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CONGRUENCES IN PARTIAL ABELIAN SEMIGROUPS

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Abstract. A *partial abelian semigroup* (PAS) is a structure $(L, \perp, \oplus)$, where $\oplus$ is a partial binary operation on L with domain $\perp$, which is commutative and associative (whenever the corresponding elements exist). A class of congruences on partial abelian semigroups are studied such that the corresponding quotient is again a PAS. If M is a subset of a PAS L, we say that $x, y \in L$ are perspective with respect to M, if there is $z \in L$ such that $x \oplus z \in M$ and $y \oplus z \in M$. A subset M is called weakly algebraic if perspectivity with respect to M is a congruence. Some conditions are shown under which a congruence coincides with a perspectivity with respect to an appropriate set M. Especially, conditions under which the corresponding quotient is a D-poset are found. It is also shown that every congruence of MV-algebras and orthomodular lattices is given by a perspectivity with respect to an appropriate set M.

0. Introduction

In this paper, we will study some congruences on partial abelian semigroups, where a *partial abelian semigroup* is a structure $(L, \perp, \oplus)$, where $\oplus$ is a partial binary operation on L with domain $\perp$, which is commutative and associative in a restricted sense ([W]). Similar structures have already appeared in the literature and found applications in several fields.

A general theory of universal partial algebras can be found in [Bu]. A special case of *partial grupoids* - algebras with a partially defined binary operation - have been studied in [L,E].

A (commutative) *minimal clan*, as a common abstraction of Boolean rings and commutative lattice-ordered groups was introduced by Schmidt in 1985 [Sch]. A commutative minimal clan is a set E with a set S of pairs of summable elements of E, a partial addition $+ : S \to E$, and an order relation $\leq$ such that $(E, S, +, \leq)$ is a commutative lattice-ordered partial semigroup having the cancellative property and the difference property - a property which is related to the existence of relative complements in Boolean rings and to modular law in commutative lattice-ordered groups.

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A special case of a commutative minimal clan is an *MV-algebra* introduced by Chang [Ch] in 1957, where MV is supposed to suggest many-valued logic (see also [M], [Mu], [Ch,K] for alternative definitions of MV-algebras).

In 1987, Belavkin [B] introduced a notion of a *quasilogic* as a partially ordered set $\mathcal{N}$ with a partial operation of substraction such that $a \leq b \implies \exists b - a \leq b$, subject to some monotonicity properties. The motivation was to find an abstract model for the set of all positive self-adjoint operators on a Hilbert space, in particular, for the set of all *effects* (i.e., self-adjoint operators between 0 and I by the usual partial ordering of self-adjoint operators), as a special case.

With the same motivation, based on the abstraction of the set of all effects and unsharp quantum measurements, Giuntini and Greuling [G,G] in 1989 introduced the notion of a *weak orthoalgebra* as an algebra with a partially defined binary operation $+$ on pairs of orthogonal elements, a totally defined unary operation of orthocomplementation and two constants 0 and 1.

The notion of a weak orthoalgebra is a generalization of the notion of an *orthoalgebra*, introduced by Foulis and Randall [F,R] as an algebraic structure which plays a role of a "logic" in the theory of *manuals* (or *test spaces*). An orthoalgebra is defined as a partial binary algebra with a sum (operation) $\oplus$ on a set with two special elements. (See [Go], [F,G,R,], [R], [Gu] for the axioms and basic properties of orthoalgebras).

In [K,Ch], Kôpka and Chovanec (1992) independently of [B], introduced a *difference* (operation) on a partially ordered set (poset) P as a partial binary operation $\ominus$ on P such that $b \ominus a$ is defined if and only if $a \leq b$ satisfying some conditions. For example, $b \wedge a'$ is such an operation in an orthomodular poset. A *difference poset*, *D-poset*, (DP) is a bounded poset equipped with a difference operation. For example, every orthoalgebra (which is a natural generalization of an orthomodular lattice or poset, [Be], [K], [P,P]), and also every orthomodular poset is a difference poset. An exact relationship between difference posets and orthoalgebras was pointed out in [N,P]. A description of an orthoalgebra in terms of a difference operation on a poset is given there. A generalization of orthoalgebras as structures with a partial $\oplus$ operation, not necessarily containing a greatest element, can be found in [Kr]. For a generalization of test spaces where the logic becomes a D-poset, see [D,P1],[DP2], [W], [P,W]. In [Ch,K], there is shown that every MV-algebra is a D-poset (a D-lattice, more precisely), and conditions under which a D-lattice becomes an MV-algebra are given.

A description of an orthomodular poset, resp. a difference poset in terms of a sum operation on a set is given in [B,M], resp. [P] and [F,B 2]. In the last paper, the term *effect algebra* has been used. Yet more general approach is used in [K,R] when considering a difference operation on an arbitrary set with a special element (so-called *RI-set*, RI stands for *relative inverse*).

In [H,P], a *generalized difference poset* as a poset with a smallest element and with a difference operation subject to an additional cancellativity condition in such a way that every order ideal of a difference poset is a generalized difference poset. It is proved that every generalized difference poset P is an order ideal of a difference poset $\hat{P}$. Similar results were obtained in [J1] and [M-I] for generalized orthomodular lattices (see also [Be], [K]) and (weak) generalized orthomodular posets, respectively.

A very important property (from the mathematical point of view as well as from the point of view of physical applications) is the existence of tensor products of D-posets. These tensor products were studied in [F,B], [D], [D,P1], [D,P2].

Some more details about relations among the above mentioned structures will be given elsewhere.

In [W], a class of "faithful" congruences of partial abelian semigroups was studied, and it was shown that some of them can be described as perspectivity relations with respect to algebraic subsets. In the present paper, we will consider a more general class of congruences, and their relations to perspectivities with respect to weakly algebraic subsets. While faithful congruences on D-posets (effect algebras) are only the trivial ones, there are nontrivial examples of congruences of the more general type. In fact, every congruence of an MV-algebra and of an orthomodular lattice is of that type, and any MV-filter, resp. any orthomodular filter, is a weakly algebraic subset. Moreover, there are close connections between the factorisation with respect to a central ideal (resp. central element, [G,F,P]) and the congruences.

1. Basic definitions and results

We will start with basic definitions (see [W]).

By *partial abelian semigroup* (PAS) we mean a structure $(L, \perp, \oplus)$ where $\perp$ is a binary relation on L and $\oplus$ is a partially defined binary operation with domain $\perp$ satisfying

(1) $p \oplus q = q \oplus p$
(2) $(p \oplus q) \oplus r = p \oplus (q \oplus r)$

These identities are to be understood as asserting that if the term on either side is defined, so is that on the other, and the two are equal. Thus, (1) and (2) place sharp restrictions on the binary relation $\perp$: (1) requires that $\perp$ to be symmetric, and (2) implies

(3) $p \perp q$ and $r \perp (p \oplus q) \implies r \perp q$ and $(q \oplus r) \perp p$.

We say that a PAS L is *cancellative* iff for all $a, b, c \in L$, $a \perp c \perp b$ and $a \oplus c = b \oplus c$ entail $a = b$. A *zero* in L is an element 0 such that $p \perp 0$ for all $p \in L$ and $p \oplus 0 = p$. If L is cancellative, it has at most one zero. The formal adjunction of a zero to a PAS presents no difficulty; we shall therefore assume that every PAS possesses a zero. We shall say that L is *positive* iff for all $p, q \in L$, $p \oplus q = 0$ only if $p = q = 0$. Note that if L contains no zero, then the formal adjunction of a zero yields a positive PAS. A *unit* in a PAS L is an element 1 in L such that for every $a \in L$ there is $b \in L$ such that $a \oplus b = 1$. A *unital* PAS is a PAS with a distinguished unit 1. A cancellative positive PAS has at most one unit.

In any PAS L, we may define a relation $p \le q$ iff $\exists r \in L$ such that $p \oplus r = q$.

