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MEASURES AND STATES ON BCK-ALGEBRAS

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Abstract. We define measures and states on BCK-algebras. We show that the set of elements of measure 0 is an ideal, and the corresponding quotient BCK-algebra is commutative with a lifted original measure. We present different examples of commutative as well as positively implicative BCK-algebras on subsets of $[0,\infty)$. Finally, we characterize commutative BCK-algebras as well as positively implicative BCK-algebras via order-determining systems of state-morphisms and states, respectively.

1. Introduction

According to Imai and Iséki, [ImIs], a BCK-*algebra* is a non-empty set X with a binary operation $*$ and with a constant element 0 satisfying the following axioms: for all $x, y, z \in X$

(I) $\big((x*y)*(x*z)\big)*(z*y) = 0$;
(II) $\big(x*(x*y)\big)*y = 0$;
(III) $x*x = 0$;
(IV) $x*y = 0$ and $y*x = 0$ imply $x = y$;
(V) $0*x = 0$.

As usual, we shall write $X = (X;*,0)$. In X we can define a partial binary operation $\preccurlyeq$ by

$$x \preccurlyeq y \quad \text{iff} \quad x*y = 0. \tag{1.1}$$

Then $\preccurlyeq$ is a partial ordering on X and in any BCK-algebra $(X;*,0)$ we have, for all $x, y, z \in X$, [MeJu],

$$x \preccurlyeq y \quad \text{implies} \quad z*y \preccurlyeq z*x, \tag{1.2}$$

$$(x*y)*z = (x*z)*y, \tag{1.3}$$

$$x*0 = x.$$

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Equivalently, a poset $X = (X; \preccurlyeq, 0)$ with the least element 0 and with a unary operation $*$ is a BCK-algebra iff, for all $x, y, z \in X$, we have

(i) $(x * y) * (x * z) \preccurlyeq z * y$;
(ii) $x * (x * y) \preccurlyeq y$;
(iii) $x \preccurlyeq y$ iff $x * y = 0$.

If X possesses a greatest element 1, we say that X is *bounded*, and we shall write $X = (X; *, 0, 1)$.

Special types of BCK-algebras are MV-algebras which are many valued analogues of a two-valued logic and they have been introduced by Chang [Cha]. We recall that according to Mundici [Mun], they can be characterized as follows. An MV-*algebra* is a non-empty set L with two special elements 0 and 1 ($0 \neq 1$), with a binary operation $\oplus : L \times L \to L$ and with a unary operation $^* : L \to L$ such that, for all $a, b, c \in L$, we have

(MVi) $a \oplus b = b \oplus a$ (commutativity);
(MVii) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ (associativity);
(MViii) $a \oplus 0 = a$;
(MViv) $a \oplus 1 = 1$;
(MVv) $(a^*)^* = a$;
(MVvi) $a \oplus a^* = 1$;
(MVvii) $0^* = 1$;
(MVviii) $(a^* \oplus b)^* \oplus b = (a \oplus b^*)^* \oplus a$.

We recall that the above axioms are not the independent, e.g. (MVvii) follows from (MVvi) and (MViii) for $a = 0$.

We define the following binary operations $\odot, \vee, \wedge$ as follows:

$$a \odot b := (a^* \oplus b^*)^*, \ \ a, b \in L, \tag{1.4}$$
$$a \vee b := (a^* \oplus b)^* \oplus b, \quad a \wedge b := (a^* \vee b^*)^*, \ \ a, b \in L.$$

Then $(L; \odot, 1)$ is a semigroup written "multiplicatively" with the neutral element 1.[1] If, for $a, b \in L$, we define

$$a \leq b \ \Leftrightarrow \ a = a \wedge b,$$

then $\leq$ is a partial order on L, and $(L; \vee, \wedge, 0, 1)$ is a distributive lattice with the least and greatest elements 0 and 1, respectively, [Cha]. We recall that $a \leq b$ iff $b \oplus a^* = 1$.

We note that if we define on an MV-algebra L a binary operation $*$ via $a * b := a \odot b^*$, $a, b \in L$, then $(L; *, 0)$ is a bounded, commutative BCK-algebra.

The paper is organized as follows. In Section 2, we introduce measures, measure-morphisms, and we show that any measure on a BCK-algebra induces a commutative quotient BCK-algebra. We give their connections with MV-algebras.

In Section 3, we shall characterize all commutative BCK-algebras as well as positively implicative BCK-algebras on $[0, 1]$, $[0, 1)$, and $[0, \infty)$, which preserve the original order, and we present corresponding measures on them. Finally, in Section 4, we give characterizations of semisimple BCK-algebras via order-determining system of states.

