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Open problems more or less related to generalized Pascal triangles

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OPEN PROBLEMS MORE OR LESS RELATED TO GENERALIZED PASCAL TRIANGLES

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The presented paper is a revised (and slightly extended) text of the author's talk "Open Problems" at June 1st, 1995, in the seminar of Prof. D. Richard, LLAIC, Université d'Auvergne Clermont-Ferrand.

New information is collected as ADDENDUM at the end of the paper; also some items in References were actualized.

1. Introduction

The problems presented below mostly arised by the author's study of generalized Pascal triangles (GPT), in particular by the study of the theories of (structures related to) GPT. However, in many cases the notion of GPT could be eliminated from the formulation of the problems, and the author tries to do that in all cases when the elimination is not very artificial.

Two kind of open questions are distinguished in what follows. The first kind are "problems" which are formulated so that it will be clear whether a paper written about them solves them or not. (One possibility is such a formulation which requires yes/no answer.) "Conjecture" is only another grammatical form for "problem" (it may be false). The other kind are "proposals". A "proposal" suggests a rather narrow area of investigation, which goals are not determined clearly enough to form a "problem". Maybe, several papers can be written about, the depth of obtained results will vary very much, and their evaluation will be subjective.

A typical example of (bad) proposal is "Find necessary and sufficient conditions for X". (Distinguish it e.g. from "Find necessary and sufficient conditions for X in terms of ..."; the added part can be sometimes clear from the content.) The result can be of the form "X if and only if Y". One referee or reader can say that a very interesting characterization of X was found, and another one can say that Y is only a trivial modification of X. Much worse is that both evaluations can be made by the same referee, according to his relation to the author or simply according to his momental mood. Maybe, this is not typical, but it is surely possible. Of course, similar situation can happen also with solutions of "problems", but it is much less probable. Therefore the author prefers "problems". However, sometimes the form of "problems" is too restrictive, and "proposals" are more suitable. In particular, "proposals" are more suitable when the form of expected results is not clear yet.

"Claims" show the results (known to the author) related to given problems, proposals or conjectures. (While a false conjecture ought not be considered as a mistake of the author, a false claim can be.)

$\mathbb{N}$ will denote the set of natural numbers (including 0), $\mathbf{s}$ the successor, $+$, $\times$ $\leq$ the usual arithmetical operations and the usual order. A^+ will denote the set of all nonempty words in the alphabet A. Now we shall define generalized Pascal triangles; this notion will explicitly occur in some problems below, and was implicitly used in some of the others.

Definition 1. *For every finite algebra $\mathcal{A} = \langle \mathbf{A}; *, \mathrm{o} \rangle$ (with $\mathbf{A} \subseteq \mathbb{N}$ to avoid some technical problems) and every word $w = w_0 \dots w_{|w|-1} \in \mathbf{A}^+$ the function $G = \mathrm{GPT}(\mathcal{A}, w)$ will be the partial binary operation on $\mathbb{N}$ defined by*

$$G(x,y) = \begin{cases} \text{undefined} & \text{if} \quad x+y < |w|-1, \\ w_x & \text{if} \quad x+y = |w|-1, \\ \mathrm{o} * G(0, y-1) & \text{if} \quad x=0, \ y \geq |w|, \\ G(x-1, 0) * \mathrm{o} & \text{if} \quad y = 0, \ x \geq |w|, \\ G(x-1, y) * G(x, y-1) & \text{if} \quad x+y \geq |w|, \ x > 0, \ y > 0. \end{cases}$$

The functions of the form $\mathrm{GPT}(\mathcal{A}, w)$ ($\mathcal{A}$ finite) will be called GPT.

Definition 2. *Let us denote $B_n^w = \mathrm{GPT}(\mathcal{N}_n, w)$, where $\mathcal{N}_n$ denotes the cyclic group modulo n and $w \in \{0, 1, \dots, n-1\}^+$; these functions will be called GPT modulo n.*

Further, let $B_n = B_n^1$ denote the Pascal triangle modulo n;

Notice that $B_n(x,y) = \binom{x+y}{x} \mathrm{MOD}\, n$ for all $x, y \in \mathbb{N}$.

Some further definitions are given below. Missing ones can be found in [K85a], [K89] (older versions, with algebras of signature $(0,1,1,2)$) or [K93b], [K94]. The reference list is a small bibliography; it shows also related papers which are not cited in the paper.

"Definability" means first order (or: elementary) definability unless something else is explicitly mentioned. If definability in a class of structures is considered then the defining formula must be the same for all structures (we could speak about *uniform* definability). WMSO abbreviates "weak monadic second order".

2. Weakly nonstandard GPT

Problem 2.1. Let us consider the elementary theory of all GPT as structures $\langle \mathbb{N}; \mathbf{s}, G \rangle$ where G is a GPT. Has this theory a model of the form $\langle \mathbb{N}; \mathbf{s}, H \rangle$, where the range of H is infinite?

Notice that GPT are associated to *finite* algebras, hence the range of any GPT is finite. The model above will be called *a weakly nonstandard model* of the theory of GPT ("weakly" because $\langle \mathbb{N}; \mathbf{s} \rangle$ remains standard). Of course, the mentioned theory has nonstandard models (e. g. uncountable ones).

