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Probability theory on some ordered structures

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PROBABILITY THEORY ON SOME ORDERED STRUCTURES

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Abstract. The weak law of large numbers and the central limit theorems are proved in a type of posets. This type includes MV algebras, D-posets of fuzzy sets, F-quantum spaces and orthomodular posets.

Introduction

There are various mathematical models of quantum mechanical systems. The most known model is based on orthomodular posets (see e.g. [9]). Recently fuzzy sets have been used (e.g. [10], [11]) and then some more general structures, e.g. D-posets ([6]) and MV algebras ([8], [4]).

For to building probability theory for observables in structures mentioned above there is necessary to construct a calculus with observables. It was done for orthomodular posets ([9], [1]), D-poset of fuzzy sets ([7]), F-quantum spaces ([2], [3]) and recently for MV algebras ([13], [14]), of course, only in a special case. Namely, if x and y are independent observables in an MV algebra M, $x, y\colon \mathcal{B}(R) \to M$, then $x + y$ is an observable satisfying the following equality

$$(*) \qquad m\big((x+y)((-\infty,t))\big) = m_x \times m_y\big(\{(u,v);\ u+v<t\}\big)$$

holding for every $t \in R$. Here $m\colon M \to \langle 0,1\rangle$ is a given state, $m_x \times m_y\colon \mathcal{B}(R^2) \to \langle 0,1\rangle$ is the product of measures m_x, m_y defined on $\mathcal{B}(R)$ by the equalities

$$m_x(A) = m\big(x(A)\big),\ m_y(B) = m\big(y(B)\big)\,.$$

In [5] the weak law of large numbers has been proved for a sequence of independent observables with values in an MV algebra. Of course, the proof is based on the equality $(*)$ only. Therefore we try to prove some probability assertions in a general ordered structure by the help of $(*)$ only. As special cases we obtain the law of large numbers and the central limit theorem in all above mentioned structures. They are new in the case of an MV algebra.

Recall that in the classical case of a probability space $(\Omega, \mathcal{S}, P)$ and any random variables $\xi, \eta\colon \Omega \to R$ we have

$$\begin{aligned} &P\big(\{\omega \in \Omega,\ \xi(\omega)+\eta(\omega) \in D\}\big) = \\ &\qquad = P_T\big(\{(u,v) \in R^2;\ u+v \in D\}\big)\,. \end{aligned}$$

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for any $D \in \mathcal{B}(R)$, where $T = (\xi, \eta)\colon \Omega \to R^2$, $P_T\colon \mathcal{B}(R^2) \to \langle 0,1\rangle$ is defined by the formula

$$\begin{aligned} P_T(A) &= P\big(T^{-1}(A)\big) = \\ &= P\big(\{\omega \in \Omega;\ \big(\xi(\omega),\, \eta(\omega)\big) \in A\}\big)\,. \end{aligned}$$

If ξ, η are independent, then $P_T = P_\xi \times P_\eta$, where $P_\xi, P_\eta\colon \mathcal{B}(R) \to \langle 0,1\rangle$ are the probability distributions defined by $P_\xi(C) = P\big(\xi^{-1}(C)\big)$, $P_\eta(D) = P\big(\eta^{-1}(D)\big)$. Therefore

$$P\big((\xi+\eta)^{-1}((-\infty,t))\big) = P_\xi \times P_\eta\big(\{(u,v); u+v<t\}\big)\,.$$

This equality justifies $(*)$.

States and observables

Definition 1. There is given a partially ordered set M with the least element 0 and the greatest element 1 and with a partial commutative binary operation $+$.

Definition 2. There is given a state $m\colon M \to \langle 0,1\rangle$, i.e. a mapping satisfying the following conditions:

(i) $m(1) = 1$, $m(0) = 0$.
(ii) If $a, b, c \in M$, $b + c$ is defined and $a = b + c$, then $m(a) = m(b) + m(c)$.
(iii) If $(a_n)_n \in M$, $a \in M$ and $a_n \nearrow a$, then $m(a_n) \nearrow m(a)$.