[1] We remark that in the literature by an MV-algebra is assumed the structure $(L; \oplus, \odot, ^*, 0, 1)$, where $\odot$ is defined by (1.4).

2. Measures and Commutative BCK-algebras

Let $X = (X; *, 0)$ be a BCK-algebra.

If $X = [0, a)$ or $X = [0, a]$, where $a \in (0, \infty)$, we denote by $*_R$ a binary operation on X such that

$$s *_R t = \max\{0,\ s - t\},\ \ s, t \in X, \tag{2.1}$$

where $s - t$ denotes the usual subtraction of real numbers s, t. Then $(X; *_R, 0)$ is a commutative BCK-algebra.

A mapping $m : X \to [0, \infty)$ such that, for all $x, y \in X$,

(i) $m(x * y) = m(x) - m(y)$ whenever $y \preccurlyeq x$ is said to be a *measure*;
(ii) if $1 \in X$, m is a measure with $m(1) = 1$, then m is said to be a *state*;
(iii) $m(x * y) = \max\{0,\ m(x) - m(y)\}$ is said to be a *measure-morphism*;
(iv) if $1 \in X$ and m is a measure-morphism with $m(1) = 1$, m is said to be a *state-morphism*.

A non-void subset I of a BCK-algebra X is said to be an *ideal* of X if

(i) $0 \in I$;
(ii) $x * y \in I$ and $y \in I$ imply $x \in I$.

For any $x, y \in X$ denote

$$x \curlywedge y := y * (y * x).$$

Obviously, $x \curlywedge y$ is a lower bound of x and y, and $x \curlywedge x = x$, $x \curlywedge 0 = 0 \curlywedge x = 0$, and

$$x * (y \curlywedge x) = x * y. \tag{2.2}$$

But in general, $x \curlywedge y \neq y \curlywedge x$. X is said to be *commutative* if $x \curlywedge y = y \curlywedge x$ for any $x, y \in X$. We recall that any MV-algebra can be organized into a bounded commutative BCK-algebra.

Proposition 2.1. *Let $(X; *, 0)$ be a BCK-algebra.*

(i) *Any measure-morphism on X is a measure on X.*
(ii) *$m(0) = 0$ whenever m is a measure on X.*
(iii) *$m(x) \le m(y)$ whenever $x \preccurlyeq y$ and m is a measure on X.*
(iv) *$x \preccurlyeq y$ implies $m\big(y * (y * x)\big) = m(x)$ whenever m is a measure on X.*
(v) *Let m be a measure on X, then $I_m := \{x \in X : m(x) = 0\}$ is an ideal of X.*

Proof. (ii). Since $0 \preccurlyeq 0$, we have $m(0) = m(0 * 0) = m(0) - m(0) = 0$. The rest of properties is simple.

(iv). We have

$$m\big(y * (y * x)\big) = m(y) - m(y * x) = m(y) - m(y) + m(x) = m(x).$$

(v). By (ii), $0 \in I_m$. Let $m(x*y) = 0$ and $m(y) = 0$. Due to (2.2), $0 = m(x*y) = m\big(x * (y \curlywedge x)\big) = m(x) - m(y \curlywedge x) = m(x)$, when we have used (iii) and $y \curlywedge x \preccurlyeq y$. Hence, $x \in I_m$. □

A measure (a state) on a BCK-algebra X is said to be a *σ-measure* (a *σ-state*), if for a sequence $\{x_n\}$ of elements from X with $\{x_n\} \nearrow x$, $x \in X$, (i.e., $x_1 \preccurlyeq x_2 \preccurlyeq \cdots \preccurlyeq x$ and $x = \bigvee_n x_n$) we have $\lim_n m(x_n) = m(x)$.

Proposition 2.2. *Let m be a σ-measure on X. Then if $\{x_n\} \searrow x$ (i.e., $x_1 \succcurlyeq x_2 \succcurlyeq \cdots \succcurlyeq x$ and $\bigwedge_n x_n = x$) we have $m(x_n) \searrow m(x)$.*

Proof. Let $\{x_n\} \searrow x$. Put $y_n = x_1 * x_n$, $y = x_1 * x$. We claim that $\bigvee_n y_n = y$. It is clear that $y_n \preccurlyeq y$ for any $n \geq 1$. Let $y_n \preccurlyeq z$ for some $z \in X$ and any $n \geq 1$. Then $x_1 * x_n \preccurlyeq z$ so that $x_1 * z \preccurlyeq x_n$ for any n. Hence $x_1 * z \preccurlyeq x$, and, consequently, $x_1 * x \preccurlyeq z$ so that $\bigvee_n y_n = x_1 * x$.

