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Abstract

We find a fifth approximation of the Just Intonation which generalizes Equal Temperament. The intervals causing a dilemma are the second and the minor seventh and the tritone because they are unambiguous in Just Intonation (the relative frequencies 10/9, 9/8, 8/7 and 7/4, 16/9, 18/10 and 45/32, 64/45, respectively). If we do not consider the second and seventh with the relative frequencies 8/7 and 7/4, respectively, all the music intervals in this approximation either coincide with he Just Intonation interval values (the octave, fifth, fourth, second (9/8) and the minor seventh (16/9)) or are exactly the one comma distant from the corresponding Just Intonation intervals. This comma is 32 805/32 768 $\approx$ 1.00112915, which is less than the ratio of frequencies of the perfect and the equal tempered fifths ($\approx$ 1.00112989).

1 Introduction

Mathematicians have solved problems connected with tuning since antiquity. It seems that in these considerations can be applied various mathematical structures on real line. Connections between mathematics and music are mutually enriching, depending on the progress in mathematics or music. Therefore problems having their origin in music attract the interest of scientists and musicians throughout the history up to present days.

Pythagorean Tuning, cf. e.g. [4], was created as a sequence of ratios, i.e. products $2^p 3^q$, where p and q are integers, cf. Table 1. This tuning was established about five hundred years B. C. and used in the Western music up to the 14th century.

The gradual development of polyphony led to the introduction of thirds (5/4 and 6/5) and sixths (8/5 and 5/3). After that the problem of why musical ratios of integers was considered in the context of attempts to provide a strict mathematical definition of consonance. The key figures in these studies were Gioseffo Zarlino (1517 – 1590), Simon Stevin (1548 – 1620), Johann Kepler (1571 – 1630). The practical result of this period was the creation of the *Just Intonation* set, cf. Table 2. Just Intonation is based on the intervals which are integer exponents of numbers 2, 3, 5, and 7.

The further progress in tuning continued by *Equal Temperament* (Andreas Werkmeister, 1698) which is represented by the sequence

$$W = \{1, \sqrt[12]{2}, (\sqrt[12]{2})^2, \ldots, (\sqrt[12]{2})^{12} = 2\} = \{C^{(W)}, C_\sharp^{(W)}, \ldots, B^{(W)}, C'^{(W)}\}.$$

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Key words: Many valued coding, Equal temperament, Just Intonation, Pythagorean Tuning, Music acoustic, Perception

Denotation	Expressions		
$C^{(P)}$	$2^0 3^0$	1/1	1.0
$C_\sharp^{(P)}$	$2^8 3^{-5}$	256/243	1.053497942
$D_\flat^{(P)}$	$2^{-11} 3^7$	2187/2048	1.067871094
$D^{(P)}$	$2^{-3} 3^2$	9/8	1.125
$D_\sharp^{(P)}$	$2^5 3^{-3}$	32/27	1.185185185
$E_\flat^{(P)}$	$2^{-14} 3^9$	19683/16384	1.201354981
$E^{(P)}$	$2^{-6} 3^4$	81/64	1.265625
$F^{(P)}$	$2^2 3^{-1}$	4/3	1.333333333
$F_\sharp^{(P)}$	$2^{10} 3^{-6}$	1024/729	1.404663923
$G_\flat^{(P)}$	$2^{-9} 3^6$	729/512	1.423828125
$G^{(P)}$	$3^1 2^{-1}$	3/2	1.5
$G_\sharp^{(P)}$	$2^7 3^{-4}$	128/81	1.580246914
$A_\flat^{(P)}$	$2^{-12} 3^8$	6561/4096	1.601806641
$A^{(P)}$	$2^{-4} 3^3$	27/16	1.6875
$A_\sharp^{(P)}$	$2^4 3^{-2}$	16/9	1.777777777
$B_\flat^{(P)}$	$2^{-15} 3^{10}$	59049/32768	1.802032473
$B^{(P)}$	$2^{-7} 3^5$	243/128	1.898437528
$C'^{\ (P)}$	2	2/1	2.0

