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EQUAL TEMPERAMENT AND PYTHAGOREAN TUNING: A GEOMETRICAL INTERPRETATION IN THE PLANE

Abstract

We show that an image of Equal Tempered Scale is the projection of the image of Pythagorean Tuning to a line in the plane.

1. Introduction. Pythagorean Tuning, cf. e.g. [1], Table 0.1, was created as a sequence of numbers of the form $2^p 3^q$, where p, q are integers. This tuning was established about five hundred years B. C. and used in the Western music up to the 14th century.

Equal Tempered Scale was invented by Andreas Werkmeister in "Erweiterte und verbesserte Orgel - Probe", 1698, and commonly spread since the appearance of the collection of compositions "Das wohltemperierte Klavier", 1721, by Johann Sebastian Bach.

In this paper we show that an image of Equal Tempered Scale is the projection of the image of Pythagorean Tuning to a line in the plane.

2. Preliminaries. The vector (the ratios of pitch frequencies relatively to the frequency of the first tone in the scale)

$$W = \{1, \sqrt[12]{2}, (\sqrt[12]{2})^2, \ldots, (\sqrt[12]{2})^{12} = 2\}$$

represents *Equal Tempered Scale* (or simply, *Equal Temperament*) in music. Analogously, *Pythagorean Tuning* is represented by the following sequence of 18 rational numbers (ordered increasingly):

$$P = \{1/1, 2^8/3^5, 3^7/2^{11}, 3^2/2^3, 2^5/3^3, 3^9/2^{14}, 3^4/2^6, 2^2/3, 2^{10}/3^6,$$
$$3^6/2^9, 3/2, 2^7/3^4, 3^8/2^{12}, 3^3/2^4, 2^4/3^2, 3^{10}/2^{15}, 3^5/2^7, 2/1\}.$$

Denote by $\mathcal{N}, \mathcal{Z}, \mathcal{Q}, \mathcal{R}, \mathcal{C}$ the sets of all natural, integer, rational, real, and complex numbers, respectively. Equal Tempered Scale can be generalized as follows:

Definition 1 Let $\{x, \ldots, y \in \mathcal{R}; 1 < x \leq \ldots \leq y < 10/9\}$ be a set of N numbers. Let $m_0, m_1, \ldots, m_M; \ldots; n_0, n_1, \ldots, n_M$ be $M \times N$ nonnegative integers, such that

$$0 = m_0 \leq m_1 \leq \ldots \leq m_M, \ldots, 0 = n_0 \leq n_1 \leq \ldots \leq n_M$$

and

$$m_j + \ldots + n_j = j, j = 0, 1, \ldots, M.$$

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Then $S = \{x^{m_0} \dots y^{n_0}, x^{m_1} \dots y^{n_1}, \dots, x^{m_M} \dots y^{n_M}\}$ is called to be a *M-ton N-interval scale*. Particularly, a 12-ton N-interval scale S is a *12-ton 2-interval* (2/1, 3/2)*-scale* if

$$x^{m_{12}} y^{n_{12}} = 2/1, x^{m_7} y^{n_7} = 3/2.$$

Clearly, Equal Tempered Scale is a 12-ton 1-interval scale which is not a 12-ton (2/1, 3/2)-scale. In this paper we deal with 12-ton 2-interval (2/1, 3/2)-scales where the both intervals $x, y \in \mathcal{Q}$ in Definition 1.

3. Pythagorean Tuning. In this section we describe the structure of Pythagorean Tuning.

Theorem 1 *The unique two rational ratio intervals in Definition 1 for 12-ton 2-interval (2/1, 3/2)-scales are*

$$x = 2^8/3^5, y = 3^7/2^{11}.$$

Proof. According to the symmetry,

$$A_2 = \begin{pmatrix} 7 & 5 \\ 4 & 3 \end{pmatrix}$$

is the unique solution of the Diophantine equation

$$\det \begin{pmatrix} m_{12} & n_{12} \\ m_7 & n_7 \end{pmatrix} = 1, \begin{cases} 0 \leq m_7 \leq m_{12} \\ 0 \leq n_7 \leq n_{12} \end{cases}, \begin{cases} m_{12} + n_{12} = 12, \\ m_7 + n_7 = 7, \end{cases}$$

where $m_{12}, n_{12}, m_7, n_7 \in \mathcal{Z}$.

