for any $m \in \mathcal{S}$.

Let X be a non-empty set. By 0_X and 1_X we understand the constant functions $0_X(x) = 0$ and $1_X(x) = 1$ for each $x \in X$. For two fuzzy sets f and g of X we write $f \le g$ iff $f(x) \le g(x)$ for any $x \in X$. Similarly $f + g$ and $f - g$ are defined pointwisely.

We say that two effect algebras L and P are *isomorphic* if there exists a one-to-one mapping ϕ from L onto P such that $\phi(1) = 1$, and $a \oplus b \in L$ iff $\phi(a) \oplus \phi(b) \in P$ and $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$. If L and P are σ-effect algebras (complete effect algebras) we assume in addition for any isomorphism ϕ that it preserves all sums of all countable (arbitrary) $\bigoplus$-orthogonal systems in L.

Theorem 2.3. *Let a system of fuzzy sets $L \subseteq [0, 1]^X$, $X \ne \emptyset$, satisfy the following conditions:*

(i) $1_X \in L$;
(ii) $1_X - f \in L$ *whenever* $f \in L$;
(iii) $f + g \in L$ *whenever* $f + g \le 1_X$.

Then $L = (L, \oplus, 0_X, 1_X,)$, where $f \oplus g$ is defined if and only if $f + g \le 1_X$, $(f, g \in L)$ and we put $f \oplus g = f + g$, is an effect algebra. In addition, the system $\mathcal{S} = \{m_x : x \in X\}$, where $m_x : L \to [0, 1]$ is defined via $m_x(f) := f(x)$, $f \in L$, is an order determining system of states.

Conversely, let $L = (L, \oplus, 0, 1)$ *be an arbitrary effect algebra with an order determining system of states* $\mathcal{S}$. *Then* L *is isomorphic with the system of fuzzy sets* $\bar{L} \subseteq [0,\,1]^{\mathcal{S}}$, *where* $\bar{L} = \{\bar{a} \in [0,\,1]^{\mathcal{S}} : \bar{a}(m) := m(a),\ m \in \mathcal{S},\ a \in L\}$, *and* $\bar{L}$ *satisfies the conditions* (i)–(iii).

Proof. An easy calculation shows that the partial binary operation $\oplus$ defined in Theorem satisfies all conditions to be $(L, \oplus, 0_X, 1_X)$ an effect algebra. We recall that that $f^{\perp} := 1_X - f,\ f \in L$. It is evident that $\{m_x : x \in X\}$ is an order determining system of states on L.

Conversely, let $L = (L, \oplus, 0, 1)$ be an effect algebra with an order determining system of states, and define $\bar{L}$ and $\bar{a}$. Then $\bar{1} \in \bar{L}$, and if $a \in L$, we define $\overline{a^{\perp}} = \bar{1} - \bar{a}$. Since $\bar{a} + \bar{b} \leq \bar{1}$ iff $m(a) + m(b^{\perp}) \leq 1$, we have that this is equivalent to $a \leq b^{\perp}$ so that $a \oplus b$ is defined in L. Then $\overline{a \oplus b} = \bar{a} + \bar{b} \in \bar{L}$.

The mapping $a \mapsto \bar{a}$ is an isomorphism between $(L, \oplus, 0, 1)$ and $(\bar{L}, \oplus, \bar{0}, \bar{1})$. $\square$

The proofs of the following two theorems are similar to the proof of Theorem 2.3.

Theorem 2.4. *Let a system of fuzzy sets* $L \subseteq [0,\,1]^X$, $X \neq \emptyset$, *satisfy the conditions* (i)–(iii) *of Theorem* 2.3 *and*

(iv) *if* $f_1 \leq f_2 \leq \cdots,\ f_i \in L,\ i \geq 1$, *then* $\sup_i f_i \in L$.

Then $L = (L, \oplus, 0_X, 1_X,)$, *where* $\oplus$ *is defined as that in Theorem* 2.3, *is a* σ*-effect algebra. In addition, the system* $\mathcal{S} = \{m_x : x \in X\}$, *where* $m_x : L \to [0,\,1]$ *is defined via* $m_x(f) := f(x),\ f \in L$, *is an order determining system of* σ*-additive states.*

Conversely, let $L = (L, \oplus, 0, 1)$ *be an arbitrary* σ*-effect algebra with an order determining system of* σ*-additive states* $\mathcal{S}$. *Then* L *is isomorphic with the system of fuzzy sets* $\bar{L} \subseteq [0,\,1]^{\mathcal{S}}$, *where* $\bar{L} = \{\bar{a} \in [0,\,1]^{\mathcal{S}} : \bar{a}(m) := m(a),\ m \in \mathcal{S},\ a \in L\}$, *and* $\bar{L}$ *satisfies the conditions* (i)–(iv).

Theorem 2.5. *Let a system of fuzzy sets* $L \subseteq [0,\,1]^X$, $X \neq \emptyset$, *satisfy the conditions* (i)–(iii) *of Theorem* 2.3 *and*

(iv^*) *if, for a system of fuzzy sets* $\{f_i : i \in J\}$ *from* L *with* $\sum_{i \in J} f_i \in L$ *for any finite subset* J *of* I, *then* $\sum_{i \in I} f_i \in L$.

