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COEXISTENT OBSERVABLES AND EFFECTS IN QUANTUM MECHANICS

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Abstract. A pair of coexistent observables which can be expressed as a function of a third observable is characterised in terms of positive operator bimeasures and jointmeasures. A characterisation of sets of coexistent observables in terms of consistent families of finite (nonorthogonal) resolutions of identity is worked out.

1. Introduction.

It is by now well-known that the proper mathematical representation of the notion of observable in the Hilbert space formulation of quantum mechanics is the one given by a positive operator valued measure. The extension of the mathematical representation of observable from that of a self-adjoint operator to such a measure has proved highly useful not only in treating mathematical and conceptual problems of quantum mechanics but also in analysing concrete experiments.[1–6] One of the most striking features of this new concept of observable is that it allows for the possibility of measuring together noncommuting observables. It is the concept of *coexistence* of observables which has been introduced to formalise the possibility of measuring together two or more observables.[2] Apart from some case studies[1,4,6] a general theory of coexistent observables has not yet been worked out. The present paper offers the first results in that aim. Theorem 3.1 of this paper gives a characterisation of the pairs of functionally coexistent observables, a form of coexistence which arises in a direct way from the quantum theory of measurement (as explained in Sect. 4.5). This characterisation is given in terms of positive operator bimeasures and joint observables and it is obtained on the basis of the recent work of Ylinen.[7] Coexistent sets of effects (positive operators bounded by one) are characterised in Theorem 5.12 by a consistent system of finite (nonorthogonal) resolutions of identity. This characterisation applies the Neumark dilation of a nonorthogonal resolution of identity to an orthogonal one and it builds on an inductive limit of the associated Hilbert spaces. This result serves also to characterise families of coexistent observables.

The structure of the paper is as follows. Section 2 gives the basic definitions and facts applied in this work. The functionally coexistent observables are characterised in Section 3, Section 4 providing some examples and illustrations of that result. Section 5 gives a characterisation of coexistent sets of effects leading also to a characterisation of coexistent families of observables in terms of consistent systems of simple observables.

2. Notations, definitions and basic facts.

2.1. Let $\mathcal{H}$ be a complex separable Hilbert space, with the inner product $\langle\ |\ \rangle$, and let $\mathcal{L}(\mathcal{H})$ denote the set of bounded operators on $\mathcal{H}$. Let $(\Omega, \mathcal{B}(\Omega))$ denote the Borel space $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$ for some $n = 1, 2, \cdots$.

A function $E : \mathcal{B}(\Omega) \to \mathcal{L}(\mathcal{H})$ is a *positive operator valued measure*, a POV measure for short, if

$$E(X) \geq E(\emptyset) = O \quad \text{for all } X \in \mathcal{B}(\Omega), \tag{1a}$$

$$E(\cup X_i) = \sum E(X_i) \quad \text{for all disjoint sequences } (X_i) \subset \mathcal{B}(\Omega), \tag{1b}$$

where the series converges in the weak (or, equivalently, in the strong) operator topology of $\mathcal{L}(\mathcal{H})$.

We call a POV measure $E : \mathcal{B}(\Omega) \to \mathcal{L}(\mathcal{H})$ an *observable* (of a quantum system associated with the Hilbert space $\mathcal{H}$) if, in addition, E is normalised,

$$E(\Omega) = I. \tag{1c}$$

We say that an observable E is *regular* if for each $X \in \mathcal{B}(\Omega)$, such that $O \neq E(X) \neq I$,

$$E(X) \not\leq \frac{1}{2}I, \ \frac{1}{2}I \not\leq E(X). \tag{1d}$$

An observable E is regular if and only if its range $\mathrm{ran}\,(E) := \{E(X) \,|\, X \in \mathcal{B}(\Omega)\}$ is a Boolean subset of the set of positive operators on $\mathcal{H}$ [8].

We say that an observable E is *sharp* if it is projection valued:

$$E(X)^2 = E(X) \quad \text{for all } X \in \mathcal{B}(\Omega). \tag{1e}$$

A sharp observable is regular. We recall [9] that for a POV measure E condition (1e) is equivalent to the multiplicativity condition

$$E(X \cap Y) = E(X)E(Y) \quad \text{for all } X, Y \in \mathcal{B}(\Omega). \tag{1f}$$

This shows that a sharp observable E is also *commutative*,

$$E(X)E(Y) = E(Y)E(X) \quad \text{for all } X, Y \in \mathcal{B}(\Omega), \tag{2}$$

though a commutative observable need not be sharp nor regular [10].

Finally, we say that an observable $E : \mathcal{B}(\Omega) \to \mathcal{L}(\mathcal{H})$ is *simple* if there is a finite subset $\{\omega_1, \cdots, \omega_n\}$ of Ω such that $\sum_1^n E(\omega_i) = I$. Then for any observable E, if $\mathcal{R} = (X_i)_1^n$ is a finite partition of Ω into mutually disjoint Borel subsets, then $i \mapsto E_i^{\mathcal{R}} := E(X_i)$, $i = 1, \cdots, n$, defines a simple observable $E^{\mathcal{R}}$, which is a coarse-graining of E.

2.2. Observables $E_1 : \mathcal{B}(\Omega_1) \to \mathcal{L}(\mathcal{H})$ and $E_2 : \mathcal{B}(\Omega_2) \to \mathcal{L}(\mathcal{H})$ are *coexistent* if there is an observable $E : \mathcal{B}(\Omega) \to \mathcal{L}(\mathcal{H})$ such that its range contains the ranges of E_1 and E_2, that is,

$$\mathrm{ran}\,(E_1) \cup \mathrm{ran}\,(E_2) \subset \mathrm{ran}\,(E). \tag{3}$$

2.3. We recall that for any two observables E_1 and E_2, if one of them is sharp, then their coexistence implies their mutual commutativity [4],

$$E_1(X)E_2(Y) = E_2(Y)E_1(X) \text{ for all } X \in \mathcal{B}(\Omega_1), Y \in \mathcal{B}(\Omega_2). \tag{4}$$

2.4. We say that observables E_1 and E_2 are functions of an observable E if there are Borel functions $f_1 : \Omega \to \Omega_1$ and $f_2 : \Omega \to \Omega_2$ such that

$$\begin{aligned} E_1(X) &= E(f_1^{-1}(X)) \text{ for all } X \in \mathcal{B}(\Omega_1), \\ E_2(Y) &= E(f_2^{-1}(Y)) \text{ for all } Y \in \mathcal{B}(\Omega_2). \end{aligned} \tag{5}$$

In that case we write $E_i = E^{f_i}$, $i = 1, 2$, and say that E_1 and E_2 are *functionally coexistent*, or *f-coexistent* for short. From (5) one observes that

$$\operatorname{ran}(E_1) \cup \operatorname{ran}(E_2) \subset \operatorname{ran}(E),$$

showing that f-coexistent observables are coexistent.