Lemma 1.1. ([W]) *Let L be a PAS. For $p, q \in L$, let $p \le q$ iff $\exists r \in L : p \oplus r = q$. Then*

(i) *The relation $\le$ is reflexive and transitive, i.e., a preorder.*
(ii) *If the relatiom $\le$ is a partial ordering, then S is positive.*
(iii) *If L is cancellative and positive, then $\le$ is a partial ordering.*

Let $(P, \le)$ be a poset with the smallest element 0 and let $\ominus$ be a partial binary operation on P such that $b \ominus a$ is defined iff $a \le b$. Then $\ominus$ is a difference on

$(P, \leq)$ if and only if the following two conditions are satisfied for all $a, b, c \in P$ (cf. [K,Ch]):

(i) $a \ominus 0 = a$.
(ii) If $a \leq b \leq c$ then $c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

In [H,P] the notion of a generalized difference poset has been introduced as follows.

A difference $\ominus$ is called *cancellative* if the following condition is satisfied:

(C) If $a \leq b, c$ and $b \ominus a = c \ominus a$, then $b = c$.

Definition 1.2. ([H,P]) Let $(P, \leq, \ominus)$ be a poset with a cancellative difference containing a smallest element 0. The system $(P, \leq, 0, \ominus)$ is called a *generalized difference poset* (GDP).

The following notion was introduced in [K,Ch] (see also [N,P]).

Definition 1.3. ([K,Ch]) Let $(P, \leq, \ominus)$ be a poset with a difference and let 0 and 1 be the smallest and greatest elements in P, respectively. The structure $(P, \leq, 0, 1, \text{ff})$ is called a *difference poset (D-poset, DP)*.

We note that condition (C) is satisfied in every D-poset ([H,P]).

Let $(P, \leq, \ominus)$ be a poset with a difference satisfying condition (C). This means that for every $a, b \in P$ there is at most one $c \in P$ such that $a = c \ominus b$. Thus property (C) enables us to define a *sum operation* on P, that is, a partial binary operation $\oplus$ on P given by $(a, b, c \in P)$:

(S) $a \oplus b$ is defined and $a \oplus b = c$ if and only if $c \ominus b$ is defined and $a = c \ominus b$.

By [H,P], Corollary 13, there is a one to one correspondence between posets with a cancellative difference having a smallest element 0 and partial binary algebras with a sum operation and with a special element 0 having the following properties:

(1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$ (commutativity).
(2) If $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, then $b \oplus c$ and $a \oplus (b \oplus c)$ are defined and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ (associativity).
(3) If $a \oplus b$ and $a \oplus c$ are defined and $a \oplus b = a \oplus c$, then $b = c$ (cancellativity).
(4) $a \oplus 0 = a$ (zero property).
(5) $a \oplus b = 0 \implies a = b = 0$ (positivity).

Consequently, cancellative, positive PAS with a zero coincides with a generalized difference poset (and also with an abelian RI-set in the sense of [K,R]), and a unital cancellative positive PAS with zero coincides with a difference poset.

Let S be a PAS. For $a \in S$, define $n.a$ $(n \in \mathbb{N})$ as follows: $1.a = a$, $(n+1).a = n.a \oplus a$. Let $\mu(a)$ denote the greatest n such that $n.a$ exists (possibly $\mu(a) = \omega$). If S is positive and cancellative PAS and $\oplus$ is totally defined, then $\mu(a) = \omega$ for all $a \in S$.

The following statement is straightforward.

Lemma 1.4. *If S is cancellative unital PAS and $\mu(a)$ is finite for all nonzero $a \in S$, then S is positive, hence a D-poset.*

We recall that a D-poset S is an *orthoalgebra* iff $\mu(a) \in \{0, 1\}$ for all nonzero $a \in S$ ([F,G,R], [Go], [R] for the basic axioms of orthoalgebras,, [N,P], [P], [F,B 2] for the relations between orthoalgebras and effect algebras).

2. Homomorphisms and congruences

Definition 2.1. If L and M are two PASs, a function $f : L \to M$ is a *homomorphism* iff $p \perp q \implies f(p) \perp f(q)$ and $f(p \oplus q) = f(p) \oplus f(q)$. A homomorphism f is *full* iff $f(p) \perp f(q)$ and $f(p) \oplus f(q) \in f(L)$ imply that there are p_1, q_1 in L such that $p_1 \perp q_1$, $f(p) = f(p_1)$, $f(q) = f(q_1)$ and $f(p) \oplus f(q) = f(p_1 \oplus q_1)$. A homomorphism f is *faithful* (or *closed* by [Bu], or strong, by [L,E]) iff $f(p) \perp f(q) \implies p \perp q$.

Note that an isomorphism (i.e., a homomorphism with a two-sided inverse) is the same thing as a bijective faithful homomorphism. The trivial homomorphism $f : L \to \{0\}$ is faithful iff L is a semigroup, i.e., iff $a \perp b$ for all $a, b \in L$.

Definition 2.2. Let L be a PAS. A binary relation $\sim$ on L is a *congruence* iff

(i) $\sim$ is an equivalence relation,
(ii) $a_1 \sim b_1$, $a_2 \sim b_2$, $a_1 \perp a_2$, $b_1 \perp b_2$ imply $a_1 \oplus a_2 \sim b_1 \oplus b_2$,
(iii) $a \sim b$, $b \perp c \implies \exists d$ such that $d \sim c$ and $a \perp d$.

A binary relation $\sim$ on L is a *faithful congruence* ([W]) iff

(i') $\sim$ is an equivalence relation,
(ii') $a \sim b$ and $a \perp c$ imply $b \perp c$ and $a \oplus c \sim b \oplus c$.

We note that a faithful congruence is sometimes called *closed* ([Bu]). It easy to see that a faithful congruence is a congruence. If L is a semigroup, then every congruence is faithful.

Let $\sim$ be a congruence and denote by $[a]$ the equivalence class of $a \in L$ under $\sim$. The set $L/\sim$ of all congruence classes is the *quotient* of L.

Proposition 2.3. *Let $\sim$ be a congruence on L. Then the quotient $L/\sim$ is a PAS.*

Proof. Define $[a] \perp [b]$ iff there are a_1, b_1 such that $a \sim a_1$, $b \sim b_1$, $a_1 \perp b_1$ and put $[a] \oplus [b] = [a_1 \oplus b_1]$. According to (ii) of Definition 2.2, $[a] \oplus [b]$ is well defined.

Commutativity of the operation $\oplus$ is clear. To prove associativity, assume $[a] \perp [b]$, $([a] \oplus [b]) \perp [c]$. Then there are a_1, b_1, $a_1 \perp b_1$ and $[a] \oplus [b] = [a_1 \oplus b_1]$. By definition, $[c] \perp [a_1 \oplus b_1]$ implies that there are $c_1 \sim c$, $d \sim a_1 \oplus b_1$ such that $c_1 \perp d$ and $([a] \oplus [b]) \oplus [c] = [a_1 \oplus b_1] \oplus [c] = [d \oplus c_1]$. Now $d \sim a_1 \oplus b_1$, $d \perp c_1$ imply by (iii) that there is $c_2 \sim c_1$, $c_2 \perp a_1 \oplus b_1$. Therefore,

$$\begin{aligned}[a_1 \oplus b_1] \oplus [c_1] &= [d \oplus c_1] \\ &= [(a_1 \oplus b_1) \oplus c_2] = [a_1 \oplus (b_1 \oplus c_2)] \\ &= [a_1] \oplus [b_1 \oplus c_2] = [a] \oplus [b \oplus c].\end{aligned}$$

□

We note that in [Bu], a congruence relation is defined as a binary relation satisfying (i) and (ii) of Definition 5. Owing to the specific property (3) of the relation $\perp$ we need additional properties to make $L/\sim$ a PAS.