Definition 3. A mapping $x\colon \mathcal{B}(R) \to M$ is called a weak observable, if the following conditions are satisfied:

(i) $m\big(x(R)\big) = 1$.
(ii) If $A, B \in \mathcal{B}(R)$, $A \cap B = \emptyset$, then $x(A) + x(B)$ exists and $x(A \cup B) = x(A) + x(B)$.
(iii) If $A_n \in \mathcal{B}(R)$, $n = 1, 2, \dots$ and $A_n \nearrow A$, then $x(A_n) \nearrow x(A)$.

Proposition 1. *If x is a weak observable and $m_x\colon \mathcal{B}(R) \to \langle 0,1\rangle$ is defined by the formula $m_x = m \circ x$, then m_x is a probability measure.*

Proof. Evidently m_x is continuous from bellow.
Let $A, B \in \mathcal{B}(R)$, $A \cap B = \emptyset$. Then $x(A) + x(B)$ exists and $x(A \cup B) = x(A) + x(B)$. Therefore

$$\begin{aligned} m \circ x(A \cup B) &= m\big(x(A \cup B)\big) = m\big(x(A)\big) + m\big(x(B)\big) \\ &= m_x(A) + m_x(B)\,, \end{aligned}$$

Example 1. Let $(M, \le, \perp, 0, 1)$ be an orthomodular poset. We define $a + b = a \vee b$, whenever $a \le b^\perp$. If $m\colon M \to \langle 0,1\rangle$ is a state and $x\colon \mathcal{B}(R) \to M$ an observable, then all assumptions stated above are satisfied. A very well known example of an orthomodular poset is a the family of all linear subspaces of a Hilbert space, where $\le$ is the inclusion and $a^\perp$ is the orthogonal complement of a subspace a.

Example 2. Let $(M, \leq, \setminus, 0, 1)$ be a D-poset of fuzzy sets, e.g. the set of all mappings $f\colon \Omega \to \langle 0,1\rangle$, $\leq$ is the usual ordering, 0 the zero function 1 the unit function and $f \setminus g$ is the difference $f - g$ defined only in the case that $g \leq f$. We define $f + g$ only in the case that $f \geq 1 \setminus g$ by the formula $f + g = 1 \setminus \big[(1 \setminus g) \setminus f\big]$. Of course, $f + g$ is the usual sum of functions in the special case stated above. Also in this case $+$ is only a partial binary operation.

Example 3. Let $(M, \leq, \vee, \perp, 1)$ be an F-quantum space, e.g. the set of all mappings $f\colon \Omega \to \langle 0,1\rangle$, $\leq$ is the usual ordering, $f^\perp = 1 - f$, $f \vee g = \max(f, g)$. We define $f + g$ if and only if $f \leq g^\perp$ and in this case $f + g = \max(f, g)$.

Example 4. Let $(M, \oplus, \perp, 0, 1)$ be an MV algebra, e.g. the set of all mappings $f\colon \Omega \to \langle 0,1\rangle$, $f \oplus g = \min(f + g, 1)$, $f^\perp = 1 - f$. In this case $+$ is a binary operation and $f + g = f \oplus g$ (of course $f + g$ is the usual sum, if $f \leq g^\perp$.)

Weak independence

Definition 4. Observables $x_1, \dots, x_n\colon \mathcal{B}(R) \to M$ are called to be weakly independent, if there is an observable $y_n\colon \mathcal{B}(R) \to M$ such that

$$m \circ y_n = (m_{x_1} \times \dots \times m_{x_n}) \circ g_n^{-1},$$

where $g_n\colon R^n \to R$ is defined by the formula $g_n(u_1, \dots, u_n) = u_1 + \dots + u_n$.

Definition 5. Observables $x_1, \dots, x_n\colon \mathcal{B}(R) \to M$ are called to be strongly independent, if there is a mapping $h_n\colon \mathcal{B}(R^n) \to M$ satisfying the following properties:

(i) $m\big(h_n(R^n)\big) = 1$.