Indeed, by assumption, $\det A_2 = m_{12}n_7 - m_7 n_{12} = 1, m_{12} = 12 - n_{12}, m_7 = 7 - n_7$. So, $12n_7 - n_{12}n_7 - 7n_{12} + n_{12}n_7 = 1$.

From the possibilities $n_{12} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ only $n_{12} = 5$ satisfies the equality $12n_7 - 7n_{12} = 1$ in $\mathcal{Z}$. Consequently, $n_7 = 3, m_{12} = 7, m_7 = 4$.

It is easy to see that x, y are of the form $2^\alpha 3^\beta, \alpha, \beta \in \mathcal{Z}$. The concrete values of x, y we can obtain from Definition 1, cf. [2]. $\square$

For $x = 2^8/3^5, y = 3^7/2^{11}$, denote by $c = 1, c_\sharp = x, d_\flat = y, d = xy, d_\sharp = x^2 y, e_\flat = xy^2, e = x^2y^2, f = x^3y^2, f_\sharp = x^4y^2, g_\flat = x^3y^3, g = x^4y^3, g_\sharp = x^5y^3, a_\flat = x^4y^4, a = x^5y^4, a_\sharp = x^6y^5, b_\flat = x^5y^5, b = x^6y^5 c' = x^7y^5$.

The proof of the following theorem is easy and omitted.

Theorem 2 *Let*

$$P_{r,s,t,u,v} = \{c, r, d, s, e, f, t, g, u, a, v, b, c'\},$$

where $r = c_\sharp, d_\flat; s = d_\sharp, e_\flat; t = f_\sharp, g_\flat; u = g_\sharp, a_\flat; v = a_\sharp, b_\flat$. *Then*

(1) $P_{r,s,t,u,v}$ *are 12-ton 2-interval (2/1, 3/2)-scales,*

(2)

$$P = \bigcup_{r,s,t,u,v} P_{r,s,t,u,v}.$$

Theorem 3 *For* $x = 2^8/3^5, y = 3^7/2^{11}$,

$$P = \{(x^4y^3)^k(x^4y^2)(\mathrm{mod}\ \ x^7y^5); k = 0, 1, 2, \ldots, 16\}.$$

Proof. $(x^4y^2)\cdot(x^4y^3)^0 = f_\sharp, (x^4y^2)\frac{x^4y^3}{x^7y^5} = x = c_\sharp, x\cdot(x^4y^3) = x^5y^3 = g_\sharp, (x^5y^3)\cdot x^4y^3x^7y^5 = x^2y = d_\sharp, (x^2y)\cdot(x^4y^3) = x^6y^4 = a_\sharp, (x^6y^4)\cdot\frac{x^4y^3}{x^7y^5} = x^3y^2 = f, (x^3y^2)\cdot\frac{x^4y^3}{x^7y^5} = 1 = c, 1\cdot(x^4y^3) = g, (x^4y^3)\cdot\frac{x^4y^3}{x^7y^5} = xy = d, (xy)\cdot(x^4y^3) = x^5y^4 = a, (x^5y^4)\cdot\frac{x^4y^3}{x^7y^5} = x^2y^2 = e, (x^2y^2)\cdot(x^4y^3) = x^6y^5 = b, (x^6y^5)\cdot\frac{x^4y^3}{x^7y^5} = x^3y^3 = g_\flat, (x^3y^3)\cdot x^4y^3x^7y^5 = y = d_\flat, y\cdot(x^4y^3) = x^4y^4 = a_\flat, (x^4y^4)\cdot\frac{x^4y^3}{x^7y^5} = xy^2 = e_\flat, (xy^2)\cdot(x^4y^3) = x^5y^5 = b_\flat$. $\square$

