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WEAK OBSERVABLES IN MV ALGEBRAS

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Abstract. A notion of a weak observable is defined and a construction of a weak observable is examined. By the help of the construction the sum of weak observables is realized as well as the upper and lower limits of a sequence of weak observables.

1. Introduction

The notion of an MV algebra is interesting and inspiring from various points of view: multivalued logic, fuzzy sets, operator theory, quantum models and ℓ-groups (namely every MV algebra can be represented by the help of an ℓ-group). Let us present a simple, but very instructive example of an MV algebra M. Let M consists of all functions $f\colon \Omega \to <0,1>$ (or of all functions $f\colon \Omega \to <0,1>$ measurable with respect to a σ-algebra of subsets of Ω). We define two binary operations $\oplus$ and $\odot$ on M by the formulas

$$f \oplus g = \min(f+g, 1)\,,$$
$$f \odot g = \max(f+g-1, 0)$$

and a binary operation $*$ defined by

$$f^* = 1 - f\,.$$

Generally [1–2], [4–11] an MV algebra is an algebraic system $(M, \oplus, \odot, *, 0, 1)$, where $\oplus, \odot$ are binary operations, $*$ is a unary operation and $0, 1$ fixed elements such that the following identities are satisfied: $a \oplus b = b \oplus a$, $a \oplus (b \oplus c) = (a \oplus b) \oplus c$, $a \oplus 0 = a$, $a \oplus 1 = 1$, $(a^*)^* = a$, $0^* = 1$, $a \oplus a^* = 1$, $(a^* \oplus b)^* \oplus b = (a \oplus b^*)^* \oplus a$, $a \odot b = (a^* \oplus b^*)^*$. If we define $a \vee b = (a \odot b^*) \oplus b$, $a \wedge b = (a \oplus b^*) \odot b$ and $a \leq b$, if $a \vee b = b$, then we obtain a distributive lattice.

By the Mundici representation theorem ([7]) there is a commutative lattice ordered group G and a strong unit u in G such that

$$M = <0, u>, \;\; a \oplus b = (a+b) \wedge u\,,$$
$$a \odot b = (a + b - u) \vee 0, \;\; a^* = u - a$$

and u plays the role 1 in M and the neutral element 0 of G plays the role 0 in M. We may work with elements of an MV algebra as elements of a commutative group. E.g., if $a \leq b$, $a, b \in M$, then there is in G the element $b - a$ (i.e.,

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such element that $a + (b - a) = b$) and, because of $a \leq b$, the element $b - a \in < 0, 1 >= M$. Similarly in the following definition of an observable, if $A \cap B = \emptyset$, then $x(A) + (B) = x(A \cup B) \in < 0, u >$, hence $x(A) \oplus x(B) = \big(x(A) + x(B)\big) \wedge u = x(A) + x(B)$. So, we can work with the group operations $+$ instead of the MV algebra operation $\oplus$.

A state ([2]) is a mapping $m\colon M \to < 0, 1 >$ such that $m(1) = 1$, $m(a_n) \nearrow m(a)$, whenever $a_n \nearrow a$ and $m(a) = m(b) + m(c)$, whenever $a = b \oplus c$, $b \leq c^*$. An observable is a mapping $x\colon \mathcal{B}(R) \to M$ such that $x(R) = 1$, $x(A_n) \nearrow x(A)$ whenever $A_n \nearrow A$ and $x(A \cup B) = x(A) + x(B)$ whenever $A \cap B = \emptyset$.

In this article we shall consider weak observables instead of observables replacing the condition $x(R) = 1$ by a weaker condition $m\big(x(R)\big) = 1$.

In Section 2 we construct a weak observable from a function $F\colon R \to M$. Of course, we need a special property of M, so-called weak σ-distributivity ([3], [12]). Recall that in the case that M is a vector lattice, the weak σ-distributivity is a necessary and sufficient condition for the extendability of any M-valued measure from a ring to the generated σ-ring.

Using these results we construct a sum of two (independent) weak observables. We prove the commutative and the associative law.

Finally in Section 4 we define upper and lower limits of a sequence of weak observables as a weak observable.

2. Construction of a weak observable

We shall assume that M is an MV-algebra satisfying the following two conditions:

(i) M is σ-complete, i.e., every sequence of elements in M has the supremum and the infimum.