Calculate $\lim_n m(x_1 * x_n) = m(x_1 * x)$ so that $m(x_1) - \lim_n m(x_n) = m(x_1) - m(x)$ and $m(x_n) \searrow m(x)$. $\square$

We note that any measure on a commutative BCK-algebra, which is continuous from above, is σ-additive. For general BCK-algebras this is not true (see (ii) of Example 3.2 below, where the measure is continuous from above but it is not σ-additive).

A system $\mathcal{S}$ of measures (states, measure-morphisms, or state-morphisms, respectively) is an *order-determining system* on a BCK-algebra X if $m(x) \leq m(y)$ for any $m \in \mathcal{S}$ imply $x \preccurlyeq y$.

Example 2.3. Let Ω be an arbitrary non-empty set and let X be the set of all positive-valued functions on Ω. For $f, g \in X$, we define

$$(f * g)(\omega) = \begin{cases} 0 & \text{if } f(\omega) \leq g(\omega), \\ f(\omega) - g(\omega) & \text{if } g(\omega) < f(\omega), \end{cases} \quad \omega \in \Omega. \tag{2.3}$$

Then $(X; *, 0_\Omega)$ is a commutative BCK-algebra, where $0_\Omega(\omega) = 0$ for any $\omega \in \Omega$, and, for any fixed $\omega \in \Omega$, the mapping $m_\omega : X \to [0, \infty)$ defined via $m_\omega(f) := f(\omega)$, $f \in X$, is a measure-morphism on X. In addition, $\mathcal{S} = \{m_\omega : \omega \in \Omega\}$ is an order-determining system of measure-morphisms.

Proof. It is simply to verify that $(X; *, 0_\Omega)$ is a commutative BCK-algebra. In addition, $m_\omega(f * g) = (f * g)(\omega) = \max\{0, f(\omega) - g(\omega)\} = m_\omega(f) *_R m_\omega(g)$. Since $f * g = 0_\Omega$ iff $f(\omega) \leq g(\omega)$ for any $\omega \in \Omega$, $\mathcal{S}$ is an order-determining system of measure-morphisms. $\square$

Example 2.4. Let $X \subseteq [0, \infty)^\Omega$ be such a system that

(i) $0_\Omega \in X$;
(ii) if $f, g \in X$, then $f * g \in X$,

where $*$ is defined via (2.3). Then $(X; *, 0_\Omega)$, is a commutative BCK-algebra with an order-determining system of measure-morphisms defined in Example 2.3.

Theorem 2.5. *Let $X \subseteq [0, 1]^\Omega$ be a system of fuzzy sets on Ω such that*

(i) *$1_\Omega \in X$, where $1_\Omega(\omega) = 1$ for any $\omega \in \Omega$;*
(ii) *if $f, g \in X$, then $f * g \in X$,*

where $*$ is defined via (2.3), is a bounded commutative BCK-algebra such that the system $\mathcal{S}$ from Example 2.3 is an order-determining system of state-morphisms.

Conversely, let $(X; *, 0, 1)$ be a bounded BCK-algebra with an order-determining system of state-morphisms $\mathcal{S}$. Then $(X; *, 0, 1)$ is a commutative BCK-algebra, and $(X; *, 0, 1)$ is isomorphic with a system $\overline{X} \subseteq [0, 1]^{\mathcal{S}}$, where $\overline{X} := \{\bar{x} \in [0, 1]^{\mathcal{S}} : \bar{x}(m) := m(x),\ m \in \mathcal{S}$ and $x \in X\}$, and $\overline{X}$ satisfies the conditions of Example 2.4.

Proof. If $\mathcal{S}$ is an order-determining system of measure-morphisms, then

$$\begin{aligned} m\big(x*(x*y)\big) &= m(x) *_R m(x*y) = m(x) *_R \big(m(x) *_R m(y)\big) \\ &= m(y) *_R \big(m(y) *_R m(x)\big) = m\big(y*(y*x)\big) \end{aligned}$$

which entails $x*(x*y) = y*(y*x)$ for any $x, y \in X$.

Conversely, we have $(\overline{x*y})(m) = m(x*y) = m(x) *_R m(y) = (\bar{x} * \bar{y})(m)$ for any $m \in \mathcal{S}$, so that $\bar{x} * \bar{y} \in \overline{X}$. It is clear that $\overline{0} = 0_{\mathcal{S}}$ and $\bar{1} = 1_{\mathcal{S}}$. In addition, $\bar{x} = \bar{y}$ implies $m(x) = m(y)$ for any $m \in \mathcal{S}$, so that $x = y$. □

Theorem 2.6. *Let $(X;*,0)$ be a BCK-algebra ($(X;*,0,1)$ be a bounded BCK-algebra). If X possesses an order-determining system of measures (an order-determining system of states), then X is commutative.*

