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Sylvia Pulmannová

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ON CONNECTIONS AMONG SOME ORTHOMODULAR STRUCTURES

Sylvia Pulmannová

Mathematical Institute
Slovak Academy of Sciences
Štefánikova 49, 814 73 Bratislava, Slovakia

Abstract. Connections among (generalized) D-posets, positive commutative minimal clans, MV-algebras and semi-MV-algebras are studied.

1. Introduction

By an orthomodular structure we mean a poset with (partially) defined operations $\ominus, \oplus$ such that $a \le b \implies b = a \oplus (b \ominus a)$, which can be considered as a generalized form of orthomodular law. Orthomodular structures as partial algebras have been studied in [B,M] and [P]. We study connections between the following orthomodular structures.

Difference posets (effect algebras, alternatively), have been found a useful tool into pursuit of quantum mechanical constructions. Difference posets have been introduced in [K,Ch 1] as a generalization of orthoalgebras (see also [B,F], [G,G]). An important example of a D-poset is the set of all effects (i.e., s.a. operators A with $0 \le A \le I$ on a Hilbert space), which play an important role in unsharp quantum measurements ([B,L,M]).

The notion of a commutative minimal clan as a common abstraction of Boolean rings and commutative lattice-ordered groups was introduced by Schmidt [S].

The notion of an MV-algebra was introduced by Chang [Ch] in 1957, where MV is supposed to suggest many-valued logic (see also [M], [Mu], [K,Ch 2] for alternative definitions of MV-algebras).

According to Abbott [A], a meet semi-Boolean algebra is a meet semilattice in which every principal ideal is a Boolean algebra. Abbott has shown that meet semi-Boolean algebras are equationally defined as substraction algebras.

In [L,H 1], a close relation between manuals (hence orthoalgebras) and semi-Boolean algebras with special properties was shown. In what follows we bring a generalization of of the latter result replacing orthoalgebras by effect algebras, and semi-Boolean algebras by semi-MV-algebras.

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX

2. Posets with a difference

Definition 2.1. ([K,Ch 1]) Let $(P, \leq)$ be a partially ordered set (poset). A partial binary operation $\ominus$ on P such that $b \ominus a$ is defined if and only if $a \leq b$ is called a *difference* on $(P, \leq)$ if the following conditions are satisfied for all $a, b, c \in P$:

(D1) If $a \leq b$ then $b \ominus a \leq b$ and $b \ominus (b \ominus a) = a$.
(D2) If $a \leq b \leq c$ then $c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

Proposition 2.2. [K,Ch 1], [H,P] *Let $(P, \leq, \ominus)$ be a poset with a difference and let $a, b, c, d \in P$. The following assertions are true:*

(i) *If $a \leq b \leq c$ then $b \ominus a \leq c \ominus a$ and $(c \ominus a) \ominus (b \ominus a) = c \ominus b$.*
(ii) *If $b \leq c$ and $a \leq c \ominus b$ then $a \leq c$ and $b \leq c \ominus a$, and $(c \ominus b) \ominus a = (c \ominus a) \ominus b$.*
(iii) *If $a \leq b \leq c$ then $b \ominus a \leq c$ and $a \leq c \ominus (b \ominus a)$, and $(c \ominus (b \ominus a)) \ominus a = c \ominus b$.*
(iv) *If $a, b \leq c$ and $c \ominus a = c \ominus b$ then $a = b$.*
(v) *If $d \leq a, b \leq c$ and $c \ominus a = b \ominus d$ then $c \ominus b = a \ominus d$.*
(vi) *If $a \leq b, c \leq d$ and $b \ominus a = c \ominus a$ then $b = c$.*

It has been shown in [H,P] that every poset P with a difference can be written as a disjoint union of posets with a difference, each of which contains a smallest element. Therefore we may assume that a smallest element is contained in P.

Lemma 2.3. [H,P] *Let $(P, \leq, \ominus)$ be a poset with a difference. If 0 is the smallest element in P then for all $a \in P$, $a \ominus a = 0$ and $a \ominus 0 = a$. Moreover, for all $a, b \in P$ with $a \leq b$ we have $b \ominus a = 0$ iff $a = b$ and $b \ominus a = b$ iff $a = 0$. If 1 is the greatest element in P then $1 \ominus 1$ is the smallest element in P.*

A poset P with a difference is called a *generalized difference poset* ([H,P]) (a GDP or a GDL if P is a lattice) if

(GD1) There is a smallest element 0 in P (zero existence).
(GD2) If $a, b, c \in P$, $a \leq b$, $a \leq c$ and $b \ominus a = c \ominus a$, then $b = c$ (cancellation property).

In a GDP, a partial binary operation $\oplus$ can be defined as follows:

(S) $a \oplus b$ is defined and $a \oplus b = c$ iff $a \leq c$ and $c \ominus a = b$.

(GD2) guarantees that c is well-defined. We say that a, b are *orthogonal* ($a \perp b$) if $a \oplus b$ exists. In [H,P], Corollary 1.3, the following result was obtained.

Proposition 2.4. ([H,P] *There is a one-to-one correspondence between posets with a cancellative difference poset P having a smallest element 0 and partial binary algebras with a sum operation and with a special element 0 having properties (S1)-(S5) below (where $\leq$ is defined by $a \leq b$ iff $\exists c : b = a \oplus c$):*

(S1) *If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$ (commutativity).*
(S2) *If $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, then $b \oplus c$ and $a \oplus (b \oplus c)$ are defined and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ (associativity).*
(S3) *If $a \oplus b$ and $a \oplus c$ are defined and $a \oplus b = a \oplus c$, then $b = c$ (cancellativity).*
(S4) *$a \oplus 0$ is defined for any $a \in P$, and $a \oplus 0 = a$.*
(S5) *$a \oplus b = 0$ iff $a = 0 = b$.*

Let $(P, \leq, \ominus)$ be a poset with difference and let 0 and 1 be the smallest and greatest elements in P, respectively. The structure $(P, \leq, \ominus)$ is called a *difference* *poset (D-poset, DP)*, or a *difference lattice (D-lattice, DL)* if P is a lattice. In every D-poset, the cancellation property (GD2) is always satisfied by Proposition 2.2 (vi).