2.5. Simple observables E_1 and E_2 are coexistent if and only if they are f-coexistent. Indeed, assume that E_1 and E_2 are coexistent, and let $E : \mathcal{B}(\Omega) \to \mathcal{L}(\mathcal{H})$ be an observable such that $E_1(\omega_i^1) = E(X_i)$, $i = 1, \cdots, n$, and $E_2(\omega_j^2) = E(Y_j)$, $j = 1, \cdots, m$. For any $X \in \mathcal{B}(\Omega)$, we put $X^1 = X$ and $X^0 = \Omega \backslash X$. The sets $X_1^{g(1)} \cap \cdots \cap X_n^{g(n)} \cap Y_1^{g(n+1)} \cap \cdots \cap Y_m^{g(n+m)}$, with g being a $\{0, 1\}$-valued function on the index set $\{1, \cdots, n, n+1, \cdots, n+m\}$, form a finite partition $\mathcal{R}$ of Ω into mutually disjoint Borel sets. The map $Z \mapsto E^{\mathcal{R}}(Z) := E(Z)$, $Z \in \mathcal{R}$, constitutes a simple observable $E^{\mathcal{R}}$ for which $\operatorname{ran}(E_1) \cup \operatorname{ran}(E_2) \subset \operatorname{ran}(E^{\mathcal{R}})$. Choosing a function $f_1 : \Omega \to \{\omega_1^1, \cdots, \omega_n^1\}$ such that $f(\omega) = \omega_i^1$, for $\omega \in X_i$, we get $E_1 = E^{f_1}$; similarly one gets $E_2 = E^{f_2}$.

2.6. For any two sharp observables the notions of coexistence and f-coexistence are equivalent. Indeed, by 2.3, coexistent sharp observables are mutually commutative. It is a classical result that such observables can be expressed as Borel functions of a sharp observable, see e.g. [11].

2.7. We say that observables E_1 and E_2 have a *joint observable* if there is an observable $E : \mathcal{B}(\Omega_1 \times \Omega_2) \to \mathcal{L}(\mathcal{H})$ such that

$$\begin{aligned} E_1(X) &= E(X \times \Omega_2) \text{ for all } X \in \mathcal{B}(\Omega_1), \\ E_2(Y) &= E(\Omega_1 \times Y) \text{ for all } Y \in \mathcal{B}(\Omega_2). \end{aligned} \tag{6}$$

2.8. A positive operator valued function $E : \mathcal{B}(\Omega_1) \times \mathcal{B}(\Omega_2) \to \mathcal{L}(\mathcal{H})$ is a POV-*bimeasure* if for each $X \in \mathcal{B}(\Omega_1)$ the function

$$E_X : \mathcal{B}(\Omega_2) \to \mathcal{L}(\mathcal{H}),\ Y \mapsto E_X(Y) := E(X, Y), \tag{7a}$$

and for each $Y \in \mathcal{B}(\Omega_2)$ the function

$$E^Y : \mathcal{B}(\Omega_1) \to \mathcal{L}(\mathcal{H}),\ X \mapsto E^Y(X) := E(X, Y), \tag{7b}$$

are POV measures.

3. A characterisation of f-coexistent observables.

3.1 Theorem. For any two observables $E_1 : \mathcal{B}(\Omega_1) \to \mathcal{L}(\mathcal{H})$ and $E_2 : \mathcal{B}(\Omega_2) \to \mathcal{L}(\mathcal{H})$ the following statements are equivalent:

a) there is a POV-bimeasure $E : \mathcal{B}(\Omega_1) \times \mathcal{B}(\Omega_2) \to \mathcal{L}(\mathcal{H})$ such that

$$E_1(X) = E(X, \Omega_2) \text{ for all } X \in \mathcal{B}(\Omega_1),$$
$$E_2(Y) = E(\Omega_1, Y) \text{ for all } Y \in \mathcal{B}(\Omega_2);$$

b) there is an observable $F : \mathcal{B}(\Omega_1 \times \Omega_2) \to \mathcal{L}(\mathcal{H})$ such that

$$E_1(X) = F(X \times \Omega_2) \text{ for all } X \in \mathcal{B}(\Omega_1),$$
$$E_2(Y) = F(\Omega_1 \times Y) \text{ for all } Y \in \mathcal{B}(\Omega_2);$$

c) there are an observable $G : \mathcal{B}(\Omega) \to \mathcal{L}(\mathcal{H})$ and Borel functions $f_i : \Omega \to \Omega_i$, $i = 1, 2$, such that

$$E_1(X) = G(f_1^{-1}(X)) \text{ for all } X \in \mathcal{B}(\Omega_1),$$
$$E_2(Y) = G(f_2^{-1}(Y)) \text{ for all } Y \in \mathcal{B}(\Omega_2).$$

Proof. If *a*) is satisfied, then according to Thm 4.2 in [7], there exists a Hilbert space $\mathcal{K}$, a bounded linear map $T : \mathcal{H} \to \mathcal{K}$, and two sharp observables $P_1 : \mathcal{B}(\Omega_1) \to \mathcal{L}(\mathcal{K})$, $P_2 : \mathcal{B}(\Omega_2) \to \mathcal{L}(\mathcal{K})$ such that for all $X \in \mathcal{B}(\Omega_1), Y \in \mathcal{B}(\Omega_2)$,

$$E(X, Y) = T^* P_1(X) P_2(Y) T,$$
$$P_1(X) P_2(Y) = P_2(Y) P_1(X).$$

Due to the commutativity of P_1 and P_2 the function $X \times Y \mapsto P_1(X)P_2(Y)$ extends to a sharp observable $P : \mathcal{B}(\Omega_1 \times \Omega_2) \to \mathcal{L}(\mathcal{K})$. Then $F(Z) := T^* P(Z) T$ defines an observable $\mathcal{B}(\Omega_1 \times \Omega_2) \to \mathcal{L}(\mathcal{H})$, for which $F(X \times Y) = T^* P_1(X) P_2(Y) T = E(X, Y)$, so that *b*) is satisfied.

The implication *b*) $\Rightarrow$ *c*) is obvious, with the choice $G = E$ and f_1 and f_2 are the coordinate projections $\Omega_1 \times \Omega_2 \to \Omega_1$ and $\Omega_1 \times \Omega_2 \to \Omega_2$, respectively.

Assume finally that *c*) holds true, and let

$$E(X, Y) := G(f_1^{-1}(X) \cap f_2^{-1}(Y)), \; X \in \mathcal{B}(\Omega_1), \; Y \in \mathcal{B}(\Omega_2).$$

Then E is a POV-bimeasure, and it has the required properties

$$E(X, \Omega_2) = G(f_1^{-1}(X)) = E_1(X),$$
$$E(\Omega_1, Y) = G(f_2^{-1}(Y)) = E_2(Y).$$

3.2 Corollary. Observables $E_1 : \mathcal{B}(\Omega_1) \to \mathcal{L}(\mathcal{H})$ and $E_2 : \mathcal{B}(\Omega_2) \to \mathcal{L}(\mathcal{H})$ are f-coexistent if and only if there is a Hilbert space $\mathcal{K}$, a bounded linear map $T : \mathcal{K} \to \mathcal{H}$ and two mutually commuting sharp observables $P_1 : \mathcal{B}(\Omega_1) \to \mathcal{L}(\mathcal{K})$ and $P_2 : \mathcal{B}(\Omega_2) \to \mathcal{L}(\mathcal{K})$ such that $E_1(X) = T^* P_1(X) T$, $X \in \mathcal{B}(\Omega_1)$, and $E_2(Y) = T^* P_2(Y) T$, $Y \in \mathcal{B}(\Omega_2)$.

