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X-MINIMAL PATTERNS AND A GENERALIZATION OF SHARKOVSKIĬ'S THEOREM

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Abstract. We study the law of coexistence of different types of cycles for a continuous map of the interval. For this we introduce the term of eccentricity of a pattern and characterize those patterns with a given eccentricity that are simplest from the point of view of the forcing relation. We call these patterns X-minimal. We obtain a generalization of Sharkovskiĭ's Theorem where the notion of period is replaced by the notion of eccentricity.

0. Introduction

The question of coexistence of different types of cycles (or periodic orbits) arises in the theory of discrete dynamical systems. In dimension one for interval maps pattern seems to be the finest classification of cycles for this question. A law of coexistence of different patterns, now usually called the forcing relation, has been proved (see [B], [ALM]). However the exact structure of this forcing relation is not known. Better results can be obtained if one defines types as bigger collections of patterns. The period of a pattern is the first natural thing that can determine a type. This was used by Sharkovskiĭ in his powerful theorem. To state it we need to introduce the Sharkovskiĭ ordering:

$$3 \succ 5 \succ 7 \succ \dots \succ 2\cdot 3 \succ 2\cdot 5 \succ 2\cdot 7 \succ \dots \succ 2^2\cdot 3 \succ 2^2\cdot 5 \succ 2^2\cdot 7 \succ \dots$$
$$\succ 2^\infty \succ \dots \succ 2^3 \succ 2^2 \succ 2 \succ 1$$

The terms 'pattern' and 'forces' used here will be defined later.

Sharkovskiĭ's Theorem. ([S])

(i) *A pattern with period m forces some pattern with period n for any $n \in \mathbb{N}$ such that $m \succ n$.*

(ii) *For any $m \in \mathbb{N} \cup \{2^\infty\}$ there is a continuous map $f : I \to I$ such that it has a cycle of period $n \in \mathbb{N}$ if and only if $m = n$ or $m \succ n$.*

The aim of this paper is to get better results about the structure of the forcing relation. In order to achieve this we consider a different notion of the "type" of a pattern. We consider the position of a fixed point whose existence is implied by a pattern, or more precisely the ratio of the number of points of the cycle on each

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side of such fixed point. Patterns with the same ratio will be said to be of the same type. Our main aim is to find which patterns of a given type are simplest in terms of the forcing relation. Let us state this in a more rigorous way. The terminology used here is mainly the same as in [ALM].

Let $P = \{p_1, \dots, p_n\} \subset \mathbb{R}$ and $\varphi : P \to P$. Then (P, φ) is a *periodic orbit* (or *cycle*) if φ is a cyclic permutation of P. We will usually omit φ and simply say that P is a cycle. The *period* of a cycle P is $\operatorname{per}(P) = n$.

Two periodic orbits $(P, \varphi), (Q, \psi)$ are equivalent if there exists a homeomorphism $h : \operatorname{conv}(P) \to \operatorname{conv}(Q)$ such that $h(P) = Q$ and $\psi \circ h|_P = h \circ \varphi|_P$. An equivalence class of this relation will be called a *pattern.* If A is a pattern and $(P, \varphi) \in A$ we say that the cycle P has a pattern A (or P is a representative of A) and we will use the symbol $[P]$ to denote the pattern A. The period of the pattern A is $\operatorname{per}(A) = \operatorname{per}(P)$.

We consider the space $C(I, I)$ of all continuous maps $f : I \to I$, where I is a closed interval. A function $f \in C(I, I)$ has a cycle (P, φ) if $f|_P = \varphi$. We shall say that f *exhibits* the pattern $[P]$. Now we can define the forcing relation between patterns.

Definition. A pattern A *forces* a pattern B if all maps in $C(I, I)$ exhibiting A exhibit also B.

We have the following information about the forcing relation

Theorem. ([B], [ALM]) *The forcing relation is a partial order relation.*

Now we will define our notion of "type" of a pattern. Let $(P = \{p_1, \dots, p_n\}, \varphi)$ be a cycle with spatial labeling (so $p_1 < \dots < p_n$). If

$$(*) \qquad (p_i - \varphi(p_i)) \cdot (p_{i+1} - \varphi(p_{i+1})) < 0$$

then any continuous function with cycle P has a fixed point in the open interval (p_i, p_{i+1}). On the other hand if $(*)$ is not true, then there is a function with the cycle P that does not have a fixed point in the interval (p_i, p_{i+1}). Hence we can give the following

Definition. A cycle (P, φ) has *eccentricity* $r \in \mathbb{Q}$ if for any map $f \in C(I, I)$ with the cycle P there is a fixed point $c \in \operatorname{Fix}(f)$ such that $\frac{\#\{x \in P;\ x \le c\}}{\#\{x \in P;\ x \ge c\}} = r$. Note that a cycle $(h(P), h^{-1} \circ \varphi \circ h)$ where $h(x) = -x$ has eccentricity $\frac{1}{r}$ and so we define the eccentricity of a pattern $[P]$ as an eccentricity of a representative whose eccentricity is not smaller than one.

Remark. Note that a pattern (or cycle) can have more than one eccentricity (see Fig. 1).

In order to quickly demonstrate the connection between Sharkovskiĭ's Theorem and forcing relation based on the eccentricity of pattern take the Sharkovskiĭ ordering on odd numbers

$$3 \succ 5 \succ 7 \succ 9 \succ 11 \succ \cdots \succ 1$$

and rewrite it into the corresponding form

$$\frac{2}{1} \succ \frac{3}{2} \succ \frac{4}{3} \succ \frac{5}{4} \succ \frac{6}{5} \succ \cdots \succ 1\,.$$

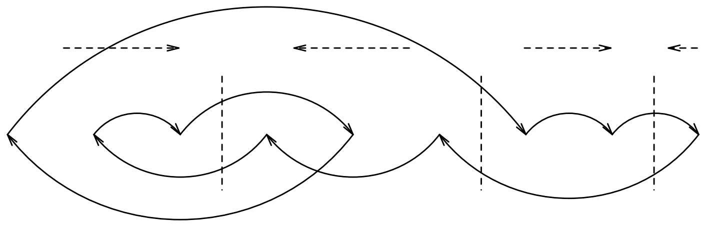


FIGURE 1. An example of a cycle P with eccentricities $\frac{1}{2}$, $\frac{2}{1}$ and $\frac{8}{1}$. The pattern $[P]$ has eccentricities $\frac{2}{1}$ and $\frac{8}{1}$.

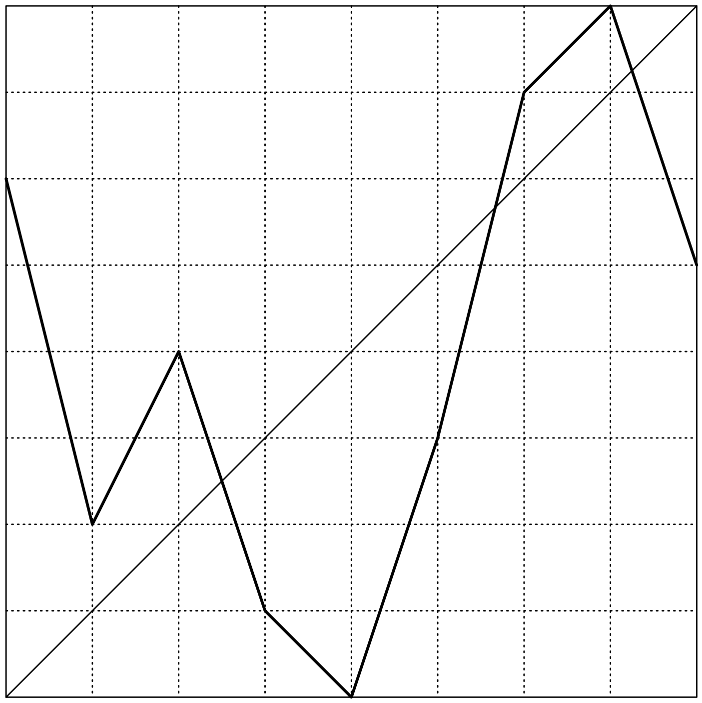


FIGURE 2. The graph of the function f_P.

The basic idea of the proof of Sharkovskiĭ's Theorem is to show that every pattern of period $2k+1$ forces a pattern with the same period and eccentricity $\frac{k+1}{k}$ and that these patterns form a chain corresponding to the order above.

So you can see that Sharkovskiĭ ordering is defined only on a small part of rational numbers. We of course have natural order on all rational numbers and in this paper we prove corresponding generalization of Sharkovskiĭ's Theorem for this order.

The crucial role in the proof of Sharkovskiĭ's Theorem was played by so called Štefan patterns. In this paper similar role is played by X-minimal patterns defined below.

We will fix an $r \in \mathbb{Q}$ and consider the set of all patterns with eccentricity r. We can look at the forcing relation restricted to this set. Some of the patterns may not force any other pattern from this set. These will be called *X-minimal patterns* with eccentricity r (*X-minimal r-patterns*) and for their representatives we shall use term *X-minimal cycles* (*X-minimal r-cycles*).

Remark. Note that in fact X-minimal patterns do not have to exist. Because there are infinitely many patterns with any given rational eccentricity it could be possible that they can be arranged into the infinite chain of patterns where each one forces the next one. We will prove that this is not the case and that there are indeed X-minimal patterns for any given eccentricity. Also note that a pattern with eccentricities $r, q \in \mathbb{Q}$ can theoretically be an X-minimal r-pattern but not an X-minimal q-pattern. Our results show that this is not possible either.

The structure of the paper is as follows.

In section 1 we give some basic notation, definitions and lemmas used throughout the paper.

In section 2 we study the forcing relation between patterns with different eccentricity. The main result of this section is in Theorem 2.10.

Section 3 is devoted to the characterization of X-minimal patterns. The main result of this section is in Theorem 3.7. Reader can also have a look at the easy geometrical condition given in Lemma 3.9 and the important property concerning the period of X-minimal patterns is proved in Lemma 3.4.

In section 4 we prove the existence of X-minimal patterns and we give a simple algorithm for constructing all X-minimal patterns. The main result proving the existence of X-minimal patterns is in Theorem 4.5.

Finally in section 5 we define a new notion of the "type" of a pattern and using it we prove our generalization of Sharkovskiĭ's Theorem in the Theorem 5.1.

Remark. Results very similar to those obtained in this paper can be also found in the papers [Bl] and [BM]. They have been obtained by different methods and independently on this paper. The method used there is based on the theory of rotation numbers for maps on the circle while in this paper we use only elementary combinatorial arguments.

1. Background

By $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$ we denote the sets of real, rational, integer and positive integer numbers respectively. By $\operatorname{conv}(X)$ we denote the convex hull of a set X. We will put sets inside $\{\}$ brackets and for us the set is a collection of elements without multiple occurrence. An ordered collection of elements with possible multiple occurrence will be called sequence and we use $\langle\rangle$ to put it in. By f^i we denote ith iterate of a function f. A point p is a periodic point of a function f if $f^n(p) = p$ for some $n \in \mathbb{N}$. The least such n is called the period of p. A cycle given by a periodic point p and a function f is (P, φ) where $P = \{f^i(p);\ i \in \mathbb{N}\}$ and $\varphi = f|_P$. A point p is a fixed point of a function f if $f(p) = p$. The set of all periodic points of a function f will be $\operatorname{Per}(f)$ and the set of all fixed points will be $\operatorname{Fix}(f)$. For a cycle $(\{p_1, \dots, p_n\}, \varphi)$ we will use usually one of the following labelings. A spatial labeling when $p_1 < p_2 < \cdots < p_n$ and a dynamical labeling when $\varphi(p_i) = p_{i+1}$ for $i = 1, \dots, n-1$ and $\varphi(p_n) = p_1$

We shall use some standard notions and techniques from combinatorial dynamics. The most important is the notion of P-linear map.

Definition. Let (P, φ) be a periodic orbit and $I = \operatorname{conv}(P)$. Then $f_P \in C(I, I)$ such that $f_P|_P = \varphi$ and $f_P|_J$ is linear for any interval $J \subset I$ such that $J \cap P = \emptyset$.

The function f_P is the piecewise linear function given by the cycle P and sometimes it is called the connect-the-dot map (see Fig. 2).

Very often we will use the following basic fact.

Lemma 1.1. (Theorem 2.6.13 from [ALM]) *Let (P, φ) be a cycle. If f_P exhibits pattern B then $[P]$ forces B.*

We say that an interval J P-covers an interval L if $L \subset f_P(J)$. We will denote this by $J \xrightarrow{P} L$. A sequence of closed intervals $\mathcal{A} = \langle I_k \rangle_{k=1}^{m}$ is called P-cyclic if $I_1 \xrightarrow{P} I_2 \xrightarrow{P} \dots \xrightarrow{P} I_m \xrightarrow{P} I_1$. Note that a P-cyclic sequence is in fact a cycle of intervals and therefore we will consider two P-cyclic sequences equal if they form the same cycle and have the same length. This will allow us to start a cyclic sequence wherever we want by simply rotating it. The P-cyclic sequences and cycles of the function f_P are in close relation. Namely we have

Lemma 1.2. ([B2], [BGMY], Lemma 1.2.7 from [ALM]) *Let P be a periodic orbit and $\mathcal{A} = \langle I_k \rangle_{k=0}^{m-1}$ be P-cyclic. Then there is a periodic point $x \in \mathrm{Per}(f_P)$ such that $f_P^k(x) \in I_k$ for $k = 0, \dots, m-1$ and $f_P^m(x) = x$. The period of the cycle given by the point x and the function f_P divides m.*

We will say that a cycle obtained from a P-cyclic sequence $\mathcal{A}$ using the Lemma 1.2 is *contained* in $\mathcal{A}$.