Claim 2.2. The answer to Problem 2.1 would be negative if $\mathbf{s}$ is replaced by $\leq$ or by $+$. It also would be negative if the condition $\mathbf{A} \subseteq \mathbb{N}$ (in Definition 1) is replaced by "$\mathbf{A}$ is an initial segment of $\mathbb{N}$".

In each of these cases the property "range of G is finite" can be expressed by a first order formula (provided $\langle \mathbb{N}; \mathbf{s} \rangle$ remains standard).

3. Frequencies and asymptotical densities in GPT

Proposal 3.1. Investigate the growth of functions

$$Occ_{n,k}(z) = \mathrm{card}\{(x,y) \mid x+y < z \wedge B_n(x,y) = k\}.$$

By [Sin] we have $\lim_{z\to\infty} Occ_{n,k}(z)/z^2 = 0$ for all $0 < k < n$.

Conjecture 3.2. It holds $\lim_{z\to\infty} Occ_{6,5}(z)/Occ_{6,2}(z) = 0$.

Proposal 3.3. Investigate asymptotical densities of elements in GPT of small algebras.

Already algebras of cardinality 3 are interesting. If the cardinalities of algebras are not fixed then the (upper and lower) densities of some elements are almost arbitrary; only some weak recursivity conditions must be fulfilled.

Let $G = \mathrm{GPT}(\mathcal{A}, w)$ be a GPT and $\mathbf{A} = \{a_1, \ldots, a_m\}$, $a_1 < \cdots < a_n$. The *frequency sequence* of G will consists of n-dimensional vectors. Its kth number will be $(\frac{m_1}{m}, \ldots, \frac{m_n}{m})$, where m_i denotes the number of occurrences of a_i in the first k rows of G and $m = m_1 + \cdots + m_n$.

Proposal 3.4. Investigate frequency sequences of GPT and sets of their limit points.

Main results of [K86a] can be formulated as theorems about projections of sets of limits points to n coordinate axis. These projections are segments whose endpoints are the lower and the upper asymptotical density of the corresponding $a_i \in \mathbf{A}$.

4. Definability of $+$, $\times$ from cubes and Pascal triangle modulo 2

Problem 4.1. Are the operations $+$, $\times$ definable in the structure $\langle \mathbb{N}; B_2, \mathrm{Cub} \rangle$, where $\mathrm{Cub} = \{i^3 \mid i \in \mathbb{N}\}$?

If $+$ is definable then $\times$ is definable, too.

Claim. The answer to Problem 4.1 would be positive if Cub is replaced by the set of squares and negative if Cub is replaced by the set of 4-th powers.

Problem 4.2. Solve the exponential Diophantine equation

$$2^{6x_3+4} + 2^{6x_2+3} + 2^{6x_1+1} + 1 = y^3.$$

I conjecture that the unique solution is $y = 3$, $x_i = 0$.

If there are only finitely many solutions then the answer to Problem 4.1 is positive. (An explicit bound for y is necessary if the defining formula ought to be written.) The equation in (4.2) can be extended to a system of equations which looks more complicated but can be easier to solve.

Another statement which would suffice to solve Problem 4.1 is:

Conjecture 4.3. For every $k > 0$ there are odd cubes x, y such that in the octal number system

(1) every of x, y has the only odd digit (the last one);
(2) the kth digit of x is 2 or 6;
(3) the kth digit of y is 4 or 6.

A more general conjecture is given below; it is easier because the structure contains f itself, and not the range of f.

Conjecture 4.4. If $f(x)$ is a polynomial of degree at least 2 which maps $\mathbb{N}$ into $\mathbb{N}$ then $+$, $\times$ are definable in the structure $\langle \mathbb{N}; B_2, f \rangle$.

Remark. There are many further problems concerning Pascal triangle modulo 2 (or, more generally, modulo n). However, they will not be listed there because many of them were already solved by the author and by A. Bès. Maybe, a separate list concerning the theories of related structures will be prepared.

5. Why is the arithmetic undecidable

Is the theory of $\langle \mathbb{N}; +, \times \rangle$ undecidable simply because of simultaneous presence of the operations $+$, $\times$ or because of special relationship between these operations? More precisely:

Problem 5.1. Are there binary operations $\oplus$, $\otimes$ on the set $\mathbb{N}$ such that $\langle \mathbb{N}; \oplus \rangle$ is isomorphic with $\langle \mathbb{N}; + \rangle$, the structure $\langle \mathbb{N}; \otimes \rangle$ is isomorphic with $\langle \mathbb{N}; \times \rangle$ and the elementary theory of $\langle \mathbb{N}; \oplus, \otimes \rangle$ is decidable?

The formulation in 5.1 is rather strong. For some weaker formulation the answer could be positive. For example, $\langle \mathbb{N}; \times \rangle$ has decidable theory, and contains two disjoint substructures isomorphic to $\langle \mathbb{N}; + \rangle$ and $\langle \mathbb{N}; \times \rangle$. An example with two-sorted structure is given below ($\mathbb{N}^{\mathrm{x}}$, $\mathbb{N}^{\mathrm{y}}$ are copies of $\mathbb{N}$).

Claim 5.2. The elementary theory of two-sorted structure $\langle \mathbb{N}^{\mathrm{x}}, \mathbb{N}^{\mathrm{y}}; +, \times \rangle$, where $+$ operates on $\mathbb{N}^{\mathrm{x}}$ and $\times$ operates on $\mathbb{N}^{\mathrm{y}}$, is decidable.