Table 1: **Pythagorean Tuning**

Denotation	Expressions		
$C^{(JI)}$	$2^0 3^0 5^0$	1/1	1.0
$C_\sharp^{(JI)}, D_\flat^{(JI)}$	$2^4 3^{-1} 5^{-1}$	16/15	1.066666666
$D^{(JI)}$	$2^1 3^{-2} 5^1$	10/9	1.111111111
	$2^{-3} 3^2$	9/8	1.125
	$2^3 7^{-1}$	8/7	1.14285714
$D_\sharp^{(JI)}, E_\flat^{(JI)}$	$2^1 3^1 5^{-1}$	6/5	1.2
$E^{(JI)}$	$2^{-2} 5^1$	5/4	1.25
$F^{(JI)}$	$2^2 3^1$	4/3	1.333333333
$F_\sharp^{(JI)}$	$2^{-5} 3^2 5^1$	45/32	1.40625
$G_\flat^{(JI)}$	$2^6 3^{-2} 5^{-1}$	64/45	1.422222222
$G^{(JI)}$	$2^{-1} 3^1$	3/2	1.5
$G_\sharp^{(JI)}, A_\flat^{(JI)}$	$2^3 5^{-1}$	8/5	1.6
$A^{(JI)}$	$3^{-1} 5^1$	5/3	1.666666666
$A_\sharp^{(JI)}, B_\flat^{(JI)}$	$2^{-2} 7^1$	7/4	1.75
	$2^4 3^{-2}$	16/9	1.777777777
	$3^2 5^{-1}$	9/5	1.8
$B^{(JI)}$	$2^{-3} 3^1 5^1$	15/8	1.875
$C'^{(JI)}$	2^1	2/1	2.0

Table 2: **Just Intonation**

There have been many theoretical and experimental works concerned – both *pro* and *contra* – with the assumed universality of diatonic scales, cf. [4]. However, our aim is not to judge such non-mathematical problem. In the present paper, we find a fifth approximation of the Just Intonation which generalizes Equal Temperament. The intervals causing a dilemma are the second and the minor seventh because they are unambiguous in Just Intonation (the relative frequencies 10/9, 9/8, 8/7 and 7/4, 16/9, 18/10, respectively). If we do not consider the second and seventh with the relative frequencies 8/7 and 7/4, respectively, all the music intervals in this approximation either coincide with he Just Intonation interval values (the octave, fifth, fourth, second (9/8) and the minor seventh (16/9)) or are exactly the one comma distant from the corresponding Just Intonation intervals. This comma is 32 805/32 768 $\approx$ 1.00112915, which is less than the ratio of frequencies of the perfect and the equal tempered fifths ($\approx$ 1.00112989).

2 Preliminaries

Denote by $\mathcal{N}, \mathcal{Z}, \mathcal{Q}, \mathcal{R}, \mathcal{C}$ the sets of all natural, integer, rational, real, and complex numbers, respectively.

Denote by $L = ((0,\infty), \cdot, 1, \leq)$ the usual multiplicative group with the usual order on $\mathcal{R}$. So, if $a \leq b, a, b \in (0,\infty)$, then b/a is an L-length of the interval (a, b). Since this terminology is not obvious, we borrow the usual musical terminology, i.e. we simply say that b/a is an interval. This inaccurateness does not lead to any misunderstanding because the therm "interval" is used only in this sense in this paper.

The proof of the following lemma is easy.

Lemma 1 *Let $q = const \in (0,\infty)$.*

Then $\rho_q(u,v) = \left|\log_2 \sqrt[q]{u/v}\right|, u, v \in (0,\infty)$, is a metric on L.

