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Preprint 15/1996

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On connections between BCK-algebras and difference posets

June 12, 1996

PREPRINT SERIES

ON CONNECTIONS BETWEEN BCK-ALGEBRAS AND DIFFERENCE POSETS

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Abstract. Equivalent relations between BCK-algebras and posets with difference, and relations among bounded commutative BCK-algebras, difference posets, MV-algebras, quantum MV-algebras and orthoalgebras, respectively, are studied.

1. Introduction

In 1966, Imai and Iséki [ImIs] introduced a notion of *BCK-algebras.* This notion was originated from two different ways: (1) set theory, and (2) classical and non-classical propositional calculi. Today BCK-algebras have been studied by many authors and they have been applied to many branches of mathematics, such as group, functional analyse, probability theory, topology, fuzzy set theory, and so on.

Investigation of mathematical foundations of quantum mechanics showed that Kolmogorovian model of probability theory, holding for classical mechanics, fails in the case of quantum mechanics. Birkhoff and von Neumann [BiNe] introduced a *quantum logic*, i.e., an algebraic system describing propositional system of quantum mechanics. This system is more general than Boolean algebras, and the most important example of quantum logics is a quantum logic $\mathcal{L}(H)$ consisting of all closed subspaces of a real, complex or quaternionic Hilbert space H. More general structures are *orthoalgebras* introduced by Foulis and Randall [FoRa]. Recently Kôpka and Chovanec [KoCh], students of one of the present authors, introduced *difference posets* (D-posets in abbreviation) which combine both algebraic and fuzzy sets ideas. The most important examples are the interval [0, 1] and the system $\mathcal{E}(H)$ of all effect operators, i.e., of all Hermitian operators A of a real, complex or quaternionic Hilbert spaces H such that $O \leq A \leq I$, where I is the identity. An equivalent structure is an *effect algebra* introduced originally by Giuntini and Greuling [GiGr] as a weak orthoalgebra, and Foulis and Bennett [FoBe], for more details on effect algebras and measure theory on them, see [DLP].

The primary notion of D-posets is a *difference* which has some similar properties as the difference $*$ defined in BCK-algebras. Therefore, in the present paper we shall investigate connections between posets with difference and BCK-algebras, and among D-posets and BCK algebras, MV-algebras introduced by Chang [Cha] and

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1991 *Mathematics Subject Classification.* Primary 06F35, 03G12.

Key words and phrases. BCK-algebra, commutative BCK-algebra, poset with difference, D-poset, MV-algebra, quantum MV-algebra, orthoalgebra.

$*$ This research is partially supported by the grant G-229/94 of the Slovak Academy of Sciences, Slovakia.

quantum MV-algebras presented by Giuntini [Giu 1], and orthoalgebras, respectively.

2. BCK-algebras

According to Imai and Iséki, a BCK-*algebra* is a non-empty set X with a binary operation $*$ and with a constant element 0 satisfying the following axioms: for all $x, y, z \in X$

(I) $\big((x*y)*(x*z)\big)*(z*y)=0$;
(II) $\big(x*(x*y)\big)*y=0$;
(III) $x*x=0$;
(IV) $x*y=0$ and $y*x=0$ imply $x=y$;
(V) $0*x=0$.

As usual, we shall write $X=(X;*,0)$. In X we can define a partial binary operation $\preccurlyeq$ by

$$x \preccurlyeq y \quad \text{iff} \quad x*y=0. \tag{2.1}$$

Then $\preccurlyeq$ is a partial ordering on X and in any BCK-algebra $(X;*,0)$ we have, for all $x,y,z\in X$, [MeJu],

$$x \preccurlyeq y \quad \text{implies} \quad z*y \preccurlyeq z*x, \tag{2.2}$$

$$(x*y)*z=(x*z)*y, \tag{2.3}$$

$$x*0=x.$$

For any $x,y\in X$ denote

$$x \curlywedge y := y*(y*x).$$

Obviously, $x \curlywedge y$ is a lower bound of x and y, and $x\curlywedge x = x$, $x \curlywedge 0 = 0 \curlywedge x = 0$, and

$$x*(y\curlywedge x)=x*y.$$

But in general, $x\curlywedge y \neq y \curlywedge x$. X is said to be *commutative* if $x \curlywedge y = y \curlywedge x$ for any $x,y\in X$. In this case, X is a lower semilattice $\preccurlyeq$ with respect to the partial order, i.e., for all $x,y\in X$, the greatest lower bound of x and y, $x\wedge y$, exists in X, in this case $x\wedge y = x\curlywedge y$. We can easily see that any BCK-algebra $(X;*,0)$ is commutative if and only if $x\preccurlyeq y$ implies $x=y*(y*x)$.

If there is an element 1 of a BCK-algebra X such that $x\preccurlyeq 1$ for all $x\in X$, the element 1 is called a *unit* of X. A BCK-algebra with unit is said to be *bounded*. We recall that any BCK-algebra can be extended to a bounded BCK-algebra by adding a special element $1\notin X$, [MeJu].

Example 2.1. Let $X=\{0,1,2,\dots\}$ be ordered with the natural order. We define, for any $x,y\in X$,

$$x*y=\begin{cases} 0 & \text{if } x\le y,\\ 1 & \text{if } y<x,\ y\neq 0,\\ x & \text{if } y<x,\ y=0.\end{cases}$$

Then $(X;*,0)$ is a non-bounded non-commutative BCK-algebra.

3. Difference Posets

Let L be a non-empty poset. A partial binary operation $\ominus$ is called a *difference* on L if the element $b \ominus a$ is defined in L iff $a \leq b$, and such that the following axioms hold for any $a, b, c \in L$:

(DPi) $b \ominus a \leq b$;
(DPii) $b \ominus (b \ominus a) = a$;
(DPiii) $a \leq b \leq c \;\Rightarrow\; c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

The structure $L = (L; \leq, \ominus)$ is said to be a *poset with difference*. If there is a greatest element $1 \in L$, $(L; \leq, \ominus, 1)$ is said to be a *difference poset* (D-*poset*, in abbreviation); if the least element 0 belongs to $(L; \leq, \ominus)$, we shall write $(L; \leq, \ominus, 0)$.