We give a related claim for one-sorted (i. e. usual) structure but special arithmetical formulae. Let us call an arithmetical formula $\{+, \times\}$-*separated* if all variables occurring in it can be divided into two disjoint groups V_+, $V_\times$ so that every atomic subformula

either contains variables from V_+ only and does not contain $\times$
or contains variables from $V_\times$ only and does not contain $+$.

(Defined symbols are not allowed, or special conditions for the definitions are requested.)

Claim 5.3. The set of true $\{+, \times\}$-separated formulae of arithmetic is decidable.

Claim 5.3 generalizes both decidability of Presburger arithmetic (the theory of $\langle \mathbb{N}; + \rangle$) and decidability of Skolem arithmetic (the theory of $\langle \mathbb{N}; \times \rangle$).

The following problems related to the above are more general, and probably easier.

Conjecture 5.4. If $A = \{a_0, a_1, a_2, \dots\} \subseteq \mathbb{N}$ is a recursive set and $a_{n+1} > n + a_n$ for all $n \in \mathbb{N}$ then the elementary theory of $\langle \mathbb{N}; \mathbf{s}, A \rangle$ is decidable.

Conjecture 5.5. There are binary relations R, S on the set $\mathbb{N}$ such that:

1. The theories of the structures $\langle \mathbb{N}; R \rangle$ and $\langle \mathbb{N}; S \rangle$ are decidable and
2. The theory of $\langle \mathbb{N}^{\mathrm{x}}, \mathbb{N}^{\mathrm{y}}; R, S \rangle$, where $R \subseteq \mathbb{N}^{\mathrm{x}} \times \mathbb{N}^{\mathrm{x}}$ and $S \subseteq \mathbb{N}^{\mathrm{y}} \times \mathbb{N}^{\mathrm{y}}$, is undecidable.

Similar problem can be asked for higher arities, and also for finite systems of relations on $\mathbb{N}$.

Claim 5.6. Conjecture 5.4 implies Conjecture 5.5.

As a reaction to Problem 5.1, P. Pudlák formulated the following problem:

Problem 5.7. Are there two decidable structures such that if one of them is arbitrarily put over the other then the obtained structure is undecidable?

(P. Pudlák's formulation was similarly informal and in Czech; a strict formulation can be like in Problem 5.1.) Conjecture 5.5 (if true) gives also positive answer to 5.7.

6. Undecidability caused by unbounded number of quantifiers

Problem 6.1. Is there a structure $\mathcal{S} = \langle \mathbb{N}; \mathbf{s}, R_1, \dots, R_k \rangle$ (of finite signature, $\mathbf{s}$ the successor) such that

(i) the elementary theory of $\mathcal{S}$ is undecidable and

(ii) for every n the set of all Σ_n formulae true in $\mathcal{S}$ is decidable?

Σ_n formulae are defined as formulae in prenex normal form with n alternating groups of quantifiers, existential first.

For $\mathcal{S}$ without $\mathbf{s}$ (successor) L. Heindorf (Berlin) and A. Pinus (Novosibirsk) independently claimed the positive answer with $k = 1$ and a linear order R_1 which type is $\dots + \omega^{n_3} + \omega^{n_2} + \omega^{n_1} + \omega^{n_0}$, where $(n_0, n_1, n_2, n_3, \dots)$ is the increasing list of elements of a non-recursive set of positive integers.

7. Strength of binary relations in Presburger arithmetics

Conjecture 7.1. Let $R_1, \dots, R_k$ be symmetric binary relations on $\mathbb{N}$ which are definable in $\langle \mathbb{N}; \mathbf{s} \rangle$. Then $\mathbf{s}$ is not definable in $\langle \mathbb{N}; R_1, \dots, R_k \rangle$.

Conjecture 7.2. There are no semilinear binary relations $R_1, \dots, R_k$ such that $+$ is definable in $\langle \mathbb{N}; R_1, \dots, R_k \rangle$.

However, there are unary functions (i.e. special binary relations) $f_1, \dots, f_k$ definable in $\langle \mathbb{N}; +, \times \rangle$ such that $+, \times$ are definable in $\langle \mathbb{N}; f_1, \dots, f_k \rangle$.

We shall show also some formulations with GPT. Let the word *semilinear* will be understood as "definable in Presburger arithmetic", i.e. in the theory of $\langle \mathbb{N}; + \rangle$.

Conjecture 7.3. If the successor function $\mathbf{s}$ is definable in a structure $\langle \mathbb{N}; G \rangle$, where G is a commutative GPT (i.e., always $G(x, y) = G(y, x)$) then G is not definable in $\langle \mathbb{N}; \mathbf{s} \rangle$.

Conjecture 7.4. For no semilinear GPT G the addition is definable in $\langle \mathbb{N}; G \rangle$.

Conjecture 7.5. Every binary semilinear relation is definable in a semilinear GPT (but not all of them in one such GPT).

Remark. Some of problems of this section probably will be solved soon in common paper of J. Tomasik and the author.

8. Low level definability

Proposal 8.1. Investigate the definability power of the structure $\langle \mathbb{N}; \mathrm{Neib} \rangle$ where $\mathrm{Neib} = \{(i, j) \mid i = \mathbf{s}(j) \vee j = \mathbf{s}(i)\}$.