4. Examples and illustrations of f-coexistent observables.

4.1 Example. Assume that the observables E_1 and E_2 are mutually commuting, that is, $E_1(X)E_2(Y) = E_2(Y)E_1(X)$ for all $X \in \mathcal{B}(\Omega_1), Y \in \mathcal{B}(\Omega_2)$. The map $E : \mathcal{B}(\Omega_1) \times \mathcal{B}(\Omega_2) \to \mathcal{L}(\mathcal{H})$, defined as $E(X,Y) := E_1(X)E_2(Y)$, is then a POV-bimeasure. Indeed, the positivity of $E(X,Y)$ follows from the commutativity (and positivity) of $E_1(X)$ and $E_2(Y)$, whereas the measure properties of E_1 and E_2 and the (weak) continuity of the operator product give the conditions (7a) and (7b). The two mutually commuting observables, sharp or not, do have a joint observable and they are f-coexistent. The mutual commutativity of E_1 and E_2 is, however, not necessary for any of the conditions of Theorem 3.1. This becomes evident from Example 4.3.

4.2. Let $\mathcal{T}(\mathcal{H})$ denote the set of trace class operators on $\mathcal{H}$, and let $\mathcal{S}(\mathcal{H})$ denote its subset of positive trace one operators, the *states* of the quantum system associated with $\mathcal{H}$. Let $\mathcal{L}(\mathcal{T}(\mathcal{H}))$ denote the set of (trace norm) bounded linear operators on $\mathcal{T}(\mathcal{H})$, which is a complex Banach space with respect to the trace norm. A function $\mathcal{I} : \mathcal{B}(\Omega) \to \mathcal{L}(\mathcal{T}(\mathcal{H}))$ is a *state transformer*, or an instrument, if for all $T \in \mathcal{S}(\mathcal{H})$,

(8a) $\quad \mathrm{tr}\big[\mathcal{I}(X)(T)\big] \geq 0$ for all $X \in \mathcal{B}(\Omega)$,

(8b)

$$\mathrm{tr}\big[\mathcal{I}(\cup X_i)(T)\big] = \sum \mathrm{tr}\big[\mathcal{I}(X_i)(T)\big] \quad \text{for all disjoint sequences } (X_i) \subset \mathcal{B}(\Omega),$$

(8c) $\quad \mathrm{tr}\big[\mathcal{I}(\Omega)(T)\big] = 1.$

It follows that the function $X \mapsto E(X)$, defined through

$$\mathrm{tr}\big[TE(X)\big] := \mathrm{tr}\big[\mathcal{I}(X)(T)\big], \ X \in \mathcal{B}(\Omega), T \in \mathcal{S}(\mathcal{H}), \tag{9}$$

is an observable $\mathcal{B}(\Omega) \to \mathcal{L}(\mathcal{H})$, the *associate observable* of $\mathcal{I}$. It is another matter of fact that each observable E is the associate observable of some state transformers $\mathcal{I}$; such state transformers are called E-compatible. (For further details of state transformers and their physical importance the reader may wish to consult [1] or [6]).

4.3 Example. Consider any two state transformers $\mathcal{I}_1 : \mathcal{B}(\Omega_1) \to \mathcal{L}(\mathcal{T}(\mathcal{H}))$ and $\mathcal{I}_2 : \mathcal{B}(\Omega_2) \to \mathcal{L}(\mathcal{T}(\mathcal{H}))$, and let E_1 and E_2 be their associate observables. For each $T \in \mathcal{S}(\mathcal{H})$ the function

$$\mu_T : \mathcal{B}(\Omega_1) \times \mathcal{B}(\Omega_2) \to [0,1], \ (X,Y) \mapsto \mu_T(X,Y) := \mathrm{tr}\big[\mathcal{I}_1(X)\big(\mathcal{I}_2(Y)(T)\big)\big] \tag{10}$$

is a bimeasure. By the duality $\mathcal{T}(\mathcal{H})^* \cong \mathcal{L}(\mathcal{H})$ the bimeasures μ_T, $T \in \mathcal{T}(\mathcal{H})$, define a POV-bimeasure $E : \mathcal{B}(\Omega_1) \times \mathcal{B}(\Omega_2) \to \mathcal{L}(\mathcal{H})$ such that

$$\mathrm{tr}\big[TE(X,Y)\big] = \mu_T(X,Y) \tag{11}$$

for all $T \in \mathcal{S}(\mathcal{H}), X \in \mathcal{B}(\Omega_1), Y \in \mathcal{B}(\Omega_2)$. The associated POV measures E_{Ω_1} and E^{Ω_2} are easily seen to be the observables

$$E_{\Omega_1}(Y) = E_2(Y), \quad Y \in \mathcal{B}(\Omega_2), \tag{12a}$$

$$E^{\Omega_2}(X) = \mathcal{I}_2(\Omega_2)^*(E_1(X)), \quad X \in \mathcal{B}(\Omega_2), \tag{12b}$$

where we have introduced the dual transformation $\mathcal{I}_2(\Omega_2)^* : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ of the state transformation $\mathcal{I}_2(\Omega_2) : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$; for all $T \in \mathcal{S}(\mathcal{H}), A \in \mathcal{L}(\mathcal{H})$

$$\mathrm{tr}\big[T\mathcal{I}_2(\Omega_2)^*(A)\big] := \mathrm{tr}\big[\mathcal{I}_2(\Omega_2)(T)A\big]. \tag{13}$$

(We recall that using the dual transformer $\mathcal{I}^* : X \mapsto \mathcal{I}(X)^*, X \in \mathcal{B}(\Omega)$, the associate observable E of $\mathcal{I}$ can be expressed as $E(X) = \mathcal{I}(X)^*(I)$, $X \in \mathcal{B}(\Omega)$; see, e.g. [1])

4.4 Example. The above example shows that any two state transformers $\mathcal{I}_1$ and $\mathcal{I}_2$ give rise to a pair of observables for which any of the conditions of Theorem 3.1 is satisfied. These observables depend on the order in which the state transformers are applied:

$$\mathrm{tr}\big[TE^{12}(X \times Y)\big] := \mathrm{tr}\big[\mathcal{I}_1(X)\big(\mathcal{I}_2(Y)(T)\big)\big], \tag{14a}$$

$$\mathrm{tr}\big[TE^{21}(X \times Y)\big] := \mathrm{tr}\big[\mathcal{I}_2(Y)\big(\mathcal{I}_1(X)(T)\big)\big]. \tag{14b}$$

In the first case these observables are those given in (12a,b), in the second case they are

$$(E^{21})_{\Omega_1}(Y) = \mathcal{I}_1(\Omega_1)^*(E_2(Y)), \quad Y \in \mathcal{B}(\Omega_2), \tag{15a}$$

$$(E^{21})^{\Omega_2}(X) = E_1(X), \quad X \in \mathcal{B}(\Omega_2). \tag{15b}$$

Usually, the sequential observables E^{12} and E^{21} are different. However, it may happen that they are the same, that is, $E^{12} = E^{21}$. In such a case the observables E_1 and E_2 are clearly f-coexistent.