The following proposition has been proved in [KoCh]:

Proposition 3.1. *Let a, b, c, d be elements of a poset L with difference. Then*

(i) $a \leq b \leq c \;\Rightarrow\; b \ominus a \leq c \ominus a$ *and* $(c \ominus a) \ominus (b \ominus a) = c \ominus b$,
(ii) $b \leq c,\ a \leq c \ominus b \;\Rightarrow\; b \leq c \ominus a$ *and* $(c \ominus b) \ominus a = (c \ominus a) \ominus b$,
(iii) $a \leq b \leq c \;\Rightarrow\; a \leq c \ominus (b \ominus a)$ *and* $(c \ominus (b \ominus a)) \ominus a = c \ominus b$.

Let b be an element of a poset L. The set $A(b) := \{a \in L : a \leq b\}$ is called an *initial segment* of b. A poset $(L; \leq)$ is said to be *directed downwards* if, for any $x, y \in L$, there is $z \in L$ such that $z \leq x,\ z \leq y$. In particular, any lower semilattice is directed downwards.

Proposition 3.2. *Let $(L; \leq, \ominus)$ be a poset with difference. Then, for any $a \in A(b)$, we have*

(i) $(b \ominus b) \leq a$ *and* $a \ominus (b \ominus b) = a$,
(ii) $a \ominus a = b \ominus b$.
(iii) *A poset $(L; \leq, \ominus)$ with difference possesses a least element 0 if and only if $(L; \leq)$ is a downwards directed poset. In this case, $b \ominus b$ is the least element of L for any $b \in L$.*

Proof. (i). Let $a \in A(b)$. According to (DPi), $b \ominus a \in A(b)$, and due to (DPiii) and (DPii) we have $b \ominus b \leq b \ominus (b \ominus a) = a$. Since $a \ominus a \leq a \leq b$, we conclude that $b \ominus b \leq a \ominus a$ because $b \ominus b$ is the least element in $A(b)$. From (DPii) and (DPiii) we have

$$a = a \ominus (a \ominus a) \leq a \ominus (b \ominus b) \leq a$$

which implies (i).

(ii). From (i) we have $a \ominus a = a \ominus \big(a \ominus (b \ominus b)\big) = b \ominus b$.

(iii). If L possesses a least element, then it is evidently directed downwards.

Suppose now that L is a downwards directed poset. Denote $0_b := b \ominus b$. We claim that $0_b \leq a$ for any $a \in L$. Put $0_a := a \ominus a$. According to hypothesis, there is an element $x \in L$ with $x \leq 0_b,\ x \leq 0_a$. Since $x \in A(a) \cap A(b)$, we have by (i), $x \leq 0_a \leq x$ and $x \leq 0_b \leq x$, so that $0_a = 0_b \leq a$. □

We recall that if $(L; \leq, \ominus)$ is a poset with difference, then the least element 0 can fail in L. Indeed, let $L = \{x, y\}$, where x and y are incomparable. Define $x \ominus x = x$ and $y \ominus y = y$, then L is a poset with difference without the least element. If L is a

D-poset, then $L = A(1)$ and hence $1 \ominus 1$ is the least element of L by Proposition 3.2-(i). Eventhough a poset with difference $(L; \leq, \ominus)$ has the least element, the poset $(L; \leq)$ has not necessary the greatest element even if it is an upper semilattice or a lattice.

The following two examples are prototypes of D-posets.

Example 3.3. Let H be a Hilbert space. Then the set $\mathcal{E}(H)$ of all Hermitian operators A on H such that $O \leq A \leq I$, where I is the identity operator on H, is a D-poset where a partial ordering $\leq$ is defined by $A \leq B$ iff $(Ax, x) \leq (Bx, x)$, $x \in H$, and $C = B \ominus A$ iff $(Ax, x) - (Bx, x) = (Cx, x)$, $x \in H$.

This set plays an important role for unsharp measurement of quantum mechanics.

Example 3.4. Let the closed interval $[0,\ 1]$ be ordered by the natural way, and, for two numbers $a, b \in [0,\ 1]$, we define $a \oplus b$ iff $a + b \leq 1$ and we put then $a \oplus b = a + b$. Then $[0,\ 1]$ is a totally ordered distributive lattice.

4. BCK-algebras and Posets with Difference

In the present section, we study the relationships between BCK-algebras and posets with difference.

Theorem 4.1. *Let $(X; *, 0)$ be a BCK-algebra. We define a partial binary operation $\div$ on X such that, for $x, y \in X$, $y \div x$ is defined if and only if $x \preccurlyeq y$, and in this case*

$$y \div x := y * x. \tag{4.1}$$

Then

(i) $y \div x \preccurlyeq y$ *if* $x \preccurlyeq y$,
(ii) $y \div (y \div x) \preccurlyeq x$ *if* $x \preccurlyeq y$,
(iii) *if* $x \preccurlyeq y \preccurlyeq z$, *then* $z \div y \preccurlyeq z \div x$ *and* $(z \div x) \div (z \div y) \preccurlyeq y \div x$.

The partial binary operation $\div$ is a difference on $(X; \preccurlyeq)$ if and only if $(X; *, 0)$ is a commutative BCK-algebra.

Proof. (i). According to (2.3), $(y * x) * y = (y * y) * x = 0 * x = 0$, so that $y * x \preccurlyeq y$.
(ii). By (II), $y \div (y \div x) = y * (y * x) \preccurlyeq x$.
(iii) In view of (2.2), $x \preccurlyeq y$ implies $z \div y \preccurlyeq z \div x$ and (I) entails that $(z \div x) \div (z \div y) \preccurlyeq y \div x$.