Its definability power is weaker than that of $\langle \mathbb{N}; \mathbf{s} \rangle$ because $\mathbf{s}$ (successor) is not definable in $\langle \mathbb{N}; \mathrm{Neib} \rangle$.

Conjecture 8.2. $\langle \mathbb{N}; \mathrm{Neib} \rangle$ is a weakest structure where every natural number is definable.

The structures are compared in the sense of (elementary) definability; so a quasiordering is obtained. However, also after the usual factorization (when a partial order arises) "weakest" must be understood in the sense "minimal", not "smallest" (therefore "a weakest" instead of "the weakest" is used in 8.2).

If suitable, restrict the considerations to the structures weaker than $\langle \mathbb{N}; \mathbf{s} \rangle$.

9. Computer arithmetics

Let $\mathbb{N}_n = \{0, 1, \ldots, n-1\}$. Let $\oplus$, $\otimes$ denote the arithmetical operations modulo n, and let $<$ denote the restriction of the usual order to the set $\mathbb{N}_n$. Further, let $\mathrm{Pow}_2 = \{2^k \mid k \in \mathbb{N}\}$.

Conjecture 9.1. The elementary theory of $\{\langle \mathbb{N}_n; \oplus, \otimes \rangle \mid n \in \mathrm{Pow}_2\}$ is decidable.

Claim 9.2. Every infinite subclass of $\{\langle \mathbb{N}_n; \otimes, < \rangle \mid n > 1\}$ (where $\otimes$ denotes multiplication modulo n and $<$ is the restriction of usual ordering to $\mathbb{N}_n$) has undecidable elementary theory.

As easy consequences we can obtain:

Claim 9.3. The elementary theory of $\{\langle \mathbb{N}_n; \oplus, \otimes, < \rangle \mid n \in \mathrm{Pow}_2\}$ is undecidable.

Conjecture 9.4. $<$ is not definable in the theory from 9.1.

There are good reasons to call the theories from 9.1 and 9.3 *computer arithmetics*; various computers (and compilers) use $n = 2^k$ for $k = 8, 16, 24, 32, 37, 64$ (sometimes several values simultaneously). Then 9.3 says that computer arithmetic with $<$ is undecidable and 9.1 says that computer arithmetic without $<$ is decidable.

Compare with a result from [Ax] that the class $\{\langle \mathbb{N}_p; \oplus, \otimes \rangle \mid p \text{ prime}\}$ has decidable elementary theory. See [Ye, Th. 7 on p. 290]. However, if we add $<$ we obtain by 9.2 an undecidable theory. Therefore $<$ is not uniformly definable in the considered class of fields.

Remarks. (1) The structures $\langle \mathbb{N}_n; \oplus, \otimes, < \rangle$ are not ordered rings. Nevertheless, we can replace $<$ by a set Pos of "positive" elements, and then define $<$ using Pos (and $\oplus$).

(2) Computers usually work with the sets $\{x \in \mathbb{N} \mid -\frac{n}{2} \leq x < \frac{n}{2}\}$ instead of $\mathbb{N}_n$. Moreover, the computer $<$ differs from the above, and even is not transitive. However, the differences are not substantial from definability and decidability point of view.

10. Embeddability of partial groupoids into finite semigroups

Problem 10.1. Is the following algorithmic problem decidable?

> *For any finite partial groupoid decide whether it can be embedded into a finite semigroup.*

Proposal 10.2. Give (effective and simple) necessary and/or sufficient conditions for embeddability of a partial groupoid into a finite semigroup.

Results could be useful in solving decidability of symbol (or word) occurence problem for GPT of associative algebras. We show one of such problems.

Problem 10.3. Is the following algorithmic problem decidable?

> *For any associative finite algebra $\mathcal{A} = \langle \mathbf{A}; *, \circ \rangle$ and $w, x \in \mathbf{A}$ determine whether x occurs in* GPT$(\mathcal{A}, w)$.

Claim. The same problem for commutative finite algebras is undecidable.

Many similar problems were considered in [K85a] and [K86].

11. Special algebraic integers

A special set $\mathbb{K}$ of reals was introduced in [K93a]. It is used in a theorem that if $\alpha, \beta \in \mathbb{K}$, $b \in \mathbb{N}$ and $0 < \alpha\beta < b$ then there is a one-dimensional cellular automaton with speeds of configuration growth equal to $\frac{\log \alpha}{\log b - \log \beta}$. We give the definition of $\mathbb{K}$.

Definition 11.1. (a) *A finite sequence* $(\beta_1, \beta_2, \dots, \beta_n)$ *of positive reals will be called a* $\mathbb{K}$*-system if there are nonnegative integers* a_{ijk}, $i, j, k \in \{1, 2, \dots, n\}$, *such that for all* $i, k \in \{1, 2, \dots, n\}$ *it holds* $\beta_i \cdot \beta_k = \sum\limits_{j=1}^{n} a_{ijk} \cdot \beta_j$

(b) *A* $\mathbb{K}$*-system* $(\beta_1, \beta_2, \dots, \beta_n)$ *will be called a* $\mathbb{K}$*-system for a real* α *if there are nonnegative integers* $b_1, b_2, \dots, b_n$ *such that* $\alpha = b_1 \cdot \beta_1 + b_2 \cdot \beta_2 + \dots + b_n \cdot \beta_n$

(c) $\mathbb{K}$ *is the set of all reals for which a* $\mathbb{K}$*-system exists.*

Proposal 11.2. Investigate the properties of $\mathbb{K}$.