Then $L = (L, \oplus, 0_X, 1_X,)$, *where* $\oplus$ *is defined as that in Theorem* 2.3, *is a complete effect algebra. In addition, the system* $\mathcal{S} = \{m_x : x \in X\}$, *where* $m_x : L \to [0,\,1]$ *is defined via* $m_x(f) := f(x),\ f \in L$, *is an order determining system of completely additive states.*

Conversely, let $L = (L, \oplus, 0, 1)$ *be an arbitrary complete effect algebra with an order determining system of completely additive states* $\mathcal{S}$. *Then* L *is isomorphic with the system of fuzzy sets* $\bar{L} \subseteq [0,\,1]^{\mathcal{S}}$, *where* $\bar{L} = \{\bar{a} \in [0,\,1]^{\mathcal{S}} : \bar{a}(m) := m(a),\ m \in \mathcal{S},\ a \in L\}$, *and* $\bar{L}$ *satisfies the conditions* (i)–(iv^*).

Example 2.6. Let $\mathcal{E}(H)$ be that from Example 2.1. Define $X(H) := \{x \in H : ||x|| = 1\}$ and put $\mathcal{S}(H) = \{\mu_x : x \in X(H)\}$, where $\mu_x(A) := (Ax, x),\ A \in \mathcal{E}(H)$. Then $\mathcal{S}(H)$ is an order determining system of completely additive states on $\mathcal{E}(H)$ and its fuzzy representation is $\overline{\mathcal{E}}(H) := \{\bar{A} \in [0,\,1]^{X(H)} : \bar{A}(x) := (Ax, x),\ x \in X(H)\}$.

Example 2.7. The effect algebra $L = [0,\ 1]$ from Example 2.2 has a unique state, namely $m(t) := t,\ t \in [0,\ 1]$. The singleton consisting of this state is an order determining system of states. The effect algebra $L = [0,\ 1]$ has itself as a fuzzy set representation in view of Theorems 2.3 and 2.5.

3. Fuzzy Set Representation of Orthoalgebras

A smaller class of effect algebras are orthoalgebras introduced by Foulis and Randall [FoRa]. We recall that an *orthoalgebra* is a set L with two particular elements 0, 1 ($0 \neq 1$), and with a partial binary operation $\oplus :\ L \times L \to L$ such that for all $a, b, c \in L$ we have

(OAi) if $a \oplus b \in L$, then $b \oplus a \in L$ and $a \oplus b = b \oplus a$ (commutativity);
(OAii) if $b \oplus c \in L$ and $a \oplus (b \oplus c) \in L$, then $a \oplus b \in L$ and $(a \oplus b) \oplus c \in L$, and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$ (associativity);
(OAiii) for any $a \in L$ there is a unique $b \in L$ such that $a \oplus b$ is defined, and $a \oplus b = 1$ (orthocomplementation);
(OAiv) if $a \oplus a$ is defined, then $a = 0$ (consistency).

If the assumptions of (OAii) are satisfied, we write $a \oplus b \oplus c$ for the element $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ in L. Similarly as for effect algebras we introduce a partial order $\leq$ on L, and we write $a \perp b$ iff $a \oplus b$ exists in L. A state and an ordering system of states on orthoalgebras are defined as these for effect algebras as well as a σ-orthoalgebra and a complete orthoalgebra.

Example 3.1. (Firefly in a box) According to Cohen [Coh], consider a firefly in a three–chamber box pictured in Figure 1. The firefly is free to roam among the three chambers and to light up to will. The sides of the box are windows with vertical lines down their centers. We make three experiments, A, B, C, –three windows. For each experiment E we record l_E, r_E, n_E if we see, respectively, a light to the left, right, of the center line or no light. It is clear that we can identify $r_A = l_C =: a$, $r_C = l_B =: c$, $r_A = l_B =: e$ but do not identify $f := n_A$, $b := n_B$, $d := n_C$.

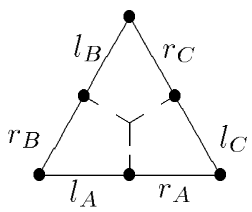

Fig. 1.

The propositional logic of this model has the corresponding Hasse and Greechie diagrams given by Figure 2 and Figure 3, which is an orthoalgebra, called the Wright triangle, being no orthomodular posets. It is the most simple case of an orthoalgebra which is not an orthomodular poset.

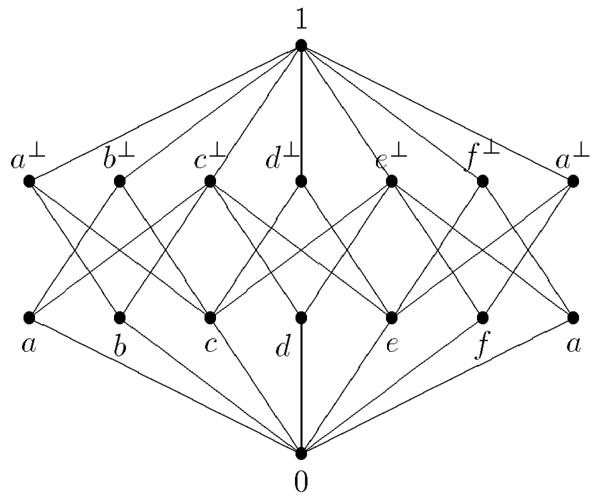

Fig. 2.

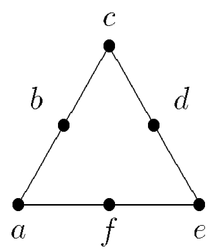

Fig. 3.