From the definition of the relation $\perp$ on the quotient $L/\sim$ it follows that the canonical mapping $a \mapsto [a]$ is a full and surjectve homomorphism from L onto $L/\sim$.

Proposition 2.4. *Let S and L be PASs and let $f : L \to S$ be a surjective homomorphism. Define $a \sim^f b$ iff $f(a) = f(b)$. The following are equivalent:*

(1) *f is full and $\sim^f$ is a congruence,*
(2) *condition (c) is satisfied, where*
(c) $f(a) \perp f(b) \implies \exists b_1 \in L,\ f(b_1) = f(b)$ *and* $b_1 \perp a$

(symmetrically, $\exists a_1 \in L,\ f(a_1) = f(a)$ and $a_1 \perp b$).

Moreover, $S \equiv L/{\sim^f}$.

Proof. (1) $\implies$ (2). Let $\{(a, b) : f(a) = f(b), a, b \in L\}$ be a congruence. This implies that

$$a \sim^f b,\ b \perp c \implies \exists d \sim^f c :\ a \perp d.$$

Assume $f(a) \perp f(b)$. Since f is full and surjective, there are $a_1, b_1 \in L$, $a_1 \perp b_1$, $f(a) = f(a_1)$, $f(b) = f(b_1)$. Therefore,

$$a \sim^f a_1,\ a_1 \perp b_1 \implies \exists b_2 \sim^f b_1 \sim^f b,\ b_2 \perp a.$$

(2) $\implies$ (1). Now assume that condition (c) is satisfied. Then, clearly, f is full and $\sim^f$ is an equivalence relation. If $a_1 \sim^f b_1$, $a_2 \sim^f b_2$ and $a_1 \perp a_2$, $b_1 \perp b_2$ then, clearly, $f(a_1 \oplus a_2) = f(a_1) \oplus f(a_2) = f(b_1) \oplus f(b_2) = f(b_1 \oplus b_2)$, hence $a_1 \oplus a_2 \sim^f b_1 \oplus b_2$.

Finally, let $a \sim^f b$, $b \perp c$. Then $f(a) = f(b)$, $f(b) \perp f(c)$. By (c), there is $c_1 \sim^f c$ such that $c_1 \perp a$. This proves that $\sim^f$ is a congruence.

Let $a, b \in S$, $a \perp b$. Let $a_1, b_1 \in L$ be any elements such that $a = f(a_1)$, $b = f(b_1)$. Then there is $b_2 \in L$, such that $f(b) = f(b_1) = f(b_2)$ and $b_2 \perp a_1$. Hence $a \oplus b = f(a_1) \oplus f(b_1) = f(a_1) \oplus f(b_2) = f(a_1 \oplus b_2)$. Conversely, if $[a] \perp [b]$, $a, b \in L$, then there are a_1, b_1 in L, $a_1 \sim^f a$, $b_1 \sim^f b$ and $a_1 \perp b_1$. Then $f(a_1) = f(a)$, $f(b_1 = f(b)$ and $f(a) = f(a_1) \perp f(b_1) = f(b)$. This shows that $(S, \oplus, 0)$ is isomorphic with $(L/{\sim^f}, [0])$ in the isomorphism $a \to [a_1]$. □

Proposition 2.5. *Let S and L be PASs and let $f : L \to S$ be a faithful and surjective homomorphism. Then $\sim^f$ is a faithful congruence and $S \equiv L/{\sim^f}$.*

Conversely, if $\sim^f$ is a faithful congruence, then $a \to [a]$ is a faithful and surjective homomorphism.

Proof. Let f be faithful. Assume $a \sim^f b$, $b \perp c$. Then $f(a) = f(b) \perp f(c)$ imply $a \perp c$. The remaining part is clear. □

3. Algebraic sets and perspectivities

Let S be a PAS and let M be any subset of S. We will say that $a, c \in S$ are *complements* of each other with respect to M if $a \perp c$ and $a \oplus c \in M$.

We will say that $a, b \in S$ are *perspective* with respect to M if they share a common complement, that is, there is $c \in S$ such that $a \oplus c \in M$, $b \oplus c \in M$. We will write $a \sim_c^M b$, or $a \sim^M b$, $a \sim_c b$, $a \sim b$ if c or M or both are not emphasized.

We will say that a subset M of S is *dominating* if for every $a \in S$ there is a $c \in S$ such that $a \oplus c \in M$. In other words, every $a \in S$ has at least one complement with respect to M.

We will say that $M \subseteq S$ is *weakly algebraic* if the perspectivity relation $\sim^M$ is a congruence. We will say that M is *algebraic* if $\sim^M$ is a faithful congruence.

Lemma 3.1. *Let S be a PAS and let M be a weakly algebraic subset of S. Then*
(i) *M is dominating,*
(ii) *$S/\sim^M$ is a cancellative unital PAS,*
(iii) *$a, b \in M$, $r \in S$ and $b = a \oplus r$ iff $r \sim 0$.*

Proof. (i) The perspectivity $\sim^M$ is an equivalence, therefore $a \sim a$, hence there is $b \in S$ with $a \oplus b \in M$. Hence M is dominating. (ii) Let $1 \in S/\sim^M$ denote the class containing M. Since M is dominating, to every $a \in S$ there is $b \in S$ with $a \oplus b \in M$. This implies $[a] \oplus [b] = 1$, hence 1 is a unit. Let $[a] \oplus [b] = [a] \oplus [c]$. Then there are $a_1 \sim a$, $b_1 \sim b$, $a_1 \perp b_1$ and $[a] \oplus [b] = [a_1 \oplus b_1]$. Similarly, there are $a_2 \sim a$, $c_1 \sim c$, $a_2 \perp c_1$ and $[a] \oplus [c] = [a_2 \oplus c_1]$. Moreover, $a_1 \sim a_2$, $a_2 \perp c_1$ imply that there is c_2, $c_2 \sim c_1$, $c_2 \perp a_1$ and $[a_1 \oplus c_2] = [a_2 \oplus c_1]$. Therefore, $a_1 \oplus b_1 \sim a_1 \oplus c_2$ implies that there is $d \in S$ with $a_1 \oplus b_1 \oplus d \in M$ and $a_1 \oplus c_2 \oplus d \in M$. But this gives $b_1 \sim c_2$, hence $b \sim c$, i.e., $[b] = [c]$. Cancellativity of $S/\sim^M$ is proved. (iii) Let $r \sim 0$. Then there is $b \in S$ such that $r \oplus b \in M$ and $0 \oplus b \in M$, hence $b \in M$. Conversely, let $a \in M$ and $a \oplus r \in M$. Then $a \sim a \oplus r$ imply $r \sim 0$ by (ii). $\square$

We will say that a subset M of a PAS S is *closed* if $x \in M$, $y \sim x$ imply $y \in M$. For any set $M \subseteq S$, the set $M' := \{x \in S : \exists y \in M, x \sim y\}$ will be called the *closure* of M. We will say that M is an *order filter* if $x \in M$, $a \perp x$ imply $a \oplus x \in M$. A subset M of S is said to be *irredundant* if $x, y \in M$, $y = x \oplus r \implies r = 0$. Observe that in a cancellative PAS, irredundancy is equivalent with $x, y \in M, x \leq y \implies x = y$.

Lemma 3.2. *Let M be a subset of a PAS S.*
(i) *If M is weakly algebraic then*

$$M' = \{x \in S : \exists r \in S \text{ such that } x \oplus r \in M \text{ and } r \sim 0\}.$$

(ii) *Let M be weakly algebraic and closed. Then $S/\sim^M$ is positive (hence a D-poset) if and only if M is an order filter.*
(iii) *Let M be irredundant and weakly algebraic. If S is positive then S/M is a D-poset. Moreover, $x \sim 0 \implies x = 0$.*

Proof. (i) Let $y \in M$, $x \sim y$. Since M is dominating, there is $r \in S$ such that $x \oplus r \in M$. Then $x \oplus r \sim y \sim x$ imply $r \sim 0$. Conversely, if $r \sim 0$ and $x \oplus r \in M$, then $x \oplus r \sim x$ since $\sim$ is a congruence.