Conjecture 11.3. If $a, b \in \mathbb{K}$ and $a + b \in \mathbb{N}$ then $a, b \in \mathbb{N}$.

Conjecture 11.4. If $a, b \in \mathbb{K} \setminus \{0\}$ and $ab \in \mathbb{N}$ then $a = \sqrt[n]{x}$ for some $x, n \in \mathbb{N}$.

Let $\mathbb{Q}$ denote the set of rationals and $\mathbb{Q}(a)$ the field generated by a.

Conjecture 11.5. If $a, b \in \mathbb{K}$ and $\mathbb{Q}(a) = \mathbb{Q}(b)$ then $\mathbb{Q}(a + b) = \mathbb{Q}(a)$.

Problem 11.6. Does $\mathbb{K}$ contain $3 - \sqrt{2}$?

Of course, the latest problem is not very important itself, but the solution will probably help to find some properties of $\mathbb{K}$. For example, positive answer to 11.6 would give a counterexample to 11.3–11.5. Some known properties of $\mathbb{K}$ are given below; (a)–(c) show that $\mathbb{K}$ is sufficiently rich, (d)–(e) show that $\mathbb{K}$ is not too rich.

Claim 11.7. (a) The set $\mathbb{K}$ contains all nonnegative (rational) integers.

(b) The set $\mathbb{K}$ is closed under addition and multiplication.

(c) If $\alpha_0, \alpha_1, \dots, \alpha_{n-1} \in \mathbb{K}$ and γ is the positive root of the equation $\gamma^n = \alpha_{n-1}\gamma^{n-1} + \dots + \alpha_1\gamma + \alpha_0$ then $\gamma \in \mathbb{K}$.

(d) The set $\mathbb{K}$ contains no negative real and no real x, $0 < x < 1$.

(e) All elements of $\mathbb{K}$ are algebraic integers.

Remark. For many elements $\alpha \in \mathbb{K}$ there is a $\mathbb{K}$-system (for α) of the form

$$(1, \alpha, \alpha^2, \dots, \alpha^{m-1});$$

in the simplest case m is the degree of α. However, in some cases m must be greater; this happen e.g. for the positive real α satisfying $\alpha^3 = 6\alpha^2 - 2\alpha + 6$, where $m = 4$ is necessary. There are $\beta \in \mathbb{K}$ for which no $\mathbb{K}$-system of the above form exists. Examples can be found in [K93a].

12. Clones of partial functions with universal partial functions

In the computational praxis primitive recursive functions suffice (and, maybe, even this class is too large). In the theory of computable functions universal partial functions are useful. Also closeness under composition probably ought to be requested. Is the set of partial recursive functions the smallest set with all these properties? More precisely:

Problem 12.1. Is there a proper subclone X of the clone of partial recursive functions which contains all primitive recursive functions and a binary partial function F which is universal for the set $X^{(1)}$ of all unary elements of X (i.e. $X^{(1)} = \{\lambda x.F(a, x) \mid a \in \mathbb{N}\}$).

Remark. A *clone* of partial functions on a set X is a set of partial functions which contains all projections (on X) and is closed under composition. Since projections on $\mathbb{N}$ are primitive recursive the word "subclone" in 12.1 means only "closed under composition".

13. Formalization of the notion of tape

Proposal 13.1. Formalize the notion of tape so that every tape will be a set of hereditarily finite sets. Is such formalization possible?

Hereditarily finite sets are elements of $\bigcup_{n\in\mathbb{N}} p_n$, where $p_0 = \emptyset$ and $p_{n+1} = \mathrm{P}(p_n)$ (the set of all subsets of p_n). Informally, they are finite sets whose elements are finite sets, whose elements are finite sets etc.

A tape consists of windows ordered into the type of $\mathbb{Z}$ (or: $\omega^* + \omega$), but are not enumerated; no "origin" exists. Each window contains a letter from an alphabet. By usual formalization tapes are minimal nonempty sets of functions with the domain $\mathbb{Z}$ closed under shifts. Hence a tape is a set of infinite sets.

Take the reals for explanation. Let rational numbers are constructed as hereditarily finite sets (e.g., ordered triples (sign,m,n) where $n \neq 0$ and $m, n \in \mathbb{N}$ are relatively prime). A real can be thought either as a set of Cauchy sequences of rational numbers or as (the lower part of) a cut of rational numbers. In the later construction reals are hereditarily finite sets, in the former they are not.

Proposal 13.2. Do the same for two-dimensional or higher-dimensional tapes.

If the requested formalization is found we can directly speak about (infinite) configurations of cellular automata already in Gödel–Bernays theory of finite sets (they will be proper classes).