We recall that $\oplus$ is defined as follows:

$$a \oplus b = d \oplus e = c^\perp, \qquad a \oplus c = b^\perp,$$
$$b \oplus c = f \oplus e = a^\perp, \qquad c \oplus e = d^\perp,$$
$$a \oplus f = c \oplus d = e^\perp, \qquad a \oplus e = f^\perp.$$

Also for orthoalgebras there is a fuzzy set representation similar to Theorem 2.3.

Theorem 3.2. *Let a system of fuzzy sets $L \subseteq [0,\,1]^X$, $X \neq \emptyset$, satisfy the conditions* (i)–(iii) *of Theorem 2.3 and*

(iv) $f + f \leq 1_X$, *for $f \in L$, implies $f = 0_X$.*

Then $L = (L, \oplus, 0_X, 1_X)$, where $\oplus$ is defined as that in Theorem 2.3, is an orthoalgebra, with an order determining system of states $\mathcal{S} := \{m_x : x \in X\}$, where $m_x(f) := f(x)$, $f \in L$.

Conversely, let $L = (L, \oplus, 0, 1)$ be an arbitrary orthoalgebra with an order determining system of states $\mathcal{S}$. Then L is isomorphic with the system of fuzzy sets $\bar{L} \subseteq [0,\,1]^{\mathcal{S}}$, where $\bar{L} = \{\bar{a} \in [0,\,1]^{\mathcal{S}} : \bar{a}(m) := m(a),\ m \in \mathcal{S},\ a \in L\}$, and $\bar{L}$ satisfies the conditions (i)–(iv).

Proof. The conditions (i)–(iii) imply by Theorem 2.3 that $L = (L, \oplus, 0_X, 1_x)$ is an effect algebra. The condition (iv) implies that L is in fact an effect algebra.

The converse statement follows the same ideas as that in Theorem 2.3. $\square$

Example 3.3. The orthoalgebra in Example 3.1 has an order determining states of states as follows

$$\left|\begin{array}{ccccccc} & a & b & c & d & e & f \\ m_1 & 1 & 0 & 0 & 1 & 0 & 0 \\ m_2 & 0 & 0 & 1 & 0 & 0 & 1 \\ m_3 & 0 & 1 & 0 & 0 & 1 & 0 \\ m_4 & 0 & 1 & 0 & 1 & 0 & 1 \\ m_5 & 1/2 & 0 & 1/2 & 1/4 & 1/4 & 1/4 \end{array}\right|.$$

Example 3.4. The Fano plane, illustrated by the Greechie diagram in Figure 4 is an orthoalgebra having no fuzzy set representation because it possesses only a unique state m, namely $m(x) = 1/3,\ x \in \{a, b, c, d, e, f, g\}$.

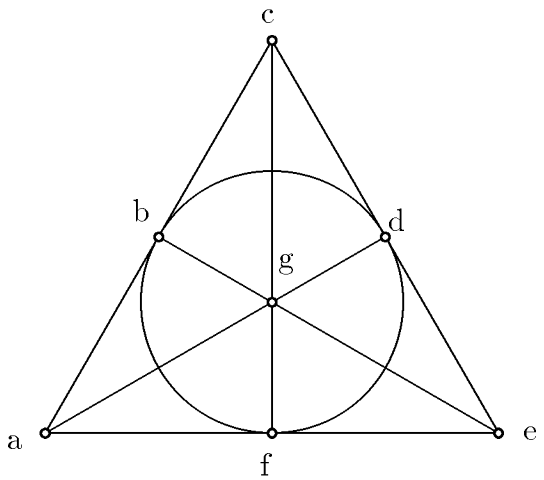

FIG. 4.

4. Fuzzy Set Representation of Orthomodular Posets

An orthomodular poset is a poset $L = (L, \leq, ^\perp, 0, 1)$ $(0 \neq 1)$ satisfying the axioms (OMi)–(OMv) from Section 1. If we change the axiom (OMiv) to

$$\bigvee_{i=1}^{\infty} a_i \in L, \text{ whenever } a_i \perp a_j,\ i \neq j,\ a_i \in L,\ i \geq 1,$$

we call L a *quantum logic.*

We recall that any orthomodular poset $L = (L, \leq, ^\perp, 0, 1)$ can be viewed as an orthoalgebra. Indeed, we define $a \oplus b := a \vee b$ iff $a \leq b^\perp$. Then a state on an orthomodular poset L is a mapping $m : L \to [0, 1]$ such that $m(1) = 1$ and $m(a \vee b) = m(a) + m(b)$ whenever $a \leq b^\perp$. Similarly we define a σ-additive state on a quantum logic L, i.e. $m : L \to [0, 1]$ such that (i) $m(1) = 1$, and (ii) $m(\bigvee_{i=1}^{\infty} a_i) = \sum_{i=1}^{\infty} m(a_i)$ whenever $a_i \leq a_j^\perp$ for $i \neq j$.

Example 4.1. Consider again a physical system, which is a firefly in a box with a clear plastic window at the front and another one on the side pictured in Figure 5.

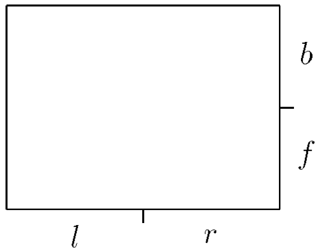

Fig. 5.

Suppose each window has a thin vertical line drawn down the center to divide the window in half. We shall consider two experiments on the system: The experiment A: Look at the front window. The experiment B: Look at the side window. The outcomes of A and B are: See a light in the left half (l_A, l_B), right half (r_A, r_B) of window or see no light (n_A, n_B). It is clear that $n_A = n_B =: n$ and we put $l_A =: l$, $r_A =: r$, $l_B =: f$, $r_B =: b$ (f for the front, b for the back).