We will use the following simple notation:

If $\mathcal{A} = \langle a_1, \dots, a_n \rangle$ and $\mathcal{B} = \langle b_1, \dots, b_m \rangle$ then $\mathcal{A} + \mathcal{B} = \langle a_1, \dots, a_n, b_1, \dots, b_m \rangle$.

Finally if $A, B \subset \mathbb{R}$ then we say that $A < B$ if $A \neq B$ and $a \leq b$ for any $a \in A$ and $b \in B$. If $x \in \mathbb{R}$ then we say that $x < A$ ($x > A$) if $\{x\} < A$ ($\{x\} > A$).

2. Unipatterns and forcing between patterns with different eccentricity

A cycle P with unique eccentricity will be called a *unicycle*[1]. We shall denote the eccentricity of a unicycle P by $E(P)$. A unicycle P with $E(P) = r$ will be called an *r-unicycle*. Similarly we shall use the terms *unipattern*, $E([P])$ and *r-unipattern*. Note that if a cycle P is not a unicycle then the pattern $[P]$ has at least two different eccentricities. Therefore a representative of a unipattern is a unicycle.

We will show that an X-minimal pattern must be a unipattern.

Lemma 2.1. *Suppose the cycle P is not a unicycle. Then f_P has an r-unicycle for any positive $r \in \mathbb{Q}$.*

Proof. Because P is not a unicycle the function f_P has least two fixed points (in fact at least three). If $r = 1$ then a fixed point gives an r-cycle. So we can assume that $r > 1$ (the case $r < 1$ is similar).

Let $z_1 < z_2$ be two rightmost fixed points of f_P. Hence $f_P(x) > x$ for $x \in (z_1, z_2)$ and $f_P(x) < x$ for $x > z_2$. Let $a \in (z_1, z_2)$ be such that $f_P(a) \geq f_P(x)$ for any $x \in (z_1, z_2)$. Clearly $f_P(a) > z_2$ (otherwise the interval $[z_1, z_2]$ would be

[1] Unfortunately there is no bicycle. However according to A. Manning there may be a little comfort for cyclists. Actually the object we study (a periodic orbit and a fixed point "in" it) consists of two cycles so it is a bicycle ☺

f_P-invariant which is impossible because it contains a point from P). Now let $J_1 = [z_1, a]$ and $J_2 = [z_2, f_P(a)]$. We have that $J_1 \stackrel{P}{\to} J_1$ and $J_1 \stackrel{P}{\to} J_2$. Moreover $J_2 \stackrel{P}{\to} J_1$ (otherwise interval $[z_1, f_P(a)]$ would be f_P-invariant) (see Fig. 3). Hence for any $m \geq n$

$$\overbrace{\langle J_1, \dots, J_1 \rangle}^{m-n \text{ times } J_1} + \overbrace{\langle J_1, J_2, \dots, J_1, J_2 \rangle}^{n \text{ times } J_1, J_2}$$

is a P-cyclic sequence. Using Lemma 1.2 we obtain a cycle Q with period $m+n$. So it does not contain any fixed point. But $Q \subset [z_1, f_P(a)]$ and $(z_1, f_P(a)) \cap \mathrm{Fix}(f_P) = \{z_2\}$ so Q is a unicycle. Finally because $J_1 < z_2 < J_2$ we have that Q is an $\frac{m}{n}$-unicycle and we can choose $\frac{m}{n} = r$. ■

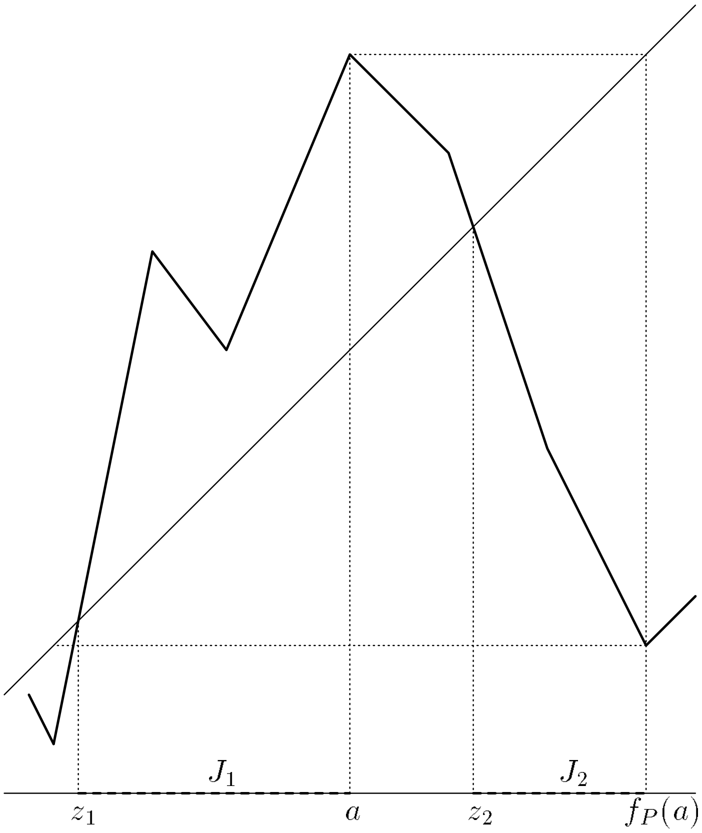


FIGURE 3

Hence we have the straightforward

Corollary 2.2. *An X-minimal pattern is a unipattern.*

Proof. If an r-pattern A is not a unipattern then its representative P is not a unicycle. By Lemma 2.1 function f_P exhibits an r-unipattern B and by Lemma 1.1 pattern A forces B. But $A \neq B$ (one is a unipattern and the other is not) and so A is not an X-minimal r-pattern. ■

Now we would like to find all patterns forced by a unipattern $[P]$.

First note that f_P has a unique fixed point and therefore every cycle it has is a unicycle. So a unipattern can force only unipatterns. Later we will often use this fact without mentioning it.

A possible way to find patterns forced by a unipattern $[P]$ is to find all P-cyclic sequences and using Lemma 1.1 and 1.2 we get some of the patterns forced by $[P]$. But in general if we have a P-cyclic sequence then we have no information about the eccentricity of patterns forced by this sequence. Fortunately, for some special P-cyclic sequences we can get this information.

Definition. Let P be a unicycle and $\mathrm{Fix}(f_P) = \{c\}$. A P-cyclic sequence $\mathcal{A} = \langle I_i \rangle_{i=1}^{a}$ will be called *separated* if $c \notin \mathrm{int}(I)$ for any $I \in \mathcal{A}$. The eccentricity of a separated P-cyclic sequence $\mathcal{A}$ will be $E(\mathcal{A}) = \frac{\#\{i; I_i \leq c\}}{\#\{i; I_i \geq c\}}$.

Lemma 2.3. *Let P be a unicycle, $c \in \mathrm{Fix}(f_P)$ and $\mathcal{A}$ be a separated P-cyclic sequence. Then f_P has an $E(\mathcal{A})$-unicycle contained in the loop $\mathcal{A}$.*

Proof. Assume that $E(\mathcal{A}) \geq 1$ (the case $E(\mathcal{A}) \leq 1$ is similar).

If there is $I \in \mathcal{A}$ such that $c \notin I$ then the cycle Q that we get from the sequence $\mathcal{A}$ by using Lemma 1.2 clearly has eccentricity $E(\mathcal{A})$.

Assume that $c \in I$ for all $I \in \mathcal{A}$. If $E(\mathcal{A}) = 1$ then the fixed point c gives such a cycle. So we can assume that $E(\mathcal{A}) > 1$. Then $\mathcal{A} = \langle \dots, I, J, K \dots \rangle$ where $I, J < c$ and $K > c$ ($I \neq \{c\}$ because $E(\mathcal{A}, c) \neq 1$). Now there are two possibilities.

1. $f_P(x) \geq c$ for any $x \in J$ or
2. there is an $a \in J$ such that $f_P(a) = c$ and $a \neq c$.

In case 1 we have that $J \subset I$. So there is a point $a \in I$ such that $f_P(a) = \inf J$ and a point $b \in I$ such that $b \neq c$ and $f_P(b) = c$. Hence we can replace interval I by $I^* = \mathrm{conv}(\{a, b\})$ and we have again a separated P-cyclic sequence $\mathcal{B}$.

In case 2 let $b \in J$ such that $f_P(b) = \sup\{f_P(x); x \in J\}$. Now we can replace interval J by $J^* = \mathrm{conv}(\{a, b\})$ to get a separated P-cyclic sequence $\mathcal{B}$.

In both cases we obtain a new separated P-cyclic sequence $\mathcal{B}$ with eccentricity $E(\mathcal{A})$. But now there is an interval in $\mathcal{B}$ (I^* or J^*) that does not contain the point c. Hence using the argument from above there is a $E(\mathcal{A})$-unicycle Q in f_P. ∎

Now the question is how we can tell whether we have picked up all possible P-cyclic sequences that can give us some information about patterns forced by $[P]$. We will show that it is enough to examine those P-cyclic sequences that have their elements only from a set of intervals given by the cycle P.

Definition. For a unicycle (P, φ) let $\mathfrak{P}$ be the partition of the interval $I = \mathrm{conv}(P)$ into intervals with endpoints in $P \cup \mathrm{Fix}(f_P)$.

In particular if $P = \{p_1, p_2, \dots, p_{k(m+n)}\}$ is a unicycle with spatial labeling where $k, m, n \in \mathbb{N}$, m, n are coprime, $E(P) = \frac{m}{n}$ and $\mathrm{Fix}(f_P) = \{c\}$, the partition $\mathfrak{P} = \{J_i\}_{i=1}^{k(m+n)}$ where

$$\begin{aligned} J_i &= [p_i, p_{i+1}] && \text{for } i < km \\ J_{km} &= [p_{km}, c] \\ J_{km+1} &= [c, p_{km+1}] \\ J_i &= [p_{i-1}, p_i] && \text{for } i > km+1. \end{aligned}$$

A P-cyclic sequence $\mathcal{A} = \langle I_i \rangle_{i=1}^{a}$ such that each $I_i \in \mathfrak{P}$ will be called a P-loop. (Note that any P-loop is separated.)

Now we can prove some kind of converse of Lemma 1.2.

Lemma 2.4. *Let P be a unicycle and $Q = \{q_1, \dots, q_m\}$ be a cycle of f_P with* $\operatorname{per}(Q) = m > 1$. *Then there is a unique P-loop $\mathcal{A} = \langle I_i \rangle_{i=1}^{m}$ such that $f_P^{i-1}(q_1) \in I_i$ for $1 \le i \le m$.*

Proof. If $Q \neq P$ then for any q_i there is a unique interval $I_i \in \mathfrak{P}$ such that $q_i \in I_i$. Moreover because f_P is linear on any interval $I \in \mathfrak{P}$ and $q_i \in \operatorname{int}(I_i)$ we have that $I_i \overset{P}{\to} I_{i+1}$ and so the sequence $\mathcal{A} = \langle I_i \rangle_{i=1}^{m}$ is a P-loop.

So assume that $Q = P = \{p_1, \dots, p_n\}$ with spatial labeling. Then there is a unique interval $I_1 \in \mathfrak{P}$ such that $p_1 \in I_1$. Assume that there is given interval $I_j \in \mathfrak{P}$ such that $f_P^{j-1}(p_1) \in I_j$. There are at most two intervals $I \in \mathfrak{P}$ such that $f_P^{j}(p_1) \in I$ but only one of them satisfies condition $I_j \overset{P}{\to} I$ (because f_P is linear on I_j and $f_P^{j-1}(p_1)$ is an endpoint of I_j). Hence there is a unique $I_{j+1} \in \mathfrak{P}$ such that $f_P^{j}(p_1) \in I_{j+1}$ and $I_j \overset{P}{\to} I_{j+1}$. Therefore there is a unique P-loop of a length $\operatorname{per}(P)$ containing the cycle P too. ■

Remark. Of course a cycle Q can be contained in more than one P-loop. But any P-loop containing Q is only a repetition of a unique P-loop $\mathcal{A}$ that has length equal $\operatorname{per}(Q)$.

Definition. We shall denote the P-loop containing cycle P by $\mathcal{A}_P$.

We will say that P-loop $\mathcal{A}$ is simple if there are no two nonempty P-loops $\mathcal{B}, \mathcal{C}$ such that $\mathcal{A} = \mathcal{B} + \mathcal{C}$.

Lemma 2.5. *A P-loop containing some interval more than once is not simple.*

Proof. After rotating we can write our P-loop as $\overbrace{\langle I, \dots, K \rangle}^{\mathcal{B}} + \overbrace{\langle I, \dots, L \rangle}^{\mathcal{C}}$ and both $\mathcal{B}$ and $\mathcal{C}$ are nonempty P-loops. ■

Now we will look at a unicycle P and its loop $\mathcal{A}_P$. There are basically two possibilities. Either $\mathcal{A}_P$ is simple or not. The next lemma shows importance of simple $\mathcal{A}_P$.