Proposition 2.5. *Any lattice with difference is a GDP.*

Proof. For any $a, b \in P$, $a \wedge b \leq a, b \leq a \vee b$. In any poset with difference, $a \leq b \implies a \ominus a = b \ominus b$ ([H,P], which implies that (GD1) is satisfied. (GD2) is satisfied by Proposition 2.2 (vi). $\square$

Lattices with a difference were studied in [K,Ch 2], where the following statements were proved.

Proposition 2.6. [K,Ch 2]

(1) *Let $(P, \leq, \ominus)$ be a poset with a difference, $a, b, c \in P$, $a \leq b \leq c$. If $a \vee b$ exists in P, then $(c \ominus a) \wedge (c \ominus b)$ exists in P and*

$$c \ominus (a \vee b) = (c \ominus a) \wedge (c \ominus b).$$

(2) *Let $(P, \leq, \ominus)$ be a lattice with a difference, $a, b, c \in P$, $a \leq c$, $b \leq c$. Then*

$$c \ominus (a \wedge b) = (c \ominus a) \vee (c \ominus b).$$

(3) *In a lattice with a difference*

$$(a \vee b) \ominus (a \wedge b) = ((a \vee b) \ominus a) \vee ((a \vee b) \ominus b).$$

(4) *Let $(P, \leq, \ominus)$ be a lattice with a difference, $a, b, c \in P$, $c \leq a$, $c \leq b$. Then*

$$(a \wedge b) \ominus c = (a \ominus c) \wedge (b \ominus c).$$

The following two propositions were proved in [K,Ch 2] for D-posets (D-lattices), but they hold true also in more general situations.

Proposition 2.7. *Let P be a poset with difference having a smallest element 0. If $a \vee b \in P$, then $((a \vee b) \ominus a)) \wedge ((a \vee b) \ominus b) \in P$ and $((a \vee b) \ominus a)) \wedge ((a \vee b) \ominus b) = 0$.*

Proof. In Proposition 2.6 (1) put $c = a \vee b$. $\square$

Proposition 2.8. *Let P be a lattice with difference, $c \leq a$, $c \leq b$. Then $(a \ominus c) \vee (b \ominus c) = (a \vee b) \ominus c$.*

Proof. From $c \leq a \leq a \vee b$, $c \leq b \leq a \vee b$ we get $a \ominus c \leq (a \vee b) \ominus c$, $b \ominus c \leq (a \vee b) \ominus c$. Let $w \in P$ be such that $a \ominus c$, $b \ominus c \leq w$. Then $a \ominus c \leq w \wedge ((a \vee b) \ominus c) \leq (a \vee b) \ominus c$ imply $((a \vee b) \ominus c) \ominus (w \wedge ((a \vee b) \ominus c)) \leq ((a \vee b) \ominus c) \ominus (a \ominus c) = ((a \vee b) \ominus a)$; and similarly, $((a \vee b) \ominus c) \ominus (w \wedge ((a \vee b) \ominus c)) \leq ((a \vee b) \ominus b)$. Therefore $((a \vee b) \ominus c) \ominus (w \wedge ((a \vee b) \ominus c)) \leq ((a \vee b) \ominus a) \wedge ((a \vee b) \ominus b) = 0$ by Proposition 2.6 (1) Hence $(a \vee b) \ominus c \leq w$, which gives the desired result. $\square$

Corollary 2.9. *Let P be a lattice with difference. Then $(a\ominus(a\wedge b))\wedge(b\ominus(a\wedge b)) = 0$.*

Proof. In Proposition 2.6 (2) put $c = a \wedge b$. $\square$

From the above properties, the following statement can be derived.

Proposition 2.10. *Let P be a lattice with difference. If $x \oplus y$ and $x \oplus z$ exist, then*

(1) $x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$.
(2) $x \oplus (y \vee z) = (x \oplus y) \vee (x \oplus z)$.

Proof. (1) By Proposition 2.6 (2) , $((x\oplus y)\wedge(x\oplus z))\ominus x = ((x\oplus y)\ominus x)\wedge((x\oplus z)\ominus x) = y \wedge z$, whence by (S), $(x \oplus y) \wedge (x \oplus z) = x \oplus (y \wedge z)$.

(2) By Proposition 2.6 (3), $((x\oplus y)\vee(x\oplus z))\ominus x = ((x\oplus y)\ominus x)\vee((x\oplus z)\ominus x) = y\vee z$, whence by (S), $(x \oplus y) \vee (x \oplus z) = x \oplus (y \vee z)$. $\square$

In [K], the notion of compatibility of two elements in a D-poset P was introduced as follows: $a, b \in P$ are *compatible* (written $a \leftrightarrow b$) if there are $u, v \in P$ such that $v \leq a \leq u$, $v \leq b \leq u$ and $u \ominus a = b \ominus v$. We say that elements $a, b, c \in P$ are $\oplus$*-orthogonal* if $a \oplus b \oplus c$ exists.