The Hasse diagram of the corresponding propositional logic is given by Figure 6 which an orthomodular poset and which is not a Boolean algebra.

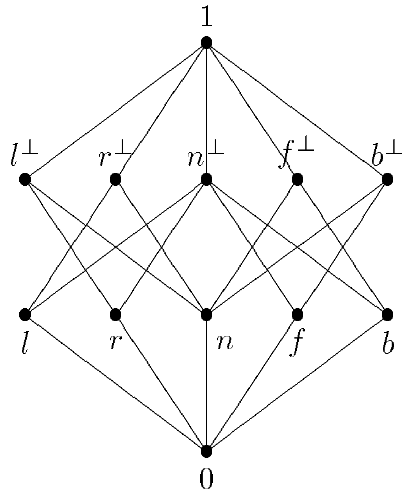

Fig. 6.

The following theorem is due to Mączyński and Traczyk [MaTr].

Theorem 4.2. *Let a system of fuzzy sets $L \subseteq [0,\, 1]^X$, $X \neq \emptyset$, satisfy the following conditions:*

(i) $1_X \in L$;
(ii) $1_X - f \in L$ *whenever* $f \in L$;
(iii) *if* $f+g,\ g+h,\ f+h \le 1_X$ *for* $f,g,h \in L$, *then* $f+g+h \in L$.

Then $L = (L, \le, ^\perp, 0_X, 1_X,)$, *where* $\le$ *is the natural ordering of fuzzy sets,* $f^\perp := 1_X - f,\ f \in L$, *is an orthomodular poset such that* $f \vee g = f + g$ *whenever* $f + g \le 1_X$.

In addition, the system $\mathcal{S} = \{m_x : x \in X\}$, where $m_x : L \to [0,\,1]$ is defined via $m_x(f) := f(x)$, $f \in L$, is an order determining system of states.

Conversely, let $L = (L, \leq, ^\perp, 0, 1)$ be an arbitrary orthomodular poset with an order determining system of states $\mathcal{S}$. Then L is isomorphic with the system of fuzzy sets $\bar{L} \subseteq [0,\,1]^{\mathcal{S}}$, where $\bar{L} = \{\bar{a} \in [0,\,1]^{\mathcal{S}} : \bar{a}(m) := m(a),\ m \in \mathcal{S},\ a \in L\}$, and $\bar{L}$ satisfies the conditions (i)–(iii).

Proof. Let $L \subseteq [0,\,1]^X$ satisfy (i)–(iii) of Theorem. Suppose that $f + g \leq 1_X$. Then $f, g, 0_X$ are mutually orthogonal, i.e. $f + 0_X \leq 1_X, g + 0_X \leq 1_X$, so that by (ii) $f + g = f + g + 0_X \in L$. We assert that $f \vee g = f + g$. Indeed, we have $f, g \leq f + g$. If $f, g \leq h$ for some $h \in L$, then $f + (1_X - h) \leq 1_X$, $g + (1_X - h) \leq 1_X$ which by (iii) means that $f + g + (1_X - h) \leq 1_X$, i.e. $f + g \leq h$, so that $f \vee g$ exists in L and it is equal to $f + g$. In particular, $f \vee (1_X - f) = f + (1_X - f) = 1_X$.

An easy calculation shows that $^\perp : f \mapsto f^\perp := 1_X - f$ satisfies conditions (OMi)–(OMiv). We now verify the orthomodular law (OMv). Let $f \leq g$, $f, g \in L$. Then $f + (1_X - g) \in L$ and $1_X - (g - f) = f + (1_X - g) = f \vee g^\perp \in L$, i.e., $g - f \in L$. Since $g - f \leq f^\perp$, we have $g = (g - f) + f = f \vee (g - f)^\perp$.

The rest is now clear. $\square$

The following theorem is essentially due to Mączyński [Mac a].

Theorem 4.3. *Let a system of fuzzy sets $L \subseteq [0,\,1]^X$, $X \neq \emptyset$, satisfy the following conditions:*

(i) $1_X \in L$;
(ii) $1_X - f \in L$ *whenever* $f \in L$;
(iii) *if* $f_i + f_j \leq 1_X$ *for* $i \neq j$, $f_i \in L$ *for* $i \geq 1$, *then* $\sum_{i=1}^\infty f_i \in L$.

Then $L = (L, \leq, ^\perp, 0_X, 1_X,)$, where $\leq$ is the natural ordering of fuzzy sets, $f^\perp := 1_X - f$, $f \in L$, is a quantum logic such that $\bigvee_{i=1}^\infty f_i = \sum_{i=1}^\infty f_i$ whenever $f_i + f_j \leq 1_X$ for $i \neq j$. In addition, the system $\mathcal{S} = \{m_x : x \in X\}$, where $m_x : L \to [0,\,1]$ is defined via $m_x(f) := f(x)$, $f \in L$, is an order determining system of σ-additive states.

Conversely, let $L = (L, \leq, ^\perp, 0, 1)$ be an arbitrary quantum logic with an order determining system of σ-additive states $\mathcal{S}$. Then L is isomorphic with the system of fuzzy sets $\bar{L} \subseteq [0,\,1]^{\mathcal{S}}$, where $\bar{L} = \{\bar{a} \in [0,\,1]^{\mathcal{S}} : \bar{a}(m) := m(a),\ m \in \mathcal{S},\ a \in L\}$, and $\bar{L}$ satisfies the conditions (i)–(iii).