Lemma 2.6. *Let P be a unicycle with* $\operatorname{per}(P) > 2$ *and simple loop $\mathcal{A}_P$. Then for each P-loop $\mathcal{A}$ there is a unique cycle contained in $\mathcal{A}$.*

Proof. Assume the contrary. Let $\mathcal{A} = \langle I_i \rangle_{i=1}^{m}$ be a P-loop and let $x < y \in \operatorname{Per}(f_P)$ be such that $f_P^{i-1}(x), f_P^{i-1}(y) \in I_i$ for $1 \le i \le m$ and $f_P^m(x) = x$ and $f_P^m(y) = y$. Hence $f_P^m|_{[x,y]}$ is linear and therefore the identity. We will take the smallest possible $a \in \mathbb{N}$ such that $f_P^a|_{[x,y]}$ is the identity and that means that x, y have either period a or period $\frac{a}{2}$.

Take the points $x^*, y^* \in I_1$ such that $x^* \le x < y \le y^*$, $f_P^n(x^*), f_P^n(y^*) \in P \cup \operatorname{Fix}(f_P)$ for some $n \in \mathbb{N}$, $f_P^a|_{[x^*,y^*]}$ is linear and $(x^*, y^*) \cap (P \cup \operatorname{Fix}(f_P)) = \emptyset$.

So we have that $f_P^a|_{[x^*,y^*]}$ is the identity and therefore $x^*, y^* \in P \cup \operatorname{Fix}(f_P)$. Hence $f_P^i([x^*, y^*]) \in \mathfrak{P}$ for any $i \ge 0$ and if $\{x^*, y^*\} \cap \operatorname{Fix}(f_P) \neq \emptyset$ then $\operatorname{per}(P) \le$

2. Therefore $x^*, y^* \in P$. Moreover a is the smallest possible number such that $f_P^a|_{[x^*,y^*]}$ is the identity and therefore $\mathrm{per}(P) = a$.

Take a sequence $\mathcal{A}^* = \langle f_P^i([x^*, y^*])\rangle_{i=1}^a$. Clearly $\mathcal{A}^*$ is a P-loop and moreover it contains the cycle P. Therefore from Lemma 2.4 we have $\mathcal{A}^* = \mathcal{A}_P$. But it is easy to see that $\mathcal{A}^*$ is not simple (it contains the interval $[x^*, y^*]$ twice - once covering the point x^* and second time covering the point y^*) – a contradiction. ∎

Now we will investigate the forcing relation between patterns with different eccentricities.

Lemma 2.7. *Let P be a unicycle with $E(P) \geq 1$ and $\mathcal{A}_P$ be not simple. Then f_P has a unicycle Q such that per$(Q) < \mathrm{per}(P)$ and $E(Q) \geq E(P)$.*

Proof. Because $\mathcal{A}_P$ is not simple there are P-loops $\mathcal{B}, \mathcal{C}$ such that $\mathcal{A}_P = \mathcal{B} + \mathcal{C}$. But either $E(\mathcal{B}) \geq E(\mathcal{A}_P)$ or $E(\mathcal{C}) \geq E(\mathcal{A}_P)$ and they are both shorter than loop $\mathcal{A}_P$. Hence we are finished by Lemmas 2.3 and 1.2. ∎

Lemma 2.8. *A unipattern A forces some unipattern B such that $E(A) \leq E(B)$,* $\mathrm{per}(A) \geq \mathrm{per}(B)$ *and Q, representative of B, has a simple loop $\mathcal{A}_Q$.*

Proof. If P a representative of A has simple loop then $B = A$. If not then by Lemma 2.7 the pattern A forces a unipattern A^* such that $E(A) \leq E(A^*)$, $\mathrm{per}(A) > \mathrm{per}(A^*)$. Because $\mathrm{per}(A)$ is finite after repeating this finitely many times we must get our unipattern B. ∎

Lemma 2.9. *Let A be an r-unipattern and let its representative P have a simple loop $\mathcal{A}_P$. Then the pattern A forces some q-unipattern for each $q \in \mathbb{Q}$ such that $r \geq q \geq 1$.*

Proof. We may assume that $E(P) = r > 1$ (the case $r = 1$ is trivial) and $\mathrm{per}(P) = k(m + n)$ where $\frac{m}{n} = r$ (m, n are coprime).

Because $\mathcal{A}_P$ is simple it contains every interval from the partition $\mathfrak{P}$. So we may assume that the loop $\mathcal{A}_P$ starts with interval J_{km}. Moreover $J_{km} \xrightarrow{P} J_{km+1}$ and $J_{km+1} \xrightarrow{P} J_{km}$. Hence

$$\mathcal{B} = \overbrace{\langle J_{km}, J_{km+1}, \ldots, J_{km}, J_{km+1}\rangle}^{a \text{ times } J_{km}, J_{km+1}} + \overbrace{\mathcal{A}_P + \cdots + \mathcal{A}_P}^{b \text{ times } \mathcal{A}_P}$$

is a P-loop with eccentricity $\frac{a+bm}{a+bn}$. If $q = \frac{r}{s}$ then we can choose $a = ms - rn$ and $b = r - s$. So $\frac{a+bm}{a+bn} = q$. Hence from Lemma 2.3 we get that f_P has a q-unicycle and by Lemma 1.1 the pattern A forces a q-unipattern. ∎

Now we can easily get the final statement of this section.

Theorem 2.10. *Let $r, q \in \mathbb{Q}$ satisfy $r \geq q \geq 1$. Then any r-pattern forces a q-unipattern.*

Proof. Let A be an r-pattern. If A is not a unipattern then, by Lemma 2.1 and Lemma 1.1, A forces a q-unipattern. If A is a unipattern then, by Lemma 2.8, it forces a unipattern B such that $E(B) \geq r$ and P a representative of B has a simple loop $\mathcal{A}_P$. By Lemma 2.9 the pattern B forces a q-unipattern but because forcing relation is transitive we have that A forces a q-unipattern. ∎

3. X-minimal patterns

First we recall the

Definition. An r-pattern is *X-minimal* if it does not force any other r-pattern.

Now we would like to find all X-minimal r-patterns. We already have some information about such patterns. More precisely we have

Lemma 3.1. *An X-minimal pattern is a unipattern and its representative P has a simple loop $\mathcal{A}_P$.*

Proof. If $[P]$ is an X-minimal pattern then by Corollary 2.2 it is a unipattern. If it does not have simple loop $\mathcal{A}_P$ then by Lemma 2.8 it forces a unipattern $[Q]$ with a simple loop $\mathcal{A}_Q$ and $E(Q) > E(P)$. So $[P] \neq [Q]$ and by Lemma 2.9 the pattern $[Q]$ forces a pattern with eccentricity $E(P)$. Finally because forcing relation is antisymetrical we have that $[P]$ is not X-minimal - a contradiction. ∎

Definition. Let P be a unicycle and $c \in \mathrm{Fix}(f_P)$. A sequence $Q = \langle q_i \rangle_{i=0}^{a}$ will be called a *P-semicycle* if

$$\begin{aligned} q_i &\in P \\ f_P(q_{i-1}) &= q_i \qquad \text{for} \quad 1 \leq i \leq a \\ q_0 &\neq q_a \\ q_0 &\in \mathrm{conv}\{q_a, c\}. \end{aligned}$$

Eccentricity of the P-semicycle Q will be $E(Q) = \frac{\#\{i>0; q_i<c\}}{\#\{i>0; q_i>c\}}$. (See Fig. 4.)

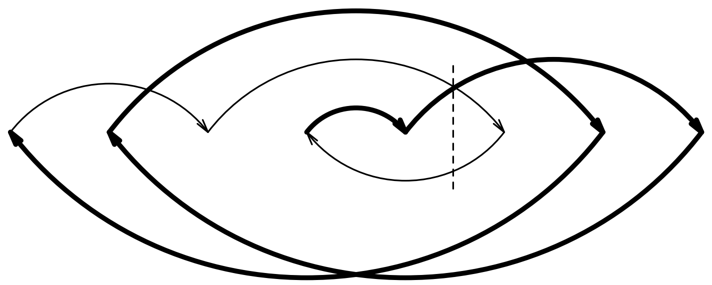


Figure 4. A cycle P with a semicycle Q (thick lines). $E(Q) = \frac{3}{2}$.

Lemma 3.2. *Let P be a unicycle with a P-semicycle Q. Then f_P has a $E(Q)$-cycle R such that* $\mathrm{per}(R) \neq \mathrm{per}(P)$.

Proof. Let $Q = \langle q_i \rangle_{i=0}^{a}$ and $I_i = \mathrm{conv}\{q_i, c\}$. Clearly $I_0 \xrightarrow{P} I_1 \xrightarrow{P} \dots \xrightarrow{P} I_a$ and $I_0 \subset I_a$. Therefore $\langle I_i \rangle_{i=1}^{a}$ is a separated P-cyclic sequence with eccentricity $E(Q)$. By Lemma 2.3 the function f_P has an $E(Q)$-cycle R. Moreover a is not divisible by $\mathrm{per}(P)$ and therefore $\mathrm{per}(R) \neq \mathrm{per}(P)$. ∎

Definition. Let P be an $\frac{m}{n}$-unicycle where $m \geq n \in \mathbb{N}$ are coprime and $c \in \mathrm{Fix}(f_P)$. Define the coding $K_P : P \to \mathbb{Z}$ by

$$\begin{aligned} K_P(p_1) &= 0 \\ K_P(f_P(p_i)) &= K_P(p_i) + n \qquad \text{for} \quad p_i < c \\ K_P(f_P(p_i)) &= K_P(p_i) - m \qquad \text{for} \quad p_i > c. \end{aligned}$$

We say that P has monotone code if and only if either $\mathrm{per}(P) = 1$ or $E(P) > 1$ and for any $p, q \in P$ such that $p \neq q$ and $q \in \mathrm{conv}(\{p, c\})$ we have $K_P(q) > K_P(p)$ (see Fig. 5).

If P has monotone code we also say that the pattern $[P]$ has monotone code.

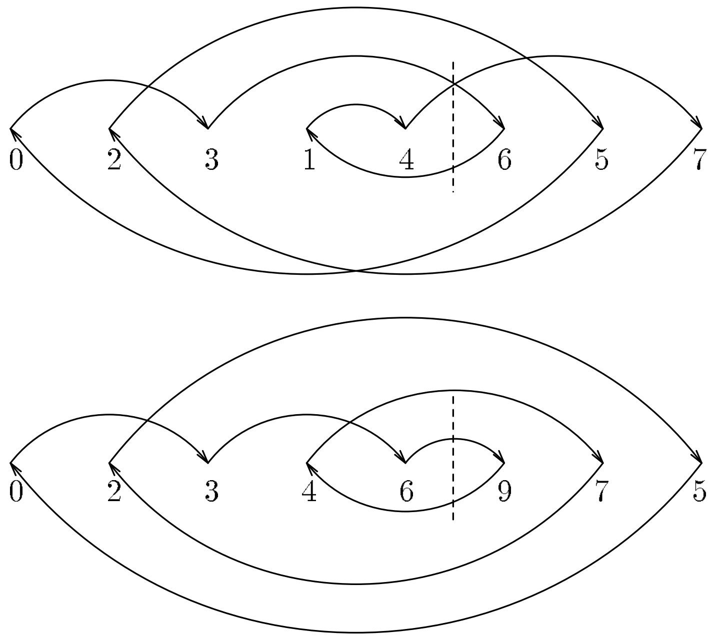


FIGURE 5. An example of a cycle without (top) and with (bottom) monotone code.

Lemma 3.3. *An X-minimal pattern has monotone code.*

Proof. Let A be an X-minimal $\frac{m}{n}$-pattern (m, n are coprime) and an $\frac{m}{n}$-cycle P be a representative of A. From Lemma 2.1 we have that P is a unicycle.

Assume that P does not have monotone code. Then there are two different $p, q \in P$ such that $q \in \mathrm{conv}(\{p, c\})$ and $K_P(q) \leq K_P(p)$. Set $Q = \langle q_j \rangle_{j=0}^{a}$ where $q_0 = q$, $q_{j+1} = f_P(q_j)$ and $q_a = p$. Clearly Q is a semicycle and we can estimate $E(Q)$. From definition of K_P we have that

$$K_P(q_a) = K_P(q_0) + n\#\{j;\ 0 \leq j < a,\ q_j < c\} - m\#\{j;\ 0 \leq j < a,\ q_j > c\}.$$

Hence $E(Q) = \frac{\#\{j;\ 0 \leq j < a,\ q_j < c\}}{\#\{j;\ 0 \leq j < a,\ q_j > c\}} \geq \frac{m}{n}$. Using Lemma 3.2 and Lemma 1.1 we get that A forces a pattern $B \neq A$ such that $E(B) \geq \frac{m}{n}$ and from Theorem 2.10 we have that B forces an $\frac{m}{n}$-pattern C. But $A \neq C$ because the forcing relation is antisymetric and so A is not an X-minimal $\frac{m}{n}$-pattern - a contradiction. ■

So we have proved that an X-minimal pattern is a unipattern with monotone code. Now we are going to get more information about a unicycle with monotone code.

Let $(P = \{p_1, p_2, \ldots, p_{k(m+n)}\}, \varphi)$ be an $\frac{m}{n}$-unicycle with spatial labeling, monotone code ($m \geq n$ are coprime) and $c \in \mathrm{Fix}(f_P)$. From monotonicity of code we immediately have that $\varphi(p_i) < c$ for $i > km$.