Theorem 2.11. [K] *Let P be a D-poset, $a, b \in P$. The following statements are equivalent:*

(1) $a \leftrightarrow b$.
(2) *There exists $u \in P$ such that $a \leq u$, $b \leq u$, $u \ominus a \leq b$ and $u \ominus b \leq a$.*
(3) *There exists $v \in P$ such that $v \leq a$, $v \leq b$, $b \ominus v \leq 1 \ominus a$ and $a \ominus v \leq 1 \ominus b$.*
(4) *There exists a $\oplus$-orthogonal triple $\{a_1, b_1, c\} \subset P$ such that $a = a_1 \oplus c$, $b = b_1 \oplus c$*

Proof. First observe that $a \leftrightarrow b$ iff $1 \ominus a \leftrightarrow 1 \ominus b$. Indeed, $v \leq a \leq u$, $v \leq b \leq u$ are equivalent with $1 \ominus u \leq 1 \ominus a \leq 1 \ominus v$, $1 \ominus u \leq 1 \ominus b \leq 1 \ominus v$, and $(1 \ominus v) \ominus (1 \ominus a) = a \ominus v = u \ominus b = (1 \ominus b) \ominus (1 \ominus u)$. (1) $\Longrightarrow$ (2) follows immediately, (2) $\Longrightarrow$ (3) follows by the transition $x \mapsto 1 \ominus x$. (3) $\Longrightarrow$ (4): $a = v \oplus (a \ominus v)$, $b = v \oplus (b \ominus v)$. It remains to prove $\oplus$-orthogonality of $v, a \ominus v, b \ominus v$. Recall that in a D-poset, elements x, y are orthogonal iff $x \leq 1 \ominus y$. Then $v \oplus (a \ominus v) = a$, and $b \ominus v \leq 1 \ominus a$ imply the desired result. (4) $\Longrightarrow$ (1): put $v = c$, $u = c \oplus a_1 \oplus b_1$. $\square$

The $\oplus$- orthogonal triple in (4) is called a *Mackey decomposition*. We note that the definition of compatibility makes sense also in a generalized difference poset, and the equivalence of (1) and (4) of Theorem 2.11 holds true. We introduce the following strengthening of the definition of compatibility.

Definition 2.12. Let P be a GDP. We say that $a, b \in P$ are *strongly compatible* if there exists a $\oplus$-orthogonal triple $\{a_1, b_1, c\}$ in P such that $a = a_1 \oplus c$, $b = b_1 \oplus c$, and $a_1 \wedge b_1 = 0$.

Theorem 2.13. *If a, b are strongly compatible elements of a GDL P, then $a \wedge b = c$, $a \vee b = a_1 \oplus b_1 \oplus c$.*

Proof. We have $c \leq a, b$. If $d \in P$ is such that $d \leq a, b$, then $c \leq c \vee d \leq a, b$ imply $(c \vee d) \ominus c \leq a \ominus c = a_1, b \ominus c = b_1$, hence $(c \vee d) \ominus c \leq a_1 \wedge b_1 = 0$, whence $d \leq c$.

Similarly we prove the second statement. □

We note that in orthomodular posets, compatibility coincides with strong compatibility, and Mackey decompositions are uniquely defined as in Theorem 2.13. In orthoalgebras compatibility coincides with strong compatibility too, but several Mackey decompositions may occur [L,H 2]. In more general D-posets strong compatibility and compatibility need not coincide, as we can see from the following simple example. Let P be a chain $0 \leq a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5$, where $a_i \oplus a_j = a_{i+j}$, iff it is defined. For a_2, a_4 we have two Mackey decompositions, $a_2 = a_2 \oplus 0$, $a_4 = a_2 \oplus a_2$ (strong compatibility), and $a_2 = a_1 \oplus a_1$, $a_4 = a_1 \oplus a_3$. In [K,Ch3], there has been proved that in a D-lattice P, $a \leftrightarrow b$ if and only if $a \ominus (a \wedge b) = (a \vee b) \ominus b$, and the proof can be transferred also to a GDL. This enables to obtain the following result.

Proposition 2.14. *In a GDL P, if two elements are compatible, they are also strongly compatible.*

3. Minimal clans

A *minimal clan* is a set E with a relation $S \subseteq E \times E$, a map $\oplus : S \to E$, and an order relation $\leq$ such that

(MC-1) there exists an element 0 in E satisfying $(0, x) \in S$, $(x, 0) \in S$, and $0 \oplus x = x = x \oplus 0$ for all $x \in E$;

(MC-2) for all $x, y, z \in E$, $(x, y) \in S$ and $(x \oplus y, z) \in S$ if and only if $(y, z) \in S$ and $(x, y \oplus z) \in S$, and in this case $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;

(MC-3) $x = y$ holds for all $x, y \in E$ satisfying $u \oplus x \oplus v = u \oplus y \oplus v$ for some $u, v \in E$ satisfying $(u, x) \in S$, $(u \oplus x, v) \in S$, and $(u \oplus y, v) \in S$;

(MC-4) $u \oplus x \oplus v \leq u \oplus y \oplus v$ holds for all $x, y \in E$ satisfying $x \leq y$ and and for all $u, v \in E$ satisfying $(u, x) \in S$, $(u \oplus x, v) \in S$, $(u, y) \in S$, and $u \oplus y, v) \in S$;

(MC-5) $x \vee y$ and $x \wedge y$ exist for all $x, y \in E$;

(MC-6) for all $x, y \in E$, there exist $u, v \in E$ satisfying $0 \leq u, 0 \leq v$, $(u, v) \in S$, $(x, v) \in S$, $(u, x \wedge y) \in S$, $(x \wedge y, v) \in S$, $u \oplus x = x \vee y = x \oplus v$, and $u \oplus (x \wedge y) = y = (x \wedge y) \oplus v$.

