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Chaotic expansion of elements of the universal enveloping algebra of a Lie algebra associated with a quantum stochastic calculus

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Chaotic expansion of elements of the universal enveloping algebra of a Lie algebra associated with a quantum stochastic calculus*

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Abstract

In classical stochastic analysis the stochastic derivative of a polynomial function f of integrator process $\mathbf{\Lambda} = (\Lambda_1, \cdots, \Lambda_N)$ whose differentials satisfy a closed Ito multiplication table can be expressed as

$$df(\mathbf{\Lambda}) = f(\mathbf{\Lambda} + d\mathbf{\Lambda}) - f(\mathbf{\Lambda}),$$

and leads to a chaotic decomposition of $f(\mathbf{\Lambda})$. In quantum theories the Ito table becomes noncommutative but the same formula and the corresponding chaotic decomposition hold formally for elements of the universal enveloping algebra $\mathcal{U}$ of the Lie algebra formed by the Ito differentials, and acquire rigorous meaning in terms of the coproduct structure of $\mathcal{U}$.

Keywords: quantum stochastic calculus, universal enveloping algebra, chaotic decomposition.

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1 Introduction

Ito's formula for the stochastic differential of a smooth function f of Brownian motion B and time T is

$$df(B,T) = f'(B,T)dB + (\dot{f}(B,T) + \frac{1}{2}f''(B,T))dT \tag{1.1}$$

where $'$ and $\dot{}$ denote partial differentiation with respect to B and T respectively. The meaning to be ascribed to (1.1) is that, at an arbitrary time t, $f(B(t),t)$ is given by the stochastic integral expression

$$f(B(t),t) = f(0,0)+\int_{0<s<t}\{f'(B(s),s)dB(s)+(\dot{f}(B(s),s)+\frac{1}{2}f''(B(s),s))ds\}. \tag{1.2}$$

When f is a polynomial function, the formula (1.1) can also be expressed formally as

$$df(B,T) = f(B+dB,T+dT) - f(B,T) \tag{1.3}$$

where dB and dT are to be multiplied according to the Ito table

$$\begin{array}{c|cc} & dB & dT \\ \hline dB & dT & 0 \\ dT & 0 & 0 \end{array} \tag{1.4}$$

as may readily be seen by expanding $f(B+dB,T+dT)$ as a Taylor series about (B,T). Correspondingly (1.2) becomes

$$f(B(t),t) = f(0,0) + \int_{0<s<t}\{f(B(s)+dB(s),s+ds) - f(B(s),s)\} \tag{1.5}$$

where the apparent non-linearity in $dB(s)$ and ds of the integrand is removed using the table (1.4). An advantage of (1.5) over (1.1) is that it is easily iterated to yield a "chaotic decomposition"

$$\begin{aligned} f(B(t),t) &= f(0,0) \\ &\quad + \textstyle\int_{0<t_1<t}\{f(dB(t_1),dt_1) - f(0,0)\} \end{aligned}$$

$$\begin{aligned} &+ \textstyle\int_{0<t_1<t_2<t}\{f(dB(t_1)+dB(t_2),dt_1+dt_2) - f(dB(t_1),dt_1) \\ &+ - f(dB(t_2),dt_2) + f(0,0)\} + \cdots \end{aligned} \tag{1.6}$$

Exactly as in the calculus of finite differences, and for the same algebraic reason, the iterative expansion (1.6) terminates with an n-fold iterated integral, where n is the degree of the polynomial f. Note that (1.6) differs from the chaotic decomposition common in probabilistic literature by the inclusion of time as an integrator.

More generally, given a collection of stochastic integrator processes $\mathbf{\Lambda} = (\Lambda_0, \cdots, \Lambda_N)$ whose differentials satisfy an Ito multiplication rule (we may think of the case of N-dimensional Brownian motion when $\mathbf{\Lambda} = (T, B_1, \cdots, B_N)$ or the case of the compensated Poisson process Λ, for which $d\Lambda d\Lambda = d\Lambda$)

$$d\Lambda_j d\Lambda_k = \sum_{l=0}^{N} c_{jk}^l d\Lambda_l \tag{1.7}$$

we may write, for any polynomial function f,

$$df(\mathbf{\Lambda}) = f(\mathbf{\Lambda} + d\mathbf{\Lambda}) - f(\mathbf{\Lambda}) \tag{1.8}$$

and hence, for any time t,

$$f(\mathbf{\Lambda}(t)) = f(\mathbf{0}) + \int_{0<s<t} (f(\mathbf{\Lambda}(s) + d\mathbf{\Lambda}(s)) - f(\mathbf{\Lambda}(s))) \tag{1.9}$$

and hence obtain the chaotic decomposition

$$f(\mathbf{\Lambda}(t) = \sum_{r=0}^{deg f} \int_{0<s_1<\cdots<s_r<t} \sum_{A\subseteq\{1,\cdots,r\}} (-1)^{r-|A|} f(\sum_{\alpha\in A} d\mathbf{\Lambda}(s_\alpha)). \tag{1.10}$$

Our purpose in this paper is to examine the generalization of (1.8), (1.9) and (1.10) to theories of quantum stochastic calculus in which the multiplication (1.7) is noncommutative. It is a feature of most such theories [1] that the differentials of the integrator processes and the integrator processes themselves both form realisations of one and the same abstract Lie algebra $\mathcal{L}$; thus, for $L, K \in \mathcal{L}$ we write

$$d\Lambda_L d\Lambda_K - d\Lambda_K d\Lambda_L = d\Lambda_{[L,K]} \tag{1.11}$$

$$\Lambda_L \Lambda_K - \Lambda_K \Lambda_L = \Lambda_{[L,K]} \tag{1.12}$$

[1] The exceptions are $\mathbb{Z}_2$-graded (in particular Fermionic) theories where the Lie algebra becomes a Lie-superalgebra [EyH].

even though the processes do *not* realise the same associative algebra as that represented by the Ito multiplication table itself.

(In actual fact, in the simplest theories (which we shall use here) of quantum stochastic calculus (1.12) must be replaced by relations of the form

$$\begin{aligned}&\langle \Lambda_L^\dagger(t)\psi, \Lambda_K(t)\chi\rangle - \langle \Lambda_K^\dagger(t)\psi, \Lambda_L(t)\chi\rangle \\ &= \langle \psi, \Lambda_{[L,K]}(t)\chi\rangle\end{aligned} \tag{1.13}$$

for arbitrary $t \in \mathbb{R}_+$ and ψ, χ in a dense subspace because the processes in general consist of unbounded Hilbert space operators acting (with their adjoints) on a common dense domain $\mathcal{E}$ which they do not leave invariant, so that the products making up the left hand side of (1.12) are not meaningful.)

It is the coexistence of two realisations (1.11) and (1.12) (more strictly (1.13)) which makes possible the generalisation of (1.8), (1.9) and (1.10) in terms of the universal enveloping algebra of $\mathcal{L}$ and, in particular, its coproduct structure.

We now review briefly the parts of the theory of quantum stochastic calculus which we will require. We begin with the Fock space over a Hilbert space $\mathfrak{h}$, which it is most convenient to define abstractly as the Gel'fand pair corresponding to the positive definite kernel $(f, g) \mapsto \exp\langle f, g\rangle$ over $\mathfrak{h}$. Thus it comprises a pair $(\mathcal{H}, e)$ comprising a Hilbert space $\mathcal{H}$ and a map $e : \mathfrak{h} \to \mathcal{H}$, $f \mapsto e(f)$ such that, for all $f, g \in \mathfrak{h}$,

$$\langle e(f), e(g)\rangle_{\mathcal{H}} = \exp\langle f, g\rangle_{\mathfrak{h}}$$

and the range $\{e(f) : f \in \mathfrak{h}\}$ is total in $\mathcal{H}$. The unit vector $e(0)$ is called the *vacuum* and the $e(f)$ are *exponential vectors*.

We now specialise to the case

$$\mathfrak{h} = L^2(\mathbb{R}_+, \mathbb{C}^N) = L^2(\mathbb{R}_+) \otimes \mathbb{C}^N$$

and denote by $\mathcal{E}$ the dense subspace of $\mathcal{H}$ comprising finite linear combinations of exponential vectors $e(f)$ for which f is locally bounded,

$$f \in \mathfrak{h} \cap L^\infty(\mathbb{R}_+, \mathbb{C}^N)_{loc}.$$

Writing $f = (f^1, \cdots, f^N)$, with $f^j \in L^\infty(\mathbb{R}_+)$, $j = 1, \cdots, N$ we define $f^0 \equiv 1$ and $f_\lambda = \bar{f}^\lambda$, $\lambda = 0, 1, \cdots, N$.

All operators in this paper will be defined on the domain $\mathcal{E}$ and will have adjoints whose domains include $\mathcal{E}$; the restriction to $\mathcal{E}$ of the adjoint T^* of such an operator T will be denoted by $T^\dagger$. We define integrator processes Λ_β^α, $\alpha, \beta = 0, 1, \cdots, N$ by

$$\Lambda_0^0(t) = te(f)$$

$$\Lambda_0^j(t)e(f) = \int_0^t f^j(s)ds e(f)$$

$$\Lambda_k^0(t)e(f) = \frac{d}{dz} e(f + z\chi_{[0,t]} \otimes \varepsilon_k)|_{z=0}$$

$$\Lambda_k{}^j(t)e(f) = \frac{d}{dz}((M_{\chi_{(0,t)}} \otimes e^{z|\varepsilon_k\rangle\langle\varepsilon_j|} + M_{\chi_{(t,\infty)}} \otimes 1)f|_{z=0}$$

for $j, k = 1, \cdots, N$. Here χ_S is the indicator function of $S \subseteq \mathbb{R}_+$, $\varepsilon_k, k = 1, \cdots, N$ is the canonical basis in $\mathbb{C}^N$, $|\varepsilon_k\rangle\langle\varepsilon_j|$ the Dirac dyad whose matrix in this basis has 1 in the $(j,k)^{th}$ position and 0 elsewhere, and M_χ means the operator of multiplication by χ. Then we have

$$\Lambda_\beta^\alpha(t)^\dagger = \Lambda_\alpha^\beta(t), \ \alpha, \beta = 0, 1, \cdots, N$$

$$\langle e(f), \Lambda_\beta^\alpha(t)e(g)\rangle = \int_0^t f_\beta(s)g^\alpha(s)\langle e(f), e(g)\rangle$$

$$\langle \Lambda_\beta^\alpha(t)e(f), \Lambda_\delta^\gamma(t)e(g)\rangle$$

$$= \{\int_{0<t_1<t_2<t} (f_\alpha(t_1)g^\beta(t_1)f_\delta(t_2)g^\gamma(t_2) + f_\delta(t_1)g^\gamma(t_1)f_\alpha(t_2)g^\beta(t_2))dt_1dt_2$$

$$+ \int_{0<t<s} \hat{\delta}_\delta^\beta f_\alpha(t)g^\gamma(t)dt\}\langle e(f), e(g)\rangle$$

where $\hat{\delta}_\delta^\beta$ is the Evans [Ev] delta which is 1 if $\beta = \delta \neq 0$ and 0 otherwise.