Proof. It follows the same ideas as the proof of Theorem 4.2. It is necessary only to verify that $\bigvee_{i=1}^\infty f_i = \sum_{i=1}^\infty f_i$ whenever $f_i + f_j \leq 1_X$ for $i \neq j$. We have $f_i \leq f := \sum_{i=1}^\infty f_i \in L$ for any $i \geq 1$. If, in addition, $f_i \leq g$ for any i and some $g \in L$, then $f_i + (1_X - g) \leq 1_X$. By (iii), $(1_X - g) + f_1 + f_2 + \cdots \leq 1_X$ so that $f \leq g$. $\square$

Proposition 4.4. *Let a system of fuzzy sets $L \subseteq [0,\,1]^X$, $X \neq \emptyset$, satisfy the conditions of Theorem 2.3 (Theorem 3.2, Theorem 4.2 or Theorem 4.3). Choose $f \in L$, $f \neq 0_X$ and define $L_f := \{g \in L : g(x) \leq f(x),\ x \in X\}$. Then $(L_f, \oplus_f, 0_X, f)$ is an effect algebra (orthoalgebra, orthomodular poset or quantum logic) when we define $g^{\perp_f} := f - g$, $\oplus_f$ is defined iff $g + h \leq f$, then $g \oplus_f h := g + h$, and $\{m_x : x \in supp(f)\}$, where $m_x(g) := g(x), g \in L_f$ and $supp(f) := \{x \in X : f(x) > 0\}$, is an order determining system of states (σ-additive states) on L_f. An isomorphic* *system of fuzzy sets is given by* $X = supp(f)$ *and* $\overline{L_f} = \{\tilde{g} \in [0,\,1]^X : \ \tilde{g} := g|X,$ *where* $g \in L_f\}$.

Proof. It is straightforward. $\square$

Example 4.5. Let S be a real, complex or quaternionic inner product space and let $L = E(S) := \{M \subseteq S : M + M^\perp = S\}$ and $X(S) := \{x \in S : ||x|| = 1\}$. Then $(E(S), \subseteq, ^\perp, \{0\}, S)$ is an orthomodular poset which is not necessarily a quantum logic. It is a quantum logic if and only if S is complete ([Dvu, Thm 4.1.6]). Then $L \cong \{\overline{M} \in [0,\,1]^{X(S)} : \ \overline{M}(x) := (P_M x, x), \ x \in X(S), \ M \in E(S)\}$, where P_M is the orthogonal projection from S onto M.

Example 4.6. Let $L = L(H)$ be the set of all closed subspaces of a real, complex or quaternionic Hilbert space H. Then $(L(H), \subseteq, ^\perp, \{0\}, H)$ is a complete lattice quantum logic. Then $L(H) \cong \{\overline{M} \in [0,\,1]^{X(H)} : \ \overline{M}(x) := (P_M x, x), \ x \in X(H), \ M \in L(H)\}$, where $X(H) := \{x \in H : ||x|| = 1\}$ and P_M is an orthogonal projection from H onto $M \in L(H)$.

Let f and g be two fuzzy sets from $[0,\,1]^X, X \neq \emptyset$. Original Zadeh's fuzzy connectives of f and g are

$$(f \cap g)(x) := \min\{f(x), g(x)\}, \quad (f \cup g)(x) := \max\{f(x), g(x)\}, \ x \in X, \quad \text{and } 1_X - f.$$

Today we use a more general approach based on triangular norms and conorms which model connectives "and" and "or". We recall that a *triangular norm* (t-norm in abbreviation) is a function $T : [0,\,1] \times [0,\,1] \to [0,\,1]$ such that, for all $x, y, z, x_1, y_1 \in [0,\,1]$, we have

(i) $T(x, y) = T(y, x)$ (commutativity);
(ii) $T\big(T(x, y), z\big) = T\big(x, (T(y, z)\big)$ (associativity);
(iii) $T(x, 1) = x$ (boundary condition);
(iv) $T(x, y) \le T(x_1, y_1)$ if $x \le x_1, \ y \le y_1$ (monotonicity).

A t-conorm is a function $S : [0,\,1] \times [0,\,1] \to [0,\,1]$ such that, for all $x, y, z, x_1, y_1 \in [0,\,1]$, we have

(i) $S(x, y) = S(y, x)$ (commutativity);
(ii) $\big(S(S(x, y), z\big) = S\big(x, (S(y, z)\big)$ (associativity);
(iii) $S(x, 0) = x$ (boundary condition);
(iv) $S(x, y) \le S(x_1, y_1)$ if $x \le x_1, \ y \le y_1$ (monotonicity).

Very important t-norms and t-conorms are due to Zadeh

$$T_Z(x, y) := \min\{x, y\}, \qquad S_Z(x, y) := \max\{x, y\}, \ x, y \in [0,\,1], \tag{4.1}$$

and due to Łukasiewicz

$$T_Ł(x, y) := \max\{x + y - 1, 0\}, \quad S_Ł(x, y) := \min\{x + y, 1\}, \ x, y \in [0,\,1]. \tag{4.2}$$

Using t-norms and t-conorms, we can define pointwisely a fuzzy intersection $\cap = \cap_T$ and a fuzzy union $\cup = \cup_S$ of two fuzzy sets f and g via

$$(f \cap g)(x) := T\big(f(x), g(x)\big), \quad (f \cup g)(x) := S\big(f(x), g(x)\big), \ x \in X. \tag{4.3}$$

By the associativity we can define $(\bigcap_{i=1}^n f_i)(x) = \Big((\bigcap_{i=1}^{n-1} f_i) \cap f_n\Big)(x),\ x \in X$ for any integer $n > 1$.