Hence we can define new cycle (P^*, ψ) where $P^* = \{p_i\}_{i=1}^{km}$ and

$$\begin{aligned} &\text{if} \quad \varphi(p_i) \in P^* \qquad \text{then} \quad \psi(p_i) = \varphi(p_i) \\ &\text{if} \quad \varphi(p_i) \notin P^* \qquad \text{then} \quad \psi(p_i) = \varphi^2(p_i). \end{aligned}$$

So we can have

Definition. Let $C_P = \langle c_i \rangle_{i=1}^{km}$ where $c_i \in \{0, 1\}$ be a code corresponding to the cycle P in the following way

$$\begin{aligned} &c_i = 0 \qquad \text{if} \quad \psi^i(p_1) = \varphi(\psi^{i-1}(p_1)) \\ &c_i = 1 \qquad \text{if} \quad \psi^i(p_1) = \varphi^2(\psi^{i-1}(p_1)). \end{aligned}$$

From the monotonicity of code K_P it can be seen that the code C_P can be also obtained from the cycle (P^*, ψ) if we start at the point p_1 and following cycle we write 0 if we move right and 1 if left (see Fig. 6).

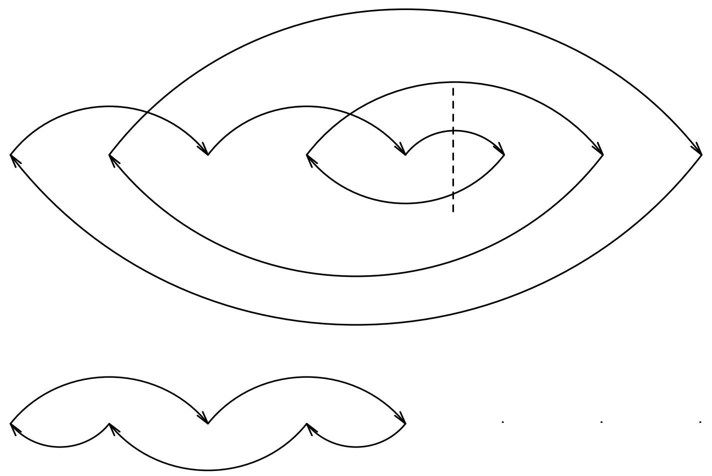


FIGURE 6. An example of a cycle (P, φ) (top) and (P^*, ψ) (bottom) with $C_P = \langle 0, 0, 1, 1, 1 \rangle$.

Note that C_P contains kn ones and $k(m-n)$ zeros. Moreover

$$\begin{aligned} &\text{if} \quad c_i = 0 \qquad \text{then} \quad K_P(\psi^i(p_1)) = K_P(\psi^{i-1}(p_1)) + n \\ &\text{if} \quad c_i = 1 \qquad \text{then} \quad K_P(\psi^i(p_1)) = K_P(\psi^{i-1}(p_1)) - m + n. \end{aligned}$$

Hence we have the following connection between K_P and C_P:

$$K_P(\psi^i(p_1)) = in - m \sum_{j=1}^{i} c_j$$

Lemma 3.4. *Let $m > n$ be coprime and P be an $\frac{m}{n}$-unicycle with monotone code. Then* $\operatorname{per}(P) = m + n$.

Proof. Assume that $P = \{p_1, \ldots, p_{k(m+n)}\}$ with spatial labeling and $k > 1$. We will study the code C_P.

Let i_j be such that $c_{i_j} = 1$ and $\sum_{i=1}^{i_j} c_i = j$ (i_j is the place of jth unit in the sequence C_P).

Because $k > 1$ we have $\psi^{i_n}(p_1) \neq p_1$ and from monotonicity of the code we have $K_P(\psi^{i_n}(p_1)) > 0$. But $K_P(\psi^{i_n}(p_1)) = ni_n - mn$ and so we have that $i_n > m$.

Moreover from monotonicity of code we have that no two points from P^* can have the same value of the code K_P. If there is a part $C^* = \langle c_i \rangle_{i=j+1}^{j+m}$ of the code C_P such that $\sum_{i=j+1}^{j+m} c_i = n$ then $K_P(\psi^{j+m}(p_1)) = K_P(\psi^j(p_1)) + (m-n)n + n(n-m) = K_P(\psi^j(p_1))$. But $\psi^{j+m}(p_1) \neq \psi^j(p_1)$ ($k > 1$) and so we have a contradiction with the monotonicity of code. So there must not be part of a sequence C_P of length m which contains $m - n$ times 0 and n times 1. Hence we get that $i_n - i_1 \geq m$ (otherwise the sequence $\langle c_i \rangle_{i=i_n-m+1}^{i_n}$ contains $m - n$ times 0 and n times 1).

Therefore $i_1 < i_{n+1} - m + 1$ and using sequence $\langle c_i \rangle_{i=i_{n+1}-m+1}^{i_{n+1}}$ as above we obtain $i_{n+1} - i_2 \geq m$. Inductively for all $j \leq (k-1)n$

$$i_{n+j} - i_{1+j} \geq m.$$

We have $c_1 = 0$ because $K_P(\psi(p_1)) \geq 0$ (monotonicity) and so $1 < i_1 < i_2 < \cdots < i_{kn-1} < i_{kn} \leq kn$. Using the inequalities above we obtain

$$km \geq 1 + \sum_{j=1}^{k} (i_{jn} - i_{(j-1)n+1}) \geq 1 + \sum_{j=1}^{k} m = 1 + km$$

which is a contradiction. ∎

Lemma 3.5. *Let P be a unicycle which is not X-minimal. Then f_P has a unicycle R such that that* $\operatorname{per}(R) < \operatorname{per}(P)$ *and* $E(R) \geq E(P)$.

Proof. If $\mathcal{A}_P$ is not simple then by Lemma 2.7 function f_P has a unicycle Q with $\operatorname{per}(Q) < \operatorname{per}(P)$ and $E(Q) \geq E(P)$. By Lemma 2.8 we have that f_P has a unicycle R such that $\operatorname{per}(R) \leq \operatorname{per}(Q)$ and $E(R) \geq E(Q)$ and so we are done.

Assume that $\mathcal{A}_P$ is simple. If $\operatorname{per}(P) = 2$ then cycle given by a fixed point is our cycle R. So we can assume that $\operatorname{per}(P) > 2$ (if $\operatorname{per}(P) = 1$ then P is X-minimal).

Because P is not X-minimal function f_P contains cycle $Q \neq P$ with $E(Q) = E(P)$. Let $\mathcal{A}$ be the unique P-loop containing cycle Q (Lemma 2.6). Because $\mathcal{A}_P$ is simple and $P \neq Q$, using Lemma 2.6 we have that $\mathcal{A} \neq \mathcal{A}_P + \cdots + \mathcal{A}_P$. Hence $\mathcal{A}$ can be written as the sum of two P-loops $\mathcal{B} + \mathcal{C}$ (the loop $\mathcal{C}$ may be empty) such that $\mathcal{B}$ is a simple P-loop, $E(\mathcal{B}) \geq E(\mathcal{A}_P)$ and $\mathcal{B} \neq \mathcal{A}_P$.

If the length of $\mathcal{B}$ is smaller then $\operatorname{per}(P)$ then cycle R given by the P-loop $\mathcal{B}$ (Lemma 1.2) is the one we are looking for (see Lemma 2.3).

So the length of $\mathcal{B}$ is at least $\operatorname{per}(P)$ and from Lemma 2.5 we have that it must be equal $\operatorname{per}(P)$. Hence both $\mathcal{A}_P$ and $\mathcal{B}$ contain all intervals from $\mathfrak{P}$. Because they are different there are intervals $I, J, K \in \mathfrak{P}$ such that $J \neq K$ and

$$\mathcal{A}_P = \langle \ldots, I, J, \ldots \rangle$$
$$\mathcal{B} = \langle \ldots, I, K, \ldots \rangle.$$

Hence after suitable rotation we can write

$$\mathcal{A}_P = \overbrace{\langle J, \dots, L\rangle}^{\mathcal{D}} + \overbrace{\langle K, \dots, I\rangle}^{\mathcal{E}}$$
$$\mathcal{B} = \langle K, \dots, I\rangle.$$

Note that $\mathcal{D}$ is nonempty and the loop $\mathcal{E}$ is P-cyclic.

If $E(\mathcal{E}) \geq E(\mathcal{A}_P)$ then the loop $\mathcal{E}$ gives us a cycle R with period smaller than $\mathrm{per}(P)$ and we are done (Lemmas 1.2 and 2.3).

Otherwise $\mathcal{D} + \mathcal{B}$ is a P-cyclic loop with $E(\mathcal{D} + \mathcal{B}) > E(\mathcal{A}_P)$. So it can be written as a sum of two P-loops such that one of them is a simple P-loop $\mathcal{F}$ such that $E(\mathcal{F}) > E(\mathcal{A}_P)$. This loop has length smaller then $\mathrm{per}(P)$ (all simple P-loops with length have eccentricity $E(\mathcal{A}_P)$) and so it will give us a cycle R with period smaller than $\mathrm{per}(P)$ (Lemmas 1.2 and 2.3). ■

Let P be an $\frac{m}{n}$-unicycle with a simple loop $\mathcal{A}_P = \langle I_i\rangle_{i=1}^{k(m+n)}$. We have $f_P^{i-1}(p_1) \in I_i$. So we may define a map $\pi : P \to \mathfrak{P}$ such that $\pi(f_P^{i-1}(p_1)) = I_i$. Because $\mathcal{A}_P$ is simple π is a bijection. Moreover if $\pi(p_j) = I_i$ then $p_j \in I_i$. Recall that $\mathfrak{P} = \{J_i\}_{i=1}^{k(m+n)}$ with spatial labeling (see the definition of $\mathfrak{P}$) and $\mathcal{A}_P$ is simple loop of the lenght $k(m+n)$. Hence there is clearly only one possibility for mapping π and that is

$$\pi(p_i) = J_i \qquad \text{for} \quad 1 \leq i \leq k(m+n)$$

(see Fig. 7).

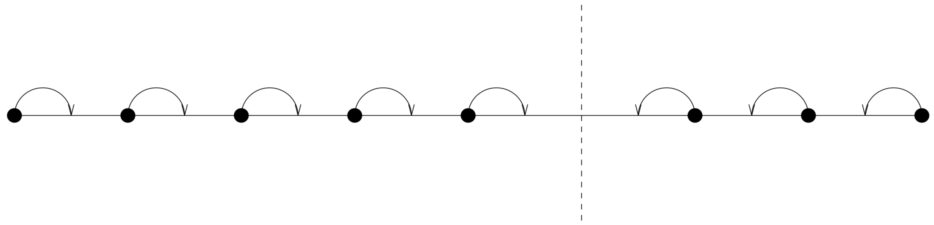


FIGURE 7. The arrows show on which intervals points of the cycle are mapped by π.

We know that $\pi(x) \xrightarrow{P} \pi(f_P(x))$ and using the bijection π we can naturally define a coding $K : \mathfrak{P} \to \mathbb{Z}$ similar to the coding $K_P : P \to \mathbb{Z}$ where

$$K(J_i) = K(\pi(p_i)) = K_P(p_i).$$

Now we shall estimate how the code of the intervals in the partition $\mathfrak{P}$ depends on the P-covering property for these intervals.

Lemma 3.6. *Let P be an $\frac{m}{n}$-unicycle with monotone code, $\mathfrak{P} = \{J_i\}_{i=1}^{m+n}$ with spatial labeling and $J_i, I \in \mathfrak{P}$ such that $J_i \xrightarrow{P} I$. If $i \leq m$ then $K(I) \geq K(J_i) + n$ and if $i > m$ then $K(I) \geq K(J_i) - m$.*

Proof. If $i = m$ than $J_i = [p_m, c]$ and so $I \subset [c, f_P(p_m)]$. From the monotonicity of the code we have $K(I) \geq K(\pi(f_P(p_m))) = K(J_i) + n$.

If $i < m$ then $J_i = [p_i, p_{i+1}]$. But f_P is linear on J_i and so if $J_i \xrightarrow{P} J_j$ then $p_j \in \text{conv}\{f_P(p_i), f_P(p_{i+1})\}$. From the monotonicity of the code we have that $K_P(p_j) \geq \min\{K_P(f_P(p_i)), K_P(f_P(p_{i+1}))\}$. But $K_P(f_P(p_i)) < K_P(f_P(p_{i+1}))$ and so $K_P(p_j) \geq K_P(f_P(p_i)) = K_P(p_i) + n$. Hence $K(I) \geq K(J_i) + n$.

Now let $i > m$. From monotonicity of code we have that $f_P(p_{j+1}) < f_P(p_j) < c$ for all $j > m$. Hence $f_P([c, p_i]) = [f_P(p_i), c]$ and so $K(I) \geq K_P(f_P(p_i)) = K(J_i) - m$. ■

Finally we are ready to prove

Theorem 3.7. *Let P be a periodic orbit. Then P is X-minimal if and only if it is a unicycle with monotone code.*

Proof. The necessity of these conditions is proved in Lemmas 2.1 and 3.3. Now we will show that they are sufficient too.

Let P be an $\frac{m}{n}$-unicycle with monotone code. If P is not X-minimal then by Lemma 3.5 there is cycle Q of f_P such that $\text{per}(Q) < \text{per}(P)$ and $E(Q) \geq E(P)$. By Lemma 2.4 there is a P-loop $\mathcal{A} = \langle I_i \rangle_{i=1}^{a+b}$ such that $E(\mathcal{A}) = E(Q) = \frac{a}{b}$ where $a = \#\{i; I_i < c\}$ and $b = \#\{i; I_i > c\}$. Finally by Lemma 3.6 we have that

$$K(I_1) = K(I_1) + \sum_{i=1}^{a} n_i - \sum_{i=1}^{b} m_i$$

where $n_i \geq n$ and $m_i \leq m$. Therefore $\frac{a}{b} \leq \frac{m}{n}$. But from Lemma 3.4 we have that $a + b < m + n$ and hence $\frac{a}{b} < \frac{m}{n}$ which contradicts $E(Q) \geq E(P)$. ■

From this Theorem we have immediately

Corollary 3.8. *A pattern is X-minimal if and only if it is a unipattern with monotone code.*

Although it is very easy to check if a pattern is a unipattern with monotone code it is still not a "look and see" (geometrical) characterization. We have at least some easy necessary geometrical condition.