(ii) M is weakly σ-distributive, i.e., for every bounded double sequence $(a_{ij})_{i,j}$ such that $a_{ij} \searrow 0$ $(j \to \infty,\ i = 1, 2, \dots)$ there is

$$\bigwedge_{\varphi \in N^N} \bigvee_{i=1}^{\infty} a_{i\varphi(i)} = 0\,.$$

By a weak observable (with respect to a state m) we mean a mapping $x\colon \mathcal{B}(R) \to M$ satisfying the following properties:

(i) $m\big(x(R)\big) = 1$

(ii) If $A, B \in \mathcal{B}(R)$, $A \cap B = \emptyset$, then $x(A \cup B) = x(A) + x(B)$.

(iii) If $A_m \in \mathcal{B}(R)$ $(n = 1, 2, \dots)$, $A_n \nearrow A$, then $x(A_n) \nearrow x(A)$.

Theorem 1. *Let $F\colon R \to M$ be a mapping satisfying the following conditions:*

(i) *If $t_1 < t_2$, then $F(t_1) \leq F(t_2)$.*

(ii) *If $t_n \nearrow t$, then $F(t_n) \nearrow f(t)$.*

(iii) $\lim_{t\to\infty} m\big(F(t)\big) = 1$.

(iv) $\lim_{t\to-\infty} m\big(F(t)\big) = 0$.

Then there exists a weak observable $x\colon \mathcal{B}(R) \to M$ such that $m\big(x((-\infty, t))\big) = m\big(F(t)\big)$ for every $t \in R$.

Proof. We shall use here two results from [8] concerning M-valued measures. An M-valued measure is a mapping $\mu\colon R \to M$ such that $\mu(\emptyset) = 0$, μ is additive and continuous. By a corollary of the Alexandrov theorem (Theorem 2 in [8]) from the conditions (i) and (ii) it follows the existence of a measure $\lambda_F\colon \mathcal{R} \to M$ defined on the ring $\mathcal{R}$ generated by all intervals of the form $< a, b >$ such that

$$\lambda_F(< a, b)) = F(b) - F(a)$$

for every $a, b \in \mathcal{R}$, $a < b$. Now the measure extension theorem ([8], Theorem 3) is applicable: There exists a measure (denote it by x) defined on the σ-algebra $\sigma(\mathcal{R}) = \mathcal{B}(R)$ with values in M and extending λ_F. We shall prove that x is a weak observable. Indeed, $x\colon \mathcal{B}(R) \to M$ is additive and continuous.

Moreover,

$$\begin{aligned} m\big(x(R)\big) &= m\Big(x\Big(\bigcup_{n=1}^{\infty} < -n, n)\Big)\Big) = m\Big(\bigvee_{n=1}^{\infty} x(< -n, n))\Big) = \\ &= m\Big(\bigvee_{n=1}^{\infty} \lambda_F(< -n, n))\Big) = m\Big(\bigvee_{n=1}^{\infty} \big(F(n) - F(-n)\big)\Big) = \\ &= m\Big(\Big(\bigvee_{n=1}^{\infty} F(n)\Big) - \Big(\bigwedge_{n=1}^{\infty} F(-n)\Big)\Big) = \\ &= m\Big(\bigvee_{n=1}^{\infty} F(n)\Big) - m\Big(\bigwedge_{n=1}^{\infty} F(-n)\Big) = \\ &= \lim_{n\to\infty} m\big(F(n)\big) - \lim_{n\to\infty} F(-n) = 1 - 0 = 1\,. \end{aligned}$$

Finally

$$\begin{aligned} m\Big(x((-\infty, t))\Big) &= m\Big(\Big(\bigcup_{n=1}^{\infty} < t - n, t\Big)\Big) = m\Big(\bigvee_{n=1}^{\infty} F(< t - n, t))\Big) \\ &= m\big(F(t)\big) - \lim_{n\to\infty} m\big(F(t - n)\big) = m\big(F(t)\big)\,. \end{aligned}$$