Proof. Let $\mathcal{S}$ be an order-determining system of measures on X. Suppose that, for $x, y \in X$ we have $x \preccurlyeq y$. According to (iv) of Proposition 2.1, we have $m\big(y*(y*x)\big) = m(x)$. Since $\mathcal{S}$ is order-determining, $y*(y*x) = x$. Applying [MeJu, Thm I.5.6 (i)], X is commutative. □

Example 2.7. Let $X = ([0,1]; *_R, 0, 1)$ be a BCK-algebra, where $*_R$ is defined via (2.1). Then $X = [0,\, 1]$ has a unique state state-morphism m, namely, $m(t) = t,\ t \in [0,\, 1]$, and $\mathcal{S} := \{m\}$ is an order-determining system of state-morphisms on X. Consequently, $[0,\, 1]$ is a commutative bounded BCK-algebra.

Let $(X;*,0,1)$ be a BCK-algebra, and define a unary operation * and binary operations $\oplus, \odot, \vee$, and $\wedge$ on X as follows

$$x^* := 1 * x,\ \ x \in X; \tag{2.4}$$

$$x \oplus y := (x^* * y)^*,\ \ x, y \in X; \tag{2.5}$$

$$x \odot y := (x^* \oplus y)^*,\ \ x, y \in X; \tag{2.6}$$

$$x \vee y := (x \odot y^*) \oplus y,\ \ x, y \in X; \tag{2.7}$$

$$x \wedge y := (x \oplus y^*) \odot y,\ \ x, y \in X. \tag{2.8}$$

Proposition 2.8. *Let $(X;*,0,1)$ be a commutative BCK-algebra and let m be a state on X. For any $x, y \in X$ we have*

$$\begin{aligned} m(x \oplus y) + m(x \odot y) &= m(x) + m(y). \\ m(x \vee y) + m(x \wedge y) &= m(x) + m(y). \\ m(x^*) &= 1 - m(x). \\ m(x \oplus y) &\le m(x) + m(y). \\ m(x \vee y) &\le m(x) + m(y). \end{aligned}$$

Proof. Since $(X;*,0,1)$ is a commutative BCK-algebra, it is well-known that $(X;\oplus,\odot,{}^*,0,1)$ is an MV-algebra and in any MV-algebra we have, for any $x, y \in X$,

$$\begin{aligned} (x \oplus y) \odot x^* &= y \odot (x \odot y)^*, \\ (x \vee y) \odot x^* &= y \odot (x \wedge y)^*. \end{aligned}$$

Since $x \odot y \preccurlyeq x \preccurlyeq x \vee y \preccurlyeq x \oplus y$, we have the statements of Proposition. □

Theorem 2.9. *Let m be a measure on a BCK-algebra $(X;*,0)$. For $x,y\in X$, we write $x\sim_m y$ if and only if $m(x*y)=0=m(y*x)$. Then $\sim_m$ is a congruence on X and $(X_m;*_m,0_m)$ is a commutative BCK-algebra, where $X_m:=X/\sim_m:=\{\tilde{x}:=\{y\in X:\ y\sim_m x\}:\ x\in X\}$, $0_m:=\tilde{0}$, and $*_m$ is a binary operation on X_m induced by $*$ and $\sim_m$. In addition, the mapping $\tilde{m}:X_m\to[0,\infty)$ defined by $\tilde{m}(\tilde{x}):=m(x)$, $\tilde{x}\in X_m$, is a well-defined measure on X_m.*

Proof. Due to (v) of Proposition 2.1, $I_m:=\{x\in X:\ m(x)=0\}$ is an ideal of X. The property $x\sim_m y$ holds iff $x*y\in I_m$ and $y*x\in I_m$, so that $\sim_m$ is a congruence on X, and $X_m:=X/\sim_m$ is a BCK-algebra, [MeJu, Thm II.1.6].

We now show that

$$x\sim_m y \quad \text{iff} \quad m(x)=m(y)=m(x\curlywedge y)=m(y\curlywedge x). \tag{2.9}$$

Indeed, let $\tilde{x}=\tilde{y}$. By (II) $x*(x*y)\preccurlyeq y$, and since $x*y\preccurlyeq x$, we have

$$m(y)\ge m\big(x*(x*y)\big)=m(x)-m(x*y)=m(x),$$

and by symmetry, $m(x)\ge m(y)$, so that $m(y)=m(y\curlywedge x)=m(x\curlywedge y)=m(x)$. The converse statement is now clear. Hence, $\tilde{m}(\tilde{x}):=m(x)$ is a well-defined mapping.