Lemma 3.9. *Let P be a representative of an X-minimal pattern with $E(P) > 1$. Then P is unicycle and for any $p, q \in P$ and $\text{Fix}(f_P) = \{c\}$ we have*

$$\begin{aligned}
&\text{if} \quad p < q < c \quad \text{and} \quad f_P(p), f_P(q) < c \quad \text{then} \quad f_P(p) < f_P(q)\\
&\text{if} \quad p < q < c \quad \text{and} \quad f_P(p), f_P(q) > c \quad \text{then} \quad f_P(p) > f_P(q)\\
&\text{if} \quad p > q > c \quad \text{then} \quad f_P(p), f_P(q) < c \quad \text{and} \quad f_P(p) < f_P(q).
\end{aligned}$$

Proof. This follows easily from Theorem 3.7 and the monotonicity of the code. ■

Unfortunately these conditions are not sufficient (see Fig. 8) and we do not know if there exists a good geometrical characterization at all.

4. Existence of X-minimal orbits

In the previous section we gave a characterization of X-minimal orbits. However, if we have a function $f \in C(I, I)$ with a periodic orbit with eccentricity r it is still

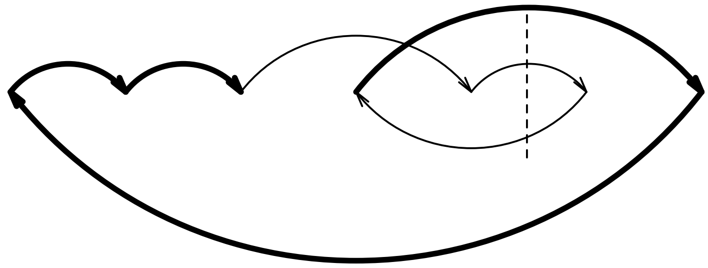


FIGURE 8. A cycle P satisfying conditions from Lemma 3.9 which is not X-minimal. It can be easy checked from code K_P or by finding a semicycle with eccentricity $\frac{3}{1}$ (thick lines).

not clear whether this map has an X-minimal r-cycle. This is because the set of all patterns with given eccentricity is infinite and so theoretically there may exist a sequence of r-patterns such that each one forces the next one and none of them forces an X-minimal r-pattern. In this section we will show that this is impossible. This question has been partially solved in [BK] for patterns with eccentricities of the form $\frac{k+1}{k}$.

Lemma 4.1. *An r-unipattern A forces an X-minimal q-pattern B such that $q \geq r$ and* $\mathrm{per}(A) \geq \mathrm{per}(B)$.

Proof. If A is an X-minimal r-pattern take $B = A$ and we are done.

If A is not an X-minimal pattern then by Lemmas 3.5 and 1.1 we have that A forces a unipattern A^* such that $E(A^*) \geq E(A)$ and $\mathrm{per}(A^*) < \mathrm{per}(A)$. Applying this finitely many times we must get an X-minimal pattern B. ■

Now we investigate X-minimal patterns more closely. Let P be an X-minimal cycle with eccentricity $\frac{m}{n}$. We would like to know what cycles does it force. If we think a little about Lemma 3.1 and the way we proved Lemma 3.5 we can see that among all the cycles forced by P only those with period lower than $\mathrm{per}(P)$ are important. Any other cycle forced by P can be obtained by a "gluing" some of these cycles together. Because P is X-minimal we have that eccentricities of these cycles depending on the period are bounded above by $\frac{\left[\frac{im}{n}\right]}{i}$ for $i = 1, \dots, n$. We now consider those that give us a maximal possible eccentricity with a minimal possible period.

Definition. Let $m, n \in \mathbb{N}$, $m > n$ coprime. The fraction $\frac{\left[\frac{im}{n}\right]}{i}$ is called an $\frac{m}{n}$-*extremal fraction* if $1 \leq i \leq n$ and $\frac{\left[\frac{im}{n}\right]}{i} > \frac{\left[\frac{jm}{n}\right]}{j}$ for all $j \in \{1, \dots, i-1\}$.

Remark. Note that $\left[\frac{im}{n}\right]$ and i are coprime for an $\frac{m}{n}$-extremal fraction $\frac{\left[\frac{im}{n}\right]}{i}$.

Lemma 4.2. *Let $\frac{p_{j-1}}{q_{j-1}} \leq \frac{p}{q} \leq \frac{p_j}{q_j}$ where $\frac{p_{j-1}}{q_{j-1}} < \frac{p_j}{q_j}$ are consecutive $\frac{m}{n}$-extremal fractions. There are nonnegative integer numbers b, c such that $p = bp_{j-1} + cp_j$ and $q = bq_{j-1} + cq_j$.*

Proof. We will use Farey series (see e.g. [HW]). We show that $\frac{p_{j-1}}{q_{j-1}}, \frac{p_j}{q_j}$ are consecutive terms of the Farey series of order q_j. If not then there is a term $\frac{p^*}{q^*}$ such that

$\frac{p_{j-1}}{q_{j-1}} < \frac{p^*}{q^*} < \frac{p_j}{q_j}$ and $\frac{p^*}{q^*}, \frac{p_j}{q_j}$ are consecutive terms of the Farey series of order q_j. So we have that $q^* \le q_j$.

If $q^* < q_j$ then $\frac{\left[\frac{q^* m}{n}\right]}{q^*}$ is $\frac{m}{n}$-extremal fraction - a contradiction with assumption that $\frac{p_{j-1}}{q_{j-1}}, \frac{p_j}{q_j}$ are consecutive $\frac{m}{n}$-extremal fractions.

If $q^* = q_j$ then from Theorem 28 [HW] we have $p_j q^* - p^* q_j = 1$. But this is possible only if $q_j = 1$ which is contradiction with definition of $\frac{m}{n}$-extremal fractions.

Hence we have that $\frac{p_{j-1}}{q_{j-1}}, \frac{p_j}{q_j}$ are consecutive terms of the Farey series of order q_j and our lemma follows from 3.3. First proof of Theorems 28 and 29. [HW] ∎

Lemma 4.3. *Let P be an X-minimal $\frac{m}{n}$-unicycle ($m > n$ coprime) and $\frac{p}{q}$ be an $\frac{m}{n}$-extremal fraction. Then f_P has a $\frac{p}{q}$-unicycle Q with* $\mathrm{per}(Q) = p + q$.

Proof. If $\frac{p}{q} = \frac{m}{n}$ then we can set $Q = P$. So we can assume that $\frac{p}{q} < \frac{m}{n}$ and let $c \in \mathrm{Fix}(f_P)$.

We will show that there is a P-semicycle with eccentricity $\frac{p}{q}$. We will define a code $C = \langle c_i \rangle_{i=1}^{(m+n)q}$ where

$$\begin{aligned} c_i &= 0 \qquad \text{if} \quad f_P^{i-1}(p_1) < c \\ c_i &= 1 \qquad \text{if} \quad f_P^{i-1}(p_1) > c. \end{aligned}$$

Note that $c_1 = 0$ and if $c_i = 1$ then $c_{i-1} = 0$ and $c_{i+1} = 0$ (if $i + 1 \le (m+n)q$). There is a close connection between C and K_P:

$$K_P(f_P^i(p_1) = in - (m+n) \sum_{j=1}^{i} c_j.$$

We will show that there is a piece of code $C^* = \langle c_i \rangle_{i=j+1}^{j+p+q}$ such that $c_{j+1} = 0$, $c_{j+p+q} = 1$ and $\sum_{i=j+1}^{j+p+q} c_i = q$.

Assume to the contrary that there is no such sequence C^*.

We will use very similar technique as in the proof of Lemma 3.4. Let i_j be such that $c_{i_j} = 1$ and $\sum_{i=1}^{i_j} c_i = j$ (i_j is position of jth unit in C).

Note that $K_P(p) \ge 0$ for all $p \in P$ because P has monotone code. If $i_q \le p + q$ then $K_P(f_P^{i_q}(p_1) < 0$ (because $\frac{p}{q} < \frac{m}{n}$) – a contradiction. Hence $i_q > p + q$.

If $i_q - i_1 < p+q$ then $\langle c_i \rangle_{i=i_q-(p+q)}^{i_q}$ would be our sequence C^*. Hence $i_q - i_1 \ge p+q$ and because $c_{i_q+1} = 0$ we have that $i_{q+1} > i_q + 1$ and so $i_{q+1} - (i_1 + 1) > p + q$. Again because there is no sequence C^* we have that $i_{q+1} - i_{1+1} \ge p + q$. Repeating this argument we obtain

$$i_{j+q} - i_{j+1} \ge p + q$$

for $1 \le j \le (n-1)q$. Now using these inequalities and the fact that

$$1 < i_1 < i_q < i_q + 1 < i_{q+1} < \cdots < i_{jq} < i_{jq} + 1 < i_{jq+1} < \cdots < i_{nq} \le (m+n)q$$

we get the inequality

$$(m+n)q \geq \sum_{j=1}^{n}(1+i_{jq}-i_{(j-1)q+1}) \geq n+n(p+q).$$

From here we have that $\frac{m}{n} \geq \frac{p+1}{q}$ which contradicts the assumption that $\frac{p}{q}$ is $\frac{m}{n}$-extremal fraction. Hence we have proved the existence of a sequence C^*.

Now we will show that sequence $A = \langle f_P^i(p_1)\rangle_{i=j+1}^{j+p+q+1}$ connected with C^* is a P-semicycle.

Because $c_{j+1} = 0$ and $c_{j+p+q} = 1$ we have that both $f_P^{j+1}(p_1), f_P^{j+p+q+1}(p_1) < c$. Moreover

$$K_P(f_P^{j+p+q+1}(p_1)) = K_P(f_P^{j+1}(p_1)) + pn - qm < K_P(f_P^{j+1}(p_1))$$

and from monotonicity of code we have that $f_P^{j+p+q+1}(p_1) < f_P^{j+1}(p_1) < c$. Therefore A is a P-semicycle. Clearly its eccentricity is $\frac{p}{q}$ and so using Lemma 3.2 the function f_P has a $\frac{p}{q}$-unicycle Q. Finally because p, q are coprime we have that $\mathrm{per}(Q) = p + q$. ■

Lemma 4.4. *Suppose m, n are coprime, p, q are coprime and $\frac{m}{n} \geq \frac{p}{q} \geq 1$. Then an X-minimal $\frac{m}{n}$-unipattern forces some $\frac{p}{q}$-unipattern with period $p + q$.*

Proof. Let A be an X-minimal $\frac{m}{n}$-unipattern.

If $\frac{p}{q} \leq \frac{p_i}{q_i} < \frac{m}{n}$ where $\frac{p_i}{q_i}$ is an $\frac{m}{n}$-extremal fraction then from Lemmas 4.1 and 4.3 the pattern A forces an X-minimal $\frac{m^*}{n^*}$-pattern A^* such that $\frac{p_i}{q_i} \leq \frac{m^*}{n^*}$ and $\mathrm{per}(A^*) < \mathrm{per}(A)$. But forcing relation is transitive and so it is enough to prove that A^* forces some $\frac{p}{q}$-unipattern with period $p + q$. We can repeat this reduction and because we decrease the period we must stop after finitely many steps.

Hence we can assume that $\frac{p_{a-1}}{q_{a-1}} < \frac{p}{q} \leq \frac{p_a}{q_a}$ where $\frac{p_{a-1}}{q_{a-1}}, \frac{p_a}{q_a}$ are two biggest $\frac{m}{n}$-extremal fractions.

From Lemma 4.2 we have that $p = bp_{a-1} + cp_a$ and $q = bq_{a-1} + cq_a$ for some nonnegative integers b, c. It is clear that $\frac{p_a}{q_a} = \frac{m}{n}$ and $c > 0$ because $\frac{p_{a-1}}{q_{a-1}} < \frac{p}{q}$.

Let P be a representative of A. By Lemma 4.3 the function f_P has $\frac{p_{a-1}}{q_{a-1}}$-unicycle P_{a-1} with period $p_{a-1} + q_{a-1}$ and by Lemma 2.4 there is a P-loop $\mathcal{A}_{a-1}$ of length $p_{a-1} + q_{a-1}$ and eccentricity $\frac{p_{a-1}}{q_{a-1}}$ connected with this cycle.

Loop $\mathcal{A}_P$ has length $p_a + q_a$ and eccentricity $\frac{p_a}{q_a}$. Moreover because P is X-minimal by Lemma 3.1 the loop $\mathcal{A}_P$ is simple and therefore it contains all the intervals from $\mathfrak{P}$.

Hence we can connect c times loop $\mathcal{A}_P$ and b times loop $\mathcal{A}_{a-1}$ into one single P-loop with length $p+q$ and eccentricity $\frac{p}{q}$. This P-loop gives us a cycle Q (Lemma 2.3) such that $\mathrm{per}(Q) = p + q$ (p, q are coprime). Finally let $B = [Q]$ and apply Lemma 1.1. ■

Theorem 4.5. *Any r-pattern forces an X-minimal r-pattern.*

Proof. By Theorem 2.10 an r-pattern forces an r-unipattern and by Lemma 4.1 an r-unipattern forces an X-minimal q-pattern for some $q \geq r$. By Lemma 4.4

this pattern forces an r-unipattern with minimal possible period. Because there are only finitely many r-patterns with this period after repeating this procedure finitely many times we must get an X-minimal r-pattern. ■

We shall end this part by an easy algorithm to construct all X-minimal pattern. Let us consider a cycle P with an eccentricity $\frac{m}{n}$. We have defined a code $K_P : P \to \mathbb{Z}$. Clearly different orbits have different codes. Moreover if we have a code $K_P : P \to \mathbb{Z}$ of an X-minimal $\frac{m}{n}$-cycle then we can easily reconstruct the function φ of the cycle (P, φ) from the given code using the following simple algorithm (assume that $P = \{1, \dots, m+n\}$).