4.5 Illustrations with measurement schemes. A *measurement scheme* for a quantum system associated with a Hilbert space $\mathcal{H}$ is a 4-tuple $\mathcal{M} := \langle \mathcal{K}, W, F, V \rangle$ consisting of a (complex separable) Hilbert space $\mathcal{K}$ (describing the measuring apparatus), a state $W \in \mathcal{S}(\mathcal{K})$ (the initial state of the apparatus), an observable $F : \mathcal{B}(\Omega) \to \mathcal{L}(\mathcal{K})$ (the pointer observable), and a state transformation $V : \mathcal{T}(\mathcal{H} \otimes \mathcal{K}) \to \mathcal{T}(\mathcal{H} \otimes \mathcal{K})$ (a positive trace preserving map which models the measurement coupling). A measurement scheme $\mathcal{M}$ determines an observable $E^{\mathcal{M}} : \mathcal{B}(\Omega) \to \mathcal{L}(\mathcal{H})$ through the relation:

$$\mathrm{tr}\big[TE^{\mathcal{M}}(X)\big] = \mathrm{tr}\big[V(T \otimes W)\, I \otimes F(X)\big], \quad T \in \mathcal{S}(\mathcal{H}), X \in \mathcal{B}(\Omega). \tag{16}$$

This observable is the observable measured by the scheme $\mathcal{M}$. It is a basic result of the quantum theory of measurement that for each observable $E : \mathcal{B}(\Omega) \to \mathcal{L}(\mathcal{H})$ there is a measurement scheme $\mathcal{M}$ such that $E = E^{\mathcal{M}}$, see e.g. [5]. A measurement scheme $\mathcal{M}$ also determines a state transformer $\mathcal{I}^{\mathcal{M}}$:

$$\mathcal{I}^{\mathcal{M}}(T) := \Pi_{\mathcal{K}}\big(V(T \otimes W)\, I \otimes F(X)\big), \quad T \in \mathcal{S}(\mathcal{H}), X \in \mathcal{B}(\Omega), \tag{17}$$

where $\Pi_{\mathcal{K}} : \mathcal{T}(\mathcal{H} \otimes \mathcal{K}) \to \mathcal{T}(\mathcal{H})$ is the partial trace over the apparatus Hilbert space $\mathcal{K}$. Clearly, $E^{\mathcal{M}}$ is the associate observable of $\mathcal{I}^{\mathcal{M}}$.

Consider now any two observables E_1 and E_2 (of the system with the Hilbert space $\mathcal{H}$). We say that E_1 and E_2 can be *measured together* if there is a measurement scheme $\mathcal{M}$ and two Borel functions (pointer functions) f_1 and f_2 such that $E_1(X) = E^{\mathcal{M}}(f_1^{-1}(X))$, $X \in \mathcal{B}(\Omega_1)$, and $E_2(Y) = E^{\mathcal{M}}(f_2^{-1}(Y))$, $Y \in \mathcal{B}(\Omega_2)$. It is an immediate observation that E_1 and E_2 can be measured together if and only if they are f-coexistent.

Consider next any two measurement schemes $\mathcal{M}_1$ and $\mathcal{M}_2$. They can be applied sequentially, in either order: first $\mathcal{M}_1$ and then $\mathcal{M}_2$, or first $\mathcal{M}_2$ and then $\mathcal{M}_1$. (For details, see [12]). This corresponds to the application of the state transformers $\mathcal{I}^{\mathcal{M}_1}$ and $\mathcal{I}^{\mathcal{M}_2}$ one after the other, in either order. The resulting sequential observables E^{21} and E^{12} are obtained from Eqs (14b,a). In general, the result of such a sequential measurement depends on the order in which the two measurements are performed. It may, however, happen that the measurements in question are commutative in the sense that their sequential application is order independet. The observables $E^{\mathcal{M}_1}$ and $E^{\mathcal{M}_2}$ determined by such measurement schemes are f-coexistent. As is well known, it is highly exceptional that two measurement schemes $\mathcal{M}_1$ and $\mathcal{M}_2$ are commutative in this sense. (For furthre details, see, e.g. [6]).

5. Coexistent effects.

5.1. Let A be a positive operator bounded by the unit operator, $O \leq A \leq I$. We call such operators *effects* and we let $\mathcal{E}(\mathcal{H})$ denote the set of effects on $\mathcal{H}$. We recall that an idempotent effect ($A^2 = A$) represents a *sharp property* of the quantum system in question, whereas other effects $A \in \mathcal{E}(\mathcal{H})$, which are regular (that is, $A \not\leq \frac{1}{2}I, \frac{1}{2}I \not\leq A$) represent *unsharp properties*. Idempotent effects are exactly those effects $A \in \mathcal{E}(\mathcal{H})$ which are *orthogonal* in the sense that $A \wedge (I - A) = O$.

The range $\mathrm{ran}\,(E)$ of an observable $E : \mathcal{B}(\Omega) \to \mathcal{L}(\mathcal{H})$ is a special subset of effects which, by the very idea of an observable, can be measured together. In agreement with this we say that a set $\mathcal{A} \subset \mathcal{E}(\mathcal{H})$ of effects is *coexistent* if $\mathcal{A}$ is contained in the range of an observable E, $\mathcal{A} \subset \mathrm{ran}\,(E)$.

5.2. A finite set of effects $\mathcal{A} = \{A_1, \cdots, A_n\}$ is coexistent if and only if there is a simple observable E such that $\mathcal{A} \subset \mathrm{ran}\,(E)$. Indeed, assume that $\mathcal{A}$ is coexistent, and let $E : \mathcal{B}(\Omega) \to \mathcal{L}(\mathcal{H})$ be an observable such that for each $i = 1, \cdots, n$, $A_i = E(X_i)$ for some $X_i \in \mathcal{B}(\Omega)$. The family $\mathcal{R} = \{X_1^{g(1)} \cap \cdots \cap X_n^{g(n)} \,|\, g : \{1, \cdots, n\} \to \{0, 1\}\}$ constitutes a finite (disjoint measurable) partition of Ω, so that $E^{\mathcal{R}}$ is a simple observable, and $\mathcal{A} \subset \mathrm{ran}\,(E^{\mathcal{R}})$.