(ii) Let M be an order filter. Assume that $[x] \oplus [y] = [0]$. Then there are x_1, y_1 such that $x_1 \perp y_1$, $x \sim x_1$, $y \sim y_1$ and $[x] \oplus [y] = [x_1 \oplus y_1]$. Hence $x_1 \oplus y_1 \sim 0$, so that for some $z \in S$, $x_1 \oplus y_1 \oplus z \in M$ and $0 \oplus z \in M$. But then $z \in M$, $x_1, y_1 \perp z$ imply (M being an order filter) that $z \oplus x_1, z \oplus y_1 \in M$, hence $x_1 \sim y_1 \sim 0$, hence $[x] = [y] = [0]$. This proves that $S/\sim^M$ is a D-poset.

Conversely, let $S/\sim^M$ be a D-poset. Assume $x \in M$ and $r \perp x$. Then $[x] = 1$, $[x \oplus r] = 1 \oplus [r]$ implies $r \sim 0$. Hence $x \oplus r \in M' = M$ since M is closed.

(iii) Let $[x] \oplus [y] = [0]$. Then there are x_1, y_1 such that $x \sim x_1$, $y \sim y_1$ and $[x] \oplus [y] = [x_1 \oplus y_1]$. Hence $x_1 \oplus y_1 \sim 0$. Then $x_1 \oplus y_1 \oplus c \in M$ for some $c \in M$. Due to irredundancy of M, $x_1 \oplus y_1 = 0$, and by positivity of S, $x_1 = y_1 = 0$, hence $[x] = [y] = [0]$. If $x \sim 0$, there is $c \in M$ such that $x \oplus c \in M$. By irredundancy of M, $x = 0$. $\square$

Lemma 3.3. *Let $M \subseteq S$ be weakly algebraic. Then M' is weakly algebraic, too, and $\sim^M \equiv \sim^{M'}$.*

Proof. It suffices to prove that $\sim^{M'} \implies \sim^M$. Let $a \sim^{M'} b$. Then there is $x \in S$ such that $a \oplus x \in M'$ and $b \oplus x \in M'$. By Lemma 3.2 (i), there are $z, w \in S$, $z \sim^M 0$, $w \sim^M 0$ and $a \oplus x \oplus z \in M$, $b \oplus x \oplus w \in M$. But then $a \oplus z \sim^M_x b \oplus w$, and since $\sim^M$ is a congruence, $a \sim^M a \oplus z$, $b \sim^M b \oplus w$. Hence $a \sim^M b$. $\square$

Corollary. [W]. *If M is algebraic, M' is algebraic, too.*

Proposition 3.4. *(a) A closed subset M of a PAS S is weakly algebraic iff*

(i) *M is dominating,*
(ii) *$a \oplus x \in M$, $b \oplus y \in M$, $x \sim^M y$ imply $a \sim^M b$,*
(iii) *$a \sim^M b$, $a \perp x \implies \exists y \sim^M x$, $b \perp y$.*

(b) A subset M of S is algebraic and closed iff M is dominating and $a \sim^M b$, $a \oplus c \in M \implies b \oplus c \in M$.

Proof. (a) If M is weakly algebraic, properties (i), (ii), (iii) follow from the fact that $\sim^M$ is a congruence.

Conversely, let (i),(ii),(iii) be satisfied for a subset M of S. We will write $\sim$ instead of $\sim^M$. Since M is dominating, for any $a \in S$ there is $a_1 \in S$ such that $a \oplus a_1 \in M$, hence $a \sim a$. Symmetry of $\sim$ is immediate. To prove transitivity, observe that (iv) is satisfied, where

(iv) $a \sim b$, $a \oplus x \in M \implies \exists y \sim x$, $b \oplus y \in M$

Indeed, by (iii) there is $y \sim x$, $y \perp b$. Since M is dominating, there is z such that $b \oplus y \oplus z \in M$. By (ii) and $a \oplus x \in M$, $a \sim b$, we have $x \sim y \oplus z$. Hence there is u, $x \oplus u \in M$, $y \oplus z \oplus u \in M$. Again by (ii) and $y \sim x$, we have $u \sim z \oplus u$, which entails $z \sim 0$, hence $b \oplus y \in M$.

Assume that $a \sim_e b$, $b \sim_f c$. Then by (iv), $a \sim b$, $b \oplus f \in M$ imply $\exists g \sim f$, $a \oplus g \in M$. By (ii), $a \oplus g \in M$, $c \oplus f \in M$, $g \sim f$ imply $a \sim c$. Hence $\sim$ is an equivalence.

Now let $a_1 \sim b_1$, $a_2 \sim b_2$, $a_1 \perp a_2$, $b_1 \perp b_2$. Since M is dominating, there are x, z such that $a_1 \oplus a_2 \oplus x \in M$, $b_1 \oplus b_2 \oplus z \in M$. By (iv), $a_1 \sim b_1$, $a_1 \oplus (a_2 \oplus x) \in M$ imply $\exists y \sim a_2 \oplus x$, $b_1 \oplus y \in M$, and similarly, $a_1 \sim b_1$, $b_1 \oplus (b_2 \oplus z) \in M$ imply $\exists u \sim b_2 \oplus z$, $a_1 \oplus u \in M$. By (ii) this entails $a_2 \oplus x \sim b_2 \oplus z$ and $a_2 \sim b_2$ gives $x \sim z$. This entails, again by (ii), $a_1 \oplus a_2 \sim b_1 \oplus b_2$. Taking (iii) into account, we proved that $\sim$ is a congruence.

(b) See [W]. $\square$

We already know from Lemma 3.1 (ii) that if M is a weakly algebraic subset of a PAS S, then $S/\sim^M$ is a cancellative unital PAS. The following proposition shows that it can be "positivized".

Proposition 3.5. *Let S be a unital cancellative PAS. The set*

$$M = \{y \in S : y = 1 \oplus r, r \in S\}$$

is algebraic and $S/\sim^M$ is positive, i.e., a D-poset.

Proof. Observe that if $1 \oplus r$ is defined then $x \oplus r$ is defined for any $x \in S$. Indeed, there is a (unique) x' in S such that $x \oplus x' = 1$. In addition, there is r_1 such that

$r\oplus r_1=0$. Indeed, by unitality, there is r_1 such that $1\oplus r\oplus r_1=1$, by cancellativity then $r\oplus r_1=0$. The set M is dominating since $1\in M$. Let $a\sim b$, $b\oplus c\in M$ Then there is $d\in S$ with $a\oplus d\in M$, $b\oplus d\in M$. Let $a\oplus d=1\oplus u$, $b\oplus d=1\oplus v$, $b\oplus c=1\oplus w$. From $a\oplus d\oplus u_1=1$ we get $d\oplus u_1=a'$. Similarly, $b'=d\oplus v_1=c\oplus w_1$. Then

$$\begin{aligned}1\oplus v_1&=(a\oplus d\oplus u_1)\oplus v_1=(a\oplus u_1)\oplus(d\oplus v_1)\\&=(a\oplus u_1)\oplus(c\oplus w_1).\end{aligned}$$

Therefore, $a\oplus c=1\oplus v_1\oplus u\oplus w\in M$. This proves that M is algebraic. By Lemma 3 (iii), $1\oplus r\in M$, $1\in M$ imply $r\sim 0$. Therefore, if $[1]\oplus[r]$ is defined, then $[1]\oplus[r]=[1\oplus r]$, and $r\sim 0$, hence $[r]=[0]$. □

The last proposition can be generalized as follows. Let S be a PAS. Recall that a subset $M\subseteq S$ is an order filter if $a\in M$, $x\in S$, $a\perp x$ imply $a\oplus x\in M$.