Corollary 1 *Pythagorean Tuning is a geometric progression of 17 rationals with the the quotient 3/2 modulo 2.*

4. Images of Equal Temperament and Pythagorean Tuning in the plane. Let $x = 2^8/3^5, y = 3^7/2^{11}$. Denote by $\mathcal{Q}_{x,y} = \{x^\alpha y^\beta; \alpha, \beta \in \mathcal{Z}\}, \mathcal{R}_{x,y} = \{x^\alpha y^\beta; \alpha, \beta \in \mathcal{R}\}$.

Consider the following map $\theta : \mathcal{Q}_{x,y} \to \mathcal{Z}^2, \theta : x^\alpha y^\beta \to (\alpha, \beta)$, and

$$(x^\alpha y^\beta)(x^\gamma y^\delta) \to (\alpha+\gamma, \beta+\delta), (x^\alpha y^\beta)^\gamma \to (\gamma\alpha, \gamma\beta),\ \ \alpha, \beta, \gamma, \delta \in \mathcal{Z}.$$

Lemma 1 *The morphism* θ *is an isomorphism.*

Proof. It is enough to show that the map $\theta : \mathcal{Q}_{x,y} \to \mathcal{Z}^2$ is an injection (the other properties of isomorphism are trivial). Suppose that there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathcal{Z}$, such that $\alpha_1 \neq \alpha_2, \beta_1 \neq \beta_2$ implies $x^{\alpha_1}y^{\beta_1} = x^{\alpha_2}y^{\beta_2}$. Then $x^{\alpha_1-\alpha_2}y^{\beta_1-\beta_2} = 1$ and $(\alpha_1 - \alpha_2) + (\beta_1 - \beta_2)\log_x y = 0$. This is possible only in the case when $\alpha_1 = \alpha_2, \beta_1 = \beta_2$. A contradiction. Consequently, $\alpha_1 = \alpha_2, \beta_1 = \beta_2$. $\square$

Embed $\mathcal{Z}^2$ identically into the complex plane $\mathcal{C}$ (for the process of the standard enlargements $\mathcal{Z} \to \mathcal{Q} \to \mathcal{R} \to \mathcal{C}$, cf. e.g. [3]). Denote this injection by η. Denote by θ_*^{-1} the extended morphism $\theta_*^{-1} : \mathcal{C} \to \mathcal{R}_{x,y}$. So we have the following commutative diagram, cf. Fig. 1, which defines the embedding $\varphi : \mathcal{Q}_{x,y} \to \mathcal{R}_{x,y}$:

$$\begin{tikzcd}
\mathcal{Q}_{x,y} \arrow[r, "\theta"] \arrow[d, hook, "\varphi"'] & \mathcal{Z}^2 \arrow[d, hook, "\eta"] \\
\mathcal{R}_{x,y} & \mathcal{C} \arrow[l, "\theta_*^{-1}"']
\end{tikzcd}$$

Fig.1. The embedding φ

In Fig. 2. we see the image of Pythagorean Tuning in the isomorphism θ.