Algorithm 1. *If $K_P(i) - K_P(j) = m$ or $K_P(j) - K_P(i) = n$ then $\varphi(i) = j$.*

We have also defined a code C_P connected with a given X-minimal cycle P. Again if we have a code $C_P = \langle c_1, \dots, c_m \rangle$ connected with a given X-minimal $\frac{m}{n}$-cycle P then we can easily reconstruct the code $K_P : P \to \mathbb{Z}$ and hence the cycle (P, φ).

Algorithm 2. *The function $K_P : P \to \mathbb{Z}$ is*

- *increasing on the set $\{1, \dots, m\}$*
  with values from $\left\{\sum_{j=1}^{k}(n - c_j m);\ k = 1, \dots, m\right\}$
- *decreasing on the set $\{m+1, \dots, m+n\}$*
  with values from $\left\{m + \sum_{j=1}^{k}(n - c_j m);\ c_k = 1\right\}$.

Now we take some code C^* which is only a rotation of the code C_P. The function K^* obtained from the code C^* using Algorithm 2 is nothing but the function K_P shifted by some negative multiple of n. Therefore if we apply Algorithm 1 to the function K^* we again obtain the cycle (P, φ).

Hence we can get any X-minimal $\frac{m}{n}$-cycle by choosing a sequence $C \in C(m,n)$ where $C(m,n) = \{\{c_i\}_{i=1}^{m} \in \{0,1\}^m;\ \sum_{i=1}^{m} c_i = n\}$, then using Algorithm 2 we get a code K^* and finally by Algorithm 1 we get an X-minimal cycle. (See Fig. 9.)

Note that you get different patterns if and only if you start from $C_1, C_2 \in C(m,n)$ such that C_1 is not a rotation of C_2. So we have the following simple

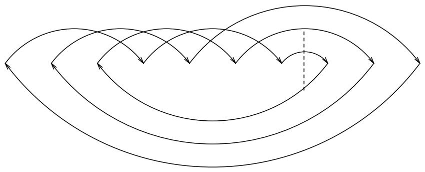
Corollary 4.6. (Proposition 4.4 from [BM]) *There are $\frac{m!}{m \cdot n! \cdot (m-n)!}$ different X-minimal $\frac{m}{n}$-patterns (m, n coprime).*

Remark. According to this Corollary there is a unique X-minimal $\frac{n+1}{n}$-pattern. This is of course the pattern of the Štefan cycle because this is the only pattern that does not force any other pattern with period $2n+1$ (see eg. [ALM]) and clearly any pattern with period $2n+1$ has eccentricity at least $\frac{n+1}{n}$.

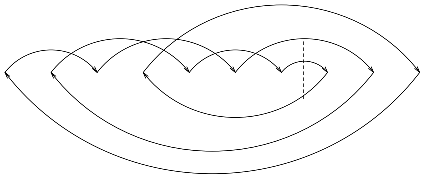
5. A Generalization of Sharkovskiĭ's Theorem

If we look back at Lemma 3.4 and Theorems 3.7, 4.5 and 2.10 then we can see that they in fact give a generalization of a part of Sharkovskiĭ's Theorem for odd periods. Indeed a pattern with a period $2k+1$ has eccentricity at least $\frac{k+1}{k}$, by Lemma 3.4 it forces a pattern with eccentricity $\frac{k+2}{k+1}$ which by Theorems 3.7, 4.5 and 2.10 forces a pattern with period $2k+3$.

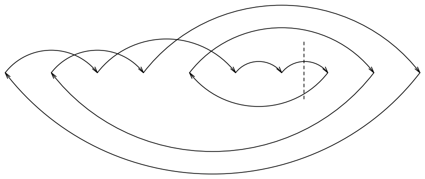
$C_P = \langle 0,0,1,0,1,0,1\rangle$



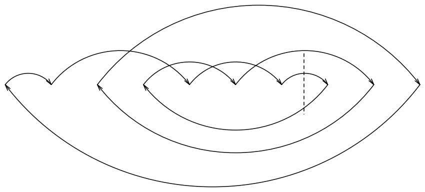
$C_P = \langle 0,0,1,0,0,1,1\rangle$



$C_P = \langle 0,0,0,1,1,0,1\rangle$



$C_P = \langle 0,0,0,1,0,1,1\rangle$



$C_P = \langle 0,0,0,0,1,1,1\rangle$

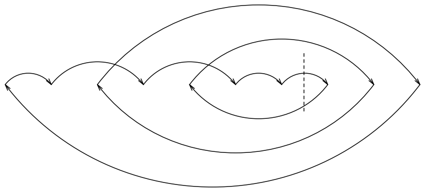


FIGURE 9. The list of all X-minimal $\frac{7}{3}$-patterns.

The part of Sharkovskiĭ's Theorem concerning even periods is somehow hidden in eccentricity equal to 1. So in order to get a full generalization we need to define a better type of patterns that will make a finer division of the set of all patterns with eccentricity 1.

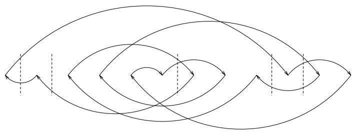
Let look a bit closer at a periodic orbit (P,φ) with $\mathrm{per}(P) > 1$ and $E(P) = 1$. It is clear that $\mathrm{per}(P)$ is even. So $P = P_1 \cup P_2$ such that (P_i, φ^2) is a periodic orbit with period $\frac{\mathrm{per}(P)}{2}$ for $i = 1,2$.

We shall say that $[(P,\varphi)]$ is a $(2,r)$-pattern if $E([(P,\varphi)]) = 1$, $\mathrm{per}(P) > 1$ and $E([(P_i,\varphi^2)]) = r$ for some $i \in \{1,2\}$. Inductively we say that $[(P,\varphi)]$ is a $(2^k,r)$-pattern for $k > 1$ if $E([(P,\varphi)]) = 1$ and $[(P_i,\varphi^2)]$ is a $(2^{k-1},r)$-pattern for some $i \in \{1,2\}$. Finally we say that an r-pattern is a $(1,r)$-pattern. (See Fig. 10.)

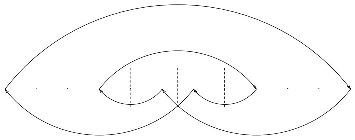
We define a space $\mathcal{X} = \{(2^k,a); k \in \mathbb{N}\cup\{0\}, a \in \mathbb{R}\cup\{\infty\}, a \geq 1\} \cup \{(2^\infty,1)\}$ and a total ordering relation on $\mathcal{X}$ such that

$$(2^{k-1},a) > (2^{k-1},b) > (2^k,a) > (2^\infty,1) > (2^k,1) > (2^{k-1},1)$$

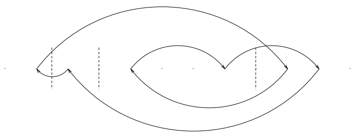
A cycle (P, φ) with eccentricities $\frac{1}{11}$, $\frac{1}{5}$, $\frac{1}{1}$, $\frac{3}{1}$ and $\frac{5}{1}$:



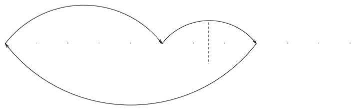
(P, φ) is a 1-cycle so we look at (P_1, φ^2) and (P_2, φ^2).
The cycle (P_1, φ^2) has eccentricities $\frac{1}{2}$, $\frac{1}{1}$ and $\frac{2}{1}$:



The cycle (P_2, φ^2) has eccentricities $\frac{1}{5}$, $\frac{1}{2}$ and $\frac{2}{1}$:



(P_1, φ^2) is a 1-cycle so we look at $(P_{1,1}, \varphi^4)$ and $(P_{1,2}, \varphi^4)$.
The cycle $(P_{1,1}, \varphi^4)$ has eccentricity $\frac{2}{1}$:



The cycle $(P_{1,2}, \varphi^4)$ has eccentricity $\frac{2}{1}$:

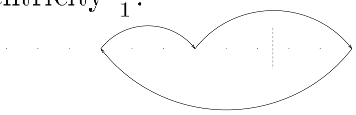


FIGURE 10. The pattern $[(P, \varphi)]$ has types $(1, \frac{11}{1})$, $(1, \frac{5}{1})$, $(1, \frac{3}{1})$, $(1, \frac{1}{1})$, $(2, \frac{5}{1})$, $(2, \frac{2}{1})$, $(2, \frac{1}{1})$ and $(4, \frac{2}{1})$.

for any $a, b \in \mathbb{R} \cup \{\infty\}$ such that $a > b > 1$ and $k \in \mathbb{N}$.

For any $(a, b) \in \mathcal{X}$ we define sets

$$\mathcal{X}(a, b) = \{(c, d) \in \mathcal{X};\ (a, b) \geq (c, d),\ c \in \mathbb{N},\ d \in \mathbb{Q}\}$$
$$\mathcal{X}_o(a, b) = \{(c, d) \in \mathcal{X};\ (a, b) > (c, d),\ c \in \mathbb{N},\ d \in \mathbb{Q}\}.$$

Now we may state

Theorem 5.1 (Generalized Sharkovskiĭ's Theorem).

(i) *Any (a, b)-pattern forces a (c, d)-pattern for any $(c, d) \in \mathcal{X}(a, b)$.*
(ii) *For any $(a, b) \in \mathcal{X}$ there is a function $f \in C(I, I)$ such that f exhibits a (c, d)-pattern if and only if $(c, d) \in \mathcal{X}(a, b)$.*
(iii) *For any $(a, b) \in \mathcal{X}$ there is a function $f \in C(I, I)$ such that f exhibits a (c, d)-pattern if and only if $(c, d) \in \mathcal{X}_o(a, b)$.*
(iv) *A $(2^k, \frac{m}{n})$-pattern with $\frac{m}{n} > 1$ (m, n coprime) forces a pattern with period $2^k(m + n)$. A $(2^k, 1)$-pattern forces a pattern with period 2^k.*

Proof. Part (i) will be proved as Lemma 5.9 and part (iv) will be proved as Lemma 5.10. Part (iii) will be proved as Lemma 5.16 and finally part (ii) will follow directly from Lemmas 5.15, 5.16 and Claim 5.17. ∎

In order to prove the theorem above we need the notion of block structure.

Let (P, φ) be a cycle of period n and $B = [(\{1, 2, \dots, m\}, \psi)]$ be a pattern of period m. Let $P = \{p_1, p_2, \dots, p_n\}$ have the spatial labeling. We say that (P, φ) has a block structure over B if $n = sm$, $P = P_1 \cup P_2 \cup \dots \cup P_m$ with $P_i = \{p_{(i-1)s+1}, \dots, p_{(i-1)s+s}\}$ for all $i = 1, 2, \dots, m$ and $\varphi(P_i) = P_{\psi(i)}$. Each of the sets P_i will be called a block of P. In other words, we could consider each block as a "fat" point and P as a "fat" cycle with pattern B (see Fig. 11).

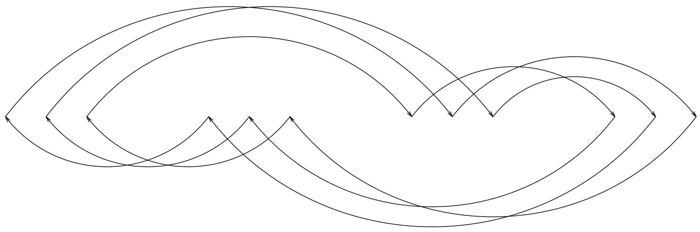


FIGURE 11. A pattern which has a block structure over a pattern with period 4.

Assume that P has a block structure over B and (Q, ψ) is a cycle with pattern B. Then we also say that P has a block structure over (Q, ψ). If P has a block structure over B (respectively over Q) we also say that pattern $[P]$ has a block structure over B (respectively over Q).

Note that if (P, φ) has a block structure over a pattern of period m, then (P_i, φ^m) is a cycle of period $\frac{\mathrm{per}(P)}{m}$ for all i.

If a cycle (pattern) has a block structure over a pattern with period 2 then we say that it has a division.

We already have enough information about patterns of type $(1, r)$ but we have no information about forcing relation for patterns of type $(2^k, r)$ where $k \in \mathbb{N}$. So take a $(2^k, r)$-pattern and let (P, φ) be its representative. Because $k \geq 1$ our pattern is also a $(2, q)$-pattern (either $q = 1$ or $q = r$). Hence $\mathrm{per}(P) = 2n$ where $n \in \mathbb{N}$. There are two possibilities. Either P has a division or not. In the case when P does not have a division we can use the following lemma.