The theory of quantum stochastic calculus [P, Me] defines processes of operators of the form

$$M(t) = \int_{0<s<t} \sum_{\alpha,\beta=0}^{N} E_\alpha^\beta(s) d\Lambda_\beta^\alpha(s) \tag{1.14}$$

where E_α^β are themselves operator valued processes satisfying certain integrability conditions. Stochastic integrals are themselves integrable processes so that, in particular, iterated integrals of the form

$$\Lambda_{\beta_1\cdots\beta_n}^{\alpha_1\cdots\alpha_n}(t) := \int_{0<t_1<\cdots<t_n<t} d\Lambda_{\beta_1}^{\alpha_1}(t_1)\cdots d\Lambda_{\beta_n}^{\alpha_n}(t_n) \tag{1.15}$$

are well defined.

The *first fundamental formula* evaluates the matrix elements of the process (1.14) between exponential vectors as

$$\langle e(f), \int_{0<s<t} \sum_{\alpha,\beta=0}^{N} E_\alpha^\beta(s) d\Lambda_\beta^\alpha(s) e(g)\rangle \tag{1.16}$$

$$= \sum_{\alpha,\beta=1}^{N} \int_{0<s<t} f_\beta(s) f^\alpha(s) \langle e(f), E_\alpha^\beta(s) e(g)\rangle ds.$$

From this it follows that, when M is given by (1.14) we have

$$M(t)^\dagger = \int_{0<s<t} \sum_{\alpha,\beta=0}^{N} (E_\alpha^\beta(s))^\dagger d\Lambda_\alpha^\beta(s) \tag{1.17}$$

so that from (1.15) we deduce that the adjoint process

$$(\Lambda_{\beta_1\cdots\beta_n}^{\alpha_1\cdots\alpha_n})^\dagger = \Lambda_{\alpha_1\cdots\alpha_n}^{\beta_1\cdots\beta_n}. \tag{1.18}$$

The *second fundamental formula* is that

$$\langle \int_{0<s<t} \sum_{\alpha,\beta=0}^{N} E_\alpha^\beta(s) d\Lambda_\beta^\alpha(s) e(f), \int_{0<s<t} \sum_{\gamma,\delta=1}^{N} F_\delta^\gamma(s) d\Lambda_\gamma^\delta(s) e(g)\rangle$$

$$= \int_{0<s<t} \sum_{\alpha,\beta=1}^{N} f_\alpha(s) g^\beta(s) \{\langle E_\alpha^\beta(s) e(f), \int_{0<s_1<s} \sum_{\gamma,\delta=0}^{N} F_\delta^\gamma(s_1) d\Lambda_\gamma^\delta(s_1) e(g)\rangle$$

$$+\langle \int_{0<s_1<s} \sum_{\gamma,\delta=0}^{N} E_\delta^\gamma(s_1) d\Lambda_\gamma^\delta(s_1) e(f), F_\beta^\alpha(s) e(g)\rangle$$

$$+ \sum_{j=1}^{N} \langle E_\alpha^j(s) e(f), F_\beta^j(s) e(g)\rangle\} ds. \tag{1.19}$$

As explained in [HP$_1$,P], this relation is summarised by the Ito multiplication rule

$$d\Lambda_\beta^\alpha d\Lambda_\delta^\gamma = \hat{\delta}_\delta^\alpha d\Lambda_\beta^\gamma. \tag{1.20}$$

Combining (1.18) and (1.16) we can infer from (1.19) that, for two iterated integrals

$$\Lambda_\beta^\alpha = \Lambda_{\beta_1,\cdots,\beta_n}^{\alpha_1,\cdots,\alpha_n}, \;\; \Lambda_\delta^\gamma = \Lambda_{\delta_1,\cdots,\delta_m}^{\gamma_1,\cdots,\gamma_m}$$

the inner product

$$\langle \Lambda^\alpha_\beta(t)e(f), \Lambda^\gamma_\delta(t)e(g)\rangle$$

can be expressed as a sum of matrix elements of the form $\langle e(f), \Lambda^\sigma_\tau e(g)\rangle$. Our first task in this paper is to make this reduction explicit (Theorem 2.1).

To do this we first introduce the *Ito algebra* $\mathcal{J}$, that is a sub-†-algebra of the associative †-algebra spanned by basic elements $d\Lambda^\alpha_\beta$, $\alpha, \beta = 0, \cdots, N$ with multiplication (1.20) and involution defined by $(d\Lambda^\alpha_\beta)^\dagger = d\Lambda^\beta_\alpha$. The iterated integral (1.15) extends to a linear map I^n from the tensor product $\bigotimes^n \mathcal{J}$ of n copies of $\mathcal{J}$. We are then able to describe explicitly the associative multiplication $*$ and involution † in the tensor space

$$\mathcal{T}(\mathcal{J}) = \mathbb{C} \oplus \mathcal{J} \oplus (\mathcal{J} \otimes \mathcal{J}) \oplus (\mathcal{J} \otimes \mathcal{J} \otimes \mathcal{J}) \oplus \cdots$$

for which

$$\langle I_t(a^\dagger)\psi, I_t(b)\chi\rangle = \langle \psi, I_t(a * b)\chi\rangle \tag{1.21}$$

for arbitrary $a, b \in \mathcal{T}(\mathcal{J})$ and $\psi, \chi \in \mathcal{E}$. Here I_t is the evaluation at $t \in \mathbb{R}_+$ of the aggregated map $I = I^{(0)} \oplus I^{(1)} \oplus I^{(2)} \oplus \cdots$ from $\mathcal{T}(\mathcal{J})$ to integrable processes. The multiplication $*$ is defined explicitly and its associativity is proved in §1 for an arbitrary finite-dimensional associative †-algebra $\mathcal{J}$; in §2 we establish (1.21) when $\mathcal{J}$ is the full Ito algebra given by (1.20) and hence also for any †-subalgebra.

By taking $\mathcal{J}$ to be a sub-†-algebra of the full Ito algebra we are able to treat many different Lie algebras. For example we may take $\mathcal{J}$ to be the span of the Λ^i_j alone, $i, j = 1, \cdots, N$ in which case we recover the stochastic analysis of representations of the Lie algebra of $u(N)$ of $U(N)$ (or equivalently $gl(N)$) [HP$_2$,HP$_3$,HP$_4$]. Taking $N = 1$ and writing

$$d\Lambda^0_0 = dT,\ d\Lambda^1_0 = dA,\ d\Lambda^0_1 = dA^\dagger,\ d\Lambda^1_1 = d\Lambda$$

we obtain the familiar quantum Ito table [HP$_1$,P]

$$\begin{array}{c|cccc} & d\Lambda & dA^\dagger & dA & dT \\ \hline d\Lambda & d\Lambda & dA^\dagger & 0 & 0 \\ dA^\dagger & 0 & 0 & 0 & 0 \\ dA & dA & dT & 0 & 0 \\ dT & 0 & 0 & 0 & 0. \end{array} \tag{1.22}$$

Omitting $d\Lambda$ from the table we obtain

$$\begin{array}{c|ccc} & dA^\dagger & dA & dT \\ \hline dA^\dagger & 0 & 0 & 0 \\ dA & dT & 0 & 0 \\ dT & 0 & 0 & 0 \end{array} \tag{1.23}$$

for which the corresponding Lie algebra is the Heisenberg algebra:

$$\begin{array}{c|ccc} [\ ,\ ] & dA^\dagger & dA & dT \\ \hline dA^\dagger & 0 & -dT & 0 \\ dA & dT & 0 & 0 \\ dT & 0 & 0 & 0 \end{array} \tag{1.24}$$

Note that the same Lie algebra is obtained from the universally invariant quantum stochastic calculus of [HL] for which the associative Ito table is

$$\begin{array}{c|ccc} & dA^\dagger & dA & dT \\ \hline dA^\dagger & 0 & \sigma^2 dT & 0 \\ dA & \gamma^2 dT & 0 & 0 \\ dT & 0 & 0 & 0 \end{array}$$

$$\gamma^2 - \sigma^2 = 1,\ \gamma^2 + \sigma^2 > 1$$

(which is embeddable in the case $N = 2$ of our original Ito algebra by setting $dA = \gamma d\Lambda_1^0 + \sigma d\Lambda_0^1$, $dA^\dagger = \sigma d\Lambda_1^0 + \gamma d\Lambda_0^1$, $dT = d\Lambda_0^0$).

In §4 we show that there is a natural homomorphism from the Lie algebra $\mathcal{L}$ derived from an associative Ito multiplication by taking commutators, to the Lie algebra of the associative algebra $(\mathcal{T}(\mathcal{J}), *)$. The canonical extension of this homomorphism to the universal enveloping algebra $\mathcal{U}$ of $\mathcal{L}$ is the *chaos map* χ, generalising that found for $u(N)$ in [HP$_2$, HP$_3$, HP$_4$].

In §5 we find an explicit formula for $\chi(U)$. The maps $j_t : \mathcal{U} \ni U \mapsto I_t \circ \chi(U)$ generalise the expression on the right hand side of (1.10) and form a family of weak representations of $\mathcal{U}$, in so far as for arbitrary $U, V \in \mathcal{U}$ and $\psi, \phi \in \mathcal{E}$

$$\langle j_t(U)^\dagger \psi, j_t(V)\phi\rangle = \langle \psi, j_t(UV)\phi\rangle.$$

In §6, as an application, we show how the conventional multiparticle structure of Fock space is obtained by applying our theory to the Heisenberg case (1.24).

The probabilistic literature on chaotic expansions is of course enormous and a number of authors [Ma, At, LMa, Me, Be] have extended these into the quantum domain. It is usual to omit time T as an integrator. By including dT we obtain a closed Ito algebra and are able to provide purely algebraic constructions which appear to be very natural.