Suppose now that $\div$ is a difference on X. Let $x \preccurlyeq y$. (DPii) entails that $y * (y * x) = y \div (y \div x) = x$. Hence, by [MeJu, Thm 5.6 (i), (ii)], X is commutative.

Conversely, let X be a commutative BCK-algebra. Since $y \div (y \div x) \preccurlyeq x$ if $x \preccurlyeq y$, it is enough to show that $x \preccurlyeq y \div (y \div x)$.

$$\begin{aligned} x * (y \div (y \div x)) &= x * (y * (y * x)) \\ &= x * (x \curlywedge y) \\ &= x * (y \curlywedge x) \\ &= x * y \\ &= 0. \end{aligned}$$

This means $x \preccurlyeq y \div (y \div x)$. Hence the partial binary operation $\div$ is a difference on $(X; \preccurlyeq)$. $\square$

Example 4.2. Let $X = \{0, 1, 2, \dots\}$ be a BCK-algebra defined in Example 2.1. Then there are a unique difference $\ominus$ on X which preserves the natural order, namely $y \ominus x := y - x$, whenever $x \le y$, $x, y \in X$, where $-$ is the usual subtraction of integers, and a unique binary operation $*_1$ on X, namely $x *_1 y := \max(0, x - y)$, $x, y \in X$, such that $(X; *_1, 0)$ is a BCK-algebra and $\div$ defined by (4.1) coincides with our $\ominus$.

Corollary 4.3. *Let $(X; *, 0)$ be a BCK-algebra and let $\div$ be defined via (4.1). The following statements are equivalent:*

(i) *$\div$ is a difference on $(X; \preccurlyeq)$.*
(ii) *$x = y \div (y \div x)$ whenever $x \preccurlyeq y$, $x, y \in X$.*
(iii) *$(X; *, 0)$ is a commutative BCK-algebra.*

Proof. (i) implies (ii) by definition of a difference. The implication (ii) $\Rightarrow$ (iii) has been established in the proof of Theorem 4.1, and the implication (iii) $\Rightarrow$ (i) follows from Theorem 4.1. □

5. BCK-algebras, D-posets and MV-algebras

In the present section, we investigate the relations among BCK-algebras, D-posets and MV-algebra. We recall that Mundici [Mun1] showed that MV-algebras are categorically equivalent to bounded commutative BCK-algebras.

MV-algebras are many valued analogues of a two-valued logic and they have been introduced by Chang [Cha]. We recall that according to Mundici [Mun], they can be characterized as follows. An MV-*algebra* is a non-empty set L with two special elements 0 and 1 ($0 \neq 1$), with a binary operation $\oplus : L \times L \to L$ and with a unary operation $^* : L \to L$ such that, for all $a, b, c \in L$, we have

(MVi) $a \oplus b = b \oplus a$ (commutativity);
(MVii) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ (associativity);
(MViii) $a \oplus 0 = a$;
(MViv) $a \oplus 1 = 1$;
(MVv) $(a^*)^* = a$;
(MVvi) $a \oplus a^* = 1$;
(MVvii) $0^* = 1$;
(MVviii) $(a^* \oplus b)^* \oplus b = (a \oplus b^*)^* \oplus a$.

We recall that the above axioms are not the independent, e.g. (MVvii) follows from (MVvi) and (MViii) for $a = 0$.

We define the following binary operations $\odot, \vee, \wedge$ as follows:

$$a \odot b := (a^* \oplus b^*)^*, \ a, b \in L, \tag{5.1}$$
$$a \vee b := (a^* \oplus b)^* \oplus b, \quad a \wedge b := (a^* \vee b^*)^*, \ a, b \in L.$$

Then $(L; \odot, 1)$ is a semigroup written "multiplicatively" with the neutral element 1.[1] If, for $a, b \in L$ we define

$$a \le b \ \Leftrightarrow \ a = a \wedge b,$$

[1] We remark that in the literature by an MV-algebra is assumed the structure $(L; \oplus, \odot, ^*, 0, 1)$, where $\odot$ is defined by (5.1).

then $\leq$ is a partial order on L, and $(L; \vee, \wedge, 0, 1)$ is a distributive lattice with the least and greatest elements 0 and 1, respectively, [Cha]. We recall that $a \leq b$ iff $b \oplus a^* = 1$.

We note that in any MV-algebra $(L; \oplus, \odot, ^*, 0, 1)$ there is a partial binary operation $\ominus$ such that $(L; \leq, \ominus, 0, 1)$ is a D-poset, [KoCh]. For that it is necessary to define $b \ominus a := (a \oplus b^*)^*$ whenever $a \leq b$.

Example 5.1. Let $L = [0,\ 1]$ and we define

$$x \oplus y := \min\{x + y, 1\}, \quad x^* := 1 - x,\ \ x, y \in [0,\ 1]. \tag{5.2}$$

Then $(L, \oplus, ^*, 0, 1)$, where 0 and 1 are the same from $[0,\ 1]$, is an MV-algebra such that

$$x \odot y = \max\{x + y - 1, 0\},\ \ x, y \in [0,\ 1],$$

$x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$, $x, y \in [0,\ 1]$, and the induced order on $[0,\ 1]$ coincides with the natural one on $[0,\ 1]$.