3. Sum of weak observables

Of course, the classical case seems to be the most illustrative. If $(\Omega, \mathcal{S}, P)$ is a probability space, then every random variable $\xi\colon \Omega \to R$, is an $\mathcal{S}$-measurable function. Putting $x(A) = \xi^{-1}(A)$, we obtain an observable $x\colon \mathcal{B}(R) \to \mathcal{S}$, $\mathcal{S}$ being an MV algebra with $A \oplus B = A \cup B$ and $A \odot B = A \cap B$. Hence in the case of the sum of observables we can be inspired by the sum $\xi + \eta$ of random variables ξ, η:

$$\begin{aligned} &(\xi + \eta)^{-1}\big((-\infty, t)\big) = \\ &\qquad = \bigcup_{n=1}^{\infty} \bigcup_{(i,j)\in\alpha_n} \xi^{-1}\Big(< \frac{i-1}{2^n}, \frac{i}{2^n})\Big) \cap \eta^{-1}\Big(< \frac{j-1}{2^n}, \frac{j}{2^n})\Big)\,, \end{aligned}$$

where $(i, j) \in \alpha_n$ if and only if $\frac{i}{2^n} + \frac{j}{2^n} < t$. In the MV algebra M valued case we shall substitute $\bigcup_{n=1}^{\infty}$ by $\bigvee_{n=1}^{\infty}$, $\bigcup_{(i,j)\in\alpha_n}$ by $\sum_{(i,j)\in\alpha_n}$ and $\cap$ by the operation $\odot$.

We have restricted our considerations to the case of operation $\odot$. The second restriction will concern the case of $\odot$-independent observables. Two weak observables x, y are called $\odot$-independent, if

$$m\big(x(A) \odot y(B)\big) = m\big(x(A)\big) \cdot m\big(y(B)\big)$$

for every $A, B \in \mathcal{B}(R)$. More generally, weak observables $x_1, \dots, x_n$ are $\odot$-independent, if

$$m\big(x_1(A_1) \odot x_2(A_2) \odot \cdots \odot x_n(A_n)\big) = m\big(x_1(A_1)\big) \cdot m\big(x_1(A_1)\big) \cdot \ldots \cdot m\big(x_1(A_n)\big)$$

for every $A_1, A_2, \dots, A_n \in \mathcal{B}(R)$.

If $m\colon M \to < 0,1 >$ is a state and $x\colon \mathcal{B}(R) \to M$ is a weak observable, then the composite mapping $m_x = m \circ x\colon \mathcal{B}(R) \to < 0,1 >$ defined by the formula $m_x(A) = m\big(x(A)\big)$, is a probability measure. Evidently

$$\begin{aligned} m\big(x(A)\big) \cdot m\big(y(B)\big) &= m_x(A) \cdot m_y(B) = \\ &= m_x \times m_y(A \times B)\,, \end{aligned}$$

where $m_x \times m_y$ is the product of the measures m_x, m_y. Therefore $\odot$-independence of x, y can be formulated by the equality

$$m\big(x(A) \odot y(B)\big) = m_x \times m_y(A \times B),\ A, B \in \mathcal{B}(R)\,.$$

Put $A_i^n = \big\langle \frac{i-1}{2^n}, \frac{i}{2^n}\big)$, $i \in Z$, $n \in N$. For a given $t \in R$ denote by $\alpha_n(t)$ the set of all pairs $(i,j) \in N \times N$ such that the corresponding rectangle $A_i^n \times A_j^n$ is a part of the semiplane $\big\{(u,v);\ u + v < t\big\}$, i.e.

$$\alpha_n(t) = \big\{(i,j);\ i + j < 2^n t\big\}\,.$$

We should want to work with the sums

$$\gamma_n(t) = \sum_{(i,j)\in\alpha_n(t)} x(A_i^n) \odot y(A_j^n)\,.$$

Of course, since the distributive law for the operations $\oplus$ and $\odot$ need not hold, we are not able to prove that $\gamma_n(t) \le \gamma_{n+1}(t)$. Therefore instead of $\gamma_n(t)$ we define first

$$\Gamma_1(t) = \sum_{(i,j)\in\alpha_1(t)} x(A_i^n) \odot y(A_j^n)$$

and then we add only those $x(A_i^2) \odot y(A_j^2)$ for which $A_i^2 \times A_j^2$ is not a part of a square $A_k^1 \times A_\ell^1$ from the preceding step, hence we put