The boundary condition entails that we can define also

$$\Big(\bigcap_{n=1}^{\infty} f_n\Big)(x) = \lim_n \Big(\bigcap_{i=1}^{n} f_i\Big)(x),\ x \in X, \tag{4.4}$$

for any sequence of fuzzy sets $\{f_n\}$ on X because the expressions in the limit on the right-hand side of (4.4) are bounded and non-decreasing. Similarly we define a fuzzy union of a finite or countable sequence of fuzzy sets.

If we need a fuzzy analogue of the complement, $c(x)$, of $x \in [0,\,1]$ such that, for $x, y \in [0,\,1]$, (i) $c(x) \in [0,\,1]$; (ii) $x < y$ implies $c(y) < c(x)$, and (iii) $c\big(c(x)\big) = x$, then by [Tri], there is a continuous, strictly increasing function $g:\ [0,\,1] \to [0,\,1]$ with $g(0) = 0$ and $g(1) = 1$ such that

$$c(x) = g^{-1}\big(1 - g(x)\big),\ x \in [0,\,1], \tag{4.5}$$

and the fuzzy complement of a fuzzy set f is defined as

$$f^c(x) = c\big(f(x)\big),\ x \in X.$$

Let c be a fuzzy complement. A t-norm T and a t-conorm S are *c-dual* if

$$T(x,y) = c\Big(S\big(c(x), c(y)\big)\Big),\ x, y \in [0,\,1].$$

Motivating by Giles [Gil], Pykacz [Pyk a] defined a *fuzzy quantum logic* as a structure $L \subseteq [0,\,1]^X$, $X \neq \emptyset$, such that

(f1) $1_X \in L$;
(f2) if $f \in L$, then $1_X - f \in L$;
(f3) if $f_i \cap f_j = 0_X$ for $i \neq j$, then $\sum_{i=1}^{\infty} f_i \in L$ and $\bigcup_{i=1}^{\infty} f_i \in L$.

In (f3), the connectives are defined by the Łukasiewicz t-norm and t-conorm, i.e.,

$$\big(f \cap g\big)(x) := \max\{f(x) + g(x) - 1, 0\}, \quad \big(f \cup g\big)(x) := \min\{f(x) + g(x), 1\},\ x \in X,$$

for $f, g \in L$.

Another structure due to Pykacz [Pyk b] is a *generalized fuzzy quantum logic* as a structure $L \subseteq [0,\,1]^X$, $X \neq \emptyset$, satisfying (f1), (f2) and

(f3*) if $f_i \cap f_j = 0_X$ for $i \neq j$, $f_i \in L$, then $\bigcup_{i=1}^{\infty} f_i \in L$;
(f4) if $f + f \leq 1_X$, then $f = 0_X$.

(Here also the Łukasiewicz t-norm and t-corm are taken for fuzzy intersection and union.)

Pykacz's [Pyk b] main result is the statement that both structures are the same things, and L with respect to the natural ordering of fuzzy sets, $0_X, 1_X$ and a unary operation $^{\perp}:\ L \to L$ defined pointwisely as $f^{\perp} = 1_X - f,\ f \in L$, is a quantum logic admitting an order determining system of σ-additive states $\{m_x:\ x \in X\}$, where $m_x(f) := f(x),\ f \in L, x \in X$, i.e. it is a Mączyński kind of quantum logics.

Mesiar [Mes a,b] investigated a so-called h-fuzzy quantum logic which is related with an Archimedean t-conorm S, i.e. S is continuous in any variation and $S(x,x) > x$ for any $x \in (0,\,1)$. If S is, in addition, a nilpotent t-conorm, then there is a generator $h:\,[0,\,1] \to [0,\,1]$ with $h(0)=0, h(1)=1$, and h is a continuous and strictly increasing function such that

$$S(x,y) = h^{-1}\big(\min\{h(x)+h(y),1\}\big),\ x,y \in [0,\,1]. \tag{4.6}$$

Its c-dual defined by h via (4.5) has the form

$$T(x,y) = c\Big(S\big(c(x),c(y)\big)\Big) = f^{-1}\big(\max\{h(x)+h(y)-1,0\}\big),\ x,y \in [0,\,1]. \tag{4.7}$$

An *h-fuzzy quantum logic* is a system $L \subseteq [0,\,1]^X$, $X \neq \emptyset$, such that

(h1) $1_X \in L$;
(h2) if $f \in L$, then $f^{\perp_h} \in L$, where $f^{\perp_h}(x) := h^{-1}\big(1-h(f(x))\big),\ x \in X$;
(h3) if $f \le f^{\perp_h}$, then $f = 0_X$;
(h4) if $f_i \le f_j^{\perp_h}$ for $i \neq j$, then $\bigcup_{n=1}^{\infty} f_n \in L$, where

$$\Big(\bigcup_{i=1}^{\infty} f_n\Big)(x) := h^{-1}\Big(\min\{\sum_{n=1}^{\infty} h\big(f_n(x)\big),\,1\}\Big),\ x \in X.$$

For example, if $h(x) = id(x) = x$, $x \in [0,\,1]$, then the id-fuzzy quantum logic of Mesiar is a Pykacz generalized fuzzy quantum logic. Mesiar [Mes a] has proved that $(L, \le, ^{\perp_h}, 0_X, 1_X)$ is a quantum logic which is isomorphic to $(L, \le, ^{\perp_{id}}, 0_X, 1_X)$. Hence, by Theorem 4.3 it has an ordering system of σ-additive states which is $\{m_x^h : x \in X\}$, where $m_x^h(f) := h(f(x)),\ f \in L,\ x \in X$.