Equal Temperament can be generalized as follows:

Definition 1. Let $\{x, \ldots, y \in \mathcal{R}; 1 < x \leq \ldots \leq y < 10/9\}$ be a set of N numbers. Let $m_0, m_1, \ldots, m_M; \ldots; n_0, n_1, \ldots, n_M$ be $M \times N$ nonnegative integers, such that $0 = m_0 \leq m_1 \leq \ldots \leq m_M, \ldots, 0 = n_0 \leq n_1 \leq \ldots \leq n_M$ and $m_j + \ldots + n_j = j, j = 0, 1, \ldots, M$. Then the set $S = \{x^{m_0} \ldots y^{n_0}, x^{m_1} \ldots y^{n_1}, \ldots, x^{m_M} \ldots y^{n_M}\}$ is said to be a *M-degree N-interval scale.* Particularly, a 12-degree N-interval scale S is a *12-degree 2-interval* (2/1, 3/2)*-scale* if $x^{m_{12}}y^{n_{12}} = 2/1, x^{m_7}y^{n_7} = 3/2$. We say that $\mathcal{S}$ is a *M-degree N-interval system ((2/1, 3/2)-system)*, if $\mathcal{S} = \bigcup S$, where S are M-degree N-interval scales ((2/1, 3/2)-scales).

Equal Tempered Scale is a 12-degree 1-interval scale which is not a 12-degree (2/1, 3/2)-scale. Pythagorean Tuning is a 12-degree 2-interval (2/1, 3/2)-system. The constructed approximation of Just Intonation is also a 12-degree 2-interval (2/1, 3/2)-system.

Lemma 2 *Let $x, y, a \in (1;\infty), m, n \neq 0, m, n \in \mathcal{R}$. Then the following equalities are equivalent: (1) $a = x^m y^n$, (2) $1 = m/\log_x a + n/\log_y a$.*

Proof. $a = x^m y^n \Longleftrightarrow \log a = m \log x + n \log y \Longleftrightarrow$

$$1 = m\frac{\log x}{\log a} + n\frac{\log y}{\log a} \Longleftrightarrow 1 = \frac{m}{\log_x a} + \frac{n}{\log_y a}.$$

□

As a consequence of Definition 1 we get the following lemma, cf. [3].

Lemma 3 *The unique two rational intervals in Definition 1 for 12-degree 2-interval (2/1, 3/2)-scales are $x = 2^8/3^5, y = 3^7/2^{11}$.*

In what follows, fix $x = 2^8/3^5, y = 3^7/2^{11}$.

Denote by $\mathcal{Q}_{x,y} = \{x^\alpha y^\beta; \alpha, \beta \in \mathcal{Z}\}, \mathcal{R}_{x,y} = \{x^\alpha y^\beta; \alpha, \beta \in \mathcal{R}\}$. Consider the following map $\theta : \mathcal{Q}_{x,y} \to \mathcal{Z}^2, \theta : x^\alpha y^\beta \to (\alpha, \beta)$, and

$$(x^\alpha y^\beta)(x^\gamma y^\delta) \to (\alpha + \gamma, \beta + \delta), (x^\alpha y^\beta)^\gamma \to (\gamma\alpha, \gamma\beta),\ \ \alpha, \beta, \gamma, \delta \in \mathcal{Z}.$$

Lemma 4 *The morphism θ is an isomorphism.*

Proof. It is enough to show that the map $\theta : \mathcal{Q}_{x,y} \to \mathcal{Z}^2$ is an injection (the other properties of isomorphism are trivial). Suppose that there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathcal{Z}$, such that $\alpha_1 \neq \alpha_2, \beta_1 \neq \beta_2$ implies $x^{\alpha_1} y^{\beta_1} = x^{\alpha_2} y^{\beta_2}$. Then $x^{\alpha_1 - \alpha_2} y^{\beta_1 - \beta_2} = 1$ and $(\alpha_1 - \alpha_2) + (\beta_1 - \beta_2)\log_x y = 0$. This is possible only in the case when $\alpha_1 = \alpha_2, \beta_1 = \beta_2$. A contradiction. Consequently, $\alpha_1 = \alpha_2, \beta_1 = \beta_2$. □