It is easy to see that $\tilde{x}\preccurlyeq_m\tilde{y}$ iff $m(x*y)=0$, where $\preccurlyeq_m$ is the BCK-order in $X/\sim_m$. Define $x_0=y\curlywedge x=x*(x*y)$. We claim that $\widetilde{x_0}=\tilde{x}$. Indeed, it is sufficient to verify (2.9) for x and x_0.

$$\begin{aligned}
m(x_0)&=\big(x*(x*y)\big)=m(x)-m(x*y)=m(x).\\
m(x_0\curlywedge x)&=m\big(x*(x*x_0)\big)=m\Big(x*\Big(x*\big(x*(x*y)\big)\Big)\Big)\\
&=m(x)-m(x)+m\big((x*(x*y)\big)=m(x)-m(x*y)=m(x).\\
m(x\curlywedge x_0)&=m\big(x_0*(x_0*x)\big)=m\Big(\big(x*(x*y)\big)*\Big(\big(x*(x*y)\big)*x\Big)\Big)\\
&=m\big(x*(x*y)\big)=m(x).
\end{aligned}$$

Since $\widetilde{x_0}=\tilde{x}$, we have $\widetilde{x_0}\preccurlyeq_m\tilde{y}$ and then

$$\begin{aligned}
\tilde{m}(\tilde{y}*_m\tilde{x})&=\tilde{m}(\tilde{y}*_m\widetilde{x_0})=\tilde{m}(\widetilde{y*x_0})\\
&=m(y*x_0)=m(y)-m(x_0)\\
&=m(y)-m(x)=\tilde{m}(\tilde{y})-\tilde{m}(\tilde{x}),
\end{aligned}$$

so that, $\tilde{m}$ is a measure on X_m.

We now show that $X/\sim_m$ is a commutative BCK-algebra. Let $\tilde{x}\preccurlyeq_m\tilde{y}$. By (II), we have $\tilde{y}*_m(\tilde{y}*_m\tilde{x})\preccurlyeq_m\tilde{x}$. We prove that $\tilde{x}*_m\big(\tilde{y}*_m(\tilde{y}*_m\tilde{x})\big)=\tilde{0}$. Calculate

$$\begin{aligned}
m\Big(x*\big(y*(y*x)\big)\Big)&=m\big(x*(x\curlywedge y)\big)=m(x)-m(x\curlywedge x)\\
&=m(x)-m(y)+m(y*x)=m(x)-m(y)+\tilde{m}(\tilde{y}*_m\tilde{x})\\
&=m(x)-m(y)+\tilde{m}(\tilde{y})-\tilde{m}(\tilde{x})=0,
\end{aligned}$$

which entails that $\tilde{x}*_m\big(\tilde{y}*_m(\tilde{y}*_m\tilde{x})\big)=\tilde{0}$, and $\tilde{x}\preccurlyeq_m\tilde{y}*_m(\tilde{y}*_m\tilde{x})$ so that $\tilde{x}=\tilde{y}*_m(\tilde{y}*_m\tilde{x})$. This property is by [MeJu, Thm I.5.6.(i)] the necessary and sufficient condition to be X_m a commutative BCK-algebra. $\square$

Corollary 2.10. Let m be a state on a bounded BCK-algebra $(X;*,0,1)$. Then $\tilde{m}$ from Theorem 2.9 is a state on a bounded commutative BCK-algebra $(X_m;*_m,0_m,1_m)$, where $1_m := \tilde{1}$. In addition, for all $x,y,z\in X$, we have (i) $m(x\oplus y)=m(y\oplus x)$; (ii) $m\big((x\oplus y)\oplus z\big)=m\big(x\oplus(y\oplus z)\big)$; (iii) $m(x\oplus 0)=m(x)$; (iv) $m(x\oplus 1)=1$; (v) $m\big((x^*)^*\big)=m(x)$; (vi) $m(x\oplus x^*)=1$; (vii) $m(0^*)=1$; (viii) $m\big((x^*\oplus y)^*\oplus y\big)=m\big((x\oplus y^*)^*\oplus x\big)$; (ix) $m(x\odot y)=m(y\odot x)$; (x) $m(x\vee y)=m(y\vee x)$; (xi) $m(x\wedge y)=m(y\wedge x)=m(x\curlywedge y)=m(y\curlywedge x)$, and $(X_m;\oplus_m,\odot_m^{*_m},0_m,1_m)$ is an MV-algebra, where $^{*_m},\oplus_m,\odot_m$ are operations on X_m induced from $^*,\oplus,\odot$ defined via (2.4)–(2.6).

Proof. It follows from Theorem 2.9 and from fact that $\tilde{m}(\tilde{1})=m(1)=1$. $\square$

3. Examples of BCK-algebras and Measures

A BCK-algebra $(X;*,0)$ is said to be (i) *positively implicative* if

$$(x*z)*(y*z)=(x*y)*z,\ \ x,y,z\in X;$$

(ii) *implicative* if

$$x=x*(y*x),\ \ x,y\in X.$$

According to [MeJu, Thm I.4.2-(b)], X is positively implicative[2] iff

$$x*y=(x*y)*y,\ \ x,y\in X, \tag{3.1}$$

and X is implicative [MeJu, Thm I.6.2] iff it is commutative and positive implicative.