In what follows a *Lie* †-*algebra* is understand to be a complex Lie algebra $\mathcal{L}$ equipped with an involution † satisfying

$$[a^\dagger, b^\dagger] = [b, a]^\dagger \ (a, b \in \mathcal{L}).$$

We shall denote the indefinite Lebesgue integral $\int_0^\cdot f(s)ds$ by $\int_{Leb} f$. If $\mathcal{A}$ is an associative algebra, $\mathcal{A}_{Lie}$ denotes the corresponding Lie algebra obtained by taking commutators,

$$[a, b] = ab - ba \ (a, b \in \mathcal{A}).$$

Theorems, Lemmas, Propositions, Corollaries and equations are labelled independently and consecutively in each chapter. Set theoretic complementation is denoted by $'$ and complex conjugation by $^-$; $|S|$ denotes the cardinal number of the set S. $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{N}^* = \{0\} \cup \mathbb{N}$. For $n \in \mathbb{N}$, $\bigotimes^n \mathfrak{h}$ is the tensor product of n copies of $\mathfrak{h}$.

When $\mathcal{H}$ is the Fock space over $\mathfrak{h} = L^2(\mathbb{R}_+)$, we denote by

$$\mathcal{H} = \mathcal{H}_t \otimes \mathcal{H}^t$$

the canonical tensor product decomposition corresponding to the direct sum decomposition

$$\mathfrak{h} = L^2[0, t] \oplus L^2(t, \infty) =: \mathfrak{h}_t \oplus \mathfrak{h}^t$$

in which exponential vectors factorise:

$$e(f) = e(f_t) \otimes e(f^t), \ f = (f_t, f^t) \in \mathfrak{h}_t \oplus \mathfrak{h}^t.$$

The vacuum vectors in $\mathcal{H}_t$ and $\mathcal{H}^t$ are denoted by $e_t(0)$ and $e^t(0)$ respectively.

Preliminary versions, without proofs, of some of the content of §1 were published in [CEyH] and of the whole paper in [H].

2 Generalization of the algebra of formal power series

Let there be given, once and for all, a finite dimensional complex associative algebra $\mathcal{J}$, equipped with an involution $\dagger$, called the Ito algebra. We consider the complex vector space $\mathcal{T}(\mathcal{J})$ of all (contravariant) tensors of finite rank over the vector space $\mathcal{J}$; thus

$$\mathcal{T}(\mathcal{J}) = \mathbb{C} \oplus \mathcal{J} \oplus (\mathcal{J} \otimes \mathcal{J}) \oplus (\mathcal{J} \otimes \mathcal{J} \otimes \mathcal{J}) \oplus \cdots$$

comprises all sequences $(a_0, a_1, a_2, \cdots)$ with $a_n \in \bigotimes^n \mathcal{J}$ and with only finitely many non-zero terms. We also consider the corresponding strong sum $\mathcal{T}_s(\mathcal{J})$ in which the sequences are not restricted to finitely many terms, and the corresponding vector spaces $\mathcal{S}(\mathcal{J})$ and $\mathcal{S}_s(\mathcal{J})$ in which each homogeneous component a_n is a symmetric tensor, $a_n \in \bigotimes^n_{sym} \mathcal{J}$. On each of the four spaces $\mathcal{T}(\mathcal{J}), \mathcal{T}_s(\mathcal{J}), \mathcal{S}(\mathcal{J}), \mathcal{S}_s(\mathcal{J})$ there is a natural involution (that is a conjugate-linear map whose square is the identity) extending $\dagger$; we denote it also by $\dagger$.

We are going to turn each of these spaces into unital associative algebras for which $\dagger$ is an involution by means of a multiplication rule $*$:

$$(a_0, a_1, a_2, \cdots) * (b_0, b_1, b_2, \cdots) = (c_0, c_1, c_2, \cdots) \tag{2.1}$$

which generalizes the multiplication of formal power series, to which it reduces in the case when $\mathcal{J}$ is the trivial one-dimensional algebra $\mathbb{C}dt$ in which all products vanish. In order to define $*$ we require some notation.

Let $n \in N$ and let A, B be subsets of $M_n := \{1, \cdots n\}$ whose union is M_n. Given two elements $u \in \bigotimes^{|A|} \mathcal{J}$, $v \in \bigotimes^{|B|} \mathcal{J}$ we define an element $w = u^A v^B$ of $\bigotimes^n \mathcal{J}$ as follows. Assume first that $u = u^1 \otimes \cdots \otimes u^{|A|}$, $v = v^1 \otimes \cdots \otimes v^{|B|}$ are product elements. Let $\sigma : M_{|A|} \to A$, $\tau : M_{|B|} \to B$, where $M_{|A|} = \{1, \cdots |A|\}, M_{|B|} = \{1, \cdots |B|\}$, be order preserving injections. Then w is the product element $w^1 \otimes \cdots \otimes w^n$ of $\bigotimes^n \mathcal{J}$ defined by

$$w^j = \begin{cases} u^{\sigma^{-1}(j)} & \text{if } j \in A \cap B', \\ v^{\tau^{-1}(j)} & \text{if } j \in A' \cap B, \\ u^{\sigma^{-1}(j)} v^{\tau^{-1}(j)} & \text{if } j \in A \cap B. \end{cases} \tag{2.2}$$

When u and v are not product elements we define w by linear extension of the rule (2.2).

Now define the multiplication (2.1) by

$$c_n = \sum_{A \cup B = M_n} a^A_{|A|} b^B_{|B|}. \tag{2.3}$$

Here the sum is over the 3^n ways of decomposing $\{1, \cdots, n\}$ into the union of an ordered pair (A, B) of subsets.

It is convenient to rewrite (2.1) as follows. Let $\bar{1}$ be an auxiliary element added to $\mathcal{J}$ which acts as a unit: $\bar{1}a = a\bar{1} = a$ for every $a \in \mathcal{J}$. Let $A = \{s_1 < s_2 < \cdots < s_{|A|}\}$ be a subset of $M_n = \{1, 2, \cdots, n\}$, $\sigma(i) = s_i$ $(i = 1, \cdots, |A|)$ and let $a_{|A|} \in \bigotimes^{|A|} \mathcal{J}$. Define $a^{A \subset M_n}_{|A|} := w^1_n \otimes \cdots \otimes w^n_n$, where

$$w^j_n = \begin{cases} a^{\sigma^{-1}(j)}_{|A|} & \text{if } j \in A \\ \bar{1} & \text{if } j \notin A; \end{cases} \tag{2.4}$$

thus $a^{A \subset M_n}_{|A|}$ is the ampliation of $a_{|A|}$ obtained by placing it in the copies of $\mathcal{J}$ corresponding to $j \in A$ and tensoring with copies of $\bar{1}$ for each $j \in A'$. Then, for $A \cup B = M_n$,

$$w = u^A_{|A|} v^B_{|B|} = u^{A \subset M_n}_{|A|} v^{B \subset M_n}_{|B|} \tag{2.5}$$

where we apply the usual multiplication rule for tensor products. It is easy to see that the right-hand side of (2.5) is a well-defined element of $\bigotimes^n \mathcal{J}$ not involving the auxiliary element $\bar{1}$, and the three possible cases $j \in A \cap B'$, $j \in A' \cap B$, $j \in A \cap B$ $(j \in M_n)$ correspond to the three cases in (2.2).

Thus we have, for example,

$$c_0 = a_0 b_0$$

$$c_1 = a_0 b_2 + a_1 b_0 + a_1 b_1$$

$$\begin{aligned} c_2 = a_0 b_2 + a_1^{\{1\}} b_1^{\{2\}} + a_1^{\{2\}} b_1^{\{1\}} + a_2 b_0 + \\ a_1^{\{1\}} b_2 + a_1^{\{2\}} b_2 + a_2 b_1^{\{1\}} + a_2 b_1^{\{2\}} + a_2 b_2. \end{aligned}$$

Note that in the trivial case $\mathcal{J} = \mathbb{C}dt$, the sum in (2.3) reduces to the 2^n terms corresponding to disjoint unions, since in that case the rule (2.2) will produce a contribution 0 if A and B are not disjoint. In this case writing each component $a_n = \alpha_n \bigotimes^n dt$, $b_n = \beta_n \bigotimes^n dt$, $c_n = \gamma_n \bigotimes^n dt$ where $\alpha_n, \beta_n, \gamma_n \in \mathbb{C}$, we have the rule for multiplying the formal power series

$$\sum_{r=0}^{\infty} \frac{\alpha_r}{r!} t^r \sum_{s=0}^{\infty} \frac{\beta_s}{s!} t^s = \sum_{n=0}^{\infty} \frac{\gamma_n}{n!} t^n,$$

namely

$$\gamma_n = \sum_{r+s=n} \frac{n!}{r!s!} \alpha_r \beta_s.$$

Theorem 2.1 *The multiplication $*$ makes $\mathcal{T}_s(\mathcal{J})$ into a complex associative algebra for which $\dagger$ is an involution and the element $(1, 0, 0, \cdots)$ is a unit element. Each of $\mathcal{T}(\mathcal{J}), \mathcal{S}_s(\mathcal{J}), \mathcal{S}(\mathcal{J})$ is a unital $\dagger$-subalgebra of $\mathcal{T}_s(\mathcal{J})$.*

Proof. It is evident from (2.3) that the composition (2.2) is bilinear. To establish associativity we write

$$((a * b) * c)_n = \sum_{D \cup C = M_n} (a * b)_{|D|}^{D \subset M_n} c_{|C|}^{C \subset M_n}$$

$$= \sum_{D \cup C = M_n} (\sum_{A \cup B = M_{|D|}} a_{|A|}^{A \subset M_{|D|}} b_{|B|}^{B \subset M_{|D|}})^{D \subset M_n} c_{|C|}^{C \subset M_n}$$

$$= \sum_{A \cup B \cup C = M_n} a_{|A|}^{A \subset M_n} b_{|B|}^{B \subset M_n} c_{|C|}^{C \subset M_n}.$$

Similarly,

$$(a * (b * c))_n = \sum_{A \cup D = M_n} a_{|A|}^{A \subset M_n} (b * c)_{|D|}^{D \subset M_n}$$

$$= \sum_{A \cup D = M_n} a_{|A|}^{A \subset M_n} (\sum_{B \cup C = M_{|D|}} b_{|B|}^{B \subset M_{|D|}} c_{|C|}^{C \subset M_{|D|}})^{D \subset M_n}$$

$$= \sum_{A \cup B \cup C = M_n} a_{|A|}^{A \subset M_n} b_{|B|}^{B \subset M_n} c_{|C|}^{C \subset M_n}$$

so that associativity is established.