Proposition 3.6. *Let S be a PAS and let M be a weakly algebraic subset of S. Then $M^1:=\{a\in S: a\sim^M x\oplus y, x\in M\}$ is also weakly algebraic, and $S/\sim M^1$ is a D-poset.*

Proof. In what follows, $\sim\equiv\sim^M$ is the perspectivity with respect to M and $\sim^1\equiv\sim^{M^1}$ is the perspectivity with respect to M^1. Similarly, $[x]$ denotes a congruence class with respect to M, $[x]_1$ denotes a congruence class with respect to M^1 containing x.

(i) Since $M\subseteq M^1$, and M is dominating, M^1 is dominating, too.

(ii) Let $a\oplus c, b\oplus d\in M^1$, and $a\sim^1 b$. Then there is $e\in S$, $a\oplus e, b\oplus e\in M^1$. Moreover, $a\oplus e\sim x\oplus x_1$, $b\oplus e\sim y\oplus y_2$, $a\oplus c\sim z\oplus z_1$, $b\oplus d\sim v\oplus v_1$, where $x,y,z.v\in M$. Since M is dominating, there are x_2,y_2,z_2,v_2 such that

$$\begin{aligned}
x\sim x\oplus x_1\oplus x_2\sim a\oplus e\oplus x_2\in M &\implies x_1\oplus x_2\sim 0\\
y\sim y\oplus y_1\oplus y_2\sim b\oplus e\oplus y_2\in M &\implies y_1\oplus y_2\sim 0\\
z\sim z\oplus z_1\oplus z_2\sim a\oplus c\oplus z_2\in M &\implies z_1\oplus z_2\sim 0\\
v\sim v\oplus v_1\oplus v_2\sim b\oplus d\oplus v_2\in M &\implies v_1\oplus v_2\sim 0.
\end{aligned}$$

Now we have

$$[x]=[x]\oplus[y_1\oplus y_2]=[x]\oplus[y_1]\oplus[y_2]=[x\oplus y_1']\oplus[y_2]$$

for a suitable $y_1'\sim y_1$, $y_1'\perp x$. Similarly, for a suitable $y_2'\sim y_2$, $y_2'\perp x\oplus y_1'$, and $x\sim x\oplus y_1'\oplus y_2'$. Since $M\subseteq M^1$, we have $\sim\subseteq\sim^1$, and from $x\in M\subseteq M^1$, $x\oplus y_1'\in M^1$, $x\oplus y_2'\in M^1$ we conclude that $y_2'\sim^1 y_1'\sim^1 0$. Similarly we prove that for any $i=1,2$, $x_i\sim^1 z_i\sim^1 v_i\sim^1 0$.

Further,

$$\begin{aligned}
[a\oplus z_2]&=[a\oplus z_2]\oplus[x_1\oplus x_2]\\
&=[a\oplus x_2]\oplus[z_2]\oplus[x_1]\\
&=[b\oplus y_2]\oplus[z_2]\oplus[x_1]\\
&=[b\oplus y_2]\oplus[z_2]\oplus[x_1]\oplus[v_1\oplus v_2]\\
&=[b\oplus v_2]\oplus[y_2]\oplus[z_2]\oplus[x_1]\oplus[v_1].
\end{aligned}$$

On the other hand,

$$a \oplus z_2 \oplus c \sim b \oplus v_2 \oplus d$$

implies

$$[d] = [c] \oplus [y_2] \oplus [z_2] \oplus [x_1] \oplus [v_1].$$

This entails $[d]_1 = [c]_1$; i.e. $d \sim^1 c$, as desired.

(iii) Let $a \sim^1 b$, $b \perp c$. Then there is z, $a \oplus z, b \oplus z \in M^1$. Since M is dominating, there are z_1, z_2, z_3 such that

$$a \oplus z \oplus z_1,\ b \oplus z \oplus z_2,\ b \oplus c \oplus z_3 \in M.$$

Hence

$$a \oplus z_1 \sim b \oplus z_2,\ c \oplus z_3 \sim z \oplus z_2.$$

Moreover, there are $x, y \in M$ and x_1, y_1 such that

$$a \oplus z \sim x \oplus x_1,\ b \oplus z \sim y \oplus y_1.$$

Then

$$[a \oplus z] = [x \oplus x_1] = [a \oplus z \oplus z_1] \oplus [x_1] \implies [x_1] \oplus [z_1] = 0,$$
$$[b \oplus z] = [y \oplus y_1] = [b \oplus z \oplus z_2] \oplus [y_1] \implies [z_2] \oplus [y_1] = 0.$$

Therefore,

$$[a \oplus z] \oplus [z_2] \oplus [y_1] = [a] \oplus [z \oplus z_2] \oplus [y_1]$$
$$= [a] \oplus [c \oplus z_3] \oplus [y_1] = [a] \oplus [c] \oplus [z_3] \oplus [y_1].$$

But $[a] \perp [c] \implies \exists c_1 \sim c$, $c_1 \perp a$ and $[a \oplus c_1] = [a] \oplus [c]$.

Since $\sim \subseteq \sim^1$, we may conclude that M^1 is algebraic. Since M^1 is obviously an order filter, $S/\sim^1$ is a D-poset. $\square$

The following example shows that some pathological situations may occur.

Example 3.7. Let $S = \{0, 1, 2\}$ be the group of positive integers mod 3. Then S is cancellative and unital with unit 0, but is not positive. The set $M = \{0\}$ is dominating. Moreover, $a \sim^M b \implies a = b$, hence M is algebraic and $S/\sim^M \equiv S$. The set $M^1 = \{0, 1, 2\}$, and $S/\sim^{M^1} = \{0\}$.

Let S be a PAS. We will say that a subset M of S is *weakly D-algebraic* (*D-algebraic*) if M is weakly algebraic (algebraic) and $L/\sim^M$ is a D-poset.

It is easy to check that if M is weakly D-algebraic, then $M^1 = M'$. Indeed, for any weakly algebraic M, if $y \sim x$, $x \in M$, then $y \in M^1$. This shows $M' \subseteq M^1$.

Conversely, let M be weakly D-algebraic and let $y \sim x \oplus c$, $x \in M$. Then $[x] \oplus [c] = 1 \oplus [c]$ implies $c \sim 0$, hence $y \sim x$.

In particular, if M is weakly algebraic and irredundant and S is positive, then $M^1 = M$. Indeed, in this case M is weakly D-algebraic and $c \sim 0 \implies c = 0$ (Lemma 3.2 (iii)).

Proposition 3.8. *Let S be a PAS and let M be a weakly D-algebraic subset of S. Then $S/{\sim^M} = \{0\}$ iff there are $a, b \in M$ such that $a \oplus b \in M$. If the operation $\oplus$ in S is totally defined, then $S/{\sim^M} = \{0\}$. If M is D-algebraic, then $S/{\sim^M} = \{0\}$ iff $\oplus$ is totally defined in S.*

Proof. Clearly, $a \sim 0$, $a \in M$ iff then there is $b \in M$ such that $b \oplus a \in M$. If the operation $\oplus$ in S is totally defined, then $a \perp b$ implies $[a] \perp [b]$ for any $a, b \in S$. Since $S/{\sim}$ is a D-poset, this implies $S/{\sim} = \{0\}$. If M is D-algebraic, then $a \perp b$ iff $[a] \perp [b]$, which yields the last statement. $\square$

The following theorem shows a relation between faithful surjective homomorphisms and algebraic sets.

Proposition 3.9. *Let S be a PAS, L a D-poset and let $f : S \to L$ be a faithful surjective homomorphism. Then $F = f^{-1}(1)$ is algebraic and $\sim^F \equiv \sim^f$. In particular, if $\sim$ is a faithful congruence on a PAS S, and $S/\sim$ is a D-poset, then $\sim \equiv \sim^M$ with $M = \{x \in S : x \in 1\}$, where 1 denotes the unit in $S/\sim$.*

Proof. Clearly, F is dominating. If $a \sim^F b$, $c \oplus b \in F$ then $f(a) = f(b)$, $f(c) \perp f(b) \implies f(c) \perp f(a)$, hence $c \perp a$. Moreover, $f(c \oplus a) = f(c) \oplus f(a) = f(c) \oplus f(b) = 1$, i.e., $c \oplus a \in F$. Hence F is algebraic. In the same way ve prove that $\sim^F \equiv \sim^f$.