Example 3.1. Let $X=\{0,1,\dots,K\}$ be integers ordered by the natural way, where K is either an integer or $\aleph_0$.

(i) X is a commutative BCK-algebra with the natural order if and only if

$$n*m=\begin{cases}0 & \text{if } n\le m,\\ n-m & \text{if } m<n.\end{cases}$$

(ii) X is a positively implicative BCK-algebra with the natural order if and only if

$$n*m=\begin{cases}0 & \text{if } n\le m,\\ n & \text{if } m<n.\end{cases}$$

If K is an integer, then there is a unique state (a unique state-morphism) m, namely for the case (i) $m(i)=i/K$, for $i=0,\dots,K$, and for the case (ii) $m(i)=0$ for $0\le i\le K-1$ and $m(K)=1$.

If K is not an integer, then there is no state-morphism, there is only a zero-valued measure-morphism.

(iii) X cannot be any implicative BCK-algebra with the natural order.

Let $(X_i,*_i,0_i)$ $(i=1,2)$ be two BCK-algebras. A mapping $h:X_1\to X_2$ is said to be a *homomorphism* if $h(x*_1 y)=h(x)*_2 h(y)$, $x,y\in X$. If h is a one-to-one homomorphism and h^{-1} is a homomorphism, too, then h is said to be an *isomorphism*.

[2] If $(X;*,0)$ is positively implicative, so is the Iséki extension of X.

Example 3.2. Let the interval of real numbers $X = [0,\,1]$ be ordered by the natural way.

(i) X is a bounded commutative BCK-algebra with the BCK-order equal to the natural one if and only if there is a continuous, increasing function $g:\ [0,\,1] \to [0,\,1]$ with $g(0) = 0$ and $g(1) = 1$ (called a generator) such that

$$x * y := g^{-1}\big(\max\{0, g(x) - g(y)\}\big), \tag{3.2}$$

for any $x, y \in [0,\,1]$. If such a generator exists, is unique. In addition, all commutative BCK-algebras on $[0,\,1]$ with the natural order are isomorphic. Any such a BCK-algebra possesses a unique state-morphism, namely $m_g(x) := g(x),\ x \in [0,\,1]$, and $\mathcal{S}_g := \{m_g\}$ is an order-determining system of state-morphisms on it.

(ii) X is a bounded positively implicative BCK-algebra with the BCK-order equal to the natural one if and only if

$$x * y := \begin{cases} 0 & \text{if } x \le y, \\ x & \text{if } x > y, \end{cases} \tag{3.3}$$

for any $x, y \in [0,\,1]$.

In this case, X possesses a unique state m, namely $m(1) = 1$ and $m(x) = 0$ for any $x \in [0,\,1)$.

(iii) X cannot be any implicative BCK-algebra with the natural order.

Proof. (i) It is easy to verify that (3.2) entails that $[0,\,1]$ is a commutative BCK-algebra. Suppose the converse, i.e., let $X = [0,\,1]$ be a commutative BCK-algebra with the natural order. If $x \le y$, then $x * y = 0$.

Define on $X = [0,\,1]$ a partial binary operation $\ominus$ such that $b \ominus a$ is defined iff $a \le b$ and then $b \ominus a := b * a$. It is not complicated to verify that, for all $a, b, c \in [0,\,1]$, we have (i) $b \ominus a \le b$ and $b \ominus (b \ominus a) = a$; (ii) if $a \le b \le c$ then $c \ominus b \le c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$ (see also [DvKi]). The structure $([0,\,1]; \oplus, 0, 1)$ is a so-called D-poset [KoCh], and by Mesiar [Mes, Thm 1], there is a generator g such that, for $x \ge y$, (3.2) holds. Hence, (3.2) holds for all $x, y \in [0,\,1]$.

Define $X_g := ([0,\,1]; *_g, 0)$ as the BCK-algebra with $*_g$ defined via (3.2) for a fixed generator g, and let X_0 be the space corresponding to $g = id_{[0,\,1]}$. Then the mapping $h_g:\ X_g \to X_0$ defined as $h_g(x) := g(x)$, $x \in X_g$, is an isomorphism in question.

Using Example 2.7 and the established isomorphism, $\{m_g\}$ satisfies our assertion.

(ii) Since X is linearly ordered, it is clear that $x * y = 0$ whenever $x \le y$. If $y = 0$ then $x * y = x$, so suppose that $1 \ge x > y > 0$. Since X is positively implicative, we have $y < x * y \le x$. We claim that $x * y = x$.