Embed $\mathcal{Z}^2$ identically into the complex plane $\mathcal{C}$ (for the process of the standard enlargements $\mathcal{Z} \to \mathcal{Q} \to \mathcal{R} \to \mathcal{C}$ cf. e.g. [3]). Denote this injection by η. Denote by θ_*^{-1} the extended morphism $\theta_*^{-1} : \mathcal{C} \to \mathcal{R}_{x,y}$. So we have the following commutative diagram which defines the embedding $\varphi : \mathcal{Q}_{x,y} \to \mathcal{R}_{x,y}$:

$$\begin{tikzcd}
\mathcal{Q}_{x,y} \arrow[r, "\theta"] \arrow[d, hook, "\varphi"'] & \mathcal{Z}^2 \arrow[d, hook, "\eta"] \\
\mathcal{R}_{x,y} & \mathcal{C} \arrow[l, "\theta_*^{-1}"']
\end{tikzcd}$$

In Figure 1, we see the image of Pythagorean Tuning in the isomorphism θ. Observe, that it is not symmetric.

Define the following Just Intonation Approximation.

Definition 2 The *Just Intonation Approximation* $\mathcal{A}$ is defined by Table 3.

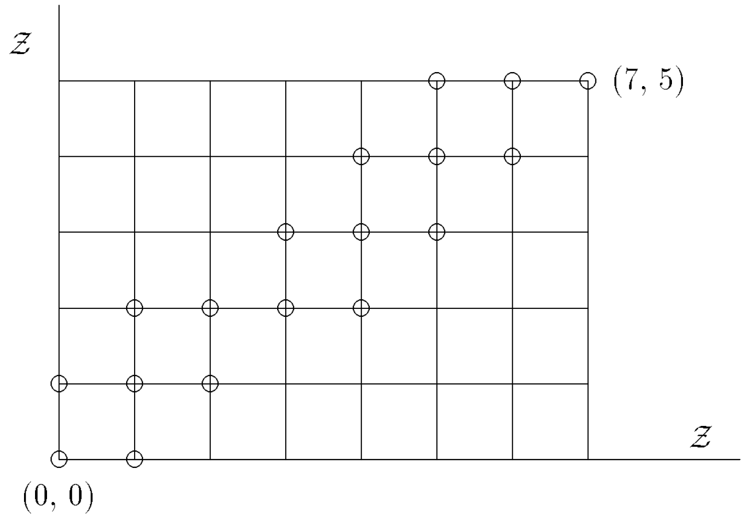

Figure 1: **Image of Pythagorean Tuning in the isomorphism θ**

3 The Just Intonation Approximation $\mathcal{A}$

The following theorem is easy to verify, cf. also Figure 2.

Theorem 1 *Let* $a_j = 2/1, 15/8, 9/5, 7/4, 16/9, 5/3, 8/5, 3/2, 64/45, 45/32, 4/3, 5/4, 6/5, 8/7, 9/8, 10/9, 16/15,$ *where* $j = 12, 11, 10, 10, 10, 9, 8, 7, 6, 6, 5, 4, 3, 2, 2, 2, 1,$ *respectively. For* $x = 2^8/3^5, y = 3^7/2^{11}$*, the system of linear equations*

$$\frac{m_j}{\log_x a_j} + \frac{n_j}{\log_y a_j} = 1, m_j + n_j = j$$

has the following integer solutions $(m_{12}, n_{12}) = (7, 5), (m_{10}, n_{10}) = (6, 4), (m_7, n_7) = (4, 3), (m_5, n_5) = (3, 2), (m_2, n_2) = (1, 1),$ *for* $a_j = 2/1, 16/9, 3/2, 4/3, 9/8,$ *where* $j = 12, 10, 7, 5, 2,$ *respectively.*

Theorem 2 *Both Just Intonation and the Just Intonation Approximation* $\mathcal{A}$ *are symmetric with respect to the point* $(7/2, 5/2)$ *in the morphism* θ_**.*

