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On the sum of observables in MV algebras of fuzzy sets

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ON THE SUM OF OBSERVABLES IN MV ALGEBRAS OF FUZZY SETS

Beloslav Riečan

Abstract. Two concepts of the sum of observables are compared for the case of MV algebras of fuzzy sets

1. Introduction

We shall consider so-called T_L-tribes ([4]) or semisimple MV algebras ([1], [3]) defined as follows.

Definition 1. A non-empty set M of functions $f\colon \Omega \to [0,1]$ is called a T_L-tribe, if the following properties are satisfied:

(i) If $f, g \in M$, then $F \oplus g = \min(f+g, 1) \in M$;
(ii) if $f \in M$, then $f^* = 1 - f \in M$;
(iii) if $f_n \in M$, $f_n \nearrow f$, then $f \in M$.

It is easy to see that for every $f, g \in M$ also

$$f \odot g = \max(f + g - 1, 0) \in M\,.$$

Indeed,

$$\max(f+g-1, 0) = 1 - (1-f) \oplus (1-g) \in M\,.$$

Further, if we define the functions 1_Ω, 0_Ω by $1_\Omega(\omega) = 1$, $0_\Omega(\omega) = 0$ for any $\omega \in \Omega$, then $1_\Omega \in M$, $0_\Omega \in M$. A little less trivial is the proof (see [2], [4], [9]) that M is closed under the product: If M contains all constant functions, then

$$f, g \in M \Rightarrow f \cdot g \in M\,.$$

Throughout the paper M will denote a fixed T_L tribe. It can be regarded as the set of all functions $f\colon \Omega \to R$ measurable with respect to a σ-algebra $\mathcal{S}$ of subsets of Ω. We shall define the notions of a state, an observable and a weak observable.

Definition 2. A state is a mapping $m\colon M \to [0,1]$ satisfying the following three conditions:

(i) $m(1_\Omega) = 1$;
(ii) if $f, g \in M$, $f \leq 1 - g$, then $m(f+g) = m(f) + m(g)$;
(iii) if $f_n \in M$, $f \in M$, $f_n \nearrow f$, then $m(f_n) \nearrow m(f)$.

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Definition 3. A weak observable (with respect to a state m) is a mapping $x\colon \mathcal{B}(R) \to M$ ($\mathcal{B}(R)$ is the σ-algebra of Borel subsets of R) satisfying the following conditions:

(i) $m\big(x(R)\big) = 1$;

(ii) if $A, B \in \mathcal{B}(R)$, $A \cap B = \emptyset$, then $x(A \cup B) = x(A) + x(B)$;

(iii) if $A_n \in \mathcal{B}(R)$, $A_n \nearrow A$, then $x(A_n) \nearrow x(A)$.

A weak observable is an observable, if $x(R) = 1_\Omega$.

Theorem 1. *Let $x\colon \mathcal{B}(R) \to M$ be a weak observable (with respect to $m\colon M \to [0,1]$). Define $m_x\colon \mathcal{B}(R) \to [0,1]$ by the formula $m_x(A) = m\big(x(A)\big)$. Then m_x is a probability measure.*

The notions of a state and an observable can be regarded as generalization of the notions of a probability measure and a random variable. If $(\Omega, \mathcal{S}, P)$ is a probability space and $\xi\colon \Omega \to R$ a random variable, then by the formula $x(A) = \chi_{\xi^{-1}(A)}$ can be defined an observable $x\colon \mathcal{B}(R) \to [0,1]^\Omega$.

We shall compare two concepts of constructing the sum $x + y$ of two weak observables x, y. The first one was introduced in [5] for MV algebras of fuzzy sets only. It is based on the notion of the joint observable of observables x, y (see Section 2). The second construction was studied in [6] as a weak observable $x \oplus y$ defined only for $\odot$-independent observables but for a lager class of MV algebras. For a T_L-tribe M both concepts are aplicable, hence we obtain two (weak) observables $x + y$, $x \oplus y$. In Theorem 5 we shall prove that $m_{x+y} = m_{x\oplus y}$, i.e., they have the same probability distribution.