Mesiar [Mes a] formulated an open problem whether any quantum logic $(L, \le, ^{\perp}, 0_X, 1_X)$ of fuzzy sets of X, where $\le$ is a natural ordering of fuzzy sets and the orthocomplementation $^{\perp} : L \to L$ is pointwisely defined, is an h-fuzzy quantum logic for a some generator h on $[0,\,1]$. In the following example we give show that the answer is negative.

Example 4.7. Let $X = [0,\,1]$ and let $L = \{0_X, 1_X, f, g\}$ where $f(x) = x$, $x \in [0,\,1]$, and $g(x) = x$ if x is a rational number from $[0,\,1]$ and $g(x) = 1-x$ if x is irrational. Define $0_X^{\perp} = 1_X, 1_X^{\perp} = 0_X,\ f^{\perp} = g, g^{\perp} = f$. Then $(L, \le, ^{\perp}, 0_X, 1_X)$ is a quantum logic where the orthocomplementation is in fact defined pointwisely, but this quantum logic is no h-fuzzy quantum logic for any generator h. It has an ordering system of (σ-additive) states, namely $\{m_x : x$ is a rational number in $[0,\,1]\}$, where $m_x(u) := u(x)$, $u \in L$. It is clear that it is a Boolean algebra isomorphic to 2^2 and it has also a sharp set representation.

5. Fuzzy Set Representation of MV-algebras

MV-algebras are many valued analogues of a two-valued logic and they have been introduced by Chang [Cha]. We recall that according to Mundici [Mun], they can be characterized as follows. An MV-*algebra* is a non-empty set L with two special elements 0 and 1 ($0 \neq 1$), with a binary operation $\oplus : M \times M \to M$ and with a unary operation $^* : M \to M$ such that, for all $a, b, c \in L$, we have

(MVi) $a \oplus b = b \oplus a$ (commutativity);
(MVii) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ (associativity);
(MViii) $a \oplus 0 = a$;
(MViv) $a \oplus 1 = 1$;
(MVv) $(a^*)^* = a$;
(MVvi) $a \oplus a^* = 1$;
(MVviii) $0^* = 1$;
(MVviii) $(a^* \oplus b)^* \oplus b = (a \oplus b^*)^* \oplus a$.

We define the following binary operations $\odot, \vee, \wedge$ as follows

$$a \odot b := (a^* \oplus b^*)^*, \ a, b \in L, \tag{5.1}$$
$$a \vee b := (a^* \oplus b)^* \oplus b, \quad a \wedge b := (a^* \vee b^*)^*, \ a, b \in L.$$

Then $(L, \odot, 1)$ is a semigroup written "multiplicatively" with the neutral element 1.[1] If, for $a, b \in L$ we define

$$a \leq b \ \Leftrightarrow \ a = a \wedge b,$$

then $\leq$ is a partial order on L, and $(L, \vee, \wedge, 0, 1)$ is a distributive lattice with the least and last elements 0 and 1, respectively, [Cha]. We recall that $a \leq b$ iff $b \oplus a^* = 1$.

Example 5.1. Let $L = [0,\ 1]$ and we define

$$x \oplus_{\text{Ł}} y := \min\{x + y, 1\}, \quad x^* := 1 - x, \ x, y \in [0,\ 1], \tag{5.2}$$

then $([0,\ 1], \oplus_{\text{Ł}}, ^*, 0, 1)$, where 0 and 1 are the same from $[0,\ 1]$, is an MV-algebra such that

$$x \odot_{\text{Ł}} y = \max\{x + y - 1, 0\}, \ x, y \in [0,\ 1],$$

$x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$, $x, y \in [0,\ 1]$, and the induced order on $[0,\ 1]$ coincides with the natural one on $[0,\ 1]$.

Example 5.2. Let $(G, +, 0)$ be an l-group written additively with a strong unit (= order unit) $u \in G$. We define $L(G) := \{x \in G : \ 0 \leq x \leq u\}$ and

$$x \oplus y := (x + y) \wedge u, \quad x^* := u - x, \ x \in L(G). \tag{5.3}$$

Then $(L(G), \oplus, ^*, 0, u,)$ is an MV-algebra with

$$x \odot y := 0 \vee (x + y - u).$$

Conversely, any MV-algebra $(L, \oplus, ^*, 0, 1)$ is isomorphic with some $(L(G), \oplus, ^*, 0, u)$, where $(G, +, 0)$ is an l-group with a strong unite u, [Mun].[2]

[1] We remark that in the literature by an MV-algebra is assumed the structure $(L, \oplus, \odot, ^*, 0, 1)$, where $\odot$ is defined by (5.1).

[2] Two MV-algebras are isomorphic if there is a bijective mapping between them preserving, $\oplus, ^*, 0$, and 1.

Example 5.3. Let $X \neq \emptyset$, and put $\mathcal{F}(X) = [0, 1]^X$. Then $(\mathcal{F}(X), \oplus, {}^*, 0_X, 1_X)$ is an MV-algebra, where

$$\big(f \oplus g\big)(x) := \min\{f(x) + g(x), 1\},\ x \in X,\ f^* := 1_X - f,$$

and it is called a *Bold algebra* of fuzzy sets on X.