If $\sim$ is a faithful congruence, then $a \mapsto [a]$ is a faithful surjective homomorphism from L onto $S/\sim$, and this gives the desired result. $\square$

Example 3.10. ([W]) Let L be a PAS, and denote by $\mathbb{Z}_+$ the set of all nonnegative integers. A functionn $f : L \to \mathbb{Z}_+$ amitting only finitely many nonzero values, is called *summable* if the element $\oplus f := \oplus_{a \in L} f(a)a$ exists (where $f(a)a$ is defined as in the end of §1). If f is summable, then necessarily $f(a) \leq \mu(a)$ $\forall a \in L$. Denote by $\mathcal{S}(L)$ the family of all summable functions on L. If $f \in \mathcal{S}(L)$ and $g : L \to \mathbb{Z}_+$ is such that $g(a) \leq f(a)$ $\forall a \in L$, then g also belongs to $\mathcal{S}(L)$. With the pointwise ordering and the restricted addition $f \oplus g := f + g$ provided that $f + g \in \mathcal{S}(L)$, $\mathcal{S}(L)$ becomes a positive, cancellative PAS. The map $f \mapsto \oplus f$ is a faithful homomorphism from $\mathcal{S}(L)$ onto L, so that every PAS is a faithful homomorphic image of a cancellative positive PAS of its summable functions ([W], Proposition 4).

In particular, if L is an effect algebra (a D-poset, equivalently), put $M := \{f \in \mathcal{S}(L) : \oplus_{a \in L} f(a)a = 1\}$. Then M is algebraic and $\mathcal{S}(L)/\sim^M \equiv L$ (see [W], [P,W]). If L is an orthoalgebra, then $f(a) \in \{0, 1\}$ for all $a \neq 0$ for any summable function f. Hence, it this case, M is simply the family of all finite orthopartitions of 1 in L, and we recover the standard representation of orthoalgebras as logics of manuals (see, [F,R 1], [F,R 2], [R,F], [F,G,R]).

4. Ideals and central ideals

A subset J of a PAS L is called *an order ideal* if $x = y \oplus z$, $x \in J \implies y, z \in J$; and J is called an *ideal* if $x = y \oplus z$, $x \in J \iff y, z \in J$.

We say that an ideal J in L is *principal* if $J = [0, x]$ for some $x \in L$.

For any subset M of L define $M^\perp := \{x \in L : x \perp y \ \forall y \in M\}$. For any M, $M^\perp$ is an order ideal. Indeed, $u \oplus v \in M^\perp \implies u \oplus v \perp x, x \in M \implies u, v \perp x, x \in M \implies u, v \in M^\perp$. Clearly, $0 \in M^\perp$ for any M.

The following is easy:

$$M \subseteq M^{\perp\perp}; M_1 \subseteq M_2 \implies M_2^{\perp} \subseteq M_1^{\perp};$$
$$(M^{\perp})^{\perp\perp} = (M^{\perp\perp})^{\perp}; M^{\perp\perp\perp} = M^{\perp}.$$

Let L be a D-poset. For any interval $[0, x]$, $x \in L$, we have

$$[0, x]^{\perp} = \{y \in L : y \perp z, \forall z \leq x\} = [0, x'].$$

A subset Z of a PAS L will be called a *central ideal* if

(i) $Z, Z^{\perp}$ are ideals in L;
(ii) $\forall x \in L$, there is a unique decomposition $x = x_1 \oplus x_2$, where $x_1 \in Z$, $x_2 \in Z^{\perp}$.

Observe that if Z is a central ideal then $Z^{\perp}$ is also a central ideal. The decomposition in (ii) will be called a *central decomposition*. It is easy to check that if a PAS $Z = Z_1 \times Z_2$ is the direct product of PASs Z_1, Z_2, then $(z_1, z_2) \mapsto (z_1, 0) \oplus (0, z_2)$ ($z_1 \in Z_1$, $z_2 \in Z_2$) is a central decomposition with respect to the central ideal $\tilde{Z}_1 := \{(z_1, 0) : z_1 \in Z_1\}$.

Lemma 4.1. *Let Z be a central ideal in a positive, cancellative PAS L. Then*

(i) $Z \cap Z^{\perp} = \{0\}$;
(ii) $Z = Z^{\perp\perp}$.

Proof. (i) Let $x \in Z \cap Z^{\perp}$, then $x = x \oplus 0 = 0 \oplus x \implies x = 0$. (ii) If Z is a central ideal, then $Z^{\perp}$ is a central ideal as well, and hence by (i), $Z^{\perp\perp} \cap Z^{\perp} = \{0\}$. Let $x \in Z^{\perp\perp}$. Then $x = x_1 \oplus x_2 \perp y$, $x_1 \in Z, x_2 \in Z^{\perp}$, $y \in Z^{\perp} \implies x_2 \perp y, y \in Z^{\perp} \implies x_2 \in Z^{\perp\perp}$, hence $x_2 = 0$, so that $x = x_1 \in Z$. $\square$

Proposition 4.2. *Let Z be a central ideal in a cancellative, positive PAS L. For $x, y \in L$, define $x \sim y$ if $x_1 = y_1$, where $x = x_1 \oplus x_2$, $y = y_1 \oplus y_2$, $x_1, y_1 \in Z$, $x_2, y_2 \in Z^{\perp}$. Then $\sim$ is a congruence.*

Proof. Clearly, $x = x_1 \oplus x_2$, $y = y_1 \oplus y_2$ and $x \perp y$ imply $x \oplus y = (x \oplus y)_1 \oplus (x \oplus y)_2 = (x_1 \oplus y_1) \oplus (x_2 \oplus y_2)$.

The relation $\sim$ is clearly an equivalence relation. Assume that $x \sim y$, $u \sim v$ and $x \perp u$, $y \perp v$. Then $x_1 = y_1$, $u_1 = v_1$ and $x_1 \perp u_1$, $y_1 \perp v_1$. Hence $x_1 \oplus u_1 = y_1 \oplus v_1$, and hence $x \oplus u \sim y \oplus v$. If $x \sim y$, $y \perp z$, then $x_1 = y_1$ and $y_1 \oplus y_2 \perp z_1 \oplus z_2$.This implies $x_1 \perp z_1$, and $x_1 \oplus z_1 \in Z$. Therefore, $x_1 \oplus z_1 \perp x_2$, and hence $x_1 \oplus x_2 \perp z_1 \oplus 0$. We obtain $z_1 \oplus 0 \sim z$ and $z_1 \oplus 0 \perp x$. This verifies axioms (i), (ii), (iii) of Definition 2.2, hence $\sim$ is a congruence. $\square$

The following proposition and corollary put into relation algebraic sets and ideals.

Proposition 4.3. *Let M be a closed, dominating subset of a PAS L, which is an order filter. Define $J := \{x \in L : x \perp y,$ for some $y \in M\}$. Then J is an ideal in L.*

Moreover, if any $x, y \in L$ have an upper bound in L, then $x \sim y$ if and only if $\exists u, v \in J$ such that $x \oplus u = y \oplus v$.

Proof. Assume that $x, y \in L$ and $x \perp y$. If $x \oplus y \in J$, then $x \oplus y \perp z$ for some $z \in M \implies x \perp z, y \perp z$, hence $x, y \in J$.