14. Definability of plus and times from a random relation

We shall consider a (very natural) complete probability measure P on the set $\mathrm{P}(\mathbb{N}^2)$ of binary relations on $\mathbb{N}$. For every two finite sets $\mathbf{X} \subseteq \mathbf{Y} \subseteq \mathbb{N}^2$ we have

$$P(\{\mathbf{R} \subseteq \mathbb{N}^2 \mid \mathbf{R} \cap \mathbf{Y} = \mathbf{X}\}) = 2^{-\mathrm{card}(\mathbf{Y})}.$$

(If $\mathbf{Y}$ is infinite then the above probability is 0.) In particular, for every pair $(i,j) \in \mathbb{N}^2$ we have

$$P(\{\mathbf{R} \subseteq \mathbb{N}^2 \mid (i,j) \in \mathbf{R}\}) = P(\{\mathbf{R} \subseteq \mathbb{N}^2 \mid (i,j) \notin \mathbf{R}\}) = \frac{1}{2}\,,$$

and the events $(i,j) \in \mathbf{R}$ for distinct (i,j) will be pairwise independent.

We shall say that a statement A *holds for almost all* $\mathbf{R} \subseteq \mathbb{N}^2$ if

$$P\big(\{\mathbf{R} \subseteq \mathbb{N}^2 \mid A \text{ holds for } \mathbf{R}\}\big) = 1.$$

Further technical details can be found in [K93].

Conjecture 14.1. The arithmetical operations $+$, $\times$ are uniformly first order definable in the structures $\langle \mathbb{N}; \mathbf{s}, \mathbf{R}\rangle$ for almost all $\mathbf{R} \subseteq \mathbb{N}^2$.

Conjecture 14.2. The arithmetical operations $+$, $\times$ are uniformly first order definable in the structures $\langle \mathbb{N}; \leq, \mathbf{R}\rangle$ for almost all $\mathbf{R} \subseteq \mathbb{N}$.

Conjecture 14.3. If for almost all relations $\mathbf{R} \subseteq \mathbb{N}^2$ a relation $\mathbf{S}$ is /uniformly/ first order definable in the structure $\langle \mathbb{N}; \leq, \mathbf{R}\rangle$ then $\mathbf{S}$ is arithmetical.

Claim 14.4. For almost all relations $\mathbf{R} \subseteq \mathbb{N}^2$ all arithmetical relations (and hence $+$, $\times$, too) are uniformly first order definable in the structure $\langle \mathbb{N}; \leq, \mathbf{R}\rangle$.

Claim 14.5. For almost all unary relations $\mathbf{R} \subseteq \mathbb{N}$ all arithmetical relations (and hence also the operation $\times$) are uniformly first order definable in the structure $\langle \mathbb{N}; +, \mathbf{R}\rangle$.

15. APPROXIMATION OF EUCLIDEAN METRICS BY A GRAPH METRICS

Proposal 15.1. Construct an infinite graph $\mathcal{G} = \langle V; E \rangle$ with the following properties:

(1) The vertices of G are (some) points of the Euclidean plane.
(2) There is $R > 0$ such that every circle with radius R contains a point from V.
(3) There is $r > 0$ such that every circle with radius r contains at most one point from V.
(4) The graph distance d is asymptotically equal to the Euclidean distance ρ.

By the latest condition d cannot reach value ∞, hence $\mathcal{G}$ must be connected. Only the condition (4) is not strict enough to obtain a problem (in the sense of introduction). By more concrete choice of (4) we can obtain e.g. the following problems.

Problem 15.2. Solve 15.1 with item (4) replaced by:

(4a) There is $K > 0$ such that for all $x, y \in V$

$$|d(x,y) - \rho(x,y)| < K.$$

Problem 15.3. Solve 15.1 with item (4) replaced by:

(4b) There is $K > 0$ such that for all $x, y \in V$

$$|d(x,y) - \rho(x,y)| < K \cdot \sqrt{\rho(x,y)}.$$

The right-hand side of the inequality can be modified in many further ways. A very weak form is:

Problem 15.4. Solve 15.1 with item (4) replaced by:

(4c) There is a function F such that $\lim_{x\to\infty} \frac{F(x)}{x} = 0$ and for all $x, y \in V$

$$|d(x,y) - \rho(x,y)| < F(d(x,y) + \rho(x,y)).$$

A positive solution can be used instead of an underlined trellis in a variant of cellular automata where "speed of light" will be the same in all directions. Of course, the 3-dinensional variant of the above problem would be still more interesting. (The one-dimensional variant is trivial.)

16. HOW TO BEAT A ONE-HAND BANDIT

An automaton works as follows:

When you come to it, you can type a positive integer $x_0 \geq 3$ and then press either O or E. If you press O you will win by odd numbers, if you press E you will win at even numbers. This is the initial 0th step. From this moment the keyboard will not work; you cannot restart the automaton.

In the nth step ($n > 0$) you can put a coin 1 Franc into the hole in the front panel. After that, the automaton computes $x_n = F(x_{n-1})$, where F is a fixed function. If x_n is odd and you pressed O, or if x_n is even and you pressed E, you obtain the winning sum 1000 Franc (in coins); otherwise you obtain nothing. This seems very advantageous, but be careful:

Claim. There is a function F for which you loose all your money if you play sufficiently long, independently of your initial capital and of the typed $x_0 \geq 3$ and the typed letter O or E.

Fortunately for you, the plans of the automaton were stolen, you bought them, and now you know that the automaton computes the function $F(x) = \lfloor \frac{4x}{3} \rfloor$. So you have a very good chance to win with it. You take 1000000 Franc, and go to play with the automaton.