A *state-morphism* on an MV-algebra L is a mapping $m:\ L \to [0, 1]$ such that

(i) $m(1) = 1$;
(ii) $m(a \oplus b) = m(a) \oplus_{\mathrm{L}} m(b),\ a, b \in L$,

We recall that, for all $a, b \in L$, (i) $m(0) = 0$; (ii) $m(a^*) = 1 - m(a)$; (iii) $m(a) \leq m(b)$ whenever $a \leq b$. A system $\mathcal{S}$ of state-morphisms on L is *order determining* if $m(a) \leq m(b)$ for any $m \in \mathcal{S}$ implies $a \leq b$.

For example, $L = [0, 1]$ in Example 5.1 has a unique state-morphism m, namely $m(t) = t,\ t \in [0, 1]$, and $\{m\}$ is order determining on $[0, 1]$.

Let $a \in L$, and n an integer $n \geq 0$. We define

$$n\,a := \begin{cases} 0 & n = 0, \\ a & n = 1, \\ (n-1)\,a \oplus a & n \geq 1. \end{cases}$$

For MV-algebras we have the following fuzzy sets representation.

Theorem 5.4. *Let a system of fuzzy sets $L \subseteq [0, 1]^X$, $X \neq \emptyset$, satisfy the following conditions:*

(i) $1_X \in L$;
(ii) $f^* := 1_X - f \in L$ *whenever* $f \in L$;
(iii) $f \oplus g \in L$ *for all* $f, g \in L$,

$$\big(f \oplus g\big)(x) := \min\{f(x) + g(x),\ 1\},\ x \in X. \tag{5.4}$$

Then $L = (L, \oplus, {}^*, 0_X, 1_X,)$, is an MV-algebra. In addition, the system $\mathcal{S} = \{m_x : x \in X\}$, where $m_x:\ L \to [0, 1]$ is defined via $m_x(f) := f(x),\ f \in L$, is an order determining system of state-morphisms.

Conversely, let $L = (L, \oplus, 0, 1)$ be an arbitrary MV-algebra with an order determining system of state-morphisms $\mathcal{S}$. Then L is isomorphic with the system of fuzzy sets $\bar{L} \subseteq [0, 1]^{\mathcal{S}}$, where $\bar{L} = \{\bar{a} \in [0, 1]^{\mathcal{S}} : \bar{a}(m) := m(a),\ m \in \mathcal{S},\ a \in L\}$, and $\bar{L}$ satisfies the conditions (i)–(iii).

Proof. An easy calculation shows that L satisfying (i)–(iii) is an MV-algebra and $\{m_x : x \in X\}$ is an order determining system of state-morphisms.

Conversely, we have (i) $\overline{a \oplus b} = \bar{a} \oplus \bar{b}$, where $\oplus$ on the right-hand side is defined on $\bar{L}$ via (5.4), $\bar{1} = 1_{\mathcal{S}}$, $\bar{a}^* := \overline{a^*} = 1_{\mathcal{S}} - \bar{a}$, for $a, b \in L$. Then $(\bar{L}, \oplus, {}^*, 0_{\mathcal{S}}, 1_{\mathcal{S}})$ is an MV-algebra being isomorphic with $(L, \oplus, {}^*, 0, 1)$. $\square$

We now present a close connection between semisimple MV-algebras and MV-algebras having an order determining system of state-morphisms. A non-empty subset I of an MV-algebra L is an *ideal* if (i) $x, y \in I$, then $x \oplus y \in I$; (ii) $x \odot y \in I$ whenever $x \in I$ and $y \in L$. An ideal I is *maximal* if $I \subseteq I_1$, where I_1 is an ideal, implies $I = I_1$ or $I_1 = L$. Any ideal I entails a congruence xIy iff $x \odot y^* \oplus y \odot x^* \in I$ and then $(L/I, \oplus, ^*, 0_I, 1_I)$, where L/I is the system of co-sets, in an MV-algebra.

An MV-algebra L is (i) *semisimple* if $Rad\ L = \{0\}$, where $Rad\ L := \bigcap\{I : I$ is a maximal ideal of $L\}$; (ii) *Archimedean* if, for any $x, y \in L$ with $n\,x \le y$ for any $n \ge 0$, we have $x \odot y = x$.

Belluce [Bel, Thm 31, 33] proved that L is a semisimple iff L is Archimedean, and in this case, there is a Bold algebra of fuzzy sets $\mathcal{F}(X)$ such that L can be embedded into $\mathcal{F}(X)$.

Theorem 5.5. *An MV-algebra L is semisimple if and only if L possesses an order determining system $\mathcal{S}$ of state-morphisms*

Proof. Assume that L is semisimple. Then there is a set $X \neq \emptyset$ such that L can be embedded into $\mathcal{F}(X)$. For any $x \in X$, we define $m_x(a) := \phi(a)(x),\ a \in L$, where ϕ is an embedding from L into $\mathcal{F}(X)$ (preserving $\oplus, ^*$ and 0,1). Then $\{m_x : x \in X\}$ is an order determining system of state-morphisms.

Conversely, suppose that, for $x, y \in L$ we have $n\,x \le y$ for any $n \ge 0$. Then $m(n\,x) \le m(y)$ so that $n\,m(x) \le m(y)$ which implies $m(x) = 0$ for any $m \in \mathcal{S}$, consequently $x = 0$. Hence $0 \odot y = 0$ and L is Archimedean, so that L is semisimple. $\square$

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