$$\begin{aligned} \beta_2(t) &= \alpha_2(t) \setminus \big\{(i,j);\ \exists (k,\ell) \in \alpha_1(t),\ A_i^2 \times A_j^2 \subset A_k^1 \times A_\ell\big\} \\ \Gamma_2(t) &= \Gamma_1(t) + \sum_{(i,j)\in\beta_2(t)} x(A_i^2) \odot y(A_j^2)\,, \end{aligned}$$

etc. So we shall consider

$$\begin{aligned} &\beta_1(t) = \alpha_1(t)\,, \\ &\beta_n(t) = \alpha_n(t) \setminus \big\{(i,j);\ \exists m < n,\ \exists (k,\ell) \in \alpha_m(t),\ A_i^n \times A_j^n \subset A_k^m \times A_l^m\big\}\,, \\ &\Gamma_n(t) = \sum_{m=1}^{n} \sum_{(i,j)\in\beta_m(t)} x(A_i^n) \odot y(A_j^n)\,. \end{aligned}$$

Theorem 2. *Let M be a σ-complete MV algebra, $x, y\colon \mathcal{B}(R) \to M$ be $\odot$-independent weak observables. Define $\Gamma_n(t)$ as above and*

$$F(t) = \bigvee_{n=1}^{\infty} \sum_{(i,j)\in\beta_n} x(A_i^n) \odot y(A_j^n)\,.$$

Then F satisfies the assumptions (i) – (iv) of Theorem 1.

Proof. First note that a semidistributive law holds:

$$(a+b)\odot c \geq (a\odot c) + (b\odot c)\,.$$

Indeed, by the Mundici theorem

$$\begin{aligned}(a\odot c)+(b\odot c) &= (a+c-u)\vee 0 + (b+c-u)\vee 0 =\\ &= \big(a+c-u+(b+c-u)\vee 0\big)\vee(b+c-u)\vee 0 =\\ &= (a+b+c-u+c-u)\vee(a+c-u)\vee(b+c-u)\vee 0 \leq\\ &\leq (a+b+c-u)\vee 0 = (a+b)\odot c\,.\end{aligned}$$

Let $t < s$. By the semidistributive law $\Gamma_n(t) \leq \Gamma_n(s)$ $(n = 1, 2, \dots)$, hence $F(t) \leq F(s)$.

Assume $t_k \nearrow t$. Denote $S = \sup\big\{i+j;\ (i,j)\in\beta_n(t)\big\}$. It is easy to see that S is an integer, $S < 2^n t$. Therefore there exists k such that $S < 2^n t_k$. Hence to every n there is k such that $\beta_n(t) \subset \beta_n(t_k)$. Denote

$$F(t) = \bigvee_n a_n,\ \ F(t_k) = \bigvee_n b_{n,k}\,.$$

Evidently $F(t_k) \leq F(t)$, hence $\bigvee_k F(t_k) \leq F(t)$. On the other hand, we have proved that to every n there is k such that $\beta_n(t)\subset\beta_n(t_k)$ hence $a_n \leq b_{n,k}$. Therefore

$$\begin{aligned}&a_n \leq b_{n,k} \leq \bigvee_n b_{n,k} = F(t_k) \leq \bigvee_k F(t_k)\,,\\ &F(t) = \bigvee_n a_n \leq \bigvee_k F(t_k)\,.\end{aligned}$$

We have proved (i) and (ii). For to prove (iii) and (iv), we first prove

$$(*)\qquad m\big(F(t)\big) = m_x \times m_y\big(\{(u,v);\ u+v<t\}\big)$$

Indeed

$$\begin{aligned}m\big(F(t)\big) &= \lim_{n\to\infty} \sum_{(i,j)\in\alpha_n} m\Big(x\big(<\frac{i-1}{2^n},\frac{i}{2^n})\big)\odot y\big(<\frac{j-1}{2^n},\frac{j}{2^n})\big)\Big) =\\ &= \lim_{n\to\infty}\sum_{m=1}^{n}\sum_{(i,j)\in\beta_m(t)} m_x\times m_y\Big(<\frac{i-1}{2^n},\frac{i}{2^n})\times<\frac{j-1}{2^n}\frac{j}{2^n})\Big) =\\ &= m_x\times m_y\Big(\bigcup_{n=1}^{\infty}\bigcup_{(i,j)\in\alpha_n(t)}\langle\frac{i-1}{2^n},\frac{i}{2^n})\times<\frac{j-1}{2^n},\frac{j}{2^n})\Big) =\\ &= m_x\times m_y\big(\{(u,v);\ u+v<t\}\big)\,.\end{aligned}$$

Since $m_x \times m_y$ is a probability measure, (iii) and (iv) follows by $(*)$.