To establish the involution property

$$(a * b)^\dagger = b^\dagger * a^\dagger$$

it is sufficient, in view of (2.3), to prove that

$$(a_{|A|}^A b_{|B|}^B)^\dagger = b_{|B|}^{\dagger B} a_{|A|}^{\dagger A}$$

for product elements $a_{|A|}, b_{|B|}$. Taking into account (2.4) and (2.5), and defining $\bar{1}^\dagger = \bar{1}$, we obtain

$$\begin{aligned} a_{|A|}^A b_{|B|}^B)^\dagger &= (a_{|A|}^{A \subseteq M_n} b_{|B|}^{B \subset M_n})^\dagger \\ &= b_{|B|}^{B \subseteq M_n \dagger} a_{|A|}^{A \subset M_n \dagger} \\ &= b_{|B|}^{B \dagger} a_{|A|}^{A \dagger}. \end{aligned}$$

That $(1,0,0,\cdots)$ is an identity element is clear from (2.3). Since, from (2.3), $(a*b)_n$ depends only on the components of a and b of ranks not exceeding n it is evident that $\mathcal{T}(\mathcal{J})$ is a subalgebra of $\mathcal{T}_s(\mathcal{J})$.

To prove that $\mathcal{S}_s(\mathcal{J})$ (and hence also $\mathcal{S}(\mathcal{J})=\mathcal{S}_s(\mathcal{J})\cap\mathcal{T}(\mathcal{J})$) is a subalgebra of $\mathcal{T}_s(\mathcal{J})$, let $a\in\mathcal{S}_s(\mathcal{J})$ and consider, for fixed $b_1\in\mathcal{J}$, the element $a*(0,b_1,0,\cdots,0)$. From (2.3)

$$[a*(0,b_1,0,\cdots,0)]_n=\sum_{j=1}^n a_n b_1^{\{j\}}+\sum_{j=1}^n a_{n-1}^{\{1,\cdots,j-1,j+1,\cdots,n\}}b_1^{\{j\}}$$

which is a symmetric tensor. It follows that, for arbitrary $b_1,c_1,d_1,\cdots\in\mathcal{J}$ $a*(0,b_1,0,\cdots,0)*(0,c_1,0,\cdots,0)*(0,d_1,0,\cdots,0)\cdots$ is a symmetric tensor. But

$$\begin{aligned}&(0,b_1,0,\cdots,0)*(0,c_1,0,\cdots,0)*(0,d_1,0,\cdots,0)*\cdots\\&=(0,0,\cdots,Sym(b_1\otimes c_1\otimes d_1\cdots),0,\cdots,0)+\text{lower order terms}\end{aligned}$$

where $Sym(b_1\otimes c_1\otimes d_1\cdots)$ is the sum over all permutations of the product tensors obtained by permuting the components. Hence, such products span $\mathcal{S}(\mathcal{J})$ and it follows that $a*b\in\mathcal{S}_s(\mathcal{J})$ for arbitrary $a\in\mathcal{S}_s(\mathcal{J})$ and $b\in\mathcal{S}(\mathcal{J})$. Since, for $b\in\mathcal{S}_s(\mathcal{J})$, $(a*b)_n$ can be formed by replacing b by its truncation at n, we have $a*b\in\mathcal{S}_s(\mathcal{J})$ for arbitrary $a,b\in\mathcal{S}_s(\mathcal{J})$ as required. $\square$

3 A higher order Ito product formula

We now take the †-algebra $\mathcal{J}$ of §1 to be the algebra of Ito differentials given by (0.20) or one of its †-subalgebras and we denote its elements by $d\Lambda_1, d\Lambda_2$, etc. The operators comprising the corresponding integrator processes act on the exponential domain $\mathcal{E}$ but do not in general leave it invariant. Thus their products are not well defined as processes consisting of operators on $\mathcal{E}$. However stochastic integrals are naturally defined as processes whose constituent operators act on $\mathcal{E}$, and are themselves always integrable processes [Pa] so that iterated stochastic integrals of all orders are well defined as operators on $\mathcal{E}$.

It is evident that the n-fold iterated integral process

$$\int_{0<t_1<t_2<\cdots<t_n<.} d\Lambda_1(t_1)\cdots d\Lambda_n(t_n)$$

is multilinear in the elements $d\Lambda_1, \cdots, d\Lambda_n$ of $\mathcal{J}$; moreover

$$(\int_{0<t_1<t_2<\cdots<t_n<.} d\Lambda_1(t_1)\cdots d\Lambda_n(t_n))^\dagger = \int_{0<t_1<\cdots<t_n<.} d\Lambda_1^\dagger(t_1)\cdots d\Lambda_n^\dagger(t_n). \tag{3.1}$$

It follows that there is a unique linear $\dagger$-map I^n from $\bigotimes^n \mathcal{J}$ to the space $\mathcal{P}$ of integrable processes comprising operators which, with their adjoints, act on $\mathcal{E}$, for which

$$I^n(d\Lambda_1 \otimes \cdots \otimes d\Lambda_n) = \int_{0<t_1<\cdots<t_n<.} d\Lambda_1(t_1)\cdots d\Lambda_n(t_n). \tag{3.2}$$

We define $I^0 : \mathbb{C} \to \mathcal{P}$ by $I^0(c) = c1$ and amalgamate the maps $I^n, n = 0, 1, 2, \cdots$ into a single *integrator map*

$$\begin{array}{l} I : \mathcal{T}(\mathcal{J}) \to \mathcal{P} \\ I(a_0, a_1, \cdots) = \sum_{n=0}^\infty I^n(a_n). \end{array} \tag{3.3}$$

The question whether I can be extended to some elements of the strong tensor space $\mathcal{T}_s(\mathcal{J})$ does not concern us here.

Our main task in this section is to prove that the map I is as near as it can be, in the absence of multiplication in $\mathcal{P}$, to being a unital $\dagger$-algebra morphism from the algebra $\mathcal{T}(\mathcal{J})$ to $\mathcal{P}$.

Theorem 3.1 *For arbitrary $\psi, \phi \in \mathcal{E}$ and $a, b \in \mathcal{T}(\mathcal{J})$*

$$\langle I(a)^\dagger\psi, I(b)\phi\rangle = \langle\psi, I(a * b)\phi\rangle. \tag{3.4}$$

Proof. By bilinearity of both sides of (3.4) in (a, b) it is sufficient to consider the case when

$$a = (0, \cdots, 0, a^1 \otimes \cdots \otimes a^m, 0\cdots),\ b = (0, \cdots, 0, b^1 \otimes \cdots \otimes b^n, 0, \cdots)$$

have only one non-zero component, which is itself a product tensor. We may also assume that $\psi = e(f)$ and $\phi = e(g)$ are exponential vectors. It is evident that (3.4) holds if either m or $n = 0$. Thus we assume $m, n \geq 1$ and introduce the notations

$$a_{m-1} = a^1 \otimes \cdots \otimes a^{m-1},\ b_{n-1} = b^1 \otimes \cdots \otimes b^{n-1}.$$

We also write

$$a^n = \sum_{\alpha,\beta=0}^{N} \lambda_\alpha^\beta d\Lambda_\beta^\alpha, \;\; b^n = \sum_{\alpha,\beta=0}^{N} \mu_\alpha^\beta d\Lambda_\beta^\alpha$$

and note that

$$a^m b^n = \sum_{\alpha,\beta=0}^{N} \sum_{j=1}^{N} \lambda_j^\beta \mu_\alpha^j d\Lambda_\beta^\alpha \tag{3.5}$$

by (1.20). By definition of I we then have

$$\begin{aligned} I(a) &= \sum \lambda_\alpha^\beta \int I(a_{m-1}) d\Lambda_\beta^\alpha \\ I(b) &= \sum \mu_\alpha^\beta \int I(b_{n-1}) d\Lambda_\beta^\alpha. \end{aligned}$$

By the second fundamental formula (1.19) [H,P$_1$] we have

$$\begin{aligned} &\langle I(a)^\dagger \psi, I(b)\phi\rangle = \langle I(a^\dagger)e(f), I(b)e(g)\rangle \\ &= \int_{Leb} \sum_{\alpha,\beta=0}^{N} \{\lambda_\alpha^\beta f_\beta g^\alpha \langle I(a_{m-1}^\dagger)e(f), I(b)e(g)\rangle \\ &+ \mu_\alpha^\beta f_\beta g^\alpha \langle I(a^\dagger)e(f), I(b_{n-1})e(g)\rangle \\ &+ \textstyle\sum_{j=1}^{N} \lambda_j^\beta \mu_\alpha^j f_\beta g^\alpha \langle I(a_{m-1}^\dagger)e(f), I(b_{n-1})e(g)\rangle\}. \end{aligned} \tag{3.6}$$

Suppose $m = n = 1$. Then $a_{n-1}^\dagger = b_{n-1} = 1$ and (3.6) becomes

$$\begin{aligned} &\int_{Leb} \sum_{\alpha,\beta=0}^{N} f_\beta g^\alpha \langle e(f), I(\lambda_\alpha^\beta b + \mu_\alpha^\beta a + \textstyle\sum_j \lambda_j^\beta \mu_\alpha^j) e(g)\rangle \\ &= \langle e(f), I(a^1 \otimes b^1 + b^1 \otimes a^1 + a^1 b^1) e(g)\rangle \\ &= \langle e(f), I(a * b) e(g)\rangle = \langle \psi, I(a * b)\phi\rangle \end{aligned} \tag{3.7}$$

using(3.5) and the first fundamental formula (1.16). Thus (3.4) holds when $m = n = 1$, hence for all values of (m,n) with $m + n \leq 2$.

Before proceeding with the proof we require a lemma.