Example 5.2. Let $(G; +, 0)$ be an l-group written additively with a strong unit (= order unit) $u \in G$. We define $L(G) := \{x \in G :\ 0 \leq x \leq u\}$ and

$$x \oplus y := (x + y) \wedge u, \quad x^* := u - x,\ \ x \in L(G). \tag{5.3}$$

Then $(L(G); \oplus, ^*, 0, u,)$ is an MV-algebra with

$$x \odot y = 0 \vee (x + y - u).$$

Conversely, any MV-algebra $(L; \oplus, ^*, 0, 1)$ is isomorphic with some $(L(G); \oplus, ^*, 0, u)$, where $(G; +, 0)$ is an l-group with a strong unit u, [Mun].[2]

Theorem 5.3. *Let $(X; *, 0, 1)$ be a bounded BCK-algebra with the induced order $\preccurlyeq$. We define*

(i) *a partial binary operation $\div$ defined via* (4.1)*;*

(ii) *a binary operation $\oslash$ on X defined via*

$$y \oslash x := y \div (x \curlywedge y),\ \ x, y \in X; \tag{5.4}$$

(iii) *a unary operation $^* : X \to X$ defined via*

$$x^* := 1 \div x,\ \ x \in X; \tag{5.5}$$

(iv) *binary operations $\oplus$ and $\odot$ on X defined via*

$$x \oplus y := (x^* \oslash y)^*,\ \ x, y \in X, \tag{5.6}$$

$$x \odot y := x \oslash y^*,\ \ x, y \in X. \tag{5.7}$$

Then, for all $x, y \in X$,

$$y \oslash x = y * x, \quad x \oplus y = (x^* * y)^*, \quad x \odot y = (x^* \oplus y^*)^*,$$

and the following statements are equivalent:

(a) *$(X; \preccurlyeq, \div, 0, 1)$ is a D-poset.*

(b) *$(X; *, 0, 1)$ is a bounded commutative BCK-algebra.*

(c) *$(X; \oplus, \odot, ^*, 0, 1)$ is an MV-algebra.*

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[2]Two MV-algebras are isomorphic if there is a bijective mapping between them preserving $\oplus, ^*, 0$, and 1.

If it is a case, then the orders determined by the BCK-algebra and the MV-algebra coincide.

Proof. $(a) \iff (b)$. Note that the greatest element of a D-poset is the greatest element of its associated BCK-algebras, and vice versa. By applying Theorem 4.3 we can see $(a) \iff (b)$.

(b) $\Rightarrow$ (c). We exhibit all axioms of MV-algebras. Let $x, y, z \in X$ be given. Then

$$(x^*)^* = (1 \div x)^* = 1 \div (1 \div x) = x,$$
$$0^* = 1 \div 0 = 1 * 0 = 1,$$
$$(x \oplus y) = (x^* \oslash y)^* = \big((1 \div x) \oslash y\big)^* = \big((1 * x) * y\big)^* = \big((1 * y) * x\big)^* = y \oplus x,$$

$$\begin{aligned}(x \oplus y) \oplus z &= (y \oplus x) \oplus z = [(y \oplus x)^* \oslash z]^* \\ &= [(y^* \oslash x) \oslash z]^* = [(y^* * x) * z]^* = [(y^* * z) * x]^* \\ &= [(y \oplus z)^* \oslash x]^* = (y \oplus z) \oplus x = x \oplus (y \oplus z),\end{aligned}$$

$$\begin{aligned}x \oplus 0 &= (x^* \oslash 0)^* = (x^* * 0)^* = (x^*)^* = x, \\ x \oplus 1 &= (x^* \oslash 1)^* = (x^* * 1)^* = 0^* = 1, \\ x \oplus x^* &= (x^* \oslash x^*)^* = (x^* * x^*) = 0^* = 1.\end{aligned}$$

To prove $(x^* \oplus y)^* \oplus y = (x \oplus y^*)^* \oplus x$, we use the equality

$$u \oslash (u \oslash v) = v \oslash (v \oslash u)$$

holding for all $u, v \in X$.

$$\begin{aligned}y \oplus (x^* \oplus y)^* &= [y^* \oslash (y \oplus x^*)^*]^* = [y^* \oslash (y^* \oslash x^*)]^* = [x^* \oslash (x^* \oslash y^*)]^* \\ &= [x^* \oslash (x \oplus y^*)^*]^* = x \oplus (x \oplus y^*)^* = (x \oplus y^*)^* \oplus x.\end{aligned}$$

It is easy to verify that $x \odot y = (x^* \oplus y^*)^*$ for all $x, y \in X$. We note that the order $\leq_M$ determined in the MV-algebra $(X; \oplus, \odot, ^*, 0, 1)$ is given by $x \leq_M y$ iff $y \oplus x^* = 1$. Suppose that $x \leq_M y$, then $(y^* \oslash x^*)^* = 1$ so that $y^* * x^* = 0$ and $y^* \preccurlyeq x^*$. Hence $1 * x^* \preccurlyeq 1 * y^*$, i.e. $x \preccurlyeq y$, and vice versa.

(c) $\Rightarrow$ (b). Since $(X; \oplus, \odot, ^*, 0, 1)$ is an MV-algebra, using the axiom (MVviii), we have $(x \oplus y^*)^* \oplus y^* = (x^* \oplus y)^* \oplus x^*$. Hence

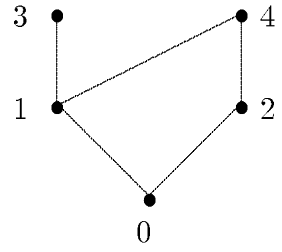
$$\begin{aligned}y^* \oplus (y^* \oplus x)^* &= x^* \oplus (x^* \oplus y)^*, \\ y^* \oplus (y * x) &= x^* \oplus (x * y), \\ \big(y * (y * x)\big)^* &= \big(x * (x * y)\big)^*, \\ y * (y * x) &= x * (x * y),\end{aligned}$$

so that $(X; *, 0, 1)$ is a bounded commutative BCK-algebra. $\square$

If X is a commutative BCK-algebra then it is a lower semilattice with respect to BCK-ordering $\leq$ [MeJu], but the converse need not be true.