Let $\epsilon >$ be an arbitrary real number less than $x - y$. Then from (1.3) we conclude that

$$\big(x * (x - \epsilon)\big) * y = (x * y) * (x - \epsilon).$$

Since the left-hand side of this equation is strictly positive, the right-hand side implies $x \ge x * y > x - \epsilon$, and hence $x * y = x$.

The converse statement is evident.

Let now m be a state on X and choose $x \in [0,\,1)$. Then $1 * x = \big(1 * x\big) * x$ and therefore $m(1*x) = m\big((1*x)*x\big)$, so that $1 - m(x) = m(1*x) - m(x) = 1 - 2\,m(x)$ which implies $m(x) = 0$. $\square$

Example 3.3. Let $X = [0,\ \infty)$ be ordered by the natural way.

(i) If $g:\ [0,\ \infty) \to [0,\ \infty)$ is an increasing and continuous function with $g(0) = 0$, then $([0,\ \infty); *_g, 0)$ is a commutative BCK-algebra with the BCK-order equal to the natural order, where

$$x *_g y := g^{-1}\big(\max\{0,\ g(x) - g(y)\}\big),\ \ x, y \in [0,\ \infty).$$

If $g_0 := id_{[0,\ \infty)}$, then any $([0,\ \infty); *_g, 0)$ is isomorphic with $([0,\ \infty); *_{g_0}, 0)$ if and only if g is unbounded. If g is bounded, then $([0,\ \infty); *_g, 0)$ is isomorphic with $([0,\ 1); *_R, 0)$, where $*_R$ is defined via (2.1).

Conversely, if $([0, \infty); *, 0)$ is a commutative BCK-algebra with the natural order, then there is an increasing and continuous function $g:\ [0, \infty) \to [0,\ \infty)$ with $g(0) = 0$ such that $* = *_g$, where $*_g$ is defined via (3.2). In addition, any measure m on $([0,\ \infty); *_g, 0)$ is of the form $m(x) = k\, g(x), x \in [0,\ \infty)$, where k is a non-negative constant.

(ii) $([0,\ \infty); *, 0)$ is a positively implicative BCK-algebra with the natural order if and only if $*$ is defined via (3.3). In addition, the zero function on $[0,\ \infty)$ is a unique measure on it.

(iii) $([0,\ \infty); *, 0)$ cannot be any implicative BCK-algebra with the natural order.

Proof. (i) The first direction is straightforward. Suppose now that $([0, \infty); *, 0)$ is a commutative BCK-algebra with the natural order. Put $X_k := [0,\ k]$, where $k \geq 1$ is an integer. The restriction of $*$ onto $X_k \times X_k$ entails that $(X_k; *, 0, k)$ is a commutative and bounded BCK-subalgebra of $[0, \infty)$. According to (i) of Example 3.2, X_k is isomorphic with X_1, for any $k \geq 1$; an isomorphism is, e.g., $h_k(x) := x/k,\ x \in X_k$. Due to the last example, there exists a unique increasing and continuous function $g_k:\ [0,\ k] \to [0,\ k]$ with $g_k(0) = 0$ and $g_k(k) = k$, such that, for all $x, y \in X_k$,

$$x * y = g_k^{-1}\big(\max\{0, g_k(x) - g_k(y)\}\big).$$

For any $k \geq 1$, we define $\hat{g}_k := g_k/g_k(1)$. We assert that the restriction of $\hat{g}_{k+1}$ onto X_k, $\hat{g}_{k+1}|X_{k+1}$, is equal to $\hat{g}_k$ for any $k \geq 1$. Indeed, due to the uniqueness of g_k's on X_k with the condition $g_k(k) = k,\ k \geq 1$, we have $k\, g_{k+1}/g_{k+1}(1)|X_k = g_k$ for any $k \geq 1$. Let $t \in X_k$ and calculate

$$\begin{aligned}
\hat{g}_{k+1}(t) &= \frac{g_{k+1}(t)}{g_{k+1}(1)} = \frac{k\, g_{k+1}(t)}{g_{k+1}(k)} \frac{g_{k+1}(k)}{k\, g_{k+1}(1)} \\
&= g_k(t) \frac{g_{k+1}(k)}{k\, g_{k+1}(1)} = \frac{g_k(t)}{g_k(1)}\, g_k(1) \left(\frac{k\, g_{k+1}(1)}{g_{k+1}(k)} \right)^{-1} \\
&= \hat{g}_k(t)\, g_k(1) \left(g_k(1)\right)^{-1} = \hat{g}_k(t).
\end{aligned}$$