Lemma 3.1

$$a * b = \sum_{\alpha,\beta=0}^{N} (\lambda_\alpha^\beta a_{m-1} * b + \mu_\alpha^\beta a * b_{n-1} + \sum_{j=1}^{N} \lambda_j^\beta \mu_\alpha^j a_{m-1} * b_{n-1}) \tag{3.8}$$

Proof of Lemma. By (2.3)

$$(a * b)_k = \sum_{A \cup B = M_k} a_{|A|}^A b_{|B|}^B = \sum_{|A|=m, |B|=n, A \cup B = M_k} a^A b^B. \tag{3.9}$$

Let us decompose this sum into three sub-sums according as to whether $k \in A \cap B'$, $k \in A' \cap B$ or $k \in A \cap B$ respectively. The sum of the terms of these three types can be expressed respectively as

$$\sum_{|C|=m-1,|B|=n,C\cup B=M_{k-1}} a^C_{m-1} b^B \otimes \sum_{\alpha,\beta=0}^{N} \lambda^\beta_\alpha d\Lambda^\alpha_\beta$$

$$\sum_{|A|=m,|D|=n-1,A\cup D=M_{k-1}} a^A b^D_{n-1} \otimes \sum_{\alpha,\beta=0}^{N} \mu^\beta_\alpha d\Lambda^\alpha_\beta$$

$$\sum_{|C|=m-1,|D|=n-1,C\cup D=M_{k-1}} a^C_{m-1} b^D_{n-1} \otimes \sum_{j=1}^{N} \sum_{\alpha,\beta=0}^{N} \lambda^j_\alpha \mu^\beta_j d\Lambda^\alpha_\beta.$$

But these are precisely the three terms of the k^{th} component of the right hand side of (3.8). This completes the proof of the Lemma.

Continuing with the proof of Theorem 2.1 we now make the inductive assumption that (3.4) holds for $m+n = 2, \cdots, k-1$. Then, from (3.6), we have, by the inductive hypothesis

$$\langle I(a^\dagger)\psi, I(b)\phi\rangle = \langle I(a^\dagger)e(f), I(b)e(g)\rangle$$

$$= \int_{Leb} \sum_{\alpha,\beta=0}^{N} f_\beta g^\alpha \langle e(f), I(\lambda^\beta_\alpha a_{m-1} * b + \mu^\beta_\alpha a * b_{n-1} +$$

$$\sum_{j=1}^{N} \lambda^j_\alpha \mu^\beta_j a_{m-1} * b_{n-1}) e(g)\rangle$$

$$= \langle e(f), I(\sum_{\alpha,\beta=0}^{N} \{\lambda^\beta_\alpha a_{m-1} * b + \mu^\beta_\alpha a * b_{n-1} + \sum_{j=1}^{N} \lambda^j_\alpha \mu^\beta_j a_{m-1} * b_{n-1}\} \otimes d\Lambda^\alpha_\beta) e(g)\rangle$$

by the first fundamental formula

$$= \langle e(f), I(a*b)e(g)\rangle = \langle \psi, I(a*b)\phi\rangle$$

by the Lemma. □

4 The chaos map

We denote by $\mathcal{L}$ the complex Lie algebra with involution realised by taking the commutator Lie bracket in the Ito algebra $\mathcal{J}$. Thus $\mathcal{L} = \mathcal{J}_{Lie}$, but it is convenient to regard $\mathcal{L}$ as an abstract Lie algebra equipped with an isomorphism $L \mapsto d\Lambda_L$ onto $\mathcal{J}_{Lie}$. We denote by $\mathcal{U}$ the universal enveloping algebra of $\mathcal{L}$. Thus $\mathcal{U}$ is a complex unital associative †-algebra equipped with an injection $\imath : \mathcal{L} \to \mathcal{U}_{Lie}$ with the following universal property: given a †-Lie algebra morphism ϕ from $\mathcal{L}$ to $\mathcal{A}_{Lie}$, where $\mathcal{A}$ is a complex unital associative †-algebra, there exists a unique complex unital associative algebra morphism j from $\mathcal{U}$ to $\mathcal{A}$ such that $j \circ \imath = \phi$.We use $\imath$ to identify $\mathcal{L}$ as a subalgebra of $\mathcal{U}_{Lie}$ and say that j *extends* ϕ.

Theorem 4.1 *The map ϕ:*

$$\mathcal{L} \ni L \mapsto (0, d\Lambda_L, 0, \cdots) \in \mathcal{T}_s(\mathcal{J}) \tag{4.1}$$

is a †-Lie algebra morphism from $\mathcal{L}$ to $\mathcal{T}_s(\mathcal{J})_{Lie}$.

Proof. ϕ is evidently a linear †-map. For arbitrary $L, M \in \mathcal{L}$ we have, using (2.3)

$$\begin{aligned}
[\phi(L), \phi(M)] &= \phi(L) * \phi(M) - \phi(M) * \phi(L) \\
&= (0, d\Lambda_L d\Lambda_M, d\Lambda_L \otimes d\Lambda_M + d\Lambda_M \otimes d\Lambda_L, 0, \cdots) \\
&\quad -(0, d\Lambda_M d\Lambda_L, d\Lambda_M \otimes d\Lambda_L + d\Lambda_L \otimes d\Lambda_M, 0, \cdots) \\
&= (0, d\Lambda_L d\Lambda_M - d\Lambda_M d\Lambda_L, 0, \cdots) \\
&= (0, [d\Lambda_L, d\Lambda_M], 0, \cdots) \\
&= (0, d\Lambda_{[L,M]}, 0, \cdots) \\
&= \phi([L, M]). \ \square
\end{aligned}$$

Corollary 4.1 *There exists a unique complex unital associative †-algebra morphism χ from $\mathcal{U}$ to $\mathcal{T}_s(\mathcal{J})$ such that, for arbitrary $L \in \mathcal{L}$, $\chi(L) = (0, d\Lambda_L, 0, \cdots)$.*

Proof. χ is the morphism which extends the Lie morphism ϕ. $\square$

We call χ the *chaos map.*

Corollary 4.2 *There exists a unique complex-linear †-map J from $\mathcal{U}$ to $\mathcal{P}$ such that* *a) for arbitrary* $L \in \mathcal{L}$, $J(L) = \Lambda_L$

b) for arbitrary $U, V \in \mathcal{U}$, $t \in \mathbb{R}_+$ *and* $\psi, \varphi \in \mathcal{E}$

$$\langle J(U_t)^\dagger \psi, J(V_t)\varphi \rangle = \langle \psi, J(UV)_t \varphi \rangle \tag{4.2}$$

J is given by

$$J(U) = I(\chi(U)). \tag{4.3}$$

Proof. It is evident from Theorem 2.1 and the morphism property of χ that the map J given by (4.3) satisfies a) and b). If $\tilde{J}$ is a second such mapping, then by a) $\tilde{J}$ coincides with J on $\mathcal{L}$. Since $\mathcal{U}$ is generated as a unital †-algebra by $\mathcal{L}$ a simple inductive argument based on (4.2) shows that $\tilde{J}$ coincides with J on $\mathcal{U}$. □

5 A formula for the chaos map

Let $\eta : \mathcal{U} \to \mathbb{C}$ be the *counit* of the algebra $\mathcal{U}$, that is the unique unital complex associative †-algebra morphism which extends the zero morphism $L \mapsto 0$ from $\mathcal{L}$ to $\mathbb{C}_{Lie}$. We denote by $\mathcal{K}$ the kernel of η; it is also the ideal $\mathcal{K} = \mathcal{U}\mathcal{L}\mathcal{U}$ in $\mathcal{U}$ generated by $\mathcal{L}$.

Proposition 5.1 *Let* $\mathcal{B}$ *be a not-necessarily unital complex associative †-algebra and let* ϕ_1 *be a Lie †-algebra morphism from* $\mathcal{L}$ *to* $\mathcal{B}_{Lie}$. *Then there exists a unique associative †-algebra morphism* j_1 *from* $\mathcal{K}$ *to* $\mathcal{B}$ *which extends* ϕ_1.

Proof. Denote by $\mathcal{A}$ the unital †-algebra based on the vector space $\mathcal{B} \times \mathbb{C}$ with multiplication

$$(b_1, z_1)(b_2, z_2) = (b_1 b_2 + z_1 b_2 + b_1 z_2, z_1 z_2) \tag{5.1}$$

and involution

$$(b, z)^\dagger = (b^\dagger, \bar{z})$$

for which $(0, 1)$ is evidently a unit. Evidently $\phi : L \mapsto (\phi_1(L), 0)$ is a Lie †-algebra morphism from $\mathcal{L}$ to $\mathcal{A}_{Lie}$. Let j be its extension to $\mathcal{U}$ and write

$$j(U) = (j_1(U), \eta_1(U)),\ U \in \mathcal{U}.$$

It is evident from (5.1) that η_1 is a $\dagger$-algebra homomorphism from $\mathcal{U}$ to $\mathbb{C}$ extending the zero map $\mathcal{L} \to \mathbb{C}_{Lie}$, thus $\eta_1 = \eta$. For $U \in \mathcal{K}$, $\eta_1(U) = \eta(U) = 0$. Consequently $j(U) = (j_1(U), 0)$. From (5.1) it is evident that the restriction of j_1 to $\mathcal{K}$ is a $\dagger$-algebra morphism which extends ϕ_1. If $\tilde{j}_1$ is a second such morphism then since every element of $\mathcal{U}$ can be expressed uniquely in form $U + z1$ where $U \in \mathcal{K}$ and $z \in \mathbb{C}$ [J] we may define $\tilde{j} : \mathcal{U} \to \mathcal{A}$ by

$$\tilde{j}(U + z1) = (\tilde{j}_1(U), \eta(z)) \ (U \in \mathcal{K}, z \in \mathbb{C}).$$

By (5.1) $\tilde{j}$ is a unital $\dagger$-algebra morphism from $\mathcal{U}$ to $\mathcal{A}$ which extends ϕ. Hence $\tilde{j} = j$ and hence also $\tilde{j}_1 = j_1$ showing that j_1 is unique. $\square$

Let γ_1 be the *coproduct* of $\mathcal{U}$, that is the extension to $\mathcal{U}$ of the $\dagger$-Lie algebra morphism from $\mathcal{L}$ to $(\mathcal{U} \otimes \mathcal{U})_{Lie}$ given by $L \mapsto L \otimes 1 + 1 \otimes L$. The map

$$\gamma_2 = (\gamma_1 \otimes id) \circ \gamma_1 = (id \otimes \gamma_1) \circ \gamma_1$$

is the extension to $\mathcal{U}$ of the Lie algebra morphism from $\mathcal{L}$ to $(\mathcal{U} \otimes \mathcal{U} \otimes \mathcal{U})_{\mathrm{Lie}}$ given by $L \mapsto L \otimes 1 \otimes 1 + 1 \otimes L \otimes 1 + 1 \otimes 1 \otimes L$. More generally we define

$$\gamma_n = (\gamma_1 \otimes id \otimes \cdots \otimes id) \circ \gamma_{n-1};$$

it is the extension to $\mathcal{U}$ of the Lie algebra morphism

$$L \mapsto \sum_{j=1}^{n} 1 \otimes \cdots \otimes L^{(j)} \otimes \cdots \otimes 1.$$

It is convenient to define $\gamma_0 : \mathcal{U} \to \mathcal{U}$ to be the identity map. We define $\kappa_1 : \mathcal{U} \to \mathcal{U} \otimes \mathcal{U}$ by

$$\kappa_1(U) = \gamma_1(U) - U \otimes 1 = \gamma_1(U) - \gamma_0^{\emptyset \subseteq M_1}(U).$$

Here $\gamma_0^{\emptyset \subseteq M_1}(U)$ indicates that $\gamma_0(U) = U$ has been ampliated as $\gamma_0^{\emptyset \subseteq M_1}(U) = U \otimes 1$ to $\mathcal{U} \otimes \mathcal{U}$.