Example 5.4. Consider the following BCK-algebra $(\{0,1,2,3,4\};*,0)$.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	2	0
3	3	1	3	0	1
4	4	4	4	4	0



The poset induced by BCK-ordering is a lower semilattice, but it is not commutative, since $3*(3*4) = 1 \neq 0 = 4*(4*3)$. Similarly, Example 2.1 gives a case of a non-commutative BCK-algebra which is a lattice with respect to the BCK-ordering.

Theorem 5.5. *Let a poset with difference $(X;\leq,\ominus)$ be a lower semilattice with respect to $\leq$. Define a binary operation $*$ on X by*

$$x*y := x\ominus(x\wedge y),\ \ x,y\in X. \tag{5.8}$$

Then it is an extension of $\ominus$, in fact,

$$x*y = x\ominus y \quad \text{if} \quad y\leq x,\ \ x,y\in X,$$

and L possesses a least element 0. Moreover, $(X;*,0)$ is a commutative BCK-algebra if and only if, for all $x,y,z\in X$, we have

$$(x*y)*z = (x*z)*y. \tag{5.9}$$

Proof. According to (iii) of Proposition 3.2, $(L;\leq,\ominus)$ possesses a least element 0. Yutani [Yu] proved that an algebra $(X;*,0)$ is a commutative BCK-algebra if and only if it satisfies (i) $x*x=0$; (ii) $x*0=x$; (iii) $x*(x*y)=y*(y*x)$ and (5.9). Since X is a poset with difference, $x*x = x\ominus(x\wedge y) = x\ominus x = 0$ and $x*0 = x\ominus(x\wedge 0) = x\ominus 0 = x$, which proves (i) and (ii). Since X is a lower semilattice and by applying (DPii) we have $x*(x*y) = x\ominus(x*y) = x\ominus(x\ominus(x\wedge y)) = x\wedge y = y\wedge x = y\ominus(y*x) = y*(y*x)$, which proves (iii). Hence if $(X;*,0)$ satisfies (5.9) then it is a commutative BCK-algebra. The converse easily follows from (2.3). $\square$

Corollary 5.6. *Let a D-poset $(X;\leq,\ominus,0,1)$ be a bounded lower semilattice. Let $*$ be a binary operation on X defined by (5.8), and let $\oplus$, $\odot$, * be defined by (5.4)–(5.7), respectively. Then the following statements are equivalent:*

(i) $(x*y)*z = (x*z)*y,\ \ x,y,z\in X$.
(ii) *$(X;*,0,1)$ is a bounded commutative BCK-algebra.*
(iii) *$(X;\oplus,\odot,^*,0,1)$ is an MV-algebra.*

In addition, if this is a case, X is a lattice with respect to $\leq$.

Proof. It follows from Theorem 5.3 and Theorem 5.5. □

We remark that Chovanec and Kôpka [ChKo] proved that a D-poset being simultaneously a lattice can be organized equivalently into an MV-algebra iff (5.9) holds. We recall that the D-poset $\mathcal{E}(H)$ from the Example 3.3 cannot be equivalently organized into neither a BCK-algebra nor into an MV-algebra because $\mathcal{E}(H)$ is not a lower semilattice.

6. BCK-algebras QMV-algebras

R. Giuntini [Giu 1] recently has introduced an interesting algebraic structure, quantum many valued algebra (QMV-algebra in abbreviation) which generalizes MV-algebras. Axiom (MVviii) is responsible for the lattice-theoretic behaviour of the corresponding operations $\vee$ and $\wedge$ defined via $a \vee b = (a^* \oplus b)^* \oplus b$ and $a \wedge b = (a \oplus b^*) \odot b$ for any $a, b \in L$.

If we take the left-hand side of (MVviii) as for defining the binary operation $\Cup$ via

$$a \Cup b = (a^* \oplus b)^* \oplus b, \ \ a, b \in L, \tag{6.1}$$

we loose important properties of MV-algebras, and to have some of interesting MV-properties, Giuntini needs some of additional properties of MV-algebras.

We recall that a *quantum MV-algebra* (*QMV-algebra*) is a non-empty set L with two special elements 0 and 1 ($0 \neq 1$), with a binary operation $\oplus : L \times L \to L$ and with a unary operation $^* : L \to L$ such that, for all $a, b, c \in L$, we have the axioms (MVi)–(MVvii) and, in addition,

(QMV8) $a \Cup (b \Cap a) = a$;
(QMV9) $(a \Cap b) \Cap c = (a \Cap b) \Cap (b \Cap c)$;
(QMV10) $a \oplus \big(b \Cap (a \oplus c)^*\big) = (a \oplus b) \Cap \big(a \oplus (a \oplus c)^*\big)$;
(QMV11) $a \oplus (a^* \Cap b) = a \oplus b$;
(QMV12) $(a^* \oplus b) \Cup (b^* \oplus a) = 1$,

(where $a \odot b := (a^* \oplus b^*)^*$, $a \Cap b := (a \oplus b^*) \odot b$ and $a \Cup b := (a \odot b^*) \oplus b$). As usually, we write $L = (L; \oplus, \odot, ^*, 0, 1)$ for QMV-algebras. Generally, $\Cap$ and $\Cup$ are not lattice-operations.

Any MV-algebra is a QMV-algebra, but the converse it is not true. A QMV-algebra L is an MV-algebra iff $a \Cap b = b \Cap a$ for any $a, b \in L$.

For all $a, b \in L$ we define

$$a \leq_Q b \quad \text{iff} \quad a = a \Cap b.$$

Then $\leq_Q$ is a partial order on L.

The following two examples are reformulated cases from [Giu 2].

Example 6.1. Let $(L; \leq, \ominus, 0, 1)$ be a D-poset. Define, for all $a, b \in L$, a binary operation $\oplus$ and a unary operation * on L as follows

$$a \oplus b := \begin{cases} 1 \ominus \big((1 \ominus b) \ominus a\big), & \text{if } a \leq (1 \ominus b), \\ 1, & \text{otherwise}, \end{cases} \qquad a, b \in L,$$

$$a^* := 1 \ominus a, \ \ a, b \in L.$$

Then $(L; \oplus, \odot, ^*, 0, 1)$ is a QMV-algebra with $\leq_Q = \leq$.