5.3. Let $\mathbf{I}_n := \{1, \cdots, n\}$. We say that a map $N : \mathbf{I}_n \to \mathcal{E}(\mathcal{H}), i \mapsto N_i$ is a *finite resolution of identity* if $\sum_1^n N_i = I$. A resolution of identity N is regular if all N_i, $i \in \mathbf{I}_n$, are regular, and it is orthogonal if all N_i are projections. If N is a finite resolution of identity, then $X \mapsto \sum_{i \in X} N_i$, $X \subset \mathbf{I}_n$, is a simple observable. Conversely, if $E : \mathcal{B}(\Omega) \to \mathcal{L}(\mathcal{H})$ is a simple observable, then $i \mapsto E_i := E(\omega_i)$ is a finite resolution of identity. Regular resolutions of identity are associated with Boolean observables (that is, observables with Boolean ranges) whereas orthogonal resolutions of identity are associated with sharp observables.

5.4. We say that a resolution of identity $N : \mathbf{I}_n \to \mathcal{E}(\mathcal{H})$ is a *covering* of a set of effects $\mathcal{A} = \{A_1, \cdots, A_m\}$ if each $A_k \in \mathcal{A}$ can be expressed as $A_k = \sum_{i \in X} N_i$ for some $X \subset \mathbf{I}_n$.

Clearly, $\mathcal{A}$ has a covering if and only if $\mathcal{A}$ is coexistent. We say that two resolutions of identity $M : \mathbf{I}_m \to \mathcal{E}(\mathcal{H})$ and $N : \mathbf{I}_n \to \mathcal{E}(\mathcal{H})$ are *equivalent* if $\mathbf{I}_n = \mathbf{I}_m$ and there is a permutation π of $\mathbf{I}_n$ such that $N_i = M_{\pi(i)}$, that is, $M = N^{\pi}$.

5.5. A resolution of identity $N : \mathbf{I}_n \to \mathcal{E}(\mathcal{H})$ is a *refinement* of a resolution of identity $M : \mathbf{I}_m \to \mathcal{E}(\mathcal{H})$ if there is a function $f_{nm} : \mathbf{I}_n \to \mathbf{I}_m$ such that for each $j \in \mathbf{I}_m$, $M_j = \sum_{\{i: f_{nm}(i)=j\}} N_i$, that is, $M = N^{f_{nm}}$. Assuming that each $M_j \neq O$ the function f_{nm} is surjective. If N is a refinement of M we write $N < M$. It may happen that $N < M$ with respect to different functions $\mathbf{I}_n \to \mathbf{I}_m$. In that case N contains an irregular effect $N_k \leq \frac{1}{2}I$, say. In particular, if $N < N$ in more than one way, then N contains, at least, one irregular element.

Observe that if N and M are equivalent, then $N < M$ and $M < N$ hold for a permutation and its inverse. It is clear that the relation $<$ does not depend on the choice of representatives of equivalence classes of resolutions of identity: if $f_{nm}(i) = j$, that is $M_j = N_i + \cdots$, and $i = \pi(k), j = \sigma(l)$, where π and σ are permutations of $\mathbf{I}_n$ and $\mathbf{I}_m$, respectively, then $f_{nm}(\pi(k)) = \sigma(l)$ and $M_{\sigma(l)} = N_{\pi(k)} + \cdots$, that is, $N^{\pi} < M^{\sigma}$ with respect to $f'_{nm} = \sigma \circ f_{nm} \circ \pi^{-1}$.

For any two finite resolutions of identity N and M, if $N < M$, we let $\mathcal{F}_{nm}$ denote the set of functions $f_{nm} : \mathbf{I}_n \to \mathbf{I}_m$ with respect to which N is a refinement of M. If $N < M$ and N is a regular resolution of identity this set contains only one element.

5.6. Let $\mathcal{N}$ be a collection of finite resolutions of identity $N : \mathbf{I}_n \to \mathcal{E}(\mathcal{H})$ and let $\mathcal{F} = \cup\big(\mathcal{F}_{nm} \,|\, N, M \in \mathcal{N}, N < M\big)$. Let $\mathcal{C} \subset \mathcal{F}$. We shall write $N <_{\mathcal{C}} M$ if there is a function $f_{nm} \in \mathcal{C}$ which makes N a refinement of M.

We say that $(\mathcal{N}, \mathcal{C}, <_{\mathcal{C}})$ is a *consistent system of finite resolutions of identity* if the following conditions are satisfied:

i) $\mathcal{N}$ contains at most one representative of each equivalence class, that is, for each $N, M \in \mathcal{N}$, if N and M are equivalent, then $N = M$;
ii) for each $N, M \in \mathcal{N}$, if $N <_{\mathcal{C}} M$, then there is exactly one function $f_{nm} \in \mathcal{C}$ such that $M = N^{f_{nm}}$;
iii) for each $N \in \mathcal{N}$, the function $f_{nn} \in \mathcal{C}$ is the identity function $id_n : \mathbf{I}_n \to \mathbf{I}_n$;
iv) for each $N, M, L \in \mathcal{N}$, if $N <_{\mathcal{C}} M$, and $M <_{\mathcal{C}} L$, then $N <_{\mathcal{C}} L$ and $f_{nl} = f_{ml} \circ f_{nm} \in \mathcal{C}$;
v) any two elements $M, N \in \mathcal{N}$ have a common $\mathcal{C}$-refinement in $\mathcal{N}$.

As an immediate consequence of this definition, if $M <_{\mathcal{C}} N$ and $N <_{\mathcal{C}} M$ then $M = N$.

5.7 Remark. Let $\mathcal{N}$ be a collection of finite regular resolutions of identity containing exactly one representative of each equivalence class, and let each $N, M \in \mathcal{N}$ have a common refinement in $\mathcal{N}$. Then $(\mathcal{N}, \mathcal{F}, <_{\mathcal{F}})$ is a consistent system.

5.8 Example. Let $E : \mathcal{B}(\Omega) \to \mathcal{E}(\mathcal{H})$ be an observable and let $\mathcal{R} = (X_i)_1^n$ be a finite partition of Ω. Since $X_i, i = 1, \cdots, n$, are mutually disjoint, $\mathcal{R}$ can be considered as a set, and we may fix one of the possible orderings of its elements. Then $E^{\mathcal{R}} : i \mapsto E_i^{\mathcal{R}} = E(X_i)$ is a finite resolution of identity. Let $\mathcal{N} = \{E^{\mathcal{R}} \,|\, \mathcal{R} \text{ a finite partition of } \Omega\}$ and let $\mathcal{F}$ denote the accompanying set of functions as described in 5.6. We consider every $\mathcal{R}$ with a fixed ordering of its elements. We show next that there is a subset $\mathcal{C} \subset \mathcal{F}$ such that $(\mathcal{N}, \mathcal{C}, <_{\mathcal{C}})$ is a consistent system.