Define a function $g:\ [0, \infty) \to [0,\ \infty)$ as follows $g(t) = \hat{g}_k(t)$ if $t \in X_k$ for some $k \geq 1$. Then g is a well-defined function with $g(0) = 0, g(+\infty) = \infty$, g is increasing and continuous, and, in addition,

$$x * y = g^{-1}\big(\max\{0, g(x) - g(y)\}\big),\ \ x, y \in [0,\ \infty).$$

This equality follows from the observation that if $x, y \in X_k$, then

$$\begin{aligned} x * y &= g_k^{-1}\big(\max\{0, g_k(x) - g_k(y)\}\big) \\ &= \hat{g}_k^{-1}\big(\max\{0, \hat{g}_k(x) - \hat{g}_k(y)\}\big) \\ &= g^{-1}\big(\max\{0, g(x) - g(y)\}\big), \end{aligned}$$

which finishes the first part of the proof.

If g is unbounded, then $h_g : [0, \infty) \to [0, \infty)$ defined via $h_g(x) := g(x)$, $x \in [0, \infty)$, is an isomorphism between $([0, \infty); *_g, 0)$ and $([0, \infty); *_{id_{[0, \infty)}}, 0)$. In addition, for any $x, y \in [0, \infty)$, there exists an $z \in [0, \infty)$ with $z *_g x = y$. Indeed, we put $z := g^{-1}\big(g(x) + g(y)\big)$.

Suppose now that g is bounded and let $k_0 := \sup\{g(x) : x \in [0, \infty)\}$. From the continuity of g we obtain that there is a $u > 0$ such that $g(u) > k_0/2$. But for $x = u$ and $y = u$ there is no z such that $z *_g u = u$ which entails that $([0, \infty); *_g, 0)$ is not isomorphic with $([0, \infty); *_{id_{[0, \infty)}}, 0)$. We claim that $([0, \infty); *_g, 0)$ is isomorphic with $([0, 1); *_R, 0)$ (see (2.1)). Indeed, the mapping $h_g(x) := g(x)/k_0$ is an isomorphism in question.

The assertion on measures follows from a simple observation that any measure m on $X = ([0, \infty); *_{id_{[0, \infty)}}, 0)$ or on $X = ([0, 1); *_R, 0)$ is of the form $m(x) = k\,x$, $x \in X$, where $k \geq 0$ is a constant.

(ii) It follows the same ideas as those in the proof of (ii) in Example 3.2. $\square$

Example 3.4. Let the interval $X = [0, 1)$ be ordered by the natural way.

(i) $([0, 1); *, 0)$ is a commutative BCK-algebra with the natural order if and only if there is an increasing and continuous function $g : [0, 1) \to [0, \infty)$ with $g(0) = 0$ such that (3.2) holds for any $x, y \in [0, 1)$. Let $*_g$ be $*$ depending on g via (3.2). If g is bounded on $[0, 1)$, then $([0, 1); *_g, 0)$ is isomorphic with $([0, 1); *_{g_0}, 0)$, where $g_0 := id_{[0, 1)}$, and if g is unbounded, then $([0, 1); *_g, 0)$ is isomorphic to $\big([0, \infty); *_{id_{[0,\infty)}}, 0\big)$.

In addition, any state on $([0, 1); *_g, 0)$ is of the form $m(x) = k\,g(x)$, $x \in [0, 1)$, where k is a non-negative constant.

(ii) $([0, 1); *, 0)$, is a positively implicative BCK-algebra with the natural order if and only if $*$ is defined via (3.3) for all $x, y \in [0, 1)$. In addition, the zero function on $[0, 1)$ is a unique measure on it, and any measure m on $([0, 1); *_g, 0)$ is of the form $m(x) := k\,g(x)$, $x \in [0, 1)$, where $k \geq 1$ is a constant.

(iii) $([0, 1); *, 0)$ cannot be any implicative BCK-algebra with the natural order.

Proof. (i) follows from [Mes, Thm 2, Cor 1] and from (ii) of Example 3.3, or it can be proved directly by way different of that used in [Mes]. Let $\{d_n\}$ be a strictly increasing sequence of positive numbers such that $d_n \nearrow 1$. Since any $X_n = [0, d_n]$ is isomorphic with $[0, 1]$, there is a unique increasing and continuous function $g_n : X_n \to X_n$ with $g_n(0) = 0$ and $g_n(d_n) = d_n$ such that $x * y = \max\{0, g_n(x) - g_n(y)\}$ if $x, y \in X_n$. If we define $\hat{g}_n := g_n/g_n(d_1)$, then similarly as in the proof of Example 3.3, we have that $\hat{g}_{n+1}|X_n = \hat{g}_n$. The function $g(x) = \hat{g}_n(x)$ if $x \in X_n$, satisfies (3.2). Comparing