Problem 16.1. Can you suggest an integer $x_0 \geq 3$ and a letter O or E so that you can **prove** that you never loose all your money in this game?

We shall repeat the problem more formally, and with arbitrary initial capital and winning sum in one step:

Problem 16.2. Can you find integers $y \geq 3$ and $d \in \{0, 1\}$ such that for the sequence $(x_0, x_1, x_2, \dots)$ given by

$$x_0 = y \quad \text{and} \quad x_{n+1} = \left\lfloor \frac{4x_n}{3} \right\rfloor \quad \text{for all} \quad n \in \mathbb{N}$$

you can **prove** that the set $M_{y,d} = \{n \in \mathbb{N} \mid x_n \equiv d \pmod 2\}$ has the lower asymptotical density positive?

Problem 16.3. Are there integers $y \geq 3$ and $d \in \{0, 1\}$ such that the set $M_{y,d}$ has the lower asymptotical density zero?

Conjecture 16.4. For every $y \geq 3$ the set $M_{y,0}$ has asymptotical density $\frac{1}{2}$.

The condition $y \geq 3$ excludes constant sequences $(x_0, x_1, x_2, \dots)$.

ADDENDUM (JULY 1996) TO

"Open problems more or less related to generalized Pascal triangles"

Problem 4.1 was solved positively (i. e., $+$, $\times$ are definable in $\langle \mathbb{N}; B_2, \mathrm{Cub} \rangle$). So a motivation to Problem 4.2 and Conjecture 4.3 is weakened (but they remain open).

Conjecture 8.2 is solved negatively. For example, the structure $\langle \mathbb{N}; \mathrm{Neib}_2 \rangle$, where $\mathrm{Neib}_2 = \{(x, y) \in \mathbb{N}^2 \mid x = 0 \wedge y = 0 \vee |x - y| = 2\}$ is strictly weaker than $\langle \mathbb{N}; \mathrm{Neib} \rangle$ (in the sense of elementary definability) and every nonnegative integer is definable in it.

An important contribution to Proposal 11.2 was made by A. Schinzel (the results are not published yet; to appear in Mathematica Slovaca). He characterized the elements of $\mathbb{K}$, in terms of absolute values of their conjugates. Using this characterization he confirmed Conjectures 11.3, 11.4, 11.5. He also solved Problem 11.6 negatively (i.e., $3 - \sqrt{2} \notin \mathbb{K}$).

References

[Ax] J. Ax, *The elementary theory of finite fields*, Ann. Math. **88** (1968), 239-271.

[Bo] B. A. Bondarenko, *Generalized Pascal triangles and pyramids, their fractals, graphs and applications (in Russian)*, "Fan", Tashkent, 1990.

[Bü] J. R. Büchi, *Weak second-order arithmetic and finite automata*, Zeitschrift für Math. Logik Grundlagen Math. **6** (1960), 66-92.

[CG] A. Černý, J. Gruska, *Modular trellises*, Acta Informaticae **9** (1986), 253-282.

[Co69] A. Cobham, *On the base dependence of sets of numbers recognizable by finite automata*, Math. System Theory **3** (1969), 186-192.

[Co72] A. Cobham, *Uniform tag sequences*, Math. System Theory **6** (1972), 164-192.

[CGS] K. Čulík II, J. Gruska, A. Salomaa, *Systolic trellis automata*, International Journal Computer Math. **15** and **16** (1984,1985), 195-212 and 3-22.

[El] C. C. Elgot, *Decision problems of finite automata design and related arithmetics*, Trans. Amer. Math. Soc. **98** (1961), 21-52.

[ER] C. C. Elgot – M. O. Rabin , *Decidability and undecidability of second (first) order theory of (generalized) successor*, J. Symbolic Logic **31** (1966), 169-181.

[Koch] M. Kochol, *Generalized Pascal triangles with maximal left periods*, Computer and Artificial Intelligence **6** (1987), no. 1, 59-70.

[K84] I. Korec, *Two kinds of processors are sufficient and large operating alphabets are necessary for regular trellis automata languages*, Bulletin EATCS **23** (June 1984), 35-42.

[K85] I. Korec, *Multiples of an integer in the Pascal triangle*, Acta Math. Univ. Comen. **46-47** (1985), 83-91.

[K85a] I. Korec, *Generalized Pascal Triangles, Decidability results*, Acta Math. Univ. Comen **46-47** (1985), 93-130.

[K86] I. Korec, *Undecidable problems concerning generalized Pascal triangles of commutative algebras*, J. Gruska – B. Rovan – J. Wiedermann (Eds.): Mathematical Foundations of Computer Science, Proceedings, 12th Symposium, Bratislava, August 25–29, 1986, Springer Lecture Notes in Computer Science **233**, 1986, pp. 458-466.

[K86a] I. Korec, *Asymptotical densities in generalized Pascal triangles*, Computers and Artificial intelligence **5** (1986), no. 3, 187-198.

[K89] I. Korec, *Generalized Pascal triangles*, Proceedings of the V Universal Algebra Symposium, Turawa, Poland, 3-7 May 1988 (K. Halkowska – B. Stawski, eds.), World Scientific, Singapore, 1989, pp. 198-218.