2. Joint observable

Definition 4. Let $x, y\colon \mathcal{B}(R) \to M$ be (weak) observables. The joint (weak) observable of x, y is a mapping $h\colon \mathcal{B}(R^2) \to M$ satisfying the following conditions:

(i) $\big(m\big(h(R^2)\big) = 1\big)$ $h(R^2) = 1_\Omega$;

(ii) if $A, B \in \mathcal{B}(R^2)$, $A \cap B = \emptyset$, then $h(A \cup B) = h(A) + h(B)$;

(iii) if $A_n \in \mathcal{B}(R^2)$, $A_n \nearrow A$, then $h(A_n) \nearrow h(A)$;

(iv) $h(A \times B) = x(A) \cdot x(B)$ for any $A, B \in \mathcal{B}(R)$.

A motivation can be taken from the classical probability theory. If $T = (\xi, \eta)\colon \Omega \to R^2$ is a random vector, then the correspondence $A \mapsto T^{-1}(A)$ is a morphism such that

$$T^{-1}(A \times B) = \xi^{-1}(A) \cap \eta^{-1}(B)\,.$$

Of course, in fuzzy set theory different connectives can be considered instead of the intersection $\cap$. In Definition 3 the product of functions was used. It was reasonable, since in the case of indicators we have

$$\chi_{C\cap D} = \chi_C \cdot \chi_D\,.$$

There are many other possibilities, e.g., we can use the equality

$$\chi_{C\cap D} = \max(\chi_C + \chi_D - 1, 0) = \chi_C \odot \chi_D\,.$$

This approach will be used in Section 3.

Theorem 2. *For any weak observables x, y there is its joint weak observable h. If x, y are observables, then h is the joint observable, too.*

Proof. For a fixed $\omega \in \Omega$ define $\mu, \nu\colon \mathcal{B}(R) \to [0,1]$ by the formulae $\mu(A) = x(A)(\omega)$, $\nu(A) = y(A)(\omega)$. Let $\mu \times \nu\colon \mathcal{B}(R^2) \to [0,1]$ be the product measure determined by the formula $\mu \times \nu(A \times B) = \mu(A) \cdot \nu(B)$, $A, B \in \mathcal{B}(R)$. Define $h\colon \mathcal{B}(R^2) \to M$ by the formula $h(C)(\omega) = \mu \times \nu(C)$.

By the help of the joint observable h the sum $x + y$ can be defined. A motivation can be taken again from the classical probability theory. If $T = (\xi, \eta)$ is a random vector and $g\colon R^2 \to R$ is given by the formula $g(u,v) = u + v$, then

$$\xi + \eta = g(\xi, \eta) = g \circ T,$$

hence

$$(\xi + \eta)^{-1} = (g \circ T)^{-1} = T^{-1} \circ g^{-1}.$$

Definition 5. If h is the (weak) joint observable of (weak) observables x, y, then its sum is the (weak) observable defined by the formula

$$x + y = h \circ g^{-1}.$$

Theorem 3. *Let $x, y\colon \mathcal{B}(R) \to M$ be two observables. Then for any $t \in R$*

$$\begin{aligned} m\big((x+y)((-\infty,t))\big) &= \lim_{n\to\infty} \sum_{(i,j)\in\alpha_n(t)} m\big(x(A_i^n) \cdot y(A_j^n)\big) = \\ &= m\big(h\big(\{(u,v);\ u+v<t\}\big)\big)\,, \end{aligned}$$

where $A_i^n = \left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)$, $\alpha_n(t) = \big\{(i,j);\ i+j < 2^n t\big\}$.