By Theorem 1 there is a weak observable z such that $m\big(z((-\infty,t))\big) = m\big(F(t)\big)$ for every $t \in R$. This function z will be called a sum of weak observables x, y and will be denote by $x+y$.

Theorem 3. *Let M be a σ-complete weakly σ-distributive MV algebra. Let $x, y, z\colon \mathcal{B}(R) \to M$ be $\odot$-independent weak (with respect to m) observables. Then the following properties are satisfied:*

(i) $m_{x+y} = m_{y+x}$ *(commutative law)*
(ii) $m_{(x+y)+z} = m_{x+(y+z)}$ *(associative law).*

Proof. By the definition of $x+y$ and the equality $(*)$ (see Proof of Theorem 2)

$$\begin{aligned} m_{x+y}\big((-\infty,t)\big) &= m\big((x+y)((-\infty,t))\big) = m\big(F(t)\big) = \\ &= (m_x \times m_y)\big(g^{-1}((-\infty,t))\big)\,, \end{aligned}$$

where $g\colon R\times R \to R$ is defined by $g(u,v) = u+v$. Therefore

$$(**) \qquad m_{x+y} = (m_z \times m_y)\circ g^{-1}\,.$$

By $(**)$ the commutative law follows. Further

$$\begin{aligned} m_{(x+y)+z} &= (m_{x+y} \times m_z)\circ g^{-1} = \\ &= \big(((m_x\times m_y)\circ g^{-1})\times m_z\big)\circ g^{-1}\,, \end{aligned}$$

hence by the Fubini theorem

$$\begin{aligned} m_{(x+y)+z}\big((-\infty,t)\big) &= (m_{x+y}\times m_z)\big(\{(u,v); u+v<t\}\big) = \\ &= \int_R m_{x+y}\big((-\infty,t-u)\big)dm_z(u) = \\ &= \int_R m_x\times m_y\big(\{(w,v); v+w<t-u\}\big)dm_z(u) = \\ &= \int_R\Big(\int_R m_x\big((-\infty,t-u-v)\big)dm_z(v)\Big)dm_z(u) = \\ &= m_x\times m_y\times m_z\big(\{(w,v,u); u+v+w<t\}\big)\,. \end{aligned}$$

By the equality

$$m_{(x+y)+z}\big((-\infty,t)\big) = m_x\times m_y\times m_z\big(\{(u,v,w); u+v+w<t\}\big)$$

the equality

$$m_{(x+y)+z}\big((-\infty,t)\big) = m_{x+(y+z)}\big((-\infty,t)\big)$$

follows, hence

$$m_{(x+y)+z} = m_{x+(y+z)}\,.$$

As a secondary product of our considerations we obtained the following formulas:

$$m_{x+y}\big((-\infty,t)\big) = m_x\times m_y\big(\{(u,v); u+v<t\}\big)$$

and

$$m_{x+y+z}\big((-\infty,t)\big) = m_x\times m_y\times m_z\big(\{(u,v,w);\ u+v+w<t\}\big)\,.$$

The generalization for the case of n independent observables is evident.

4. Upper and lower limit

We shall consider a sequence $(x_n)_n$ of weak observables and construct $\limsup\limits_{n\to\infty} x_n$ as a weak observable. We have at least two ways for a definition. The first was realized in [9] and it is motivated by the relation

$$\limsup_{n\to\infty} \xi_n(\omega) < t \Leftrightarrow \omega \in \bigcup_{p=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \xi_n^{-1}\Big(\big(-\infty, t-\frac{1}{p}\big)\Big).$$

In the paper we shall follow the second way:

$$\limsup_{n\to\infty} \xi_n(\omega) = \inf_{k\geq 1} \sup_{n\geq k} \xi_n(\omega).$$

Hence, first we shall define supremum and infimum of a sequence of observables and then $\limsup_{n\to\infty} x_n$ by a composition.