Throughout this section, let $(E, S, \oplus, \leq)$ be a minimal clan.

Two elements $x, y \in E$ are *summable* if $(x, y) \in S$, and the map $\oplus : S \to E$ is called *(partial) addition*, and the (unique element $0 \in E$ satisfying (MC-1)is said to be the *zero element* of E. Axiom (MC-3) is the *cancellation property*, and axiom (MC-6) is referred as the *difference property*; it may be used to define partial left and right substractions.

An element $x \in E$ is called *invertible* if there exist $u, v \in E$ satisfying $u \oplus x = 0 = x \oplus v$, this is equivalent with the existence of some $w \in E$ satisfying $w \oplus x = 0$ or $0 = x \oplus w$, and this is in turn equivalent with the existence of a (unique) element $x^* \in E$ satisfying $x^* \oplus x = 0 = x \oplus x^*$, which is said to be the *inverse* of x. The set of all invertible elements will be denoted by $E_\circ$. An element $x \in E$ is *positive* if $0 \leq x$. The set of all positive elements of E will be denoted by E_+. A minimal clan is *positive* if $E = E_+$. Two elements $x, y \in E$ are *disjoint* if $x \wedge y = 0$. The set of all pairs of disjoint elements of E will be denoted by $\mathcal{D}$. A minimal clan is *commutative* if $(x, y) \in S$ if and only if $(y, x) \in S$ and $x \oplus y = y \oplus x$. The following lemma collects some simple results that are satisfied in minimal clans.

Lemma 3.1. [S] *Let $(E, S, \oplus, \leq)$ be a minimal clan.*

(i) *If $u \leq v, v \leq y$, and $(x, y) \in S$, then $(u, v) \in S$.*
(ii) *If $u \leq x$ and $x \in E_\circ$, then $v \in E_\circ$.*
(iii) *For all $x \in E$ $x \vee 0 \oplus x \wedge 0 = x = x \wedge 0 \oplus x \vee 0$.*
(iv) *$E_\circ \times E \subseteq S$ and $E \times E_\circ \subseteq S$.*
(v) *If $E_\circ = E$, then $S = E \times E$.*
(vi) *$(E_\circ, E_\circ \times E_\circ, \oplus, \leq)$ is a minimal clan.*
(vii) *The following are equivalent: $E_\circ = \{0\}$ and $E_+ = E$.*
(viii) *$(E_+, (E_+ \times E_+) \cap S, \oplus, \leq)$ is a positive minimal clan.*
(ix) *If $x \wedge y = 0$, then $x \oplus y = x \vee y = y \oplus x$.*
(x) *$\mathcal{D} \subseteq S$, and if $\mathcal{D} = S$, then $E_+ = E$.*
(xi) *$E = E_+$ if and only if the following modularity property holds: For all $x, y \in E$, $(x, y) \in S$ if and only if $x \vee y, x \wedge y \in S$, and in this case $x \oplus y = (x \vee y) \oplus (x \wedge y)$.*
(xii) *$u \oplus (x \wedge y) = y = (x \wedge y) \oplus v$ if and only if $u \oplus x = x \vee y = x \oplus v$ and in this case $u \in E_+$, $v \in E_+$. Moreover, u, v are uniquely defined.*
(xiii) *If $u \oplus x \oplus v \leq u \oplus y \oplus v$, then $x \leq y$ (the order cancellation property).*

It is easy to see that every commutative, positive minimal clan is a GDL. Indeed, (S1) is satisfied by commutativity, (S2) by (MC-2), (S3) by (MC-3), (S4) by (MC-1) and (S5) by positivity and Lemma 3.1 (vii), lattice structure is guaranteed by (MC-5). Moreover, every minimal clan is a distributive lattice ([S],Th. 1.1.14).

Theorem 3.2. *Let P be a GDL. The following statements are equivalent.*

(i) *P is a positive, commutative minimal clan.*
(ii) *For any $x, y \in P$ there is a $u \in P$ such that $u \oplus x = x \vee y$ and $u \oplus (x \wedge y) = y$. Equivalently,*
$$(x \vee y) \ominus x = u = y \ominus (x \wedge y).$$
(iii) *If $x, y \in P$ and $x \wedge y = 0$ then $x \oplus y$ exists and $x \oplus y = x \vee y$.*
(iv) *Any $x, y \in P$ are strongly compatible.*
(v) *Every two orthogonal sets $(u_1, u_2, ..., u_m)$ and $(v_1, v_2, ..., v_n)$ such that*
$$\bigoplus_{i=1}^{m} u_i = \bigoplus_{j=1}^{n} v_j$$
have a common refinement, i.e., there is an orthogonal set (w_{ij}) such that $u_i = \bigoplus_{j=1}^{n} w_{ij}$, $i = 1, 2, ..., m$ and $v_j = \bigoplus_{i=1}^{m} w_{ij}$, $j = 1, 2, ..., n$ and
$$(\bigoplus_{k=i+1}^{m} w_{kj}) \wedge (\bigoplus_{l=j+1}^{n} w_{il}) = 0$$
for all $i \in \{1, 2, ..., m-1\}$ and $j \in \{1, 2, ..., n-1\}$.

Proof. (i) $\Longrightarrow$ (ii) follows by (MC-6) in the definition of the minimal clan.

(ii) $\Longrightarrow$ (iii): $x \wedge y = 0 \implies (x \vee y) \ominus y = x \ominus (x \wedge y) = x \implies x \vee y = x \oplus y$.