Example 6.2. Let $(L;\oplus,\odot,^*,0,1)$ be a QMV-algebra. Then $(L,\leq_Q,\ominus_Q,0,1)$, where $b\ominus_Q a$ is defined iff $a\leq_Q b,\ a,b\in L$ and then $b\ominus_Q a := b\odot a^*$, is a D-poset. We recall that starting from a QMV-algebra to a D-poset and from this D-poset again to a QMV-algebra, the final and starting structures are not the same. The same are only if the QMV-algebra is quasi-linear.[3]

In particular, $\mathcal{E}(H)$ is a QMV-algebra which is not an MV-algebra.

We recall that the binary operation $\Cup$ defined via (6.1) is a so-called left-join, and we write $\Cup=\Cup_L$, similarly we deal with a so-called left-meet $\Cap=\Cap_L$. If we take the right-hand side of (MVviii), we can define a so-called right-join $\Cup_R$ and a right-meet $\Cap_R$, i.e. $a\Cup_R b := (a\oplus b^*)^*\oplus a,\ a,b\in L$ and $a\Cap_R b := (a^*\oplus b)\odot a$. Then $a\Cap_L b = b\Cap_R a$ and $a\Cup_L b = b\Cup_R a$ for all $a,b\in L$. Using these notations, our QMV-algebra can be viewed as a left QMV-algebra, and we can define a so-called right QMV-algebra, which are practically the same things.

In what follows, we shall study relations between BCK-algebras and QMV-algebras.

Proposition 6.3. *Let $(X;\oplus,\odot,^*,0,1)$ be a QMV-algebra. Define a binary operation $*$ on X via*

$$x*y := x\odot y^*,\ \ x,y\in X. \tag{6.2}$$

Then, for all $x,y,z\in X$, we have

(i) $x*0=0$.
(ii) $x*x=0$.
(iii) $(x*y)*z=(x*z)*y$.
(iv) $\big(x*(x*y)\big)*y=0$.
(v) $x\Cap y = y*(y*x)$.

Proof. (i) (ii), and (v) are evident.

(iii). $(x*y)*z=(x\odot y^*)\odot z^* = x\odot(y^*\odot z^*) = x\odot(z^*\odot y^*) = (x\odot z^*)\odot y^* = (x*z)*y$.

(iv). According to (iii), $\big(x*(x*y)\big)*y = (x*y)*(x*y)=0$. ☐

Theorem 6.4. *A QMV-algebra $(X;\oplus,\odot,^*,0,1)$ is an MV-algebra if and only if, for $x,y\in X$,*

$$y\Cap x = x \quad \textit{implies} \quad x = x\Cap y. \tag{6.3}$$

Proof. First we show that $y\Cap x = x$ iff $x*y=0$. Indeed, let $x*y=0$. Then $y\Cap x = x*(x*y) = x*0 = x$. Conversely, let $y\Cap x = x$. Then

$$x*y = \big(x*(x*y)\big)*y = (x*y)*(x*y) = 0,$$

when we have used (iii) and (ii) of Proposition 6.3.

If now $(X;\oplus,\odot,^*,0,1)$ is an MV-algebra, then $(X;\leq,\wedge,\vee,0,1)$ is a distributive lattice in which

$$u\Cap v = u\wedge v = v\Cap u,\ \ u,v\in X,$$

[3] According to [Giu 2], a QMV-algebra $(L;\oplus,\odot,^*,0,1)$ with induced order $\leq_Q$ is said to be *quasi linear* if $a\not\leq_Q b$, then $a\Cap b=b$. Any QMV-algebra from Example 6.1 is quasi linear.

and the condition (6.3) is satisfied.

Conversely, let (6.3) hold. We show that $u \Cap v = v \Cap u$ for all $u, v \in X$. For that it is sufficient to establish that $(u \Cap v) * (v \Cap u) = 0$. Calculate

$$\begin{aligned}(v \Cap u) * (u \Cap v) &= \big((v \Cap u)^* \oplus (u \Cap v)\big)^* \\ &= \big((v \oplus u^*)^* \oplus u^* \oplus (u \Cap v)\big)^* \\ &= \big((v \oplus u^*)^* \oplus (u^* \oplus v)\big)^* = 1^* = 0.\end{aligned}$$

By hypothesis, $v \Cap u \leq_Q u \Cap v$. Similarly we can prove that $u \Cap v \leq_Q v \Cap u$, which gives $u \Cap v = v \Cap u$. $\square$

Remark 6.5. The condition (6.3) says that, for x, y from a QMV-algebra $(X; \oplus, \odot, {}^*, 0, 1)$

$$x \leq_Q y \quad \Leftrightarrow \quad y \Cap x = x, \tag{6.4}$$

and Theorem 6.4 says that (6.4) is a necessary and sufficient condition to be $(X; \oplus, \odot, {}^*, 0, 1)$ an MV-algebra.

Theorem 6.6. *Let $(X; \oplus, \odot, {}^*, 0, 1)$ be a QMV-algebra and define a binary operation $*$ on X via (6.2). Then $(X; *, 0)$ is a BCK-algebra if and only if $(X; \oplus, \odot, {}^*, 0, 1)$ is an MV-algebra. If it a case, then $(X; *, 0, 1)$ is a bounded commutative BCK-algebra.*

Proof. We note that in view of Proposition 6.3, (II), (II), (V) and (2.3) hold in any QMV-algebra. If $(X; \oplus, \odot, {}^*, 0, 1)$ is an MV-algebra, then $(X; *, 0, 1)$ is evidently a bounded commutative BCK-algebra.