If ξ and η are two random variables, then

$$\big(\max(\xi,\eta)\big)^{-1}(-\infty,t) = \xi^{-1}\big((-\infty,t)\big) \cap \xi^{-1}\big((-\infty,t)\big).$$

This equality leads to the following concept.

Theorem 4. *Let x, y be weak observables. Put $F(t) = x\big((-\infty,t)\big) \wedge y\big((-\infty,t)\big)$ for every $t \in R$. Then the mapping $F\colon R \to M$ satisfies the conditions (i) — (iv) from Theorem 1, hence there is an observable $z\colon \mathcal{B}(R) \to M$ (denote it by $\max(x,y)$) such that $m\big(z((-\infty,t))\big) = m\big(F(t)\big)$, $t \in R$.*

Proof. Evidently, F is non-decreasing. If $t_i \nearrow t$ then $x\big((-\infty,t_i)\big) \nearrow x\big((-\infty,t)\big)$, $y\big((-\infty,t_i)\big) \nearrow y\big((-\infty,t)\big)$, hence by the σ-continuity of M

$$F(t_i) = x\big((-\infty,t_i)\big) \wedge y\big((-\infty,t_i)\big) \nearrow x\big((-\infty,t)\big) \wedge y\big((-\infty,t)\big) = F(t).$$

Further

$$O \leq m\big(F(t)\big) \leq m\big(x((-\infty,t))\big),$$

hence

$$O \leq \lim_{t\to-\infty} m\big(F(t)\big) \leq \lim_{t\to-\infty} m\big(x((-\infty,t))\big) = 0.$$

Finally,

$$\begin{aligned} 1 \geq m\big(F(t)\big) &= m\big(x((-\infty,t)) \wedge y((-\infty,t))\big) = \\ &= m\big(x((-\infty,t))\big) + m\big(y((-\infty,t))\big) - m\big(x((-\infty,t)) \vee y((-\infty,t))\big) \\ &\geq m\big(x((-\infty,t))\big) + m\big(y((-\infty,t))\big) - 1, \end{aligned}$$

hence

$$\begin{aligned} 1 &\geq \lim_{t\to\infty} m\big(F(t)\big) \geq \\ &\geq \lim_{t\to\infty} m\big(x((-\infty,t))\big) + \lim_{t\to\infty} m\big(y((-\infty,t))\big) - 1 = 1. \end{aligned}$$

Since we have defined maximum of two weak observables, it suffices to define the supremum only for a non-decreasing sequences $(x_n)_n$ of observables. Recall that $\xi_n \leq \xi_{n+1}$ if and only if $\xi_{n+1}^{-1}\big((-\infty,t)\big) \subset \xi_n^{-1}\big((-\infty,t)\big)$ for every $t \in R$. Of course, we must assume that the corresponding supremum is finite a.e. In the classical case it can be expressed in the form $P\Big(\bigcup_{t=1}^{\infty} \bigcap_{n=1}^{\infty} \xi_n^{-1}\big((-\infty,t)\big)\Big) = 1$.

Theorem 5. *Let $(x_n)_n$ be a sequence of weak observables, $x_n\big((-\infty,t)\big) \geq x_{n+1}\big((-\infty,t)\big)$ $(t \in R$ $n = 1,2,\dots)$ and assume $m\Big(\bigvee_{t=1}^{\infty}\bigwedge_{n=1}^{\infty} x_n\big((-\infty,t)\big)\Big) = 1$. Put $F(t) = \bigvee_{p=1}^{\infty}\bigwedge_{n=1}^{\infty} x_n\big((-\infty,t-\frac{1}{p})\big)$ for every $t \in R$.*

Then the mapping $F\colon R \to M$ satisfies the conditions (i)–(iv) from Theorem 1, hence there is an observable $z\colon \mathcal{B}(R) \to M$ (denote it by $\sup x_n$) such that $m\big(z((-\infty,t))\big) = m\big(F(t)\big)$, $t \in R$.