(ii) If $A, B \in \mathcal{B}(R^n)$, $A \cap B = \emptyset$, then $h_n(A \cup B) = h_n(A) + h_n(B)$.

(iii) If $A_i \in \mathcal{B}(R^n)$ $(i = 1, 2, \dots)$, $A_n \nearrow A$, then $h_n(A_i) \nearrow h_n(A)$ $(i \to \infty)$.

(iv) $m \circ h_n = m_{m_1} \times \dots \times m_{x_n}$.

Proposition 2. *If $x_1, \dots, x_n$ are strongly independent, then they are weakly independent, too.*

Proof. Put $y_n = h_n \circ g_n^{-1}$.

Now let us consider the examples mentioned above. In orthomodular posets the sum of observables was defined in [1]:

$$(x + y)\big((-\infty, t)\big) = \bigvee_{r \in Q} x\big((-\infty, r)\big) \wedge y\big((-\infty, t - r)\big)\,.$$

In D-posets of fuzzy sets there exists a joint observable of x, y ([7]), i.e. such a σ-homomorphism $h\colon \mathcal{B}(R^2) \to M$ that

$$h(A \times B) = x(A) \cdot x(B),\ \ A, B \in \mathcal{B}(R^2)$$

(the usual product of functions). The sum $x + y$ is then defined by the equality $x + y = h \circ g^{-1}$, where $g(u, v) = u + v$, independence is defined by $m\big(x(A) \cdot x(B)\big) = m\big(x(A)\big) \cdot m\big(x(B)\big)$, hence the independence leads to the strong independence.

Similarly in F-quantum spaces there is the joint observable ([2], [3]). Finally in MV algebras the sum $x+y$ has been defined in [13] by the equality

$$(x+y)\big((-\infty,t)\big) = \bigvee_n \sum_{(i,j)\in\alpha_n} x\Big(\big\langle \tfrac{i-1}{2^n}, \tfrac{i}{2^n}\big)\Big) \cdot y\Big(\big\langle \tfrac{j-1}{2^n}, \tfrac{j}{2^n}\big)\Big),$$

where $\alpha_n = \big\{(i,j) \in N \times N;\ i+j < 2^n t\big\}$ and $+$ and $\cdot$ are the corresponding MV-algebra operations. If x, y are independent (with respect to the operation $\cdot$, i.e. if $m\big(x(A)\cdot y(B)\big) = m\big(x(A)\big)m\big(y(B)\big)$), then they are weakly independent with respect to Definition 4, too.

Since $m_x\colon \mathcal{B}(R) \to \langle 0,1\rangle$ is a probability measure, we are able to define moments, particularly the mean value and the dispersion.

Definition 6. An observable $x\colon \mathcal{B}(R) \to M$ belongs to L^1 (by other words, it is integrable), if there exists $E(x) = \int_R t\,dm_x(t)$. It belongs to L^2 if there exists $\int_R t^2 dm_x(t)$.

Proposition 3. $L^2 \subset L^1$.

Proof. If follows by the Cauchy inequality:

$$\Big|\int_R t\cdot 1 dm_x(t)\Big| \le \int_R t^2 dm_x(t) \cdot \int_R 1 dm_x(t).$$

Definition 7. If $x\colon \mathcal{B}(R) \to M$ is a weak observable belonging to L^1, then we define

$$E(x) = \int_R t\,dm_x(t),$$

if, moreover, x belongs to L^2, then we define

$$\sigma^2(x) = \int_R \big(t - E(x)\big)^2 dm_x(t).$$

Proposition 4. *If $x, y\colon \mathcal{B}(R) \to M$ are weakly independent weak observables belonging to L^1, then $x+y \in L^1$ and*

$$E(x+y) = E(x) + E(y).$$

Proof. Put $g\colon R^2 \to R$, $g(u,v) = u+v$. Then

$$\begin{aligned} E(x)+E(y) &= \int_R u\,dm_x(u) + \int_R v\,dm_y(v) = \\ &= \int_R \Big[\int_R u\,dm_x(u)\Big] dm_y(v) + \int_R \Big[\int_R v\,dm_x(u)\Big] dm_y(v) = \\ &= \iint_{R^2} (u+v)\,dm_x \times m_y(u,v) = \iint_{R^2} g\,dm_x \times m_y. \end{aligned}$$