Assume now that $x * y = 0$ implies $x = x \Cap y$. Calculate

$$(x \Cap y) * x = \big(y * (y * x)\big) * x = (y * x) * (y * x) = 0.$$

$$\begin{aligned}x * (x \Cap y) &= x \odot (x \Cap y)^* = \big(x^* \oplus (x \Cap y)\big)^* \\ &= (x^* \oplus y)^* = x \odot y^* = x * y = 0.\end{aligned}$$

when we used (QMV11). Applying (IV), we have $x = x \Cap y$. Since $x * y = 0$ iff $y \Cap x = x$, using Theorem 6.4, we have that $(X; \oplus, \odot, {}^*, 0, 1)$ is an MV-algebra, and consequently, $(X; *, 0, 1)$ is a bounded commutative BCK-algebra. $\square$

Theorem 6.7. *Let $(X; *, 0, 1)$ be a bounded BCK-algebra with the induced BCK-order $\preccurlyeq$. Define operations ${}^*, \oplus, \odot$ on X via (5.5)–(5.7). The following statements are equivalent*

(i) *$(X; \oplus, \odot, {}^*, 0, 1)$ is a QMV-algebra.*
(ii) *$(X; \oplus, \odot, {}^*, 0, 1)$ is an MV-algebra.*

Proof. It is clear that (ii) entails (i).

Suppose thus (i). If we define a binary operation $*_1$ on X via

$$x *_1 y := x \odot y^*, \ x, y \in X,$$

then $x *_1 y = x * y$ for all $x, y \in X$. Applying Theorem 6.4, we conclude that $(X; \oplus, \odot, {}^*, 0, 1)$ is an MV-algebra. $\square$

7. BCK-algebras and Orthoalgebras

A very important subfamily of D-posets are orthoalgebras introduced by Foulis and Randall [FoRa]. We recall that an *orthoalgebra* is a set $L = (L; \oplus, 0, 1)$ with two particular elements 0, 1 ($0 \neq 1$), and with a partial binary operation $\oplus : L \times L \to L$ such that for all $a, b, c \in L$ we have

(OAi) if $a \oplus b \in L$, then $b \oplus a \in L$ and $a \oplus b = b \oplus a$ (commutativity);
(OAii) if $b \oplus c \in L$ and $a \oplus (b \oplus c) \in L$, then $a \oplus b \in L$ and $(a \oplus b) \oplus c \in L$, and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$ (associativity);
(OAiii) for any $a \in L$ there is a unique $b \in L$ such that $a \oplus b$ is defined, and $a \oplus b = 1$ (orthocomplementation);
(OAiv) if $a \oplus a$ is defined, then $a = 0$ (consistency).

If the assumptions of (OAii) are satisfied, we write $a \oplus b \oplus c$ for the element $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ in L. Let a and b be two elements of an orthoalgebra L. We say that (i) a is *orthogonal* to b and write $a \perp b$ iff $a \oplus b$ is defined in L; (ii) a is *less than or equal to* b and write $a \leq b$ iff there exists an element $c \in L$ such that $a \perp c$ and $a \oplus c = b$ (in this case we also write $b \geq a$); (iii) b is the *orthocomplement* of a iff b is a (unique) element of L such that $b \perp a$ and $a \oplus b = 1$ and it is written as $a^\perp$.

If we define, for $a, b \in L$ with $a \leq b$, the unique element $c \in L$ such that $a \oplus c = b$, then we write $c := b \ominus a$, and we have

$$b \ominus a = (a \oplus b^\perp)^\perp. \tag{7.1}$$

Then $(L; \ominus, \leq, 0, 1)$ is a D-poset.

Example 7.1. An orthomodular poset is a poset $(L; \leq, {}^\perp, 0, 1)$ ($0 \neq 1$) such that

(OMi) $(a^\perp)^\perp = a$ for any $a \in L$;
(OMii) if $a \leq b$, then $b^\perp \leq a^\perp$, $a, b \in L$;
(OMiii) $a \vee a^\perp = 1$ for any $a \in L$;
(OMiv) $a \vee b \in L$ whenever $a \perp b$, $a, b \in L$;
(OMv) if $a \leq b$, $a, b \in L$, then $b = a \vee (b^\perp \vee a)^\perp = a \vee (b \wedge a^\perp)$ (orthomodular law).

If we define $a \oplus b := a \vee b$ whenever $a \perp b$, $a, b \in L$, then $(L; \oplus, 0, 1)$ is an orthoalgebra.

The D-poset $\mathcal{E}(H)$ from Example 3.3 is a case of a D-poset which is no orthoalgebra.

Example 7.2. An orthomodular lattice is an orthomodular poset which is simultaneously a lattice. In particular, any Boolean algebra is an orthomodular lattice. Let $L = \mathcal{L}(H)$ be the set of all closed subspaces of a real, complex or quaternionic Hilbert space H. Then $(\mathcal{L}(H); \subseteq, {}^\perp, \{0\}, H)$ is a complete orthomodular lattice which is no Boolean algebra.

Example 7.3. The Wright triangle is an orthoalgebra L determined by three systems of mutually orthogonal atoms $\{a, b, c\}$, $\{c, d, e\}$ and $\{a, f, e\}$ where $\oplus$ is defined as follows:

$$a \oplus b = d \oplus e = c^{\perp}, \qquad a \oplus c = b^{\perp},$$
$$b \oplus c = f \oplus e = a^{\perp}, \qquad c \oplus e = d^{\perp},$$
$$a \oplus f = c \oplus d = e^{\perp}, \qquad a \oplus e = f^{\perp}.$$

Then the Wright triangle is not an orthomodular poset because $a \vee c$ does not exists in it however $a \oplus b = c^{\perp}$.