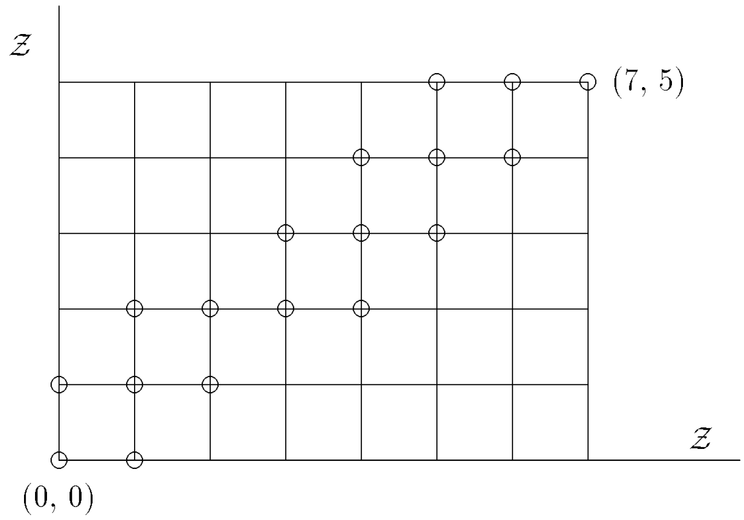

Fig.2. Image of Pythagorean Tuning in the isomorphism θ

Theorem 4 *Let p, q be two lines in the complex plane $\mathcal{C}$, such that $(0,0),(7,5) \in p; (12,0),(7,5) \in q$, respectively. Let $\pi : \mathcal{C} \to p$ be the projection of the plane $\mathcal{C}$ into the line p along the line q. Then*

$$W = \theta_*^{-1}(\pi(\eta(\theta(P)))).$$

Proof. See Fig. 3.

Denote by $w_0 = (0,0), w_1 = (\frac{7}{12}, \frac{5}{12}), w_2 = 2 \cdot (\frac{7}{12}, \frac{5}{12}), \ldots, w_{12} = 12 \cdot (\frac{7}{12}, \frac{5}{12})$.

We have:

$$\begin{array}{lcccc}
w_0 & = & \pi(\eta(\theta(c))), & \theta_*^{-1}(w_0) = x^0 y^0 & = & 1, \\
w_1 & = & \pi(\eta(\theta(c_\sharp))) = \pi(\eta(\theta(d_\flat))), & \theta_*^{-1}(w_1) = x^{\frac{7}{12}} y^{\frac{5}{12}} = (x^7 y^5)^{\frac{1}{12}} & = & \sqrt[12]{2}, \\
w_2 & = & \pi(\eta(\theta(d))), & \theta_*^{-1}(w_2) = x^{2\cdot\frac{7}{12}} y^{2\cdot\frac{5}{12}} & = & (\sqrt[12]{2})^2, \\
 & & \cdots & & & \\
w_{12} & = & \pi(\eta(\theta(c'))), & \theta_*^{-1}(w_{12}) = x^{12\cdot\frac{7}{12}} y^{12\cdot\frac{5}{12}} & = & (\sqrt[12]{2})^{12}.
\end{array}$$