(iii) $\Longrightarrow$ (iv): For any $x, y \in P$, $x = (x \wedge y) \oplus ((x \ominus (x \wedge y))$, $y = (x \wedge y) \oplus ((y \ominus (x \wedge y))$. Since $(x \ominus (x \wedge y)) \wedge (y \ominus (x \wedge y)) = 0$, it follows by (iii) that $x \ominus (x \wedge y)$ and $y \ominus (x \wedge y)$ are orthogonal. Moreover,

$$\begin{aligned} x \vee y &= (x \wedge y) \oplus ((x \vee y) \ominus (x \wedge y)) = (x \wedge y) \oplus ((x \ominus (x \wedge y)) \vee (y \ominus (x \wedge y))) \\ &= (x \wedge y) \oplus (x \ominus (x \wedge y)) \oplus (y \ominus (x \wedge y)). \end{aligned}$$

Hence, $x \wedge y$, $x \ominus x \wedge y$, $y \ominus x \wedge y$ are jointly orthogonal.

(iv) $\Longrightarrow$ (v). Our proof is an appropriate modification of the proof of [S], Theorem 1.1.12. The assertion is obvious for $m = 1$ or $n = 1$. Consider first $m = n = 2$. Then we have

$$x_1 \oplus x_2 = y_1 \oplus y_2.$$

We find z_{11}, z_{12} and z_{21} such that

$$\begin{aligned} x_1 &= z_{11} \oplus z_{12}, \\ y_1 &= z_{11} \oplus z_{21}, \\ z_{12} \wedge z_{21} &= 0. \end{aligned}$$

Now define $z = x_1 \oplus x_2 = y_1 \oplus y_2$. Then

$$z_{11} \oplus z_{12} \oplus z_{21} = x_1 \oplus z_{21} = x_1 \vee y_1 \le z$$

and we may choose $z_{22} \in P$ such that

$$\begin{aligned} z_{11} \oplus z_{12} \oplus z_{21} \oplus z_{22} &= z \\ &= x_1 \oplus x_2 \\ &= z_{11} \oplus z_{12} \oplus x_2 \end{aligned}$$

and thus

$$x_2 = z_{21} \oplus z_{22}.$$

Furthermore, we have

$$\begin{aligned} z_{11} \oplus z_{21} \oplus z_{12} \oplus z_{22} &= z \\ &= y_1 \oplus y_2 \\ &= z_{11} \oplus z_{21} \oplus y_2 \end{aligned}$$

and thus

$$y_2 = z_{12} \oplus z_{22}.$$

This proves the assertion in case $m = n = 2$.

The proof continues exactly as in [S], we give only a sketch of it. Let us consider $m \in \{3, 4, ...\}$ and assume that the assertion is true for all $m', m'' \in \{1, 2, ..., m-1\}$ and some $n \in \{2, 3, ...\}$. Letting $m' := m - 1$ and

$$x'_i = \begin{cases} x_i & i \in \{1, 2, ..., m'-1\} \\ \\ x_{m-1} \oplus x_m & i = m', \end{cases}$$

we obtain

$$\bigoplus_{i=1}^{m'} x'_i = \bigoplus_{i=1}^{m} x_i = \bigoplus_{j=1}^{n} y_j$$

and by hypothesis there exist $z'_{ij} \in E$ satisfying appropriate properties.

Furthermore, letting $m'' = 2$ and $x''_i = x_{m+i-2}$ for all $i \in \{1, 2\}$, we obtain

$$\bigoplus_{i=1}^{2} x''_i = x'_{m'} = \bigoplus_{i=1}^{n} z'_{m'j},$$

and, by hypothesis, there exist appropriate $z''_{ij} \in P$.

Now define, for all $i \in \{1, 2, ...m\}$ and $j \in \{1, 2, ..., n\}$

$$z_{ij} := \begin{cases} z'_{ij} \; i \in \{1, 2, ..., m-2\} \\ z''_{1j} \quad i = m-1, \\ z''_{2j} \quad i = m. \end{cases}$$

Then we have

$$x_i = \bigoplus_{j=1}^{n} z_{ij}$$

for all $i \in \{1, 2, ..., m\}$,

$$y_j = \bigoplus_{i=1}^{m} z_{ij}$$

for all $j \in \{1, 2, ..., n\}$, and

$$(\bigoplus_{k=i+1}^{m} z_{kj}) \wedge (\bigoplus_{l=j+1}^{n} z_{il}) = 0$$

for all $i \in \{1, 2, ..., m-1\}$ and $j \in \{1, 2, ..., n-1\}$. Interchanging the role of m and n in the inductive step completes the proof.

(v) $\Longrightarrow$ (i). It is easy to see that in any GDL P, the axioms (MC-1) - (MC-5) are satisfied, together with the positivity condition. It remains to prove condition (MC-6) which, in fact, coincides with our condition (ii). Therefore, it suffices to prove the implication (v) $\Longrightarrow$ (ii). We have

$$\begin{aligned} x \vee y &= (x \wedge y) \oplus (x \ominus (x \wedge y)) \oplus ((x \vee y) \ominus x) \\ &= (x \wedge y) \oplus (y \ominus (x \wedge y)) \oplus ((x \vee y) \ominus y), \end{aligned}$$

hence

$$(x \ominus (x \wedge y)) \oplus ((x \vee y) \ominus x) = (y \ominus (x \wedge y) \oplus ((x \vee y) \ominus y).$$

Put

$$x_1 := x \ominus (x \wedge y), \; x_2 := (x \vee y) \ominus x,$$
$$y_1 := y \ominus (x \wedge y), \; y_2 := (x \vee y) \ominus y.$$

By (v), there are $z_{11}, z_{12}, z_{21}, z_{22}$ such that

$$x_1 = z_{11} \oplus z_{12}, \;\; y_1 = z_{11} \oplus z_{21},$$
$$x_2 = z_{21} \oplus z_{22}, \;\; y_2 = z_{12} \oplus z_{22}$$

and $z_{12} \wedge z_{21} = 0$.