Proof. Evidently F is non-decreasing. Let $t_i, t \in R$, $t_i \nearrow t$. Then to every $p \in N$ there is t_i and $q \in N$ such that $t - \frac{1}{p} < t_i - \frac{1}{q}$. Therefore

$$\bigwedge_n x_n\Big(\big(-\infty,t-\frac{1}{p}\big)\Big) \leq \bigwedge_n x_n\Big(\big(-\infty,t_i-\frac{1}{q}\big)\Big) \leq$$
$$\leq \bigvee_q \bigwedge_n x_n\Big(\big(-\infty,t_i-\frac{1}{q}\big)\Big) = F(t_i) \leq \bigvee_i F(t_i)\,.$$

Since the relation holds for every $p \in N$, we obtain

$$F(t) = \bigvee_p \bigwedge_n x_n\Big(\big(-\infty,t-\frac{1}{p}\big)\Big) \leq \bigvee_i F(t_i)\,.$$

Since the opposite inequality is obvious, $F(t_i) \nearrow F(t)$.

To every $\varepsilon > 0$ there exists such $t_0 \in R$ that $m\big(x_1((-\infty,t_0))\big) < \varepsilon$. Therefore

$$m\big(F(t_0)\big) = \lim_{p\to\infty}\lim_{n\to\infty} m\Big(x_n\big((-\infty,t_0-\frac{1}{p})\big)\Big) \leq$$
$$\leq m\big(x_1((-\infty,t_0))\big) < \varepsilon\,.$$

Since $m\big(F(t)\big) < \varepsilon$ for all $t \leq t_0$, we have $\lim_{t\to-\infty} m\big(F(t)\big) = 0$. Finally,

$$m\Big(\bigvee_{p=1}^{\infty}\bigwedge_{n=1}^{\infty} x_n\Big(\big(-\infty,t-\frac{1}{p}\big)\Big)\Big) \geq m\Big(\bigwedge_{n=1}^{\infty} x_n\big((-\infty,t-1)\big)\Big)\,.$$

Therefore

$$1 \geq \lim_{t\to\infty} m\big(F(t)\big) = \lim_{t\to\infty} m\big(F(t+1)\big) \geq$$
$$\geq \lim_{t\to\infty} m\Big(\bigwedge_{n=1}^{\infty} x_n\big((-\infty,t)\big)\Big) = m\Big(\bigvee_{t=1}^{\infty}\bigwedge_{n=1}^{\infty} x_n\big((-\infty,t)\big)\Big) = 1\,.$$

Theorem 6. *Let $(x_n)_n$ be a sequence of weak observables, $x_n\big((-\infty,t)\big) \leq x_{n+1}\big((-\infty,t)\big)$ $(t \in R,$ $n = 1,2,\dots)$ and assume $m\Big(\bigvee_{t=1}^{\infty}\bigwedge_{n=1}^{\infty} x_n\big(<-t,\infty)\big)\Big) = 1$. Put $F(t) = \bigvee_{p=1}^{\infty}\bigvee_{n=1}^{\infty} x_n\big((-\infty,t-\frac{1}{p})\big)$. Then the mapping $F\colon R \to M$ satisfies* *the conditions (i)–(iv) from Theorem 1, hence there is an observable* $z\colon \mathcal{B}(R) \to M$ *(denote it by* $\inf x_n$*) such that* $m\big(z((-\infty,t))\big) = m\big(F(t)\big)$*,* $t \in R$*.*

Proof. Evidently, F is non-decreasing. If $t_i \nearrow t$, then

$$\begin{aligned}\bigvee_i F(t_i) &= \bigvee_i \bigvee_p \bigvee_n x_n\Big(\big(-\infty, t_i - \frac{1}{p}\big)\Big) = \\ &= \bigvee_p \bigvee_n \bigvee_i x_n\Big(\big(-\infty, t_i - \frac{1}{p}\big)\Big) = \\ &= \bigvee_p \bigvee_n x_n\Big(\big(-\infty, t - \frac{1}{p}\big)\Big) = F(t)\,.\end{aligned}$$

Further $F(t) \geq x_1\big((-\infty, t-1)\big)$, hence

$$1 \geq \lim_{t\to\infty} m\big(F(t)\big) \geq \lim_{t\to\infty} x_1\big((-\infty,t-1)\big) = 1\,.$$

Finally, by an assumption $m\Big(\bigwedge_t \bigvee_n x_n\big((-\infty,-t)\big)\Big) = 0$ holds, hence

$$\begin{aligned}\lim_{t\to-\infty} m\big(F(t)\big) &= \lim_{t\to-\infty} m\Big(\bigvee_p \bigvee_n x_n\big(\big(-\infty, t-\frac{1}{p}\big)\big)\Big) \leq \\ &\leq \lim_{t\to-\infty} m\Big(\bigvee_n x_n\big((-\infty,t)\big)\Big) = \\ &= m\Big(\bigwedge_{t=1}^{\infty} \bigvee_n x_n\big((-\infty,-t)\big)\Big) = 0\,.\end{aligned}$$