By the integral transformation formula we see that the integral

$$\int_R t dm_x \times m_y \circ g^{-1}(t)$$

exists and equals to the integral $\iint\limits_{R^2} g dm_x \times m_y$. Moreover $m_x \times m_y \circ g^{-1} = m_{x+y}$. Therefore $\int_R t dm_{x+y}(t)$ exists and

$$E(x+y) = \int_R t dm_{x+y}(t) = \iint\limits_{R^2} g dm_x \times m_y = E(x) + E(y)\,.$$

Proposition 5. *If $x, y\colon \mathcal{B}(R) \to M$ are weakly independent weak observables belonging to L^2, then $x + y \in L^2$ and*

$$\sigma^2(x+y) = \sigma^2(x) + \sigma^2(y)\,.$$

Proof. Put $g\colon G^2 \to R$, $g(u,v) = u + v$. Then

$$\begin{aligned}
\int_R t^2 dm_{x+y}(t) &= \int_R t^2 dm_x \times m_t \circ g^{-1}(t) = \iint\limits_{R^2} g dm_x \times m_y = \\
&= \iint\limits_{R_2} (u+v)^2 dm_x \times m_y(u,v) = \\
&= \int_R u^2 dm_x(u) + \int_R v^2 dm_y(v) + 2E(x)E(y)\,.
\end{aligned}$$

Therefore $x + y \in L^2$ and

$$\begin{aligned}
\sigma^2(x+y) &= \int_R t^2 dm_{x+y}(t) - E(x+y)^2 = \\
&= \int_R t^2 dm_{x+y}(t) - \big(E(x) + E(y)\big)^2 = \\
&= \int_R t^2 dm_x(u) - E(x)^2 + \int_R v^2 dm_y - E(y)^2 = \\
&= \sigma^2(x) + \sigma^2(y)\,.
\end{aligned}$$

Limit theorems

We shall work with a sequence $(x_n)_n$ of weak observables assuming that $(x_n)_n$ is weakly independent.

Definition 8. A sequence $(x_n)_n$ of weak observables is weakly independent, if $x_1, \dots, x_n$ are weakly independent for every n.

Therefore we can define the sum $x_1 + \dots + x_n$ as a weak observable y_n satisfying the equality

$$m_{y_n} = m_{x_1} \times \dots \times m_{x_n} \circ g_n^{-1},$$

i.e.

$$m\big(y_n((-\infty, t))\big) = m_{x_1} \times \dots \times m_{x_n}\big(\{(u_1, \dots, u_n);\ u_1 + \dots + u_n < t\}\big).$$

Of course, in limit theorems we need to use the arithmetical mean $\frac{1}{n}\sum_{i=1}^n x_i$. There is no problem to define it.

Definition 9. If $x\colon \mathcal{B}(R) \to M$ is a weak observable, $c, a \in R$, $c \neq 0$, then we define the mappings $cx\colon \mathcal{B}(R) \to M$ $(x-a)\colon \mathcal{B}(R) \to M$ by the formulas

$$(cx)(A) = x(c^{-1}A),\ (x-a)(A) = x(A+a),$$

where

$$c^{-1}A = \{u \in R;\ \exists v \in A,\ u = c^{-1}v\},\ A + c = \{u \in R;\ \exists v \in A,\ u = v + c\}.$$

Theorem 1. *Let P be the infinite product of measures $(m_{x_n})_n$ defined in R^N by the formula*

$$P\big(\{(t_n)_n \in R^N;\ (t_1, \dots, t_n) \in A\}\big) = m_{x_1} \times \dots \times m_{x_n}(A)$$