Consider first any two finite partitions $\mathcal{R}_1 = (X_i)_1^n$ and $\mathcal{R}_2 = (Y_j)_1^m$ of Ω, and assume that $\mathcal{R}_1$ is a refinement of $\mathcal{R}_2$, that is, for each $j \in \mathbf{I}_m$, $Y_j = \cup_{i \in \mathbf{I}_n(j)} X_i$ for some $\mathbf{I}_n(j) \subset \mathbf{I}_n$. Clearly, for any $j \neq k$, the sets $\mathbf{I}_n(j)$ and $\mathbf{I}_n(k)$ are disjoint and $\cup_1^m \mathbf{I}_n(j) = \mathbf{I}_n$. Therefore, the function f_{nm} associated with $\mathcal{R}_1 < \mathcal{R}_2$ is the one defined for each $i \in \mathbf{I}_n$ by $f_{nm}(i) = j$ if and only if $i \in \mathbf{I}_n(j)$, that is, if and only if $X_i \subset Y_j$. For $\mathcal{R}_1 = \mathcal{R}_2$ this function is the identity function. If $(X_i)_1^n < (Y_j)_1^m < (Z_k)_1^l$, then also $f_{nl} = f_{ml} \circ f_{nm}$. Finally, the intersection of any two finite partitions $\mathcal{R}_1$ and $\mathcal{R}_2$, $\mathcal{R}_1 \cap \mathcal{R}_2 = \{X_i \cap Y_j \,|\, i \in \mathbf{I}_n, j \in \mathbf{I}_m\}$, is a finite partition of Ω and it is a common refinement of the two. We let $\mathcal{C}$ denote the set of functions f_{nm} associated with the ordering of the finite partitions $\mathcal{R}$ of Ω. It is now easy to check that $(\mathcal{N}, \mathcal{C}, <_{\mathcal{C}})$ is a consistent system.

We shall proceed to show that coexistent sets of effects can be characterised in terms of consistent systems of finite resolutions of identity. For this we find it helpful to recall the standard construction of a Neumark dilation of a finite resolution of identity into an orthogonal one.

5.9. Let $N : \mathbf{I}_n \to \mathcal{E}(\mathcal{H})$ be a finite resolution of identity, and let $\mathcal{L} = \oplus_1^n \mathcal{H}$ be the direct sum of n copies of $\mathcal{H}$. The elements of $\mathcal{L}$ are the n-component vectors $\psi = \{\psi_i \,|\, \psi_i \in \mathcal{H}, i \in \mathbf{I}_n\} \equiv \{\psi_i\}$, and the form

$$\langle\, \psi \,|\, \psi' \,\rangle^{\sim N} := \sum_{i \in \mathbf{I}_n} \langle\, \psi_i \,|\, N_i \psi_i' \,\rangle, \quad \psi, \psi' \in \mathcal{L},$$

defines a pre-inner product on $\mathcal{L}$. The set $\mathcal{L}_0 := \{\psi \in \mathcal{L} \,|\, \langle\, \psi \,|\, \psi \,\rangle^{\sim N} = 0\}$ is a subspace of $\mathcal{L}$ and $\langle \;|\; \rangle^{\sim N}$ defines uniquely an inner product on the quotient space $\mathcal{L}/\mathcal{L}_0$. The completion of $\mathcal{L}/\mathcal{L}_0$ with respect to this inner product is a Hilbert space, which we denote by $\tilde{\mathcal{H}}^N$. The mapping $\varphi \mapsto S^N \varphi := \{\varphi : i \in \mathbf{I}_n\}$ from $\mathcal{H}$ into $\tilde{\mathcal{H}}^N$ is an isometry since for each $\varphi, \varphi' \in \mathcal{H}$,

$$\langle\, S^N \varphi \,|\, S^N \varphi' \,\rangle^{\sim N} = \sum_{i \in \mathbf{I}_n} \langle\, \varphi \,|\, N_i \varphi' \,\rangle = \left\langle\, \varphi \,\middle|\, \sum_{i \in \mathbf{I}_n} N_i \varphi' \,\right\rangle = \langle\, \varphi \,|\, \varphi' \,\rangle .$$

Identifying $\mathcal{H}$ with $S^N(\mathcal{H})$ we may assume that $\mathcal{H} \subset \tilde{\mathcal{H}}^N$. For each $j \in \mathbf{I}_n$, let E_j be the operator on $\mathcal{L}$ which cuts all the components of $\psi = \{\psi_i\}$ to zero except ψ_j, which remains unchanged. The operators $\{E_i \,|\, i \in \mathbf{I}_n\}$ define an orthogonal resolution of identity in $\tilde{\mathcal{H}}^N$. For each $\varphi \in \mathcal{H}$, $\langle\, \varphi \,|\, N_i \varphi \,\rangle = \langle\, S^N \varphi \,|\, E_i S^N \varphi \,\rangle^{\sim N}$, showing that $E : i \mapsto E_i$ is an orthogonal dilation of $N : i \mapsto N_i$.

If N and M are equivalent resolutions of identity, then the Hilbert spaces $\tilde{\mathcal{H}}^N$ and $\tilde{\mathcal{H}}^M$ are unitarily equivalent. Indeed, let $N_i = M_{\pi(i)}, i \in \mathbf{I}_n$, where π is a permutation of $\mathbf{I}_n$. Define $U\{\psi_i\} = \{\psi_{\pi(i)}\}$. Then U is a linear isometry on $\mathcal{L}$ so that it can be extended to a unitary operator $U : \tilde{\mathcal{H}}^N \to \tilde{\mathcal{H}}^M$.

5.10. Assume now that $(\mathcal{N}, \mathcal{C}, <_{\mathcal{C}})$ is a consistent system of finite resolutions of identity in $\mathcal{E}(\mathcal{H})$. According to the above construction we find, for each $M \in \mathcal{N}$, a Hilbert space $\tilde{\mathcal{H}}^M \supset \mathcal{H}$, an isometry $S^M : \mathcal{H} \to \tilde{\mathcal{H}}^M$ and an orthogonal resolution of identity E^M in $\tilde{\mathcal{H}}^M$, which is a dilation of M. Let $N : \mathbf{I}_n \to \mathcal{E}(\mathcal{H})$ be a refinement of $M : \mathbf{I}_m \to \mathcal{E}(\mathcal{H})$. Then we can find an isometry $S^{MN} : \tilde{\mathcal{H}}^M \to \tilde{\mathcal{H}}^N$ in the following way. Let $M_j = \sum_{i \in f_{nm}^{-1}(j)} N_i$, $j \in \mathbf{I}_m$, $f_{nm} \in \mathcal{C}$. For any $\{\psi_j \,|\, j \in \mathbf{I}_m\}$ in $\mathcal{L}^M$ put $S^{MN}(\{\psi_j \,|\, j \in \mathbf{I}_m\}) = \{\eta_i \,|\, i \in \mathbf{I}_n\}$, where $\eta_i = \psi_j$ iff $i \in f_{nm}^{-1}(j)$, $j \in \mathbf{I}_m$. Then S^{MN} is uniquely defined and