Lemma 5.2. (Proposition 3.4 from [LMPY]) *Let A be a pattern with $\mathrm{per}(A) = 2n$ that does not have a division (so $n > 1$). Then if n is odd, the pattern A forces a pattern with period n. If n is even it forces a pattern with period $n + 1$.*

Hence we have following simple

Corollary 5.3. *A pattern with period greater than 1 which does not have a division forces a $(1, q)$-pattern for some $q > 1$.*

Proof. This is straightforward from Lemma 5.2. ∎

Now we will look closely at the patterns that have a division. Let (P, φ) be a representative of such a pattern. Obviously P is a unicycle, $E(P) = 1$ and

$\mathrm{per}(P) > 1$. We can look at the two cycles (P_1, φ^2) and (P_2, φ^2). If we have information about the types of patterns forced by $[P_1]$ and $[P_2]$ we can deduce information about the patterns forced by $[P]$. More precisely we have

Lemma 5.4. *Let (P, φ) be a representative of a pattern with a division and $P_1 \subset P$ such that (P_1, φ^2) is a cycle. If $[P_1]$ forces an (a, b)-pattern A then $[P]$ forces a $(2a, b)$-pattern with division and period $2 \cdot \mathrm{per}(A)$.*

Proof. Suppose $[P_1]$ forces an (a, b)-pattern A. Consider the function f_P^2 and the interval $I = \mathrm{conv}(P_1)$. Because P has a division we have $f_P(P_1) = P_2$, $f_P(P_2) = P_1$ and $\mathrm{conv}(P_1) \cap \mathrm{conv}(P_2) = \emptyset$. Hence $f_P^2|_I \in C(I, I)$. But $f_P^2|_I$ exhibits the pattern $[P_1]$ (it has the cycle P_1) and therefore it has a cycle Q_1 which is a representative of the pattern A. Let $f_P(Q_1) = Q_2$ and $Q = Q_1 \cup Q_2$. Clearly $(Q, f_P|_Q)$ is a cycle. We have $Q_2 \subset \mathrm{conv}(P_2)$ and therefore Q has a division, $E(Q) = 1$ and $\mathrm{per}(Q) > 1$. Hence $[Q]$ is a $(2a, b)$-pattern with a division. Clearly $\mathrm{per}(Q) = 2\,\mathrm{per}(A)$. ■

Lemma 5.5. *A $(1, r)$-pattern with $r > 1$ forces a $(2, q)$-pattern with a division for each $q \geq 1$.*

Proof. Let A be a $(1, r)$-pattern with $r > 1$. There is a $k \in \mathbb{N}$ such that $r > \frac{k+1}{k}$. By Theorems 2.10 and 4.5 A forces an X-minimal $(1, \frac{k+1}{k})$-pattern (which must be a Štefan pattern). So we may assume that A is a Štefan pattern and $(P = \{p_1, \ldots, p_{2k+1}\}, \varphi)$ is its representative. We have

$$\begin{aligned} \varphi(p_1) &= p_{k+1} \\ \varphi(p_i) &= p_{2k+3-i} \qquad \text{for} \quad i = 2, \ldots, k+1 \\ \varphi(p_i) &= p_{2k+2-i} \qquad \text{for} \quad i = k+2, \ldots, 2k+1 \end{aligned}$$

and $p_{k+1} < c < p_{k+2}$ for $c \in \mathrm{Fix}(f_P)$. So $\mathfrak{P} = \{J_i\}_{i=1}^{2k+1}$ and

$$\begin{aligned} J_1 &\xrightarrow{P} J_j \qquad &&\text{for} \quad j = k+1, \ldots, 2k+1 \\ J_j &\xrightarrow{P} J_{2k+3-j} \qquad &&\text{for} \quad j = 2, \ldots, k+1 \\ J_j &\xrightarrow{P} J_{2k+2-j} \qquad &&\text{for} \quad j = k+2, \ldots, 2k+1 \\ J_{k+2} &\xrightarrow{P} J_{k+1}. \end{aligned}$$

Note that only J_1 and J_{k+2} P-cover more than one interval. Hence

$$\mathcal{A} = \overbrace{\langle J_{k+2}, J_{k+1}, \ldots, J_{k+2}, J_{k+1}\rangle}^{s+1 \text{ times } J_{k+2}, J_{k+1}} + \langle J_{k+2}, J_k, J_{k+3}, J_{k-1}, \ldots, J_{2k+1}, J_1\rangle$$

is a P-loop of length $2(k+s+1)$. Because the interval J_1 is only once in the loop $\mathcal{A}$ and $\mathcal{A} \neq \mathcal{A}_P$ the cycle Q given by the P-loop $\mathcal{A}$ has period $2(k+s+1)$. We can write $Q = \{q_1, q_2, \ldots, q_{2(k+s+1)}\}$ with spatial labeling. By alternating structure of $\mathcal{A}$, whenever $f_P(q_i) = q_j$ we have $c \in \mathrm{conv}(\{q_i, q_j\})$ and so Q has a division. Moreover, because f_P is monotone on the interval $[p_2, p_{2k+1}]$ we have that

$$\begin{aligned} f_P(q_1) &= q_{k+s+2} \\ f_P(q_i) &= q_{2(k+s+2)-i} \qquad &&\text{for} \quad i = 2, \ldots k+s+1 \\ f_P(q_i) &= q_{2(k+s+1)+1-i} \qquad &&\text{for} \quad i = k+s+2, \ldots, 2(k+s+1). \end{aligned}$$

Take $Q_1 = \{q_1, \dots, q_{k+s+1}\}$. Then $(Q_1, f_P^2|_{Q_1})$ is a cycle and

$$\begin{aligned} f_P^2(q_1) &= q_{k+s+1} \\ f_P^2(q_i) &= q_{i-1} \qquad \text{for} \quad i = 2, \dots, k+s+1. \end{aligned}$$

Hence the cycle Q_1 is a $\frac{k+s}{1}$-unicycle. Because we can choose s arbitrarily large we are done by Theorem 2.10 and Lemma 5.4. ■

In order to be able to use Lemma 5.5 effectively we must use patterns with a special structure.

Definition. Let A be a $(2^k, r)$-pattern and (P_1^0, φ) be a representative of A. From the definition of $(2^k, r)$-pattern we see that for $1 \le j \le k$ there are sets P_1^j, P_2^j such that $P_1^{j-1} = P_1^j \cup P_2^j$, $\varphi^{2^{j-1}}(P_1^j) = P_2^j$ and (P_1^j, φ^{2^j}) is a $(2^{k-j}, r)$-cycle. The sequence of sets $\langle P_1^j \rangle_{j=0}^k$ will be called $(2^k, r)$-*determining.* Moreover if the cycle (P_1^j, φ^{2^j}) has a division for all $j < k$ then the sequence $\langle P_1^j \rangle_{j=0}^k$ will be called *splitting.* In this case we say that pattern A has a splitting $(2^k, r)$-determining sequence (see Fig. 12).

(P_1^0, φ)

(P_1^1, φ^2)

(P_1^2, φ^4)

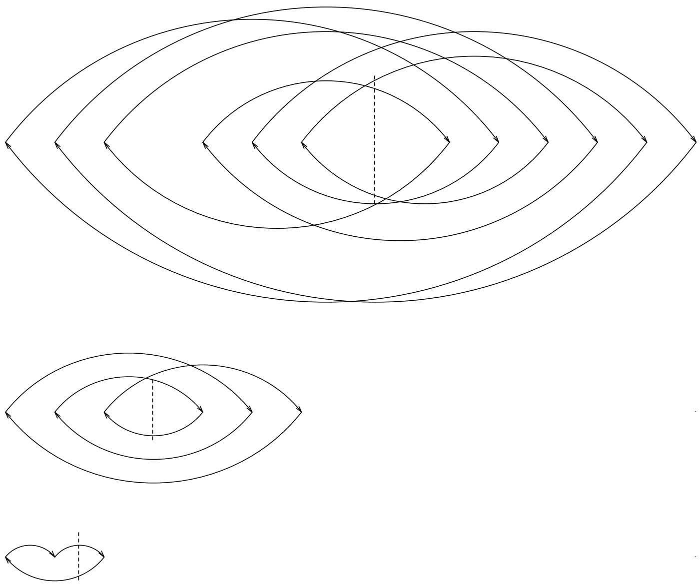


FIGURE 12. A $(2^2, \frac{2}{1})$-pattern with a splitting $(2^2, \frac{2}{1})$-determining sequence $\langle P_1^j \rangle_{j=0}^2$.

Lemma 5.6. *An (a, b)-pattern with an (a, b)-determining sequence which is not splitting forces a (c, d)-pattern A for some $c < a$ and $d > 1$ such that any (c, e)-determining sequence of A is splitting.*

Proof. Let (P_1^0, φ) be a representative of our (a, b)-pattern and $\langle P_1^j \rangle_{j=0}^k$ be an (a, b)-determining sequence which is not splitting. Take the smallest $j < k$ such that the cycle (P_1^j, φ^{2^j}) does not have a division. Using Corollary 5.3 and repeatedly Lemma 5.4 we have that our pattern forces a (c, d)-pattern such that $c = 2^j < a$ and $d > 1$. We can repeat the same procedure for new (c, d)-pattern and after finitely many steps we must get a pattern A with splitting determining sequences. ■

Lemma 5.7. *A $(2^k, r)$-pattern with $r > 1$ and a splitting $(2^k, r)$-determining sequence forces a $(2^k, q)$-pattern for any $q \in \mathbb{Q}$ such that $r \geq q \geq 1$.*

Proof. This follows easily from Theorem 2.10 and Lemma 5.4. ∎

Lemma 5.8. *A $(2^k, r)$-pattern with $r > 1$ and a splitting $(2^k, r)$-determining sequence forces a $(2^{k+1}, q)$-pattern for any $1 \leq q \in \mathbb{Q}$.*

Proof. This follows easily from Lemma 5.5 and Lemma 5.4. ∎

Now we are quite ready to prove two parts from Theorem 5.1.

Lemma 5.9. *An (a, b)-pattern forces a (c, d)-pattern for any $(c, d) \in \mathcal{X}(a, b)$.*

Proof. From Lemma 5.6 we have that an (a, b)-pattern A forces an (a^*, b^*)-pattern B with splitting (a^*, b^*)-determining sequence for some $(a^*, b^*) \geq (a, b)$. If $b^* > 1$ then using Lemmas 5.7 and 5.8 inductively we have that the pattern B forces a (c, d)-pattern for any $(c, d) \leq (a^*, b^*)$ such that $c \geq a^*$.

Now if $(c, d) \leq (a^*, b^*)$ and $c < a^*$ then $d = 1$. But from the definition it is clear that an (a^*, b^*)-pattern is a $(c, 1)$-pattern for any $(c, 1) \leq (a^*, b^*)$ such that $c < a^*$.

So we have that the pattern B forces a (c, d)-pattern for each $(c, d) \in \mathcal{X}(a^*, b^*)$. But A forces B and $(a^*, b^*) \geq (a, b)$. Hence we are done because the forcing relation is transitive. ∎

Lemma 5.10. *A $(2^k, \frac{m}{n})$-pattern with $\frac{m}{n} > 1$ (m, n coprime) forces a $(2^k, \frac{m}{n})$-pattern with period $2^k(m + n)$. A $(2^k, 1)$-pattern forces a $(2^k, 1)$-pattern with period 2^k.*

Proof. Using Lemmas 5.6 and 5.5 as in the proof of Lemma 5.9 we get that a $(2^k, \frac{m}{n})$-pattern forces a $(2^k, \frac{m}{n})$-pattern A with a splitting $(2^k, \frac{m}{n})$-determining sequence. Let (P_1^0, φ) be a representative of A and $\langle P_1^j \rangle_{j=0}^k$ be the splitting $(2^k, \frac{m}{n})$-determining sequence. So we have that $[P_1^k]$ is an $\frac{m}{n}$-pattern. Using Theorems 4.5 and 3.7 together with Lemma 3.4 we have that $[P_1^k]$ forces a $\frac{m}{n}$-pattern with period $m + n$ or period 1 if $\frac{m}{n} = 1$. Finally repeatedly applying Lemma 5.4 we are done. ∎

So we have only two parts of Theorem 5.1 left to prove. As you may already have guessed a knowledge of X-minimal (a, b)-patterns will be very useful for proving them. So first

Definition. An (a, b)-pattern which does not force any other (a, b)-pattern will be called an X-minimal (a, b)-pattern.

We will try to prove that some patterns are X-minimal (a, b)-patterns. For this we need to define a special type of block structure.

Let (P, φ) be a cycle and A, B be patterns. We say that (P, φ) is an A-extension of B if P has a block structure over B, φ is monotone on each block of P except at most one and, with the notation from the definition of block structure, we have that (P_i, φ^m) has pattern A for some $i \in \{1, 2, \ldots, m\}$ (in fact this does not depend on i). As above, if P is an A-extension of B and (Q, ψ) is a cycle with pattern B, then we say that P is an A-extension of (Q, ψ). We also say that $[P]$ is an A-extension of B (respectively of Q) if P is an A-extension of B (respectively of Q).

We define two special types of A-extension. An A-extension where $\operatorname{per}(A) = 2$ will be called a 2-extension. An A-extension where A is an X-minimal r-pattern will be called an r-extension.

A cycle will be called simple if it can be obtained from a cycle of period 1 by making 2-extensions k times and then one r-extension for some $k \in \mathbb{N}$ and $r \in \mathbb{Q}$. A pattern of a simple cycle will be called a simple pattern.

Note that a simple pattern obtained from a cycle of period 1 by making 2-extensions k times and then one r-extension will be a $(2^k, r)$-pattern. Moreover if A is a simple $(2^k, r)$-pattern and B is a simple pattern of period 2^s then an A-extension of B will be a simple $(2^{k+s}, r)$-pattern (see Fig. 13).