$\big(A \in \mathcal{B}(R^n),\ n \in N\big)$. Let $\xi_n\colon R^N \to R$ be defined by the formula $\xi_n\big((t_i)_{i=1}^\infty\big) = t_n$. Put $\zeta_n = \left(\frac{1}{n}\sum_{i=1}^n \xi_i\right) - a$. Let y_n by the observable defined by the formula $m \circ y_n = m_{x_1} \times \dots \times m_{x_n} \circ g_n^{-1}$, $z_n = \frac{1}{n}y_n - a = \left(\frac{1}{n}\sum_{i=1}^n x_i\right) - a$. Then

$$P \circ \zeta_n^{-1} = m \circ z_n\ (n = 1, 2, \dots)$$

i.e. the random variables $\left(\frac{1}{n}\sum_{i=1}^n \xi_i\right) - a$ have the same probability distributions as the weak observables $\left(\frac{1}{n}\sum_{i=1}^n x_i\right) - a$.

Proof. Let $\pi_n\colon R^n \to R^n$ be the projection defined by the equality $\pi_n\big((t_i)_{i=1}^\infty\big) = (t_1, \dots, t_n)$. Then P can be characterized by the formula

$$P \circ \pi_n^{-1} = m_{x_1} \times \dots \times m_{x_n}.$$

Put $\eta_n = \sum_{i=1}^n \xi_i = g_n \circ \pi_n$. Then

$$P \circ \eta_n^{-1} = P \circ \pi_n^{-1} \circ g_n^{-1} = m_{x_1} \times \dots \times m_{x_n} \circ g_n^{-1} = m \circ y_n.$$

If we put

$$\psi_n\colon R \to R,\ \psi_n(u) = \tfrac{1}{n}u - a,$$

we obtain

$$\zeta_n = \tfrac{1}{n}\eta_n - a = \psi_n \circ \eta_n \,.$$

Therefore

$$P \circ \zeta_n^{-1} = P \circ \eta_n^{-1} \circ \psi_n^{-1} = m \circ y_n \circ \psi_n^{-1}.$$

Of course

$$\begin{aligned} y_n\big(\psi_n^{-1}(A)\big) &= y_n\big(n(A+a)\big) = \big(\tfrac{1}{n}y_n\big)(A+a) = \\ &= \big((\tfrac{1}{n}y_n) - a\big)(A) = z_n(A) \end{aligned}$$

hence

$$P \circ \zeta_n^{-1} = m \circ z_n \,.$$

Lemma 1. *Let $(x_n)_n$ be a weakly independent sequence of weak observables. Let P be the infinite product of measures $m_{x_1}, m_{x_2}, \dots$ in the space R^N, $\xi_n \colon R^N \to R$ be defined by $\xi_n\big((t_i)_i\big) = t_n$. Then $(\xi_n)_n$ are independent random variables.*

Proof. The σ-algebra σ in R^N is generated by the family of all cylinders, i.e. sets of the form $\pi_n^{-1}(B)$, where $n \in N$, $B \in \mathcal{B}(R^n)$ and $\pi_n \colon R^N \to R^n$ is the projection, $\pi_n\big((t_i)_i\big) = (t_1, \dots, t_n)$. We must to prove that $\xi_n^{-1}(A) \in \sigma$ for every $A \in \mathcal{B}(R)$. But

$$\begin{aligned} \xi_n^{-1}(A) &= \big\{(t_i)_i \in R^N;\ \xi_n\big((t_i)_i\big) \in A\big\} = \\ &= \big\{(t_i)_i \in R^N;\ t_n \in A\big\} = \\ &= \pi_n^{-1}(R \times \cdots \times R \times A) \in \sigma \,. \end{aligned}$$