Proof. For the approximation $\mathcal{A}$ see Figure 3.
Prove the assertion for Just Intonation. Observe that

$$m_j = \frac{j \log y - \log a_j}{\log y - \log x}, n_j = \frac{\log a_j - j \log x}{\log y - \log x}$$

Denotation	**Expressions**			
$C^{(A)}$	x^0y^0	2^03^0	$1/1$	1.0
$C_\sharp^{(A)}, D_\flat^{(A)}$	x^0y^1	$2^{-11}3^7$	$2187/2048$	1.067871094
$D^{(A)}$	x^2y^0	$2^{16}3^{-10}$	$65536/59049$	1.10985791
	x^1y^1	$2^{-3}3^2$	$9/8$	1.125
	x^0y^2	$2^{-22}3^{14}$	$4782969/4194304$	1.140348673
$D_\sharp^{(A)}, E_\flat^{(A)}$	x^1y^2	$2^{-14}3^9$	$19683/16384$	1.201354981
$E^{(A)}$	x^3y^1	$2^{13}3^{-8}$	$8192/6561$	1.248590154
$F^{(A)}$	x^3y^2	2^23^{-1}	$4/3$	1.333333333
$F_\sharp^{(A)}$	x^4y^2	$2^{10}3^{-6}$	$1024/729$	1,404663923
$G_\flat^{(A)}$	x^3y^3	$2^{-9}3^6$	$729/512$	1.423828125
$G^{(A)}$	x^4y^3	$2^{-1}3^1$	$3/2$	1.5
$G_\sharp^{(A)}, A_\flat^{(A)}$	x^4y^4	$2^{-12}3^8$	$6561/4096$	1.601806641
$A^{(A)}$	x^6y^3	$2^{15}3^{-9}$	$32768/19683$	1.664786873
$A_\sharp^{(A)}, B_\flat^{(A)}$	x^7y^3	$2^{23}3^{-14}$	$8388608/4782969$	1.753849545
	x^6y^4	2^43^{-2}	$16/9$	1.777777777
	x^5y^5	$2^{-15}3^{10}$	$59049/32768$	1.802032471
$B^{(A)}$	x^7y^4	$2^{12}3^{-7}$	$4096/2187$	1.872885232
$C^{(A)}$	x^7y^5	2^13^0	$2/1$	2.0

Table 3: **The Just Intonation Approximation $\mathcal{A}$,** $x = 2^8/3^5, y = 3^7/2^{11}$