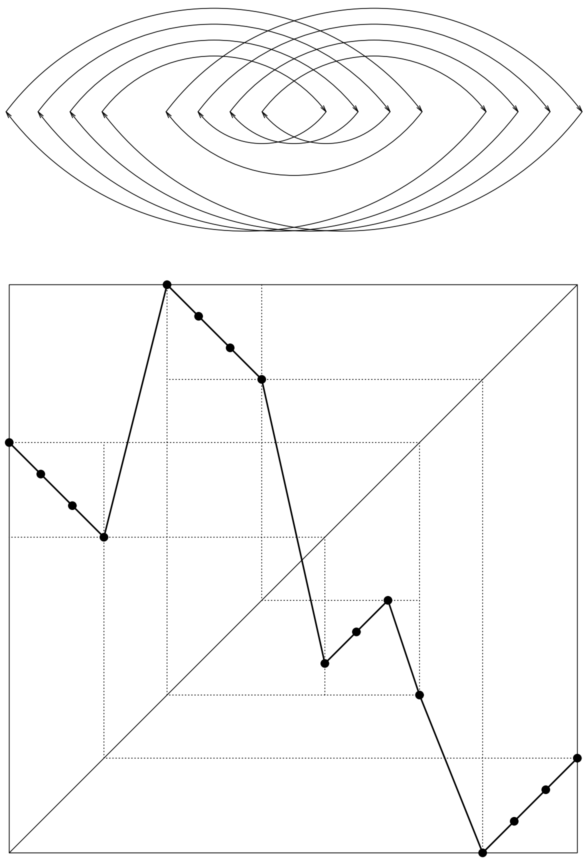


FIGURE 13. An example of a simple $(4, \frac{3}{1})$-cycle P and the function f_P.

Lemma 5.11. (Proposition 2.10.6 from [ALM]) *Let A, B, C, D be patterns such that C is an A-extension of B and C forces D. Then either B forces D or D is an A^*-extension of B for some pattern A^* forced by A. If $C \neq D$ then in the last case $A^* \neq A$.*

Lemma 5.12. (Lemma 2.11.4 from [ALM]) *Let C, B, D be patterns such that C is a 2-extension of B and C forces D. Then either $C = D$ or B forces D.*

The next lemma is only a slight modification of Lemma 2.11.5 from [ALM].

Lemma 5.13. *Let C be a simple pattern of period 2^k, $k \geq 0$. If C forces a pattern D where $D \neq C$ then D is a simple pattern of period 2^i, $i < k$.*

Proof. We use induction. For $k = 0$ this is obvious. Assume that we know it for simple patterns of period 2^{k-1}. If C is a simple pattern of period 2^k, then C is a 2-extension of a simple pattern of period 2^{k-1}. If C forces D then we are done by Lemma 5.12 and the induction hypothesis. ∎

Theorem 5.14. *A simple pattern is X-minimal.*

Proof. Let C be a simple $(2^k, r)$-pattern. Then there is an X-minimal r-pattern A and a simple pattern B of a period 2^k such that C is an A-extension of B. If $r = 1$ then we are done by Lemma 5.13. Assume now that $r > 1$ and D is a $(2^k, r)$-pattern such that C forces D. Then by Lemma 5.11 and 5.13 we have that D is an A^*-extension of B where A forces A^*. But because D is a $(2^k, r)$-pattern we have that $E(A^*) = r$. Finally because A is an X-minimal r-pattern we have that $A = A^*$ and from Lemma 5.11 we have that $D = C$. ∎

Now we are ready to prove the remaining two parts from Theorem 5.1.

Lemma 5.15. *Let $(a, b) \in \mathcal{X}(1, \infty)$ and P be a representative of a simple (a, b)-pattern. The function f_P exhibits some (c, d)-pattern if and only if $(c, d) \in \mathcal{X}(a, b)$.*

Proof. From Lemmas 1.1 and 5.9 we have that f_P exhibits a (c, d)-pattern for any $(c, d) \in \mathcal{X}(a, b)$. Moreover exactly like in the proof of Theorem 5.14 we can prove that f_P does not exhibit a (c, d)-pattern if $(c, d) \notin \mathcal{X}(a, b)$ (see also Lemma 5.18). ∎

Lemma 5.16. *For any $(a, b) \in \mathcal{X}$ there is a function $f \in C(I, I)$ such that f exhibits some (c, d)-pattern if and only if $(c, d) \in \mathcal{X}_o(a, b)$.*

Proof. Clearly there is a sequence of patterns $\langle A_i \rangle_{i=1}^{\infty}$ such that A_i is a simple (a_i, b_i)-pattern, $(a, b) > (a_{i+1}, b_{i+1}) > (a_i, b_i)$ and $(a, b) = \sup\{(a_i, b_i);\ i \geq 1\}$.

Let P_i be a representative of A_i such that $\operatorname{conv}(P_i) = [x_i, y_i] \subset (0, 1)$ and $y_i < x_{i+1}$ for all $i \in \mathbb{N}$.

Define a function $f \in C(I, I)$ where $I = [0, 1]$. Let $f(0) = 0$, $f(1) = 1$, $f|_{[x_i, y_i]} = f_{P_i}$ for $i \geq 1$ and $f|_J$ be linear for any interval $J \subset I$ such that $J \cap [x_i, y_i] = \emptyset$ for each $i \geq 1$ (see Fig. 14).

It is easy to see that outside of the intervals $[x_i, y_i]$ the function f has only fixed points. Hence if f has a (c, d)-cycle P then there is $i \geq 1$ such that $P \subset [x_i, y_i]$. So f_{P_i} exhibits $[P]$ and from Lemma 5.15 we have that $(c, d) \in \mathcal{X}(a_i, b_i) \subset \mathcal{X}_o(a, b)$. Moreover for any $(c, d) \in \mathcal{X}_o(a, b)$ there is an $i \geq 1$ such that $(a, b) > (a_i, b_i) \geq (c, d)$. By Lemma 5.15 we have that f_{P_i} exhibits a (c, d)-pattern and therefore f exhibits a (c, d)-pattern too. ∎

Finally it suffices to realize

Claim 5.17. *If $(a, b) \in \mathcal{X} \setminus \mathcal{X}(1, \infty)$ then $\mathcal{X}(a, b) = \mathcal{X}_o(a, b)$.*

Now let us study X-minimal (a, b)-patterns more closely. The first natural question is to characterize all X-minimal (a, b)-patterns. Unfortunately it is not true, as one might expect, that the only X-minimal patterns are the simple ones (this

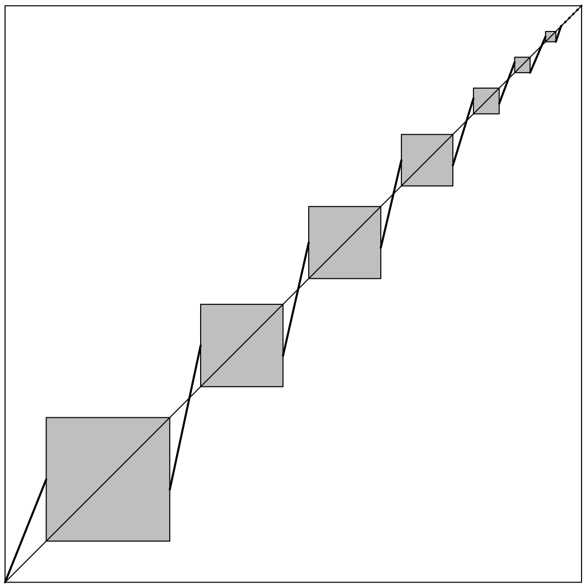
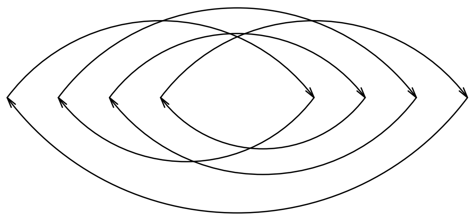


FIGURE 14. A graph of a function f. Inside the filled squares are the functions f_{P_i}.

A cycle P



The function f_P and an important part of the function f_P^2 (dotted)

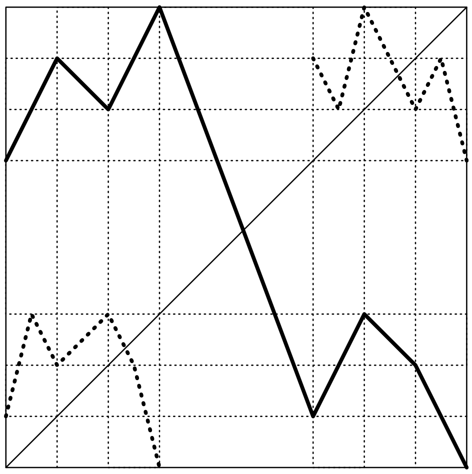


FIGURE 15. An example of an X-minimal $(2, \frac{3}{1})$-pattern which is not simple.

idea seems to be natural for someone who knows the characterization of "primary" patterns – see [ALM]) (see Fig. 15).

We can still get some more information about X-minimal (a, b)-patterns. First question is what types of patterns does an X-minimal (a, b)-pattern force? Can an X-minimal (a, b)-pattern which is not simple force a (c, d)-pattern for $(c, d) \notin \mathcal{X}(a, b)$?

Lemma 5.18. *An X-minimal (a, b)-pattern forces a (c, d)-pattern if and only if $(c, d) \in \mathcal{X}(a, b)$.*

Proof. The "if" part follows from Lemma 5.9. Assume that an X-minimal (a, b)-pattern A forces a (c, d)-pattern for some $(c, d) > (a, b)$. Clearly A has only finite set of types (less than $\mathrm{per}(A)$) and $\mathcal{X}(c, d) \setminus \mathcal{X}(a, b)$ has infinitely many elements. Using this fact, Lemma 5.9 and the fact that the forcing relation is transitive we have that A forces an (e, f)-pattern B such that $(e, f) > (a, b)$ and $A \neq B$. From Lemma 5.9 we have that B forces an (a, b)-pattern C. So A forces C and because the forcing relation is antisymetrical we have that $A \neq C$ - a contradiction. ■

Lemma 5.19. *An X-minimal $(2^k, r)$-pattern has a block structure over a simple $(2^k, 1)$-pattern.*

Proof. It suffices to realize that if a $(2^k, r)$-pattern has not a block structure over a simple $(2^k, 1)$-pattern then either it has a type $(2^n, s)$ for $n < k$ and $s > 1$ or there is a $(2^k, q)$-determining sequence that is not splitting (note that q does not have to be equal to r). Because we have an X-minimal $(2^k, r)$-pattern using Lemma 5.18 we get that the first case is not possible. In the second case by Lemma 5.6 we have that our pattern forces an (a, b)-pattern for some $(a, b) > (2^k, r)$ - again a contradiction with Lemma 5.18. ■

Lemma 5.20. *An X-minimal $(2^k, \frac{m}{n})$-pattern (m, n coprime) has period $2^k(m+n)$ if $m > n$ and period 2^k if $m = n$.*

Proof. This follows immediately from Lemma 5.10. ■

So immediately we have following simple

Corollary 5.21. *An X-minimal $(2^k, 1)$-pattern is a simple $(2^k, 1)$-pattern.*

Proof. This follows straight from Lemmas 5.19 and 5.20. ■

Now let A be an X-minimal $(2^k, r)$-pattern and (P, φ) be a representative of A. From Lemma 5.19 we have that P consists of 2^k blocks P_i. From Lemmas 2.1, 5.4 and 5.18 we have that (P_i, φ^{2^k}) is a unicycle and $E(P_i) \leq r$. Of course the pattern $[P_i]$ must not force a pattern with eccentricity greater than r. So if $E(P_i) = r$ then $[P_i]$ is an X-minimal r-pattern (see Lemma 3.5).

Now assume that A is an X-minimal $(2^k, \frac{n+1}{n})$-pattern. From Lemma 5.20 we have that the cycle (P_i, φ^{2^k}) has period $2n+1$. But the minimal possible eccentricity of a pattern with this period is $\frac{n+1}{n}$. Hence in this case for every i pattern $[(P_i, \varphi^{2^k})]$ must be an X-minimal $\frac{n+1}{n}$-pattern. But there is only one X-minimal $\frac{n+1}{n}$-pattern and that is exactly the pattern of the Štefans cycle of the period $2n + 1$.

In this special case we can prove even more. We need the following

Lemma 5.22. ([B], Theorem 2.11.1 from [ALM]) *Let A be a pattern with* $\mathrm{per}(A) = 2^k(2n + 1)$ *and B be the pattern of the Štefan cycle of period $2n + 1$. If A is not a*

B-extension of a simple $(2^k, 1)$-pattern then A forces another pattern with period $2^k(2n+1)$.

Now we can prove

Lemma 5.23. *An X-minimal $(2^k, \frac{n+1}{n})$-pattern is simple.*

Proof. By Lemma 5.20 we have that an X-minimal $(2^k, \frac{n+1}{n})$-pattern A has period $2^k(2n+1)$. If it is not simple then by Lemma 5.22 it forces another pattern B with the same period. It is easy to see that if (a,b) is a maximal (in the sense of ordering on $\mathcal{X}$) type of the pattern B then $(a,b) \geq (2^k, \frac{n+1}{n})$. Hence by Lemma 5.9 we have that B forces a $(2^k, \frac{n+1}{n})$-pattern. So the pattern A is not an X-minimal $(2^k, \frac{n+1}{n})$-pattern – a contradiction. ∎

We will end this section and so the whole paper by a hypotheses.

Conjecture. *If A is an X-minimal $(2^k, r)$-pattern and (P, φ) is a representative of A such that (P_i, φ^{2^k}) is an r-cycle for every block P_i then A is simple.*

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