By Proposition 2.7, $x_2 \wedge y_2 = 0$ and by Proposition 2.9, $x_1 \wedge y_1 = 0$. Therefore

$$0 = x_2 \wedge y_2 = (z_{21} \oplus z_{22}) \wedge (z_{12} \oplus z_{22}) \implies z_{22} = 0,$$
$$0 = x_1 \wedge y_1 = (z_{11} \oplus z_{12}) \wedge (z_{11} \oplus z_{21}) \implies z_{11} = 0.$$

Hence

$$x_1 = z_{12}, \;\; y_1 = z_{21}$$
$$x_2 = z_{21}, \;\; y_2 = z_{12}$$

so that $x_1 = y_2$, $x_2 = y_1$; and (ii) is satisfied. □

4. MV-algebras

An *MV-algebra* is an algebra $(\mathcal{A}, +, \circ, *, 0, 1)$, where $\mathcal{A}$ is a nonempty set, 0 and 1 are constants, $+$ and $\circ$ are binary operations and $*$ is a unary operation, satisfying the following axioms (cf. [M], this definition is equivalent to the definition introduced by Chang [Ch]).

(MV-1) $(a + b) + c = a + (b + c)$;
(MV-2) $a + 0 = a$;
(MV-3) $a + b = b + a$;
(MV-4) $a + 1 = 1$;
(MV-5) $(a^*)^* = a$;
(MV-6) $0^* = 1$;
(MV-7) $a + a^* = 1$;
(MV-8) $(a^* + b)^* + b = (a + b^*)^* + a$.

Given an MV-algebra (after Mangani [M]), we can define

$a \circ b = (a^* + b^*)^*$;
$a \wedge b = (a + b^*) \circ b$;
$a \vee b = (a \circ b^*) + b$.

Then $a \leq b$ iff $a = a \wedge b$ is a partial order in $\mathcal{A}$, and $(\mathcal{A}, \wedge, , \vee)$ is a distributive lattice.

If we put $b \ominus a := (a + b^*)^*$ for $a \leq b$, then we obtain that $\mathcal{A}$ is a D-poset, more exactly, a distributive D-lattice ([K,Ch 2]). Moreover, $b - a := b \ominus (a \wedge b) = b \circ a^*$ for any $a, b \in \mathcal{A}$

Conversely, in any D-lattice we can define a binary operation $b - a := b \ominus (a \wedge b)$ for any $a, b \in \mathcal{A}$.

Relations between D-lattices and MV-algebras were found in [K,Ch 2]: A D-lattice P is an MV-algebra if and only if for any $u, v, w \in P$,

$$(w - u) - v = (w - v) - u \tag{4.1}$$

The following simple example shows that not every distributive D-lattice is an MV-algebra. Let $P = \{0, a, b, 1\}$ and let the domain of the operation $\oplus$ be the binary relation $S = \{(0,0), (0,a), (0,b), (a,a), (b,b)\}$ (so-called diamond, see [G,F,P]). Indeed, we have $(1-a)-b = a \neq b = (1-b)-a$. We note that in [G,F,P], there was proved that every finite, distributive D-lattice can be written as a direct sum of diamonds and finite chains. This implies that a finite, distributive D-lattice is an MV-algebra iff it does not contain a diamond as a direct summand.

For our purposes, another characterization of MV-algebras is more appropriate (see also [K,Ch 3]). For the convenience of readers, we include the proof.

Lemma 4.1. *A D-lattice P is an MV-algebra if and only if for any $a, b \in P$,*

$$a \ominus (a \wedge b) = (a \vee b) \ominus b. \tag{4.2}$$

Proof. Define $a^* := 1 \ominus a$, $a + b := a \oplus (a^* \wedge b)$. Clearly,

$$\begin{aligned} &(a^*)^* = a, \\ &a \leq b \implies b^* \leq a^*, \\ &a \perp b \implies a + b = a \oplus b, \\ &a \leq c \implies a + b \leq c + b. \end{aligned}$$

Commutativity of the operation $+$ requires

$$a \oplus (a^* \wedge b) = b \oplus (b^* \wedge a),$$

equivalently, $a^* \ominus (a^* \wedge b) = (b \vee a^*) \ominus b$. This shows that the condition is necessary. To show sufficiency, assume that the condition is satisfied. We have to check the MV-algebra axioms. (MV-2), (MV-3), (MV-4), (MV-5), (MV-6) and (MV-7) are immediate.

(MV-1): $(a+b)+c = (a \oplus (a^* \wedge b)) \oplus (a \oplus (a^* \wedge b))^* \wedge c \leq (a \oplus (a^* \wedge c)) \oplus (a^* \wedge b) \leq (a+c)+b$; by symmetry we get the equality.

(MV-8): $a + (a+b^*)^* = a \oplus a^* \wedge (a \oplus (a^* \wedge b^*))^* = a \oplus (a^* \wedge (a^* \ominus (a^* \wedge b^*))) = a \oplus (a^* \ominus (a^* \wedge b^*) = (a^* \ominus (a^* \ominus (a^* \wedge b^*)))^* = (a^* \wedge b^*)^* = a \vee b$. By symmetry also $b + (b+a^*)^* = a \vee b$. $\square$

The following proposition puts into relation MV-algebras and minimal clans.