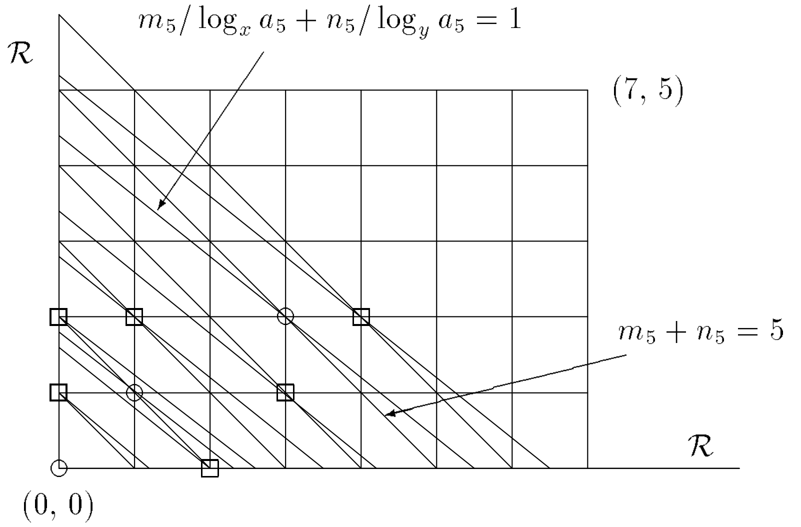

Figure 2: **Just Intonation $(C, \ldots, F_\sharp)$ in the morphism θ_***

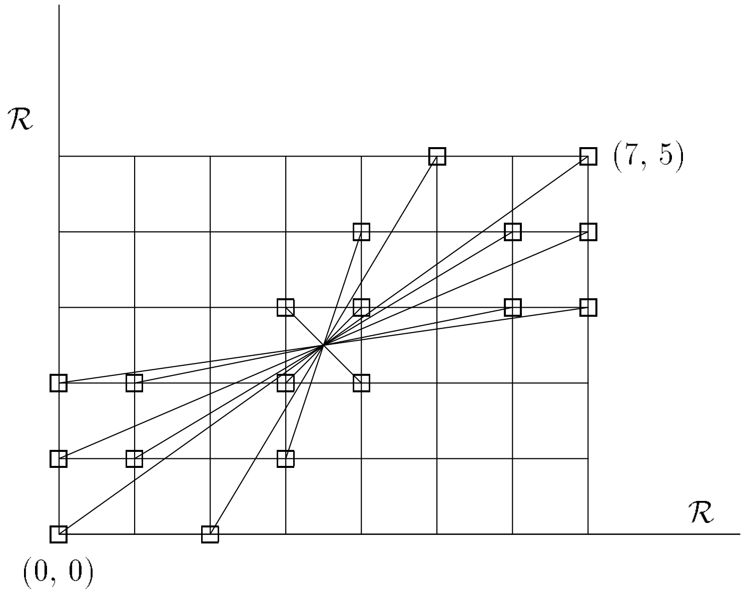

Figure 3: **Symmetry of the approximation $\mathcal{A}$ in the isomorphism θ**

is the solution of the system in Theorem 1. Further, $a_{12-j} = 2/a_j, j = 0, 1, \ldots, 12$, cf. Table 2. Since $2 = x^7y^5$, we have:

$$m_{12-j} = \frac{(12-j)\log y - \log a_{12-j}}{\log y - \log x} = \frac{12\log y - j\log y - \log(2/a_j)}{\log y - \log x} =$$

$$= \frac{-j\log y + \log a_j}{\log y - \log x} + \frac{\log y^{12} - \log x^7y^5}{\log y - \log x} = -m_j + \frac{\log(y^{12}/x^7y^5)}{\log(y/x)} = -m_j + 7.$$

Analogously, $n_{12-j} = -n_j + 5$. □

Equal Temperament is also symmetric in the morphism θ_*, cf. Figure 4, where each point of this tuning is a center of symmetry. The point $(7/2, 5/2)$ is a common center of symmetry for three tunings: Equal Temperament, Just Intonation, and the Just Intonation Approximation $\mathcal{A}$. Note that $x^{7/2}y^{5/2} = (\sqrt[12]{2})^6 = \sqrt{2}$.

We can state the following theorem which the proof is now obvious.

Theorem 3 *The Just Intonation Approximation $\mathcal{A}$ is a 12-degree 2-interval 2/1, 3/2)-system.*

The fact that the Just Intonation Approximation $\mathcal{A}$ is a generalization of Equal Temperament, confirms the following theorem which states that an image of Equal