We have proved that $\xi_n \colon (R^N, \sigma, P) \to \big(R, \mathcal{B}(R)\big)$ is a random variable for every n. Now let $A_1, \dots, A_n \in \mathcal{B}(R)$. Then

$$\begin{aligned} &P\big(\xi_1^{-1}(A_1) \cap \xi_2^{-1}(A_2) \cap \cdots \cap \xi_n^{-1}(A_n)\big) = \\ &\quad = P\big(\{(t_i);\ t_1 \in A_1, t_2 \in A_2, \dots, t_n \in A_n\}\big) = \\ &\quad = m_{x_1} \times \cdots \times m_{x_n}(A_1 \times A_2 \times \cdots \times A_n) = \\ &\quad = m_{x_1}(A_1) m_{x_2}(A_2) \dots m_{x_n}(A_n) \,. \end{aligned}$$

But $m_{x_i}(A_i) = m_{x_1}(R) \dots m_{x_{i-1}}(R) m_{x_i}(A_i) = m_{x_1} \times \cdots \times m_{x_i}(R \times \cdots \times R \times A_i) = P\big(\pi_i^{-1}(R \times \cdots \times R \times A_i)\big) = P\big(\xi_i^{-1}(A_i)\big)$. Therefore $m_{x_i}(A_i) = P\big(\xi_i^{-1}(A_i)\big)$, hence

$$P\big(\xi_1^{-1}(A_1) \cap \cdots \cap \xi_n^{-1}(A_n)\big) = P\big(\xi_1^{-1}(A_1)\big) \dots P\big(\xi_n^{-1}(A_n)\big) \,.$$

Theorem 2. *(Law of large numbers) Let (x_n) be a weakly independent sequence of weak observables of L^2 with the same probability distribution $m_{x_n} = \mu$ $(n = 1, 2, \dots)$. Denote $E(x_n) = a$ $(n = 1, 2, \dots)$. Then for every $\varepsilon > 0$ there holds*

$$\lim_{n \to \infty} m\bigg(\Big(\frac{1}{n}\sum_{i=1}^{n} x_i - a\Big)\ (-\varepsilon, \varepsilon)\big)\bigg) = 1 \,.$$

Proof. Consider the probability space (R^N, σ, P) constructed in Theorem 1. Then

$$m\left(\left(\frac{1}{n}\sum_{i=1}^{n} x_i - a\right)\ (-\varepsilon,\varepsilon)\right) = m\big(z_n((-\varepsilon,\varepsilon))\big) =$$

$$= P\circ\zeta_n^{-1}\big((-\varepsilon,\varepsilon)\big) = P\left(\left\{t\in R^N;\ \left|\frac{1}{n}\sum_{i=1}^{n}\xi_i(t)-a\right| < \varepsilon\right\}\right).$$

By the classical law of large numbers

$$\lim_{n\to\infty} P\left(\left\{t;\ \left|\tfrac{1}{n}\sum_{i=1}^{n}\xi_i(t)-a\right|\right\}\right) = 1.$$

Theorem 3. *(Central limit theorem). Let (x_n) be a weakly independent sequence of weak observables of L^2 with the same probability distribution $m_{x_n} = \mu$ ($n = 1,2,\dots$), $E(x_n) = a$, $\sigma^2(x_n) = \sigma^2 > 0$ ($n = 1,2,\dots$). Then*

$$\lim_{n\to\infty} m\left(\frac{\sqrt{n}}{\sigma}\Big(\frac{1}{n}\sum_{i=1}^{n}x_i - a\Big)\big((-\infty,t)\big)\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-\frac{u^2}{2}}du$$

for every $t\in R$.

Proof. Consider the probability space (R^n, σ, P) constructed in Theorem 1. Then

$$m\left(\frac{\sqrt{n}}{\sigma}\Big(\frac{1}{n}\sum_{i=1}^{n}x_i - a\Big)\big((-\infty,t)\big)\right) =$$

$$= P\left(\left\{u;\ \frac{\frac{1}{n}\sum_{i=1}^{n}\xi_i(u)-a}{\frac{\sigma}{\sqrt{n}}} < t\right\}\right).$$

By the classical central limit theorem, the last sequence converges to the corresponding value $\Phi(t)$ of the distribution function of the normal normed distribution.

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