Let $(L_1, \cdots, L_n)$ be a basis of $\mathcal{L}$. By the Poincaré–Birkhoff–Witt theorem [J] the elements $(L_\alpha : \alpha \in \mathbb{N}^{*n})$ where, for $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^{*n}$,

$$L_\alpha = L_1^{\alpha_1} \cdots L_n^{\alpha_n}$$

form a basis for $\mathcal{U}$. We denote $\alpha_1 + \cdots + \alpha_n$ by $|\alpha|$ and call it the *degree* of the basis element L_α.

Proposition 5.2 *κ_1 maps $\mathcal{U}$ into $\mathcal{U} \otimes \mathcal{K}$. Moreover it is degree reducing on $\mathcal{U}$, in the sense that, for each $\alpha \in \mathbb{N}^n$, there exist $\beta_1, \cdots, \beta_m \in \mathbb{N}^n$ with each $|\beta_j| < \alpha$ and elements $K_1, \cdots, K_m \in \mathcal{K}$ such that*

$$\kappa_1(L_\alpha) = \sum_{j=1}^{m} L_{\beta_j} \otimes K_j.$$

Proof. We have

$$\kappa_1(L_\alpha) = (L_1 \otimes 1 + 1 \otimes L_1)^{\alpha_1} \cdots (L_n \otimes 1 + 1 \otimes L_n)^{\alpha_n} - L_1^{\alpha_1} \cdots L_n^{\alpha_n} \otimes 1$$

and consists of the sum of $2^{|\alpha|} - 1$ terms each of which is a product of $|\alpha|$ elements of form either $L_j \otimes 1$ or $1 \otimes L_j$, in which at least one element must be of the latter type. Thus κ_1 maps the basis element L_α into $\mathcal{U} \otimes \mathcal{K}$ and reduces its degree. $\square$

Define maps $\kappa_n : \mathcal{U} \to \mathcal{U} \otimes \mathcal{K} \otimes \cdots \otimes \mathcal{K}$ inductively by

$$\begin{aligned} \kappa_n &= (\kappa_1 \otimes id \otimes \cdots \otimes id) \circ \kappa_{n-1} \\ &= (\kappa_1 \otimes id \otimes \cdots \otimes id) \circ \cdots \circ \kappa_1 \otimes id \circ \kappa_1. \end{aligned}$$

Since each application of κ_1 reduces the degree it is evident that, for each $U \in \mathcal{U}$, $\kappa_n(U) = 0$ for n sufficiently large. Iterating the formula

$$\kappa_1(U) = \gamma_1(U) - \gamma_0^{\emptyset \subseteq M_1}(U)$$

we have

$$\begin{aligned} \kappa_2(U) &= \gamma_1 \otimes id(\gamma_1(U) - \gamma_0^{\emptyset \subseteq \{1\}}(U)) - (\gamma_0^{\emptyset \subseteq \{1\}} \otimes id)(\gamma_1(U) - \gamma_0^{\emptyset \subseteq \{1\}}(U)) \\ &= \gamma_2(U) - \gamma_1^{\{1\} \subseteq \{1,2\}}(U) - \gamma_1^{\{2\} \subseteq \{1,2\}}(U) + \gamma_0^{\emptyset \subseteq \{1,2\}}. \end{aligned}$$

Here the superscripts denote, respectively, that $\gamma_1(U) \in \mathcal{U} \otimes \mathcal{U}$ has been ampliated to $\mathcal{U} \otimes \mathcal{U} \otimes \mathcal{U}$ by means of the identity of $\mathcal{U}$ in the third $(= 1 + 2^{th})$ copy, thus occupying non-trivially the zeroth and first copies of $\mathcal{U}$, that $\gamma_1(U)$ has similarly been ampliated to occupy the zeroth and second copies by an identity in the second, and that $\gamma_0(U) = U$ has been doubly ampliated to $\mathcal{U} \otimes \mathcal{U} \otimes \mathcal{U}$. Extending this notation we may write

$$\kappa_n(U) = \sum_{S \subset M_n} (-1)^{n-|S|} \gamma_{|S|}^{S \subset M_n}(U) \tag{5.2}$$

where the notation $\gamma_{|S|}^{S\subseteq M_n}(U)$ indicates that $\gamma_{|S|}(U)$has been placed in the $|S|+1$ copies of $\mathcal{U}$ in the $(n+1)$-fold tensor product $\bigotimes_{j=0}^n \mathcal{U}$ corresponding to $j=0$ and the $|S|$ elements of S and ampliated by the identity in the copies of $\mathcal{U}$ corresponding to the $n-|S|$ elements of the complement of S in M_n.

Let k be the complex associative †-algebra morphism from $\mathcal{K}$ to $\mathcal{J}_{Lie}$which extends the Lie algebra morphism $L \mapsto d\Lambda_L$ according to Proposition 4.1. Then we can form the morphism $\eta\otimes k\otimes\cdots\otimes k^{(n)}$ from $\mathcal{U}\otimes\bigotimes^n\mathcal{K}$ to $\mathbb{C}\otimes\bigotimes^n\mathcal{J}$, and we may identify the range of $\eta\otimes k\otimes\cdots\otimes k^{(n)}$ with $\bigotimes^n\mathcal{J}$. We may now state the main Theorem, giving a formula for the chaos map χ.

Theorem 5.1 *For arbitrary $U\in\mathcal{U}$*

$$\chi(U) = (\eta\circ\kappa_0(U), \eta\otimes k\circ\kappa_1(U), \cdots\ \eta\otimes\bigotimes^n k\circ\kappa_n(U), \cdots). \tag{5.3}$$

Proof. We first note that the right hand side of (5.3) has only finitely many non-zero components because $\kappa_n(U)=0$ for sufficiently large n by the degree-reduction property of κ_1. The theorem will be proved if we can show that, with $\chi(U)$ given by (5.3), the linear †-map χ satisfies

a) $\chi(1)=1$,

b) $\chi(L)=(0,dL,0,\cdots)$,

c) $\chi(U)*\chi(V)=\chi(UV)$.

Indeed this will show that χ, defined by (5.3), is the unique unital †-algebra morphism from $\mathcal{U}$ to to $\mathcal{T}(\mathcal{J})$ which extends the map $L\mapsto(0,dL,0,\cdots)$; it is then automatically a map from $\mathcal{U}$ to $\mathcal{T}_s(\mathcal{J})$, and is thus the chaos map constructed in Corollary 3.1.

To prove a) we note that $\eta\circ\kappa_0(1)=\eta(1)=1$. Furthermore, since $\kappa_n(1)=0$ for $n\geq 1$ we have $\chi(1)=(1,0,\cdots)=1$.

To establish b) we note that for $L\in\mathcal{L}$,

$$\begin{aligned} \eta\circ\kappa_o(L) &= \eta(L)=0 \\ \eta\otimes k\circ\kappa_1(L) &= \eta\otimes k(L\otimes 1+1\otimes L-L\otimes 1) \\ &= 0+\eta(1)\otimes k(L) \\ &= d\Lambda_L \end{aligned}$$

while, for $n>1$, $\eta\otimes k\otimes\cdots\otimes k\circ\kappa_n(L)=0$ by the degree reducing property.

To establish c) we shall require a combinatorial lemma.

Lemma 5.1 *Let S, T be possibly empty subsets of M_n. Then*

$$\sum_{A \supseteq S, B \supseteq T, A \cup B = M_n} (-1)^{|A|-|S|+|B|-|T|} = \delta_{S,T}(-1)^{n-|S|} \tag{5.4}$$

where $\delta_{S,T} = 1$ if $S = T$ and 0 if $S \neq T$.

Proof. We decompose the sum on the left hand side of (5.4) into two parts, $\sum_{A\cap T'\neq\emptyset} + \sum_{A\cap T'=\emptyset}$ corresponding to whether or not, in addition to the conditions $A \supseteq S, B \supseteq T, A \cup B = M_n$, $A \cap T'$ is non-empty. In the first case, fixing A and putting$N = |A \cap T'| \neq 0$, we have, since $B \supseteq A' \cup T = (A \cap T')'$, $B \setminus (A' \cup T)$ can be an arbitrary subset of $A \cap T'$. The contribution of all such terms is thus

$$\mp \sum_{k=0}^{N} \frac{N!}{k!(N-k)!}(-1)^k = \mp(1-1)^N = 0$$

since $N \neq 0$.

In the second case we have

$$S \subseteq A \subseteq T \subseteq B, \tag{5.5}$$

so that evidently $B = M_n$ whereas $A \cap S'$ can be an arbitrary subset of $S' \cap T$. Let $|S' \cap T| = \tilde{N}$. The sum of all such terms is evidently

$$\sum_{k=0}^{\tilde{N}} (-1)^k \frac{\tilde{N}!}{k!(\tilde{N}-k)!}(-1)^{n-|T|}$$

which again is 0 unless $\tilde{N} = 0$ in which case it is $(-1)^{n-|T|}$. But in the latter case it is evident from (5.5) that $S = T$ and so $(-1)^{n-|T|} = (-1)^{n-|S|}$. This completes the proof of the Lemma. □

We resume the proof of (c). By (2.3) and (5.2) we have

$$\begin{aligned}(\chi(U) * \chi(V))_n &= \textstyle\sum_{A\cup B=M_n} \chi(U)^A_{|A|}\chi(V)^B_{|B|} \\ &= \textstyle\sum_{A\cup B=M_n} \{\eta \otimes \bigotimes^{|A|} k(\sum_{S\subseteq M_{|A|}}(-1)^{|A|-|S|}\gamma^{S\subseteq M_{|A|}}_{|S|}(U))\}^A \\ &\quad \{\eta \otimes \bigotimes^{|B|} k(\textstyle\sum_{T\subseteq M_{|B|}}(-1)^{|B|-|T|}\gamma^{T\subseteq M_{|B|}}_{|T|}(V))\}^B\end{aligned} \tag{5.6}$$

We claim that the right hand side of (5.6) can be expressed as

$$\begin{array}{c}\sum_{A\cup B=M_n}\eta\otimes\bigotimes^n k\{\sum_{S\subseteq M_{|A|}}(-1)^{|A|-|S|}(\gamma_{|S|}^{S\subseteq M_{|A|}}(U))^{A\subseteq M_n}\\ \sum_{T\subseteq M_{|B|}}(-1)^{|B|-|T|}(\gamma_{|T|}^{T\subseteq M_{|B|}}(V))^{B\subseteq M_n}\}.\end{array} \tag{5.7}$$

Here the superscripts $A\subseteq M_n$, $B\subseteq M_n$ have the same meaning as before.