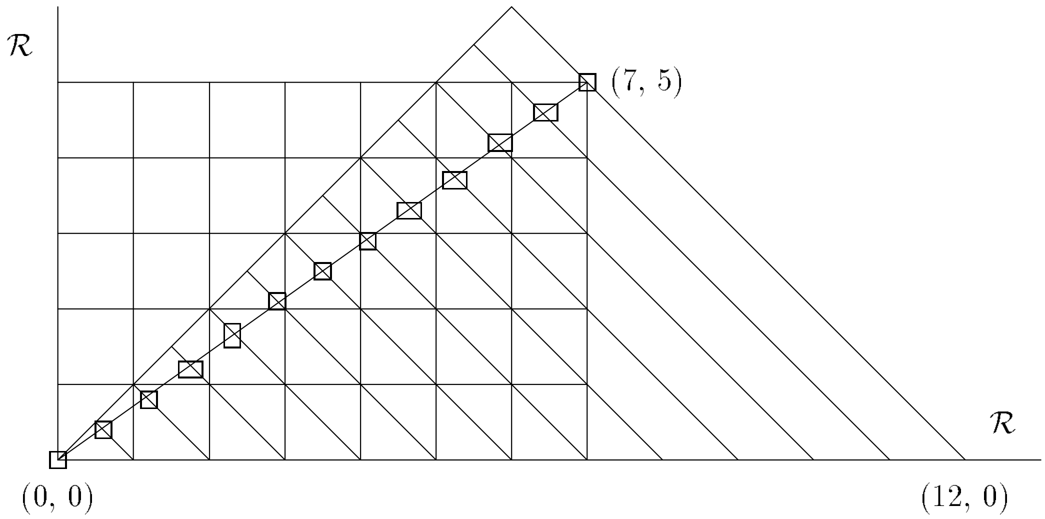

Figure 4: **Image of Equal Temperament in the morphism θ_***

Temperament is a projection of the Just Intonation Approximation to a line in the enlarged morphism θ_*.

Theorem 4 *Let p, q be two lines in the complex plane, such that $(0,0),(7,5) \in p; (12,0),(7,5) \in q$, respectively. Let $\pi : \mathcal{C} \to p$ be the projection of the plane $\mathcal{C}$ into the line p along the line q. Let $\mathcal{S}$ be an arbitrary 12-degree 2-interval (2/1, 3/2)-system. Then $W = \theta_*^{-1}(\pi(\eta(\theta(\mathcal{S}))))$.*

Proof. Cf. Figure 4. Without loss of generality, we prove the assertion for $\mathcal{S} = \mathcal{A}$.

Denote by $w_0 = (0,0), w_1 = (\frac{7}{12}, \frac{5}{12}), w_2 = 2 \cdot (\frac{7}{12}, \frac{5}{12}), \ldots, w_{12} = 12 \cdot (\frac{7}{12}, \frac{5}{12})$.
We have:

$$\begin{array}{rcrlrl}
w_0 & = & \pi(\eta(\theta(C^{(A)})), & \theta_*^{-1}(w_0) = & x^0 y^0 = & 1, \\
w_1 & = & \pi(\eta(\theta(C_\sharp^{(A)})) = \pi(\eta(\theta(D_\flat^{(A)})), & \theta_*^{-1}(w_1) = & x^{\frac{7}{12}} y^{\frac{5}{12}} = & \sqrt[12]{2}, \\
w_2 & = & \pi(\eta(\theta(D^{(A)})), & \theta_*^{-1}(w_2) = & x^{2\cdot\frac{7}{12}} y^{2\cdot\frac{5}{12}} = & (\sqrt[12]{2})^2, \\
 & & \ldots & & & \\
w_{12} & = & \pi(\eta(\theta(C'^{(A)})), & \theta_*^{-1}(w_{12}) = & x^{12\cdot\frac{7}{12}} y^{12\cdot\frac{5}{12}} = & (\sqrt[12]{2})^{12}.
\end{array}$$

$\square$

Now we show that the approximation $\mathcal{A}$ is closer to Just Intonation than Equal Temperament. As a unit q for imagination of measure values, we take $q = \log_2(G^{(W)}/G^{(JI)}) = \log_2 3 - (19/12) \approx 0.001629167$ for the metric ρ_q, cf. Lemma 1.

Denotation	$\rho_q(\cdot^{(JI)}, \cdot^{(A)})$	$\rho_q(\cdot^{(JI)}, \cdot^{(W)})$
C	0	0
$C_\sharp = D_\flat$	0.999342947	6.000641318
D 10/9	0.999342947	9.000633554
D 9/8	0	1.999995685
D 8/7	1.945816399	15.94578457
$D_\sharp, E_\flat$	0.999342947	7.000637399
E	0.999342947	7.000637399
F	0	1
$F_\sharp$	0.999342947	5.000641211

Table 4: **The Just Intonation Approximation $\mathcal{A}$ versus Equal Temperament**