[K89a] I. Korec, *Semilinear real-time systolic trellis automata*, Fundamentals of Computation Theory, Proceedings FCT'89 Szeged (Hungary), August 21–25, 1989 (J. Csirik – J. Demetrovics – F. Gécseg, eds.), Springer Lecture Notes in Computer Science **380**, 1989, pp. 267-276.

[K90] I. Korec, *Pascal triangles modulo n and modular trellises*, Computers and Artificial Intelligence **9** (1990), 105-113.

[K92] I. Korec, *The 3x+1 problem, generalized Pascal triangles and cellular automata*, Mathematica Slovaca **42** (1992), no. 5, 547-563.

[K93] I. Korec, *Definability of arithmetic operations from the order and a random relation*, Fundamenta Informaticae **18** (1993), no. 2,3,4, 287-296.

[K93a] I. Korec, *Irrational speeds of configuration growth in generalized Pascal triangles*, Theoretical Computer Science **112** (1993), 399-412.

[K93b] I. Korec, *Generalized Pascal triangles, Overview of new results* In: Algebraic methods in logic and in computer science, Banach Centre Publications, Warszawa **128** (1993), 125-134.

[K93c] I. Korec, *Definability of arithmetic operations in Pascal triangle modulo an integer divisible by two primes*, Grazer Matematische Berichte **318** (1993), 53-62.

[K94] I. Korec, *Generalized Pascal triangles, their relation to cellular automata and their elementary theories*, Development in theoretical computer science, Proceedings of the 7th IMYCS, Smolenice 92, November 16-20, 1992 (J. Dassow, A. Kelemenová, eds.), Gordon and Breach Science Publishers, Yverdon (Switzerland), 1994, pp. 59-70.

[K94a] I. Korec, *Structures related to Pascal's triangle modulo* 2 *and their elementary theories*, Mathematica Slovaca **44** (1994), no. 5, 531-554.

[K94b] I. Korec, *Substantially different theories of two structures associated to the same generalized Pascal triangle*, Summer School on General Algebra and Ordered Sets, Horní

Lipová, September 1994 (I. Chajda, F. Halaš, F. Krutský, eds.), Palacký University, Olomouc, 1994, pp. 63–78.

[K95] I. Korec, *Elementary theories of structures containing generalized Pascal triangles modulo a prime*, Discrete Mathematics and Applications, Blagoevgrad/Predel, Sept. 12-16, 1994 (S. Shtrakov and Iv. Mirchev, eds.), Blagoevgrad, 1995, pp. 91-102.

[K95a] I. Korec, *Decidable and undecidable theories of generalized Pascal triangles*, General Algebra and Discrete Mathematics, Potsdam, Sept. 27 – Okt. 1, 1993 (K. Denecke, O. Lüders, eds.), Heldermann-Verlag, Berlin, 1995, pp. 169–179.

[K95b] I. Korec, *Theories of generalized Pascal triangles*, Proceedings of the Logic Colloquium'94, Clermont-Ferrand, July 21–30, 1994 (submitted July 1994), 1-7.

[K95c] I. Korec, *Undecidable elementary theories of classes of generalized Pascal triangles*, Tatra Mountains Mathematical Journal **5** (1995), 151-168.

[Lä] H. Läuchli, *A decision procedure for the weak second order theory of linear order*, K. Schütte (Ed.): Contributions to mathematical logic, North Holand, Amsterdam, 1968, pp. 189-197.

[Mar] F. Marko, *The divisibility of the binomial coefficients near a half-line and in convex sets*, Acta Math. Univ. Comen **50-51** (1987), 267-275.

[Mo] J. D. Monk, *Mathematical logic*, Springer Verlag, New York, 1976.

[Ra] O. Rabin, *Decidability of second-order theories and automata on infinite trees*, Transactions Amer. Math. Society **141** (1969), 1-35.

[Ri] D. Richards, *Answer to a problem raised by J. Robinson: the arithmetic of positive or negative integers is definable from successor and divisibility*, Journal of Symbolic Logic **50** (1985), 927-935.

[Rob] J. Robinson, *Definability and decision problems in arithmetic*, Journal of Symbolic Logic **14** (1949), 98-114.

[Sem] A. L. Semenov, *On definability of arithmetic in their fragments (in Russian)*, Doklady Akad. nauk SSSR **263, no. 1** (1982), 44-47.

[Sie] D. Siefkes, *Büchi's monadic second order successor arithmetics*, Springer, Lecture Notes in Math. 120, 1970.

[Sin] D. Singmaster, *Notes on binomial coefficients III — Any integer divides almost all binomial coefficients*, J. London Math. Soc. **(2) 8** (1974), 555-560.

[Tho] W. Thomas, *A note on undecidable extensions of monadic second-order successor arithmetic*, Arch. math. Logik **17** (1975), 43-44.

[Vi92] R. Villemaire, *Joining k- and l-recognizable sets of natural numbers*, Proceedings STACS'92, Springer Lecture Notes in Computer Science **577**, 1986, pp. 458-466.

[Vi93] R. Villemaire, $\langle \mathbb{N}; +, V_k, V_l \rangle$ *is undecidable*, Theor. Comput. Sci **106** (1993), 337-349.

[Wo] A. Woods, *Some problems in logic and number theory, and their connection*, Ph.D. thesis, University of Manchester, 1981.

[Ye] Ju. L. Yershow, *Decidability problems and constructive models (in Russian)*, "Nauka", Moscow, 1980.

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