Proof. It is not difficult to prove that

$$\big\{(u,v);\ u+v<t\big\} = \bigcup_{n=1}^{\infty} \bigcup_{(i,j)\in\alpha_n(t)} A_i^n \times A_j^n\,.$$

Therefore

$$\begin{aligned} m\big((x+y)((-\infty,t))\big) &= m\big(h\big(g^{-1}((-\infty,t))\big)\big) = \\ &= m\big(h\big(\{(u,v);\ u+v<t\}\big)\big) = \\ &= m\Big(h\Big(\bigcup_{n=1}^{\infty} \bigcup_{(i,j)\in\alpha_n(t)} A_i^n \times A_j^n\Big)\Big) = \\ &= m\Big(\bigvee_{n=1}^{\infty} \sum_{(i,j)\in\alpha_n(t)} h(A_i^n \times A_j^n)\Big) = \\ &= \lim_{n\to\infty} \sum_{(i,j)\in\alpha_n(t)} m\big(x(A_i^n) \cdot y(A_j^n)\big)\,. \end{aligned}$$

Definition 6. Two (weak) observables are p-independent, if

$$m\big(x(A)\cdot y(B)\big) = m\big(x(A)\big)m\big(x(B)\big)$$

for any $A, B \in \mathcal{B}(R)$.

Let $m_x \times m_y \colon \mathcal{B}(R^2) \to [0,1]$ be the product of the probability distributions $m_x, m_y \colon \mathcal{B}(R) \to [0,1]$. Evidently the p-independence can be described by the formula

$$m\big(h(A\times B)\big) = m_x \times m_y(A\times B)$$

for any $A, B \in \mathcal{B}(R)$, hence

$$m\circ h = m_x \times m_y\,.$$

As an easy consequence of Theorem 3 we obtain the following assertion.

Corollary. *If $x, y \colon \mathcal{B}(R) \to M$ are p-independent (weak) observables, then for any $t \in R$*

$$m\big((x+y)\big((-\infty,t)\big)\big) = m_x \times m_y\big(\{u,v);\; u+v<t\}\big)\,,$$

hence

$$m_{x+y} = (m_x\times m_y)\circ g^{-1}\,.$$

3. $\odot$-INDEPENDENCE

Definition 7. Two weak observables x, y are $\odot$-independent, if

$$\begin{aligned} m\big((x(A)\odot y(B)\big) &= m\big(x(A)\big)\cdot m\big(y(B)\big) = \\ &= m_x\times m_y(A\times B) \end{aligned}$$

for any $A, B \in \mathcal{B}(R)$.

Theorem 4. *If x, y are $\odot$-independent weak observables, then there is a weak observable $x \oplus y \colon \mathcal{B}(R) \to M$ such that*

$$\begin{aligned} m\big((x\oplus y)\big((-\infty,t)\big)\big) &= \lim_{n\to\infty} \sum_{(i,j(\in\alpha(t)} m\big(x(A_i^n)\odot y(A_j^n)\big) \\ &= m_x\times m_y\big(\{(u,v);\; u+v<t\}\big)\,. \end{aligned}$$

Proof. [6], Theorem 1 and Theorem 2.

Theorem 5. *If x, y are $\odot$-independent weak observables, then x, y are p-independent and*

$$m_{x+y} = m_{x\oplus y}\,.$$

Proof. Put $\mu = m_x \times m_y \colon \mathcal{B}(R^2) \to [0,1]$. Put $a = x(A)$, $b = x(B)$, $c = y(C)$, where $A, B \in \mathcal{B}(R)$, $A\cap B = \emptyset$. Then

$$\begin{aligned} m\big((a+b)\odot c\big) &= m\big(x(A\cup B)\odot y(C)\big) = \\ &= \mu\big((A\cup B)\times C\big) = \mu(A\times C) + \mu(A\times C) = \\ &= m(a\odot c) + m(b\odot c) = \\ &= m(a\odot c + b\odot c)\,, \end{aligned}$$

hence

$$m\big((a+b)\odot c-(a\odot c+b\odot c)\big)=0\,.$$

But

$$\begin{aligned}
&a\odot c+b\odot c=\\
&\quad=(a+c-1)\vee 0+(b+c-1)\vee 0=\\
&\quad=\big(a+c-1+(b+c-1)\vee 0\big)\vee(b+c-1)\vee 0=\\
&\quad=(a+b+c-1+c-1)\vee(a+c-1)\vee(b+c-1)\vee 0\,,
\end{aligned}$$