To see this, we write the two inner sums occurring in (5.7) as $X^{A\subseteq M_n}$, $Y^{B\subseteq M_n}$ respectively, and we factorize the morphism

$$\eta\otimes\bigotimes^n k=\eta_0\circ k_1\circ\cdots\circ k_n \tag{5.8}$$

where

$$\begin{array}{l}\eta_0=\eta\otimes\bigotimes^n id,\\ k_j=id\otimes id\otimes\cdots\otimes k^{(j)}\otimes id\otimes\cdots\otimes id^{(n)},\quad j=1,\cdots,n.\end{array}$$

Thus the summand in (5.7) takes the form

$$\eta_0\circ k_1\circ\cdots\circ k_n(X^{A\subseteq M_n}Y^{B\subseteq M_n}).$$

We note that the maps $k_1,\cdots,k_n$ can be permuted arbitrarily.

Now consider k_j for $j\in M_n$ and distinguish the three possibilities

$$j\in A\cap B',\ j\in A'\cap B,\ j\in A\cap B,$$

exactly one of which must occur. Writing $A=\{\alpha_1<\cdots<\alpha_{|A|}\}$, $B=\{\beta_1<\cdots<\beta_{|B|}\}$ we have, in the first case, since then $Y^{B\subseteq M_n}$ is ampliated at j,

$$\begin{array}{l}k_j(X^{A\subseteq M_n}Y^{B\subseteq M_n})=\{(id\otimes\cdots\otimes k^{(r)}\otimes\cdots\otimes id X)^{A\subseteq M_n}\}Y^{B\subseteq M_n},\\ \text{where } j=\alpha_r,\end{array} \tag{5.9}$$

in the second case similarly

$$\begin{array}{l}k_j(X^{A\subseteq M_n}Y^{B\subseteq M_n})=X^{A\subseteq M_n}\{(id\otimes\cdots\otimes k^{(s)}\otimes\cdots\otimes id Y)^{B\subseteq M_n}\},\\ \text{where } j=\beta_s,\end{array} \tag{5.10}$$

while in the third case, because k_j is a homomorphism,

$$\begin{array}{l}k_j(X^{A\subset M_n}Y^{B\subset M_n})=\{(id\otimes\cdots\otimes k^{(r)}\otimes\cdots\otimes id X)^{A\subseteq M_n}\}\\ \{(id\otimes\cdots\otimes k^{(s)}\otimes\cdots\otimes id Y)^{B\subseteq M_n}\}\\ \text{where } j=\alpha_r=\beta_s.\end{array} \tag{5.11}$$

Similarly we have

$$\eta_0(X^{A\subseteq M_n}Y^{B\subseteq M_n}) = \{\eta_0(X^{A\subseteq M_n})\eta_0(Y^{B\subseteq M_n})\} \tag{5.12}$$

because η_0 is a homomorphism. It is evident that (5.9)–(5.12) remain valid if $X^{A\subseteq M_n}$ and $Y^{B\subseteq M_n}$ are replaced by arbitrary elements of the form $\hat{X}^{A\subseteq M_n}$ and $\hat{Y}^{B\subseteq M_n}$ with $\hat{X} \in \mathcal{U} \otimes \bigotimes^{|A|} \mathcal{K}$, $\hat{Y} \in \mathcal{U} \otimes \bigotimes^{|B|} \mathcal{K}$. Hence we may take over in turn each of the homomorphisms occurring in the factorization (5.8), and reconstitute products of those applicable to $X^{A\subseteq M_n}$ and $Y^{B\subseteq M_n}$ separately, to obtain the right hand side of (5.6) taking into account that $X^A_{|A|}Y^B_{|B|} = X^{A\subseteq M_n}_{|A|}Y^{B\subseteq M_n}_{|B|}$, where on the right hand side there is the usual multiplication. Thus we proved that

$$\begin{aligned}
&(\chi(U) * \chi(V))_n = \eta \otimes k \otimes \cdots \otimes k^{(n)} \\
&\{\textstyle\sum_{A\cup B=M_n}(\sum_{S\subseteq M_{|A|}}(-1)^{|A|-|S|}\gamma_{|S|}^{S\subseteq M_{|A|}}(U))^{A\subseteq M_n} \\
&(\textstyle\sum_{T\subseteq M_{|B|}}(-1)^{|B|-|T|}\gamma_{|T|}^{T\subseteq M_{|B|}}(V))^{B\subseteq M_n}\}
\end{aligned}$$

$$\begin{aligned}
&= \eta \otimes k \otimes \cdots \otimes k^{(n)} \\
&\{\textstyle\sum_{A\cup B=M_n}(\sum_{S\subseteq M_{|A|}}(-1)^{|A|-|S|}\gamma_{|S|}^{S\subseteq M_n}(U))(\sum_{T\subseteq M_{|B|}}(-1)^{|B|-|T|}\gamma_{|T|}^{T\subseteq M_n}(V))\} \\
&= \eta \otimes k \otimes \cdots \otimes k^{(n)} \\
&\{\textstyle\sum_{S,T\subseteq M_n}\sum_{A\supseteq S, B\supseteq T}(-1)^{|A|-|S|+|B|-|T|}\gamma_{|S|}^{S\subseteq M_n}(U)\gamma_{|T|}^{T\subseteq M_n}(V)\} \\
&= \eta \otimes k \otimes \cdots \otimes k^{(n)} \textstyle\sum_{S\subseteq M_n}(-1)^{n-|S|}\gamma_{|S|}^{S\subseteq M_n}(UV)
\end{aligned}$$

using the Lemma

$$\begin{aligned}
&= \eta \otimes k \otimes \cdots \otimes k^{(n)} \circ k_n(UV) \\
&= \chi(UV)
\end{aligned}$$

using the homomorphism property of $\gamma_{|S|}$ and (5.2). $\square$

It is evident from (5.44) that the process

$$j_t(U) = I_t \circ \chi(U), t \in \mathbb{R}_+$$

is the solution by iteration of the stochastic differential equation

$$dj_t(U) = j_t \otimes k \circ \kappa(U), \quad j_0(U) = \eta(U). \tag{5.13}$$

(5.13) is the generalisation of (1.8).

6 The multiparticle structure of Fock space from the point of view of quantum stochastic calculus

In this paper we have taken the point of view that the "exponential" [K] (or continuous tensor product [ArW,S]) structure of Fock space is fundamental, thus emphasising the rôle of exponential vectors, and showing that "products" of operators on the exponential domain which do not leave the domain invariant can nevertheless be defined in appropriate cases as iterated stochastic integrals.

In this section we wish to amplify this viewpoint by showing in detail how the apparatus of the previous section can be used to derive the conventional multiparticle structure of Fock space from the exponential definition of §1. For simplicity we consider the one dimensional case and take the Ito algebra to be the three dimensional †-algebra spanned by $dA^\dagger$, dA and dT with the multiplication table (1.23).

Within $\mathcal{J}$ we focus on the *creation subalgebra* $\mathcal{C}$ spanned by $dA^\dagger$ and dT alone. For $n \in \mathbb{N}$, $t \in \mathbb{R}_+$ and $dJ_1, \cdots, dJ_n \in \{dA^\dagger, dT\}$ set

$$d\mathbf{J} = dJ_1 \otimes \cdots \otimes dJ_n, \;\; \psi(\mathbf{J}, t) = I_n^t(d\mathbf{J})e(0)$$

and denote by $\mathcal{M}$ the totality of all such vectors. By convention the vacuum $e(0)$ is itself an element of $\mathcal{M}$ with $n = 0$.

We also define the *creation number*

$$|d\mathbf{J}| = \sharp\{k : dJ_k = dA_k^\dagger\}$$

and partition $\mathcal{M}$ as $\bigcup_{m=0}^{\infty} \mathcal{M}_m$, where

$$\mathcal{M}_m = \{\psi(\mathbf{J}, t) \in \mathcal{M} : |d\mathbf{J}| = m\}.$$

We can now prove

Theorem 6.1 *a) The sets $\mathcal{M}_m$, $m = 0, 1, 2, \cdots$ are mutually orthogonal,*

b) $\mathcal{M}$ is total in the Fock space $\mathcal{H}$.

Proof. a). We prove that, for $|d\mathbf{J}| \neq |d\mathbf{K}|$, $s, t \in \mathbb{R}$,

$$\psi(d\mathbf{J}, t) \perp \psi(d\mathbf{K}, s). \tag{6.1}$$

Assume first that $s = t$. Using Theorem 2.1

$$\begin{aligned}
&\langle \psi(\mathbf{J}), t), \psi(\mathbf{K}, t\rangle \\
&= \langle I^t(0, \cdots, d\mathbf{J}, 0, \cdots)e(0), I^t(0, \cdots, d\mathbf{K}, 0, \cdots)e(0)\rangle \\
&= \langle e(0), I^t((0, \cdots, d\mathbf{J}^\dagger, 0, \cdots) * (0, \cdots, d\mathbf{K}, 0, \cdots)e(0))\rangle.
\end{aligned}$$