Theorem 5 *For each interval $C, C_\sharp, \ldots, B, C'$, the Just Intonation Approximation $\mathcal{A}$ is closer to Just Intonation than Equal Temperament in the metric $\rho_q, q = \log_2 3 - (19/12)$.*

Proof. Use Theorem 2, cf. Table 4. □

The following lemma shows that the approximation $\mathcal{A}$ is characterized by a "comma". We immediately obtain:

Theorem 6 *Denote by $\kappa = 5 \cdot 3^8 \cdot 2^{-15} = 32\ 805/32\ 768 \approx 1.00112915$. Let $q = \log_2 3 - (19/12)$.*

Then $\rho_q(C_\sharp^{(A)}, C_\sharp^{(JI)}) = \rho_q(D_\sharp^{(A)}, D_\sharp^{(JI)}) = \rho_q(E^{(A)}, E^{(JI)}) = \rho_q(F_\sharp^{(A)}, F_\sharp^{(JI)}) = = \rho_q(D^{(A)}(10/9), D^{(JI)}(10/9)) = \rho_q(B^{(A)}(9/5), B^{(JI)}(9/5)) = \rho_q(G_\flat^{(A)}, G_\flat^{(JI)}) = = \rho_q(A_\flat^{(A)}, A_\flat^{(JI)}) = \rho_q(A^{(A)}, A^{(JI)}) = \rho_q(B^{(A)}, B^{(JI)}) = \log_2 \sqrt[q]{\kappa} (\approx 0.999345388) < 1 = \rho_q(G^{(W)}, G^{(JI)})$.

We collect the results of the paper into the following theorem.

Theorem 7 *Let $x = 2^8/3^5, y = 3^7/2^{11}$. Then the Just Intonation Approximation $\mathcal{A} = \{(x^4y^3)^k(x^7y^5)^{-h}; (k,h) = [-14,-9], (-10,-6), (-9,-6), (-8,-5), (-7,-5), (-6,-4), \{-2,-2\}, \{-1,-1\}, \{0,-1\}, \{0,0\}, \{1,0\}, \{2,1\}, (6,3), (7,4), (8,4), (9,5), (10,5), [14,8]\}$, where $\{,\}$ denotes approximated intervals based on numbers 2 and 3 ($B_\flat^{(A)}(16/9), F^{(A)}, C'^{(A)}, C^{(A)}, G^{(A)}, D^{(A)}(9/8)$, respectively) and the approximation $\mathcal{A}$ coincides with the Just Intonation value, $(,)$ denotes approximated intervals based on numbers 2, 3, and 5 ($D^{(A)}(10/9), A^{(A)}, E^{(A)}, B^{(A)}, F_\sharp^{(A)}, G_\flat^{(A)}, C_\sharp^{(A)}, A_\flat^{(A)}, E_\flat^{(A)}, B_\flat^{(A)}(9/5)$, respectively) and the intervals are κ-approximated. Finally, $[,]$ denotes approximated intervals based on numbers 2 and 7 ($B_\flat^{(A)}(7/4), D^{(A)}(8/7)$, respectively).*

Proof. The vector (k,h) is a solution of the following system of equations:

$$4k + (-7)h = \alpha, 3k + (-5)h = \beta,$$

where α, β are exponents of x, y, respectively, cf. Definition 2 (e.g. $(3,2)$ for F). Observe that

$$\det \begin{pmatrix} 4 & -7 \\ 3 & -5 \end{pmatrix} = 1,$$

so $k, h \in \mathcal{Z}$ and $(k,h) = (-5\alpha + 7\beta, -3\alpha + 4\beta)$. Rearrange now the solutions (k,h) as above. $\square$

Remark 1 *A tone system based on the (-14)th, (-8)th, 0th, 1st, and 9th fifths is called the Petzval tone system of the Second class, cf. [1], [2]. We see, that the approximation $\mathcal{A}$ is the (multivalued) system of this type.*

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