Finally, we shall define $\limsup_{n\to\infty} x_n$. Of course, we shall have to assume both assumptions of Theorem 5 and Theorem 6 (the finiteness of sup and inf):

$$m\Big(\bigvee_{t=1}^{\infty} \bigwedge_{n=1}^{\infty} x_n\big((-\infty,t)\big)\Big) = 1,\ \ m\Big(\bigwedge_{t=1}^{\infty} \bigvee_{n=1}^{\infty} x_n\big((-\infty,-t)\big)\Big) = 0\,.$$

These assumptions are satisfied if $(x_n)_n$ is bounded in the following sense: There are weak observables $y, z\colon \mathcal{B}(R) \to M$ such that

$$y\big((-\infty,t)\big) \leq x_n\big((-\infty,t)\big) \leq z\big((-\infty,t)\big)$$

for every $t \in R$ and every $n \in \mathbb{N}$. Indeed, in this case

$$\begin{aligned}m\Big(\bigvee_{t=1}^{\infty} \bigwedge_{n=1}^{\infty} x_n\big((-\infty,t)\big)\Big) &\geq m\Big(\bigvee_{t=1}^{\infty} y\big((-\infty,t)\big)\Big) = \\ &= \lim_{t\to\infty} m\big(y((-\infty,t))\big) = 1\,,\end{aligned}$$

$$\begin{aligned}m\Big(\bigwedge_{t=1}^{\infty} \bigvee_{n=1}^{\infty} x_n\big((-\infty,-t)\big)\Big) &\leq m\Big(\bigwedge_{t=1}^{\infty} z\big((-\infty,-t)\big)\Big) = \\ &= \lim_{t\to-\infty} m\big(z((-\infty,t))\big) = 0\,.\end{aligned}$$

Theorem 7. *Let* $(x_n)_n$ *be a sequence of weak observables such that* $m\Big(\bigvee_{t=1}^{\infty}\bigwedge_{n=1}^{\infty} x_n\big((-\infty,t)\big)\Big) = 1$, $m\Big(\bigwedge_{t=1}^{\infty}\bigvee_{n=1}^{\infty} x_n\big((-\infty,t)\big)\Big) = 0$. *Put* $F(t) = \bigvee_{p=1}^{\infty}\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty} x_n\big((-\infty,t-\frac{1}{p})\big)$. *Then there is an observable* z *(denote it by* $\limsup x_n$*) such that* $m\big(F(t)\big) = m\big(z((-\infty,t))\big)$ *for every* $t \in R$.

Proof. We shall be inspired by the classical formula $\limsup_{n\to\infty}\xi_n = \inf_{k\geq 1}\sup_{n\geq k}\xi_n = \inf_{k\geq 1}\sup_{n\geq k}\max_{k\leq i\leq n}\xi_i$. Therefore put

$$y_n = \max(x_n, x_{n+1}, \dots, x_n)\,,$$
$$z_k = \sup_{n\geq k} y_n\,,$$
$$z = \inf_{k\geq 1} z_k\,.$$

Then

$$\begin{aligned} z\big((-\infty,t)\big) &= \bigvee_{p=1}^{\infty}\bigvee_{k=1}^{\infty} z_k\Big(\big(-\infty,t-\frac{1}{p}\big)\Big) = \\ &= \bigvee_{p=1}^{\infty}\bigvee_{k=1}^{\infty}\bigvee_{q=1}^{\infty}\bigwedge_{n=k}^{\infty} y_n\Big(\big(-\infty,t-\frac{1}{p}-\frac{1}{q}\big)\Big) = \\ &= \bigvee_{r=1}^{\infty}\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty}\bigwedge_{i=k}^{n} x_i\Big(\big(-\infty,t-\frac{1}{r}\big)\Big) = \\ &= \bigvee_{r=1}^{\infty}\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty} x_n\Big(\big(-\infty,t-\frac{1}{r}\big)\Big)\,. \end{aligned}$$

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