Proposition 4.2. *Every MV-algebra is a commutative, positive minimal clan with a greatest element 1.*

Conversely, every interval $[0, x]$ in a commutative minimal clan E can be organized into an MV-algebra defining $y^{*_x} := x \ominus y$, $y +_x z := (z^{*_x} \wedge y) \oplus z$, for $0 \leq y, z \leq x$. Moreover if $0 \leq x \leq y \leq z$, then $x^{*_y} \leq x^{*_z} \wedge y$, the equality holds iff $y \wedge (z \ominus y) = 0$.

Proof. Let $\mathcal{A}$ be an MV-algebra. Put $b \ominus a := (a + b^*)^*$ iff $a \leq b$. According to [K,Ch 2], $(\mathcal{A}, \leq, \ominus)$ becomes a distributive D-lattice. Therefore, with $a \oplus b := (a^* \ominus b)^*$ for $b \leq a^*$, axioms (MC-1) - (MC-4) of the definition of a minimal clan are satisfied. (MC-5) is also satisfied, taking into account that $\mathcal{A}$ is a (distributive) lattice. (MC-6) follows by Lemma 4.1. Hence $\mathcal{A}$ is a commutative minimal clan.

Conversely, if E is a commutative minimal clan, then every interval $[0,a]$ is clearly a GDL. Define

$$x^* = a \ominus x,$$
$$x + y = x \oplus (x^* \wedge y),$$

We have $x+y = x \oplus (x^* \wedge y) = x \oplus ((a \ominus x) \wedge y) \leq x \oplus (a \ominus x) = a$, $a \ominus x \leq a$. By Theorem 3.2 (ii), condition (4.2) of Lemma 4.1 is satisfied, therefore $([0,a], +, ^*)$ is an MV-algebra.

To prove the remaining statement, assume $x \leq y \leq z$. Then $y \ominus x \leq z \ominus x$, and $z \ominus x = (y \ominus x) \oplus (z \ominus y)$. This implies $y \wedge (z \ominus x) \leq (y \ominus x) \oplus (y \wedge (z \ominus y))$, the last inequality holds by [S], Cor. 1.1.16. Taking into account that $y \ominus x \leq y \wedge (z \ominus x)$ we obtain the desired result. □

Corollary 4.3. *A D-lattice is an MV-algebra if and only if one of the equivalent conditions of Theorem 3.2 is satisfied.*

In [H,P], there was shown that every generalized difference poset is an order ideal of a difference poset. Similar results have been already obtained for particular structures: generalized orthomodular lattices (which are order ideals of orthomodular lattices) [J], [Be] and (weak) generalized orthomodular posets (which are order ideals of orthomodular posets) [M-I].

Let $(P, \leq 0, \ominus)$ be a generalized difference poset. Let $P^\sharp$ be a set disjoint from P with the same cardinality. Consider a bijection $a \mapsto a^\sharp$ from P onto $P^\sharp$ and let us denote $P \cup P^\sharp = \hat{P}$. Define a partial binary operation $\ominus^*$ on $\hat{P}$ by the following rules $(a, b \in P)$:

(i) $b \ominus^* a$ is defined iff $b \ominus a$ is defined and $b \ominus^* a = b \ominus a$.
(ii) $b^\sharp \ominus^* a$ is defined iff $a \oplus b$ is defined and $b^\sharp \ominus^* a = (a \oplus b)^\sharp$.
(iii) $b^\sharp \ominus^* a^\sharp$ is defined iff $a \ominus b$ is defined and $b^\sharp \ominus^* a^\sharp = a \ominus b$.

Define a binary relation $\leq^*$ on $\hat{P}$ as follows: $x \leq^* y$ if and only if $y \ominus^* x$ is defined. In [H,P], it was shown that $(\hat{P}, \leq^*, 0, 0^\sharp, \ominus^*)$ is a difference poset. Moreover, P is a proper order ideal closed under $\oplus^*$ in $\hat{P}$.

It follows that every minimal clan can be embedded into a D-poset.

Example 4.4. Let $\mathbb{R}^+$ denote the set of all positive real numbers. With the natural order and operation $\oplus$ defined as the usual addition, $\mathbb{R}^+$ becomes a commutative, positive minimal clan. Moreover, the $\oplus$ operation is totally defined. From the construction of $\hat{\mathbb{R}^+}$ it follows that $a \leq b^\sharp$ for any $a, b \in \mathbb{R}^+$. We obtain that $\hat{\mathbb{R}^+}$ is a D-lattice. Moreover, it is an MV-algebra. Indeed, it suffices to check (4.2). We have $(a \vee b^*) \ominus a = b^* \ominus a = b^* \ominus (b^* \wedge a)$, $(a \vee b^*) \ominus b^* = 0 = a \ominus (a \wedge b^*)$, the remaining cases are left to the reader.

It is easily seen that the latter example can be generalized as follows.

Proposition 4.5. *Let $(E, S, \oplus)$ be a positive, commutative minimal clan such that $S = E$. Then E can be embedded as an order ideal into an MV-algebra.*

5. Semi-MV-algebras

The following definition is a generalization of the notion of a semi-Boolean algebra [A].

Definition 5.1. *A meet semi-MV-algebra is a meet semilattice with a difference in which every principal ideal is an MV-algebra.*

Observe that a semi-MV-algebra is a positive, commutative minimal clan iff it is a lattice.