Since $d\mathbf{J}^\dagger$ contains $|d\mathbf{J}|$ copies of dA, which cannot both all be removed by and remove all of the $|d\mathbf{K}|$ copies of $dA^\dagger$ in $d\mathbf{K}$ via the reduction $dAdA^\dagger = dT$, it follows that the operator on the right hand side is of form

$$I_t(0, \cdots, dL_1 \otimes \cdots \otimes dL_r \otimes dA^\sharp \otimes dT \otimes \cdots \otimes dT^{(s)}, 0 \cdots)$$

where each $dL_i \in \{dA^\dagger, dA, dT\}$ and $dA^\sharp$ is either $dA^\dagger$ or dA. But

$$\begin{aligned}
&\langle e(0), I^t(0, \cdots, dL_1 \otimes \cdots \otimes dL_r \otimes dA^\sharp \otimes dT \otimes \cdots \otimes dT^{(s)}, 0, \cdots)e(0)\rangle \\
&= \langle e(0), \int_{0<t_1<\cdots<t_{r+s+1}<t} dL_1(t_1)\cdots dL_r(t_r)dA^\sharp(t_{r+1})dt_{r+2}\cdots dt_{r+s+1}e(0)\rangle \\
&= \textstyle\int_{0<t_{r+2}<\cdots<t_{r+s+1}<t}\langle e(0), \int_{0<t_1<\cdots<t_{r+2}} dL_1(t_1)\cdots dL_r(t_r)dA^\sharp(t_{r+1})e(0)\rangle \\
&= 0
\end{aligned}$$

by the first fundamental formula of §1. Thus (6.1) is proved in case $s = t$. If $s \neq t$ then we may assume without loss of generality that $s > t$. Suppose $|d\mathbf{K}| = 0$, then $\psi(d\mathbf{K}, s)$ is a multiple of the vacuum. Then also $|d\mathbf{J}| \neq 0$ and it is easily seen by writing $d\mathbf{J}$ in the form

$$d\mathbf{J} = dJ_1 \otimes \cdots \otimes dJ_r \otimes dA^\sharp \otimes dT \otimes \cdots \otimes dT$$

and dividing the integration as above that (6.1) holds. Hence we may assume that $|d\mathbf{K}| \neq 0$. We write the vector $\psi(d\mathbf{J}, t)$ in $\mathcal{H} = \mathcal{H}_t \otimes \mathcal{H}^t$ in the form$\psi_t \otimes e^t(0)$ for some ψ_t in $\mathcal{H}_t$ where $e^t(0)$ is the vacuum in $\mathcal{H}^t$. Thus

$$\begin{aligned}
&\langle \psi(d\mathbf{J}, t), \psi(d\mathbf{K}, s)\rangle \\
&= \langle \psi_t \otimes e^t(0), \textstyle\int_{0<t_1<\cdots<t_m<s} dK_1(t_1)\cdots dK_m(t_m)e(0)\rangle
\end{aligned}$$

for some $j > m$

$$= \int_{0<t_{j+2}<\cdots<t_m<t} \langle \psi_t \otimes e^t(0),$$
$$\int_{0<t_1<\cdots<t_{j+1}<t_{j+2}} dK_1(t_1)\cdots dK_j(t_j)dA^\sharp(t_{j+1})e(0)\rangle dt_{j+2}\cdots dt_m$$

for some $j < m$. We split the inner integral over t_{j+2} into the sum of the integrals over $0 < t_{j+1} < t$ and $t < t_{j+2} < s$. The first integral is of form

$$\langle \psi_t \otimes e^t(0), \chi_t \otimes e^t(0)\rangle = \langle \psi(d\mathbf{J}, t), \psi(d\tilde{\mathbf{K}}, t)\rangle$$

where $|d\tilde{\mathbf{K}}| = |d\mathbf{K}| \neq |d\mathbf{J}|$ and thus it vanishes. The second integral is of form

$$\langle \psi_t \otimes e^t(0), \int_t^s H(t_{j+1})dA^\sharp(t_{j+1})e(0)\rangle$$

for some process H, which vanishes by a simple modification of the first fundamental formula [HP$_3$].

b). Since the exponential vectors are total it is sufficient to show that each such vector $e(f)$ can be approximated by linear combinations of the vectors $\psi(\mathbf{J}, t)$. Since the map $f \to e(f)$ is continuous we may restrict f to a dense subset of $\mathfrak{h} = L^2(\mathbb{R}_+)$ which we take to be the set of functions of form $p(.)\chi_{[0,s]}$ for $s \in \mathbb{R}_+$ and $p(.)$ a polynomial. Writing

$$p(t) = a_0 + \frac{a_1 t}{1!} + \cdots + \frac{a_n t^n}{n!}, a_0, \cdots a_n \in \mathbb{C}, \tag{6.2}$$

we introduce the element of $\mathcal{J}$

$$P = (0, a_0 dA^\dagger, a_1 dT \otimes dA^\dagger, a_2 dT \otimes dT \otimes dA^\dagger, \cdots).$$

Evidently $I_t(P)e(0)$ is a linear combination of elements of $\mathcal{M}_1$, and more generally $I_t(P^m)e(0)$ is a linear combination of elements of $\mathcal{M}_m$, $m = 0, 1, 2, \cdots$ where the P^m are $*$-powers of P,

$$P^m = P * \cdots * P^{(m)}$$

since these are constructed only using $dA^\dagger$ and dT, furthermore, by part a) the vectors $I^s(P^m)e(0)$, $m = 0, 1, 2, \cdots$ are mutually orthogonal. We prove that the orthogonal series $\sum_{m=0}^\infty (m!)^{-1} I_s(P^m)e(0)$ converges to $e(f)$. To prove that it is convergent we estimate $||(m!)^{-1} I_s(P^m)e(0)||^2$ as follows. By Theorem 2.1

$$||I_s(P^m)e(0)||^2 = \langle e(0), I_s(P^{\dagger m} * P^m)e(0)\rangle. \tag{6.3}$$

We consider the *-commutator

$$\begin{array}{l} P^\dagger * P - P * P^\dagger = \\ \sum_{j,k=0}^n \bar{a}_j a_k \{dT \otimes \cdots \otimes dT^{(j)} \otimes dA * dT \otimes \cdots \otimes dT^{(k)} \otimes dA^\dagger \\ -dT \otimes \cdots \otimes dT^{(K)} \otimes dA^\dagger * dT \otimes \cdots \otimes dT^{(j)} \otimes dA\}. \end{array}$$

In the inner expression, the terms which are tensors of the highest order $j+k+2$ cancel in pairs. The surviving tensors are of order $j+k+1$, coming from the product $dAdA^\dagger = dT$; there are $\frac{(j+k)!}{j!k!}$ such terms, all equal to $dT \otimes \cdots \otimes dT^{j+k+1}$. Thus we can write

$$P^\dagger * P - P * P^\dagger = C \tag{6.4}$$

where

$$C = \sum_{j,k=0}^n \bar{a}_j a_k \frac{(j+k)!}{j!k!} dT \otimes \cdots \otimes dT^{j+k+1}. \tag{6.5}$$

Because the creation and annihilation subalgebras of $\mathcal{J}$ are both commutative it is evident that the $*$-commutators

$$C * P - P * C, \quad C * P^\dagger - P^\dagger * C \tag{6.6}$$

both vanish. One may then prove by induction on m that, in view of (6.4) and (6.6),

$$P^{\dagger m} * P^m = (P * P^\dagger + C) * (P * P^\dagger + 2C) * \cdots * (P * P^\dagger + mC). \tag{6.7}$$

Since $I_s(P)^\dagger e(0) = 0$ and from (6.5)

$$\begin{array}{rl} I_s(C) = & \sum_{j,k=0}^n \bar{a}_j a_k \frac{j+k)!}{j!k!} I_s(dT \otimes \cdots \otimes dT^{(j+k+1)}) \\ & = \int_0^s |p(t)|^2 dt1 = ||f||^2 \end{array}$$

we can write

$$||I_s(P^m)e(0)||^2 = m!||f||^{2m}$$

and hence

$$||(m!)^{-1} I_s(P^m)e(0)||^2 = \frac{||f||^{2m}}{m!}.$$

From the convergence of the exponential series $e^{||f||^2}$ it is evident that the orthogonal series $\sum_{m=0}^\infty (m!)^{-1} I_s(P^m)e(0)$ converges; denote its sum by ψ.

Since the exponential vectors of form $e(f)$ are total it is sufficient to complete the proof of the theorem to show that $\psi = e(f)$, or equivalently that

$$\langle \psi, e(g)\rangle = e^{\langle f,g\rangle}$$

for all $g \in \mathcal{H}$. This in turn is equivalent to showing that for all $m \in \mathbb{N}^*$,

$$\langle I_s(P^m)e(0), e(g)\rangle = \langle f, g\rangle^m. \tag{6.8}$$

(6.8) evidently holds when $m = 0$. Assuming its validity for $m = k$ we have, using Theorem 2.1

$$\langle I_s(P^{k+1})e(0), I_s(P^\dagger)e(g)\rangle = \langle I_s(P^k)e(0), I_s(P^\dagger e(g)\rangle =$$
$$\langle I_s(P^k)e(0), \textstyle\int_{0<t<s} I_t(\bar{a}_0, \bar{a}_1 dT, \cdots, \bar{a}_n dT \otimes \cdots \otimes dT^{(n)})dAe(g)\rangle.$$

By approximating $I_s(P^k)e(0)$ by linear combinations of exponential vectors we may apply the first fundamental formula of quantum stochastic calculus to this expression to obtain

$$\begin{aligned}
&\langle I_s(P^{k+1})e(0), e(g)\rangle \\
&= \textstyle\int_0^s \langle I_s(P^k)e(0), I_t(\bar{a}_0, \bar{a}_1 dT, \cdots, \bar{a}_n dT \otimes \cdots \otimes dT^{(n)})g(t)e(g)\rangle dt \\
&= \int_0^s \bar{f}(t)g(t)dt\langle I_s(P^k)e(0), e(g)\rangle \\
&= \langle f, g\rangle\langle f, g\rangle^k = \langle f, g\rangle^{k+1}.
\end{aligned}$$

(6.8) and, with it, the Theorem is proved. □

Theorem 5.1 shows that the Fock space $\mathcal{H}$ decomposes into the internal Hilbert space direct sum

$$\mathcal{H} = \bigoplus_{m=0}^{\infty} \mathcal{H}_m$$

where $\mathcal{H}_m$ is the closed linear span of $\mathcal{M}_m$. The spaces $\mathcal{H}_m$ are the m-particle spaces of the physical Fock space. It is not difficult to show that for arbitrary $f \in \mathfrak{h}$ of compact support in $\mathbb{R}_+$, the stochastic integrals

$$a(f) = \int \bar{f} dA, \quad a^\dagger(f) = \int f dA$$

can be extended to a domain containing all finite particle vectors (that is, elements of the weak direct sum $(\bigoplus_{m=0}^{\infty} \mathcal{H}_m)_0$) and on the latter domain they satisfy the usual canonical commutation relations

$$\begin{aligned}
&[a(f), a(g)] = [a^\dagger(f), a^\dagger(g)] = 0 \\
&[a(f), a^\dagger(g)] = \langle f, g\rangle 1.
\end{aligned}$$

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