$$
\begin{aligned}
\langle\, S^{MN}\psi \,|\, S^{MN}\psi' \,\rangle^{\sim N} &= \sum_{i \in \mathbf{I}_n} \langle\, \eta_i \,|\, N_i \eta_i' \,\rangle \\
&= \sum_{j \in \mathbf{I}_m} \left\langle\, \psi_j \,\middle|\, \sum_{i \in f_{nm}^{-1}(j)} N_i \psi_j' \,\right\rangle \\
&= \sum_{j \in \mathbf{I}_m} \langle\, \psi_j \,|\, M_j \psi_j' \,\rangle \\
&= \langle\, \psi \,|\, \psi' \,\rangle^{\sim M}
\end{aligned}
$$

for any ψ, $\psi' \in \mathcal{L}^M$. The map S^{MN} can then be extended to an isometry from $\tilde{\mathcal{H}}^M$ to $\tilde{\mathcal{H}}^N$ and we may thus assume that $\tilde{\mathcal{H}}^M \subset \tilde{\mathcal{H}}^N$.

It is easy to see that $S^{MN} \circ S^M = S^N$.

Owing to the consistency of $(\mathcal{N}, \mathcal{C}, <_{\mathcal{C}})$, the Hilbert spaces $\tilde{\mathcal{H}}^N$, $N \in \mathcal{N}$, (together with the isometries $S^{MN} : \tilde{\mathcal{H}}^M \hookrightarrow \tilde{\mathcal{H}}^N$) form a directed system of Hilbert spaces. Indeed, for any $M, N \in \mathcal{N}$, if $L \in \mathcal{N}$ is a common refinement of N and M, then $\tilde{\mathcal{H}}^M \cup \tilde{\mathcal{H}}^N \subset \tilde{\mathcal{H}}^L$. Moreover, if K is a refinement of L, then $\tilde{\mathcal{H}}^N \subset \tilde{\mathcal{H}}^L \subset \tilde{\mathcal{H}}^K$, and one may easily confirm that $S^{LK} \circ S^{NL} = S^{NK}$. There is a standard construction of the inductive limit of such a directed system. We describe this construction next.

Put $\mathcal{K} = \cup_{M \in \mathcal{N}} \tilde{\mathcal{H}}^M$, and define a binary relation $\sim$ on $\mathcal{K}$ as follows: if $\varphi, \psi \in \mathcal{K}$, $\varphi \in \tilde{\mathcal{H}}^M$, $\psi \in \tilde{\mathcal{H}}^N$, then $\varphi \sim \psi$ if and only if there is $R \in \mathcal{N}$, $R <_{\mathcal{C}} M$, $R <_{\mathcal{C}} N$ such that $S^{MR}\varphi = S^{NR}\psi$ in $\tilde{\mathcal{H}}^R$. It is easy to see that $\sim$ is an equivalence relation: reflexivity and symmetry are immediate, and the transitivity follows from the fact that the system $(\mathcal{N}, \mathcal{C}, <_{\mathcal{C}})$ is downward directed. For any $\varphi \in \mathcal{K}$, we let $[\varphi] := \{\psi \in \mathcal{K} \,|\, \psi \sim \varphi\}$ and put $\hat{\mathcal{K}} = \mathcal{K}/\sim$. The set $\hat{\mathcal{K}}$ can be endowed with a linear structure: for $\varphi \in \tilde{\mathcal{H}}^M$, $\psi \in \tilde{\mathcal{H}}^N$, and $c_1, c_2 \in \mathbf{C}$, put $c_1[\varphi] + c_2[\psi] := [c_1 S^{MR}\varphi + c_2 S^{NR}\psi]$, where R is a common refinement of N and M. If Q is any other common refinement of M and N, then there is a $T \in \mathcal{N}$, $T <_{\mathcal{C}} R, T <_{\mathcal{C}} Q$, and

$$
\begin{aligned}
S^{QT}(c_1 S^{MQ}\varphi + c_2 S^{NQ}\psi) =& c_1 S^{MT}\varphi + c_2 S^{NT}\psi \\
=& S^{RT}(c_1 S^{MR}\varphi + c_2 S^{NR}\psi)
\end{aligned}
$$

hence $c_1 S^{MQ}\varphi + c_2 S^{NQ}\psi \sim c_1 S^{MR}\varphi + c_2 S^{NR}\psi$. Similarly one proves that $c_1[\varphi] + c_2[\psi]$ does not depend on the choice of a representative of $[\varphi]$ and $[\psi]$.

Now define $\langle\, [\varphi] \,|\, [\psi] \,\rangle^{\sim} = \langle\, \varphi \,|\, \psi \,\rangle^{\sim M}$, where we choose representatives $\varphi \in [\varphi]$, $\psi \in [\psi]$ belonging to the same $\tilde{\mathcal{H}}^M$, for an appropriate $M \in \mathcal{N}$. Properties of the mappings

$S^{MN}, N <_{\mathcal{C}} M$ guarantee that $\langle \, | \, \rangle^{\sim}$ is well-defined and it is a pre-inner product. Completing $\hat{\mathcal{K}}/\mathcal{K}_0$, where $\mathcal{K}_0 = \{[\varphi] \,|\, \langle [\varphi] \,|\, [\varphi] \rangle^{\sim} = 0\}$, with respect to $\sim$, one obtains a Hilbert space $\tilde{\mathcal{H}}$. For $\varphi \in \tilde{\mathcal{H}}^M$, $\varphi \mapsto [\varphi]$ defines an isometry from $\tilde{\mathcal{H}}^M$ to $\tilde{\mathcal{H}}$. Similarly, for $\varphi \in \mathcal{H}$, $\varphi \mapsto [\varphi]$ is an isometry from $\mathcal{H}$ to $\tilde{\mathcal{H}}$.

Let N be a refinement of M, $M, N \in \mathcal{N}$. For $\varphi \in \tilde{\mathcal{H}}^N$, $\varphi = \{\varphi_i \,|\, i \in \mathbf{I}_n\}$, define, for a fixed $j \in \mathbf{I}_m$, $E_j^{MN}\varphi := \{\psi_i \,|\, i \in \mathbf{I}_n\}$, where $\psi_i = \varphi_i$ if $i \in f_{nm}^{-1}(j)$, and $\psi_i = 0$ otherwise. In other words,

$$E_j^{MN} = \sum_{i \in f_{nm}^{-1}(j)} E_i^N =: j^{MN}(E_j^M).$$

We have $j^{MN}(E_{j_1}^M + E_{j_2}^M) = j^{MN}(E_{j_1}^M) + j^{MN}(E_{j_2}^M)$, $j_1 \neq j_2$. So we can define a mapping j^{MN} which maps the projection E_j^M on $\tilde{\mathcal{H}}^M$ to a projection E_j^{MN} on $\tilde{\mathcal{H}}^N$. For every $\varphi \in \tilde{\mathcal{H}}^M$ we have $S^{MN}E_j^M\varphi = E_j^{MN}S^{MN}\varphi = j^{MN}(E_j^M)S^{MN}\varphi$. If $K <_{\mathcal{C}} N <_{\mathcal{C}} M$, $M, N, K \in \mathcal{N}$, one can check that

$$j^{NK} j^{MN}(E_j^M) = j^{MK}(E_j^M),$$

and

$$S^{NK}E_j^{MN}\varphi = E_j^{MK}S^{NK}\varphi, \ \varphi \in \tilde{\mathcal{H}}^N.$$