If $x, y \in J$, then there are $z_1, z_2 \in M$ with $x \perp z_1$, $y \perp z_2$. Since M is dominating, there is $z \in L$ such that $x \oplus y \oplus z \in M$. Moreover, since M is an order filter, we have $x \oplus z_1, y \oplus z_2 \in M$, and therefore $y \oplus z \sim^M z_1$, $x \oplus z \sim^M z_2$. The closeness of M implies $y \oplus z \in M$, $x \oplus z \in M$, which yields $z \sim^M z_1$, $z \sim^M z_2$, and closeness of M yields $z \in M$, hence $x \oplus y \in J$.

Now assume that any $x, y \in L$ have an upper bound in L. Let $x \sim^M y$, and let $z \in L$ be such that $x \oplus z \in M$, $y \oplus z \in M$. Let w be an upper bound of $x \oplus z, y \oplus z$. Then there are $u_1, u_2 \in J$ such that $x \oplus z \oplus u_1 = w = y \oplus z \oplus u_2$, so that $x \oplus u_1 = w \ominus z = y \oplus u_2$.

Conversely, let $u_1, u_2 \in J$, and $x \oplus u_1 = y \oplus u_2$. Since $u_1, u_2 \in J$, and M is an order filer, we have $u_1 \oplus z_1 \in M$, $u_2 \oplus z_2 \in M$ for some $z_1, z_2 \in M$. There is z such that $x \oplus u_1 \oplus z = y \oplus u_2 \oplus z \in M$, which implies $x \oplus z \sim^M z_1$, $y \oplus z \sim^M z_2$. Close ness of M implies $x \oplus z, y \oplus z \in M$, hence $x \sim^M y$. □

Corollary 4.4. *To every closed weakly D-algebraic subset M of a D-poset L there corresponds an ideal $J = \{x \in L : x = y', y \in M\}$ (where $x' = 1 \ominus x$), and $x \sim^M y$ if and only if $x \oplus u = y \oplus v$ for some $u, v \in J$.*

5. Congruences on D-posets

Let S be a PAS and L a unital, cancellative, positive PAS (D-poset). For $a \in L$, let $a' = 1 \ominus a$. The following result shows that there is no nontrivial faithful homomorphism $f : L \to S$.

Proposition 5.1. *Let $f : L \to S$ be faithful surjective homomorphism from a D-poset L onto a PAS S. Then f is injective, hence an isomorphism.*

Proof. The suppositions imply that S is again a D-poset, and $f(a') = f(a)'$. Thus, $f(a) = f(b)$ implies $f(a)' = f(b)' = f(b')$. By faithfulness, $a \perp b'$, which entails $a \leq b'' = b$. Symmetrically, $b \leq a$, so $a = b$. □

Corollary 5.2. *Let L be a D-poset and $M \subseteq L$ an algebraic subset. Then $M = \{1\}$, and the unique faithful congruence of L is the trivial congruence $a \sim b$ iff $a = b$.*

Proof. Follows by Proposition 5.1 and Proposition 3.9. □

Let $f : S \to L$ be a surjective full homomorphism. Let $a \sim^f b$ iff $f(a) = f(b)$. Then $\sim^f$ is a congruence iff condition (c) of Proposition 2.4 is satisfied. Put $F = f^{-1}(1)$, and let $\sim^F$ be the perspectivity with respect to F.

Clearly, $\sim^F \subseteq \sim^f$, and $\sim^F \equiv \sim^f$ iff $f(a) = f(b) \implies \exists c$: $c \perp a$, $c \perp b$, $f(c \oplus a) = f(c \oplus b) = 1$. For example, if S is a lattice and f is a lattice homomorphism, too, and $\sim^f$ is a congruence, then $\sim^f \equiv \sim^F$. Indeed, we can take $c = a' \wedge b'$. From this, the following useful result can be derived. (A D-poset L is called a *D-lattice* if L is a lattice).

Proposition 5.3. *Let L be a D-lattice, $\sim$ be a congruence on L which is also a lattice congruence, and $f : L \to L/\sim$ be the canonical mapping. Then $\sim \equiv \sim^M$, where $M = f^{-1}(1)$.*

The following example shows that in a general case the equivalence $\sim^f \equiv \sim^F$ need not hold.

Example 5.4. Let $S = MO_2$ (that is, the horizontal sum of two four-point Boolean algebras), $L = \{0, 1\}$ and let $f : S \to L$ be defined as follows: $f(1) = f(a) = f(b) = 1$, $f(0) = f(a') = f(b') = 0$. It is easy to check that $\sim^f$ is a congruence (not a lattice congruence). But the set $F = \{1, a, b\}$ is not weakly algebraic. Indeed, $a \sim b$, $a \oplus a' \in F$, $b \oplus b' \in F$, but a' and b' do not share a common complement.

In the sequel, we consider two important examples of D-lattices, MV-algebras and orthomodular lattices. We will show that congruences in these structures correspond to congruences in the sense of Definition 2.2.

An MV-algebra is an algebra $(\mathcal{A}, +, \circ, *, 0, 1)$, where $\mathcal{A}$ is a nonempty set, 0 and 1 are constants, $+$ and $\circ$ are binary operations and $*$ is a unary operation, satisfying the following axioms (cf. [M], this definition is equivalent to the definition introduced by Chang [Ch]).

(MV1) $(a + b) + c = a + (b + c)$;
(MV2) $a + 0 = a$;
(MV3) $a + b = b + a$:
(MV4) $a + 1 = 1$:
(MV5) $(a^*)^* = a$;
(MV5) $0^* = 0$:
(MV7) $a + a^* = 1$:
(MV8) $(a^* + b^*) + b = (a + b^*)^* + a$.

Given an MV-algebra (after Mangani [M]), we can define

$$a \circ b = (a^* + b^*)^*;$$
$$a \wedge b = (a + b^*) \circ b;$$
$$a \vee b - (a \circ b^*) + b.$$

Then $a \leq b$ iff $a = a \wedge b$ is a partial order in $\mathcal{A}$, and $(\mathcal{A}, \wedge, , \vee)$ is a distributive lattice.

If we put $b \ominus a := (a + b^*)^*$ for $a \leq b$, then we obtain that $\mathcal{A}$ is a D-poset, more exactly, a distributive D-lattice ([Cho]). Moreover, $b - a := b \ominus (a \wedge b) = b \circ a^*$ for any $a, b \in \mathcal{A}$

Conversely, in any D-lattice we can define a binary operation $b - a := b \ominus (a \wedge b)$ for any $a, b \in \mathcal{A}$. It was proved in [Ch,K], that a D-lattice P is an MV-algebra if and only if $(w - u) - v = (w - v) - u$ for any $u, v, w \in P$.

The following simple example shows that not every distributive D-lattice is an MV-algebra. Let $P = \{0, a, b, 1\}$ and let the domain of the operation $\oplus$ be the binary relation $S = \{(0, 0), (0, a), (0, b), (a, a), (b, b)\}$ (so-called diamond, see [G,F,P]). Indeed, we have $(1 - a) - b = a \neq b = (1 - b) - a$.

An ideal in an MV-algebra $\mathcal{A}$ is a non-empty subset I of $\mathcal{A}$ such that

(1) $a, b \in I \implies a + b \in I$;
(2) $a \in I,\ b \in \mathcal{A},\ a \leq b \implies b \in I$.

Lemma 5.5. *A subset $I \neq \emptyset$ of an MV-algebra $\mathcal{A}$ is an ideal iff for any $a, b \in \mathcal{A}$, $b \leq a^*$: $a, b \in I \iff a \oplus b \in I$, where $a \oplus b := (a^* \ominus b)^* = (a^* - b)^* = (b^* - a)^*$.*

Proof. Assume $a, b \in \mathcal{A}$, $b \leq a^*$. Let I be an ideal. Then $a, b \in I \implies a \oplus b = (a^* - b)^* = a + b \in I$. The converse implication follows trivially from $a, b \leq a \oplus b$.