hence

$$\begin{aligned}
0&=m\big((a+b)\odot c-(a\odot c+b\odot c)\big)=\\
&=m\big((a{+}b{+}c{-}1)\vee 0-\big((a{+}b{+}c{-}1{+}c{-}1)\vee(a{+}c{-}1)\vee(b{+}c{-}1)\vee 0\big)\big)=\\
&=m\big((a{+}b{+}c{-}1)\vee 0+(1{-}a{-}b{-}c{+}1{-}c)\wedge(1{-}a{-}c)\wedge(1{-}b{-}c)\wedge 0\big)=\\
&=m\Big(\big((1-c)\wedge b\wedge a\wedge(a+b+c-1)\big)\vee d\Big)
\end{aligned}$$

where

$$d=(1-a-b-c+1-c)\wedge(1-a-e)\wedge(1-b-c)\wedge 0\,.$$

Evidently $m(d)=0$. Put $B=A^c$. Then $a+b-1=m(A)+m(A^c)-1=0$, hence

$$0=m\big(y(C^c)\wedge x(A^c)\wedge x(A)\wedge y(C)\big)\,. \tag{$*$}$$

On the other hand $a\odot b\le a\wedge b$, $a\odot b\le a\cdot b$. Therefore

$$\begin{aligned}
&m\big(x(A)\cdot y(A)\big)-m\big(x(A)\odot y(C)\big)=\\
&\quad=m\big(x(A)\cdot y(C)-\big(x(A)+y(C)-1\big)\vee 0\big)=\\
&\quad=m\Big(\big(x(A)\cdot y(C)-x(A)-y(C)+1\big)\wedge\big(x(A)\cdot y(C)\big)\Big)=\\
&\quad=m\Big(\big((1-x(A))\cdot(1-y(C))\big)\wedge\big(x(A)\cdot y(C)\big)\Big)=\\
&\quad=m\Big(\big(x(A^c)\cdot y(C^c)\big)\wedge\big(x(A)\cdot y(C)\big)\Big)\le\\
&\quad\le m\big(y(C^c)\wedge x(A^c)\wedge x(A)\wedge y(C)\big)=0
\end{aligned}$$

by $(*)$. Therefore

$$m\big(x(A)\cdot y(C)\big)=m\big(x(A)\odot y(C)\big)\,.$$

We see that x,y are $\odot$-independent. Moreover, $m_{x+y}=m_{x\oplus y}$ by Theorem 3 and Theorem 4.

Example. We shall show that p-independent observables need not be $\odot$-independent. Let $M\subset[0,1]^{[0,1]}$ be the set of all measurable functions, $m\colon M\to[0,1]$ be defined by the formula

$$m(f)=\int_{[0,1]} f\,d\lambda\,,$$

where λ is the Lebesgue measure. Let $\mu\colon \mathcal{B}(R) \to [0,1]$ be the uniform probability distribution on $[0,1]$, i.e.,

$$\mu(A) = \lambda\big(A \cap [0,1]\big)\,, \;\; A \in \mathcal{B}(R)\,.$$

Let $x = y\colon \mathcal{B}(R) \to M$ be constat valued observables defined by the formula

$$x(A)(\omega) = y(A)(\omega) = \mu(A)\,, \;\; \omega \in [0,1]\,.$$

Then x, y are p-independent:

$$\begin{aligned} m\big(x(A)\cdot y(B)\big) &= \int_{[0,1]} \mu(A)\cdot\mu(B)d\lambda = \mu(A)\cdot\mu(B) = \\ &= m\big(x(A)\big)\cdot m\big(y(B)\big)\,. \end{aligned}$$

On the other hand, let $\mu(A) + \mu(B) \leq 1$. Then

$$\begin{aligned} m\big(x(A)\cdot y(B)\big) &= m\big(\max\big(x(A) + y(B) - 1, 0\big)\big) = \\ &= m(0_{[0,1]}) = 0 \neq m\big(x(A)\big)\cdot m\big(x(B)\big)\,. \end{aligned}$$

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