A BCK-algebra $(X; *, 0)$ is called to be (i) *implicative* if, for all $x, y \in X$ we have

$$x = x * (y * x);$$

(ii) *positively implicative* if, for all $x, y, z \in X$, we have

$$(x * z) * (y * z) = (x * y) * z.$$

Theorem 7.4. *Let $(X; *, 0, 1)$ be a bounded BCK-algebra with the induced order $\preccurlyeq$. Define a partial binary operation $\boxplus$ on X as follows: $x \boxplus y$ is defined in X if and only if $x \preccurlyeq 1 * y$, and then we put*

$$x \boxplus y := 1 * \big((1 * y) * x\big). \tag{7.2}$$

Define operations $^*, \oplus, \odot$ via (5.4)–(5.7). Then the following statements are equivalent:

(a) *$(X; \boxplus, 0, 1)$ is an orthoalgebra.*
(b) *$(X; *, 0, 1)$ is an implicative BCK-algebra.*
(c) *$(X, *, 0, 1)$ is a positively implicative BCK-algebra.*
(d) *$(X; \preccurlyeq, \vee, \wedge, ^*, 0, 1)$ is a Boolean algebra.*
(e) *$(X; \oplus, \odot, ^*, 0, 1)$ is an MV-algebra such that $x \oplus y = x \vee y$ for all $x, y \in X$.*

Proof. (a) $\Rightarrow$ (b). Since $(X, \preccurlyeq, \ominus, 0, 1)$ is a D-poset, where $\ominus$ is defined by (7.1) for $\oplus = \boxplus$, by (c) of Theorem 5.3, $(X, *, 0, 1)$ is a bounded commutative BCK-algebra, with $x^{\perp} := 1 \ominus x = x^*$ for any $x \in X$. We assert that, for any $x \in X$, $x \wedge x^* = 0$. Indeed, suppose $z \preccurlyeq x, x^*$. Then $z \preccurlyeq x \preccurlyeq z^*$ and therefore, $z \boxplus z$ exists in X, which by (OAiv) is possible iff $z = 0$. Applying [MeJu, (a),(d) Thm 5.13], this implies that $(X; *, 0, 1)$ is an implicative BCK-algebra.

By [MeJu, Thm 7.2], a BCK-algebra is implicative iff it is both commutative and positive implicative, so that (b) and (c) are equivalent.

(b) $\Rightarrow$ (d). It follows from [MeJu, Thm 7.5].

(d) $\Rightarrow$ (e). If $(X; \preccurlyeq, \vee, \wedge, ^*, 0, 1)$ is a Boolean algebra, then $(X; *, 0, 1)$ is a bounded commutative BCK-algebra. Calculate by (5.6)

$$x \oplus y = (x^* * y)^* = \big(x^* * (x^* * y^*)\big)^* = (x^* \wedge y^*)^* = x \vee y,$$

when we have used [MeJu, Thm 5.13 (d),(n)].

(e) $\Rightarrow$ (a). It is simple to verify that $(X; \boxplus, 0, 1)$ satisfies (OAi)–(OAiii). Let now $x \boxplus x$ exists in X. Then by (e),

$$1 = x \oplus x^* = x \vee x^* = x^*,$$

which implies $x = 0$, and $(X; \boxplus, 0, 1)$ is an orthoalgebra. $\square$

Theorem 7.5. *Let an orthoalgebra $(X;\oplus,0,1)$ be a lower semilattice. Define a binary operation $*$ on X via*

$$x * y = \left(x^{\perp} \oplus (x \wedge y)\right)^{\perp},\ \ x, y \in X.$$

Then the following statements are equivalent:

(i) $(x * y) * z = (x * z) * y,\ \ x, y, z \in X.$
(ii) $(X;*,0,1)$ *is a bounded implicative BCK-algebra.*
(iii) $(X;\oplus,\odot,^*,0,1)$ *is an MV-algebra with* $x \oplus y = x \vee y$ *for all* $x, y \in X$.
(iv) $(X;\leq,\vee,\wedge,^*,0,1)$ *is a Boolean algebra.*

In addition, if this is a case, X is a lattice with respect to $\leq$.

Proof. Since $(X,\leq,\ominus,0,1)$ is a D-poset, where $\ominus$ is defined via (7.1), we have from Corollary 5.6 that (ii) implies (i).

Suppose now (i). Then $(X;*,0,1)$ is a bounded commutative BCK-algebra. It is evident that $x^{\perp} = x^*,\ x \in X$. We assert that $x \wedge x^* = 0$ for any $x \in X$. Indeed, if $z \leq x, x^*$, then $z \leq x \leq z^{\perp}$ which implies by (OAiv) $z = 0$, which is by [MeJu, Thm 5.13] necessary and sufficient to be $(X;*,0,1)$ an implicative BCK-algebra.

The rest of the proof follows similar ideas as those in the proof of Theorem 7.4. □

Corollary 7.6. Let $(X;\leq,^{\perp},0,1)$ be an orthomodular lattice. Then it determines a BCK-algebra via Theorem 7.5 if and only if X is a Boolean algebra.

Proof. It follows from Theorem 7.5. □

8. Concluding Remarks

In the paper, we have studied BCK-algebras and their connection to quantum structures like D-posets, MV-algebra, quantum MV-algebras (QMV-algebras), orthoalgebras, and orthomodular posets. Quantum structures are various kinds of algebraic structures motivating by mathematical foundations of quantum mechanics.

We have proved that the category of BCK-algebra and the category of D-posets coincide exactly on the category of MV-algebras (Theorems 5.3 and 5.4). We have presented here also another proof of the equivalence of MV-algebras and bounded commutative BCK-algebras using the notion a difference instead of categorical argument used in [Mun 1].

Quantum MV-algebras, which are more general than MV-algebras, can be organized into a BCK-algebra iff QMV-algebra is an MV-algebra (Theorems 6.4 and 6.6).

Finally, we have established that orthoalgebras (or orthomodular lattices) can be organized into a BCK-algebra and conversely iff they are Boolean algebras (Theorems 7.4 and 7.5).

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