$\square$

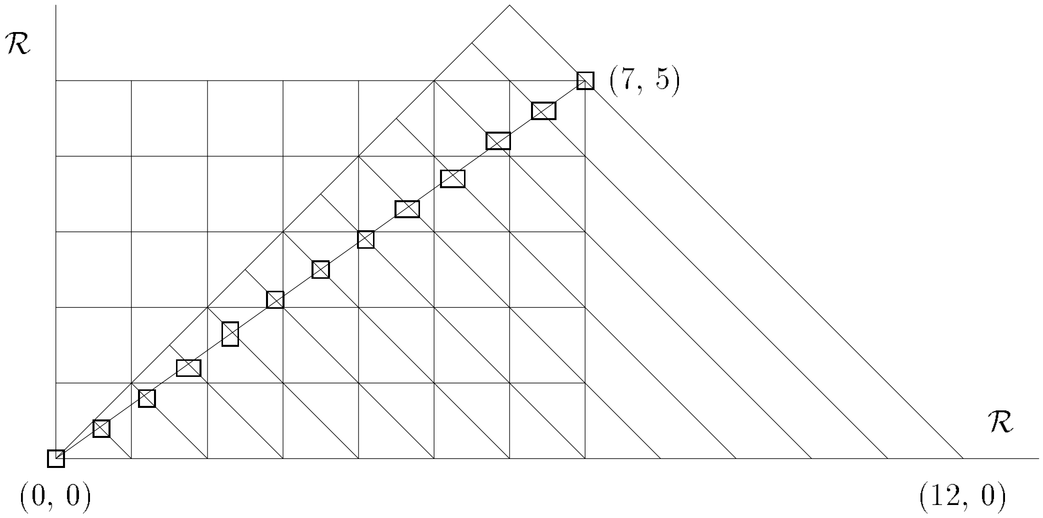

Fig. 3. Image of Equal Temperament in the plane

Corollary 2 *The image of Equal Temperament in Theorem 4 can be obtained also with the projection of the images of the whole-tone scales (the frequency ratio 9/8):* $(c, d, e, g_\flat, a_\flat, b_\flat)$ *and* $(c_\sharp, d_\sharp, f, g, a, b)$ *into the line* p *along the line* q.

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Slovak summary

Rovnomerne temperované a pytagorejské ladenie: geometrická predstava v rovine

Pytagorejské ladenie je reprezentované postupnosťou 18-tich racionálnych čísel

$$\{1/1, 2^8/3^5, 3^7/2^{11}, 3^2/2^3, 2^5/3^3, 3^9/2^{14}, 3^4/2^6, 2^2/3, 2^{10}/3^6,$$
$$3^6/2^9, 3/2, 2^7/3^4, 3^8/2^{12}, 3^3/2^4, 2^4/3^2, 3^{10}/2^{15}, 3^5/2^7, 2/1\}$$

(podiely frekvencií tónov vzhľadom na prvý tón v stupnici). Napr. pre jednočiarkovanú oktávu urobíme výpočet hodnôt frekvencií tónov tak, že uvedené hodnoty vynásobíme číslom $440 \cdot 16/27$. (výsledok je v Hz). Toto ladenie bolo vytvorené v 5. storočí pred n.l. a používalo sa v Európe až do 14. storočia.

Rovnomerne temperované ladenie bolo známe Andreasovi Werkmeistrovi, ako vidno z jeho knihy "Erweiterte und verbesserte Orgel - Probe" z r. 1698. Všeobecne sa rozšírilo po výjdení zbierky skladieb "Das wohltemperierte Klavier" v r. 1721 od Johanna Sebastiána Bacha. Dá sa charakterizovať postupnosťou čísel

$$\{1, \sqrt[12]{2}, (\sqrt[12]{2})^2, \ldots, (\sqrt[12]{2})^{12} = 2\}.$$

Teda okrem pomerových intervalov 1/1 (prima) a 2/1 (oktáva), všetky ostatné pomerové intervaly sú vyjadrené iracionálnym číslom. Odtiaľto vyplýva diskrepancia tohoto ladenia s fyzikálnou podstatou pojmu farba tónu, ktorá je daná vyššími parciálnymi harmonikami tónu.

Známy je fakt v Eulerovom hudobnom priestore, že pytagorejské ladenie je dvojrozmerné (súradnicový repér tvoria čistá kvinta a oktáva). To sa aj ľahko dá pochopiť, pretože toto ladenie je skonštruované len za pomoci čistých kvínt a oktáv. Podobne v článku, obraz pytagorejského ladenia je rovinný, dvojrozmerný. Bázické vektory roviny však uvažujeme iné ako kvinta a oktáva.

V práci je ukázané, že rovnomerne temperované ladenie, ktoré zobrazujeme jednorozmerne – na priamke, sa dá chápať ako projekcia pytagorejského ladenia do tejto priamky. Keďže projekcia originálu je obyčajne chudobnejšia ako originál, ktorý projektujeme (v našom prípade sa zmenšuje dimenzia), vzniká otázka, či pytagorejské ladenie nie je v istom zmysle zovšeobecnením rovnomerne temperovaného ladenia. Z článku vyplýva, že takto môžeme uvažovať.

Z druhej strany poznamenajme, že kým pytagorejské ladenie v nájdenom izomorfizme má jednoznačné vyjadrenie, existuje nekonečne veľa rôznych jednorozmerných obrazov rovnomerne temperovaného ladenia v rovine pytagorejského ladenia.

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