Example 5.2. Let X be a nonempty set and let $\mathcal{F}(X)$ denote the a class of positive-valued functions defined on X. Define an order on $\mathcal{F}$ pointwise and assume $\mathcal{F}$ contains any function $g : X \to \mathbb{R}^+$ such that $g \leq f$ for some $f \in \mathcal{F}$. Define the difference operation $\ominus$ by $(f \ominus g)(x) := f(x) - g(x)$ iff $g \leq f$. Clearly, $f \ominus g \leq f$, hence $f \ominus g \in \mathcal{F}$. Also, the zero function $f(x) = 0$ for all $x \in X$ is the smallest element in $\mathcal{F}$, and $f, g \in \mathcal{F} \implies f \wedge g(x) := min(f(x), g(x))$ belongs to $\mathcal{F}$, so that $\mathcal{F}$ is a semilattice. It is easy to see that the partially defined operation $\ominus$ satisfies properties (D1) and (D2) of Def. 2.1, and also (GD1) and (GD2) are satisfied, whence $\mathcal{F}$ is a GDP.

For any $h \in \mathcal{F}$, denote by $\mathcal{F}_h := \{f \in \mathcal{F} : f \leq h\}$ the principal ideal generated by h. Define on $\mathcal{F}_h$ a binary operation $+$ and a unary operation * as follows:

$$(f+g)(x) := \begin{cases} f(x) + g(x) \text{ if } f(x) + g(x) \leq h(x) \\ h(x) \ \ \text{otherwise} \end{cases}$$

and

$$f^*(x) := h(x) \ominus f(x).$$

Clearly, $f + g, f^* \in \mathcal{F}$. Moreover, $\mathcal{F}$ with these operations and with unit h forms an MV-algebra. Indeed, axioms (MV-2)-(MV-6) are immediate. (MV-1): Let $p, q, r \in \mathcal{F}_h$. Then

$$((p+q)+r)(x) = \begin{cases} p(x) + q(x) + r(x) \text{ if } p(x) + q(x) + r(x) \leq h(x), \\ h(x) \text{ otherwise.} \end{cases}$$

(MV-8): a direct calculation shows that

$$h(x) - ((h-f)+g)(x) + g(x) = min\{f(x), g(x)\}.$$

Example 5.3. Let L be a D-poset (effect algebra). For any natural number $n \in \mathbb{N}$ and $a \in L$, define $n.a$ in the following way:

$$0.a := 0, \ (n+1).a := n.a \oplus a,$$

if all desired elements are defined. We denote by $\mu(a)$ the greatest number n such that $n.a$ is defined, with the understanding that $\mu(a) = \infty$ if $n.a$ exists for any $n \in \mathbb{N}$. We say that a function $F : L \to \mathbb{N}$ is *summable* if the element $\oplus F := \oplus_{a \in L} F(a).a$ exists. That is, if F is summable then the support $N_F := \{a \in L : F(a) \neq 0\}$ is finite, $F(a) \leq \mu(a)$ for any $a \in L$.

Denote by $\mathcal{F}(L)$ the set of all summable functions on L. With usual pointwise ordering and substraction of functions, $\mathcal{F}(L)$ forms a GDP. Moreover, if F is summable and $G \leq F$, where $G : L \to \mathbb{N}$, then G is summable too, therefore $\mathcal{F}(L)$ is a meet semilattice. Similarly as in Example 5.1, we show that $\mathcal{F}(L)$ is a semi-MV-algebra.

It is easy to see that summable functions are in one-to-one relation with finite $\oplus$-summable sequences of elements of L. Partitions of unity, that is, summable functions the sum of which equals 1, are maximal elements in $\mathcal{F}(L)$. Moreover, every summable function is majorized by a maximal element.

Let us denote by $\mathcal{M}$ the set of all maximal elements in $\mathcal{F}(L)$. We say that $F, G \in \mathcal{F}(L)$ are *local complements* of each other if there is $H \in \mathcal{M}$ such that $F + G = H$. Observe that $\oplus(F+G) = \oplus F \oplus G = 1$. We will say that F and G are *perspective* (written $F \sim G$) if they share a common local complement. Observe that $F \sim G$ iff $\oplus F = \oplus G$. It follows that if F and G share one local complement then they share all local complements (i.e., they are *strongly perspective*).

Let us recall that a function $f : L \to P$ from a GDP L to a GDP P is a *homomorphism* if $\forall a, b \in L, a \perp b \implies f(a) \perp f(b)$ and $f(a) \oplus f(b) = f(a \oplus b)$. A homomorphism f is *faithful* if $f(a) \perp f(b) \implies a \perp b$. By a *faithful congruence* on a GDP L we mean an equivalence relation $\sim$ on L such that for all $a, b, c \in L$, $a \sim b$ and $a \perp c$ imply $b \perp c$ and $a \oplus c \sim b \oplus c$. Denote by $[a]$ the equivalence class of $a \in L$ under a faithful congruence $\sim$ on L, then the partial operation $[a] \oplus [b] = [a \oplus b]$ provided $a \perp b$ is well-defined, and makes $L/\sim$ a GDP. The map $[\,] : L \to L/\sim$ is a faithful homomorphism.

According to [W], perspectivity $\sim$ on $\mathcal{F}(L)$ with respect to $\mathcal{M}$ is a faithful congruence, and the quotient $\mathcal{F}(L)/\sim$ is isomorphic with L.

Using the results in [W] and [P,W], the following result can be obtained.

Proposition 5.4. *Let $\mathcal{A}$ be a semi-MV-algebra with the following properties:*

(1) *There is a set of maximal elements $\mathcal{M}$ in $\mathcal{A}$ such that every element in $\mathcal{A}$ is majorized by an element in $\mathcal{M}$.*
(2) *Every two perspective elements in $\mathcal{A}$ (with respect to $\mathcal{M}$) are strongly perspective.*

Then the perspectivity relation is a faithful congruence and the quotient $\mathcal{A}/\sim$ is a D-poset (an effect algebra).

From this we may conclude that there is a one-to-one relation between effect algebras and semi-MV-algebras with properties (1) and (2). We note that these semi-MV-algebras are not lattices, hence not minimal clans.

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