Define $\tilde{E}_j^M[\varphi] := [E_j^{MN}\varphi] = [j^{MN}(E_j^M)\varphi]$, where we choose $\varphi \in [\varphi]$ such that $\varphi \in \tilde{\mathcal{H}}^N$, $N <_{\mathcal{C}} M$. It is easy to check that $\tilde{E}_j^M$ is well-defined , $\tilde{E}_j^M\tilde{E}_j^M = \tilde{E}_j^M$ and, for $\varphi \in \tilde{\mathcal{H}}^M$, we get $\tilde{E}_j^M[\varphi] = [E_j^{MM}\varphi] = [E_j^M\varphi]$. Moreover, if $\tilde{E}_j^M[\psi] = 0$, there is a $\psi \in \tilde{\mathcal{H}}^N, N <_{\mathcal{C}} M$ such that $E_j^{MN}\psi \sim 0$. Then $\left\langle \tilde{E}_j^M[\varphi] \,|\, [\psi] \right\rangle^{\sim} = \langle E_j^{MK}\varphi \,|\, S^{NK}\psi \rangle^{\sim K} = \langle \varphi \,|\, E_j^{MK}S^{NK}\psi \rangle^{\sim K} = \langle \varphi \,|\, S^{NK}E_j^{MN}\psi \rangle^{\sim K} = 0$, where K is a refinement of N, and $\varphi \in \tilde{\mathcal{H}}^K$ is a representative of $[\varphi]$. Therefore $\tilde{E}_j^M$ defines a projection on $\tilde{\mathcal{H}}$.

If $\varphi \in \mathcal{H}$, we obtain

$$\begin{aligned} \left\langle \tilde{E}_j^M[\varphi] \,|\, [\varphi] \right\rangle^{\sim} &= \langle E_j^M S^M\varphi \,|\, S^M\varphi \rangle^{\sim M} \\ &= \langle M_j\varphi \,|\, \varphi \rangle . \end{aligned}$$

This shows that $\tilde{E}_j^M$ is an orthogonal dilation of M_j. The consistency of $(\mathcal{N}, \mathcal{C}, <_{\mathcal{C}})$ implies that for each $M_j \in \mathrm{ran}\,(M)$ and $N_i \in \mathrm{ran}\,(N)$, $M, N \in \mathcal{N}$, there is a common refinement K of M and N in $\mathcal{N}$ such that $\tilde{E}_j^M$ and $\tilde{E}_i^N$ can be written as sums of some $\tilde{E}_u^K \in \mathrm{ran}\,(\tilde{E}^K)$. This implies that $\tilde{E}_j^M$ and $\tilde{E}_i^N$ commute, for any $M, N \in \mathcal{N}$. Every set of commuting projections belongs to a Boolean sub-σ-algebra of the set of projections on $\tilde{\mathcal{H}}$, and therefore, via the Loomis-Sikorski theorem (or, by the spectral theorem, if the Boolean sub-σ-algebra is countably generated), they belong to the range of a PV measure. Projecting this observable to $\mathcal{H}$ (which can be considered as a subspace of $\tilde{\mathcal{H}}$) gives a POV measure, the range of which contains all the effects N_i of the resolutions $N \in \mathcal{N}$. This proves the coexistence of $\mathcal{N}$, and we thus have the following statement.

5.11 Theorem. Every consistent system of finite resolutions of identity is coexistent.

Together with Example 5.8 this theorem gives the following result as a corollary:

5.12 Theorem. A family $\mathcal{A} \subset \mathcal{E}(\mathcal{H})$ of effects on $\mathcal{H}$ is coexistent if and only if there is a consistent system of finite resolutions of identity $(\mathcal{N}, \mathcal{C}, <_{\mathcal{C}})$ such that for each $A \in \mathcal{A}$ there is an $N \in \mathcal{N}$ for which $A \in \operatorname{ran}(N)$.

As a further immediate corollary we get:

5.13 Theorem. A family of observables E_α, $\alpha \in \mathbf{A}$, is coexistent if and only if there is a consistent system of finite resolutions of identity $(\mathcal{N}, \mathcal{C}, <_{\mathcal{C}})$ such that for each $E_\alpha(X)$, $\alpha \in \mathbf{A}, X \in \mathcal{B}(\Omega_\alpha)$, there is an $N \in \mathcal{N}$ for which $E_\alpha(X) \in \operatorname{ran}(N)$.

Let $E_1 : \mathcal{B}(\Omega_1) \to \mathcal{L}(\mathcal{H})$, $E_2 : \mathcal{B}(\Omega_2) \to \mathcal{L}(\mathcal{H})$ be any two mutually commuting observables.

Let $(\mathcal{N}, \mathcal{C}, <_{\mathcal{C}})$ and $(\mathcal{M}, \mathcal{D}, <_{\mathcal{D}})$ be consistent systems of identity resolutions for E_1 and E_2, respectively, as given in Example 5.8. For $N \in \mathcal{N}$, $M \in \mathcal{M}$, put

$$N \times M := \{N_i M_j : i \in \mathbf{I}_n, j \in \mathbf{I}_m\}.$$

$N \times M$ is a resolution of identity with the index set $\mathbf{I}_n \times \mathbf{I}_m$. Define $N_1 \times M_1 <_{\mathcal{C}\times\mathcal{D}} N_2 \times M_2$ iff $N_1 <_{\mathcal{C}} N_2$ and $M_1 <_{\mathcal{D}} M_2$, and let $f_{n_1\times m_1 n_2\times m_2} : \mathbf{I}_{n_1} \times \mathbf{I}_{m_1} \to \mathbf{I}_{n_2} \times \mathbf{I}_{m_2}$ be defined by $f_{n_1\times m_1 n_2\times m_2} = f_{n_1 n_2} \times f_{m_1 m_2}$. Then $(\mathcal{N} \times \mathcal{M}, \mathcal{C} \times \mathcal{D}, <_{\mathcal{C}\times\mathcal{D}})$ is a consistent system of resolutions of identity in $\mathcal{E}(\mathcal{H})$, which covers $\operatorname{ran}(E_1) \cup \operatorname{ran}(E_2)$ in the sense of Theorem 5.12. Any finite system of mutually commuting observables can be treated in the same way. This enables us to prove the following statement

5.14 Corollary. Any finite system of mutually commuting observables is coexistent.

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