Now let I satisfy the above condition. If $a \in I$ and $b \leq a$, then $a = b \oplus c$, hence $b, c \in I$. If $a, b \in I$, then $a + b = (a^* - b)^* = (a^* \ominus (a^* \wedge b))^* = a \oplus (a^* \wedge b) \in I$, hence I is an ideal in $\mathcal{A}$. $\square$

Chang ([Ch]) proved that if I is an ideal of an MV-algebra $\mathcal{A}$, then

$$a \sim_I b \text{ iff } (a \circ b^*) + (a^* \circ b) \in I$$

is a congruence of $\mathcal{A}$, and the quotient $\mathcal{A}/\sim_I$ is an MV-algebra. From this we obtain the following

Proposition 5.6. *If I is an ideal of an MV-algebra $\mathcal{A}$, then $\sim_I$ is a congruence in the sense of Definition 2.2. Moreover, the set $M := \{a^* : a \in I\}$ is a weakly D-algebraic set, and $\sim_I \equiv \sim^M$.*

Proof. Let $a \sim_I b$ $b \perp c$. Then $c \wedge a^* \perp a$, and $c \wedge a^* \sim_I c$. Indeed, let $\phi : L \to L/\sim_I$ be the canonical mapping. Then $\phi(a^*) = 1$, hence $\phi(c - c \wedge a^*) = \phi(c) - \phi(c) \wedge \phi(a^*) = 0$. This proves that $\sim_I$ is a congruence in the sense of Definition 2.2.

Let $a \sim_I b$. Then $(a \circ b^*) + (a^* \circ b) \in I \implies a \ominus (a \wedge b), b \ominus (a \wedge b) \in I$, and this entails $a^* \oplus (a \wedge b)$, $b^* \oplus (a \wedge b) \in M$, whence $a^* \sim^M b^*$.

Conversely, let $a^* \sim^M b^*$. Then there is $c \perp a^*, b^*$ such that $a^* \oplus c, b^* \leq [plusc \in M$, which entails $a \ominus c, b \ominus c \in M$. Since $c \leq a, b$, we get $c \leq a \wedge b \leq a, b \implies a \ominus (a \wedge b) \leq a \ominus c$, $b \ominus (a \wedge b) \leq b \ominus c$, whence $(a \ominus (a \wedge b)) \oplus (b \ominus (a \wedge b)) = (a \circ b^*) + (b \circ a^*) \in I$, i.e., $a \sim_I b$.

Taking into account that $(a^*)^* = a$, and that $\sim_I$ is a congruence, concludes the proof. An alternative proof can be obtained making use of Proposition 5.3. $\square$

We recall that an orthomodular lattice (OML) is an ortholattice with 0 and 1 as the smallest and the greatest element, respectively, satisfying the orthomodular law $a \leq b \implies b = a \vee (a' \wedge b)$, where $a \mapsto a'$ is the orthocomplementation (see [Be], [K], [P,P]). An orthomodular lattice becomes a D-poset if we define $b \ominus a = b \wedge a'$ for $a, b \in L$, $a \leq b$.

Let L be an OML. A set $M \subset L$ is an *orthomodular filter* (dual orthomodular ideal) if $x, y \in M \implies x \wedge y \in M$ and $x \in M$, $y \in L \implies (x \wedge y') \vee y \in M$. The one-to-one correspondence between orthomodular filters and congruences of orthomodular lattices is well-known([Be], [K]). It is easy to see that an orthomodular ideal is an ideal of the PAS L, the converse need not hold.

Proposition 5.7. *Let L be an OML. Then an orthomodular filter M is a weakly D-algebraic subset of L.*

Proof. Let M be an orthomodular filter in L. Then $a \sim b$ iff $(a \wedge b) \vee (a' \wedge b') \in M$ is a congruence of L as an OML. If $a \sim b$, $b \perp c$, then put $d := c \wedge a'$. Clearly, $d = c \wedge a' \sim c \wedge b' = c$, and $d \perp a$. This proves that $\sim$ is a congruence in the sense of Definition 2.2.

We will prove that $\sim$ is perspectivity with respect to M. We have $(a \wedge b) \vee (a' \wedge b') = (a \vee (a' \wedge b')) \wedge (b \vee (a' \wedge b') \in M$ (we used Foulis-Holland theorem). Therefore, $a \vee (a' \wedge b') \in M$, $b \vee (a' \wedge b') \in M$, hence a and b share a common complement $a' \wedge b'$ which is orthogonal to both a, b. Conversely, let c be such that $a \oplus c \in M$, $b \oplus c \in M$. Then $c \leq a'$, $c \leq b'$, $a \oplus c = a \vee c \leq a \vee (a' \wedge b')$, $b \vee c \leq b \vee a' \wedge b'$.

Therefore $(a \wedge b) \vee (a' \wedge b') \in M$, i.e., $a \sim b$. An alternative proof can be obtained by applying Proposition 3.5. $\square$

In [GFP], the following definitions have been introduced. Let E be an effect algebra (a D-poset, equivalently). If $e \in E$, then e is *principal* if for $p, q \in E$, $p \perp q$ and $p, q \leq e \implies p \oplus q \leq E$. Clearly, e is principal iff $[0, e]$ is a (principal) ideal. Notice that the notion of a principal element can be easily generalized to a positive, cancellative PAS. An element $a \in E$ is *central* iff (i) a and a' are principal, (ii) $\forall p \in E$, $\exists q, r \in E$, $q \leq a, r \leq a'$, $p = q \oplus r$. It has been proved [G,F,P], that for any central element a, the elements $a \wedge p$ and $a^{\perp} \wedge p$ exist, and $p = (a \wedge p) \oplus (a^{\perp} \wedge p)$, $p \in E$, so that the decomposition in (ii) is unique.

Proposition 5.8. *In a D-poset L, every central ideal is principal.*

Proof. Let $Z \subset L$ be a central ideal. Then $1 = x_1 \oplus x_2$, $x_1 \in Z$, $x_2 \in Z^{\perp}$ is a unique decomposition of 1. Therefore, there is a unique $(x_1 =)x \in Z$ such that $x^{\perp} \in Z^{\perp}$, whence $Z = [0, x]$, $Z^{\perp} = [0, x']$. $\square$

Proposition 5.9. *Let L be D-poset. Then $x \in L$ is a central element iff $[0, x]$ is a central ideal. For $u \in L$, let $u = u_1 \oplus u_2$, $u_1 \leq x$, $u_2 \leq x^{\perp}$ be the unique central decomposition with respect to $[0, x]$. The congruence $y \sim z$ iff $y_1 = z_1$ coincides with the perspectivity with respect $M = [x, 1]$.*

Proof. The first statement is clear. To prove the second, observe that $z \sim y$ implies $y_1 = z_1 \leq x \implies x \ominus y_1 = x \ominus y_1 =: r$. Hence $y \oplus r = (y_1 \oplus r) \oplus y_2 = x \oplus y_2 \in M$, similarly, $z \oplus r = x \oplus z_2 \in M$.

Conversely, $y \sim_M z$ implies $y \oplus u \in M$, $z \oplus u \in M$ for some $u \in L$. But then $(y_1 \oplus u_1) \oplus (y_2 \oplus u_2) = x \oplus v$ for some $v \in L$. Uniqueness of the central decomposition implies $y_1 \oplus u_1 = x$, and, analogously, $z_1 \oplus u_1 = x$. By cancellativity, $y_1 = z_1$. $\square$

6. Conclusions

We studied congruences in partial abelian semigroups as equivalence relations preserving the partial operation and and having the property

$$(*) \qquad a \sim b, \; b \perp c \implies \exists d : \; d \sim c, \; d \perp a.$$

These properties quarantee that the corresponding quotient is again a PAS, in particular, the associativity condition (which corresponds to strong associativity in the sense of [L,E]) is satisfied. In some important cases, like congruences in MV-algebras and orthomodular lattices, as well as by factors in direct products, condition (*) is always satisfied. It remains an open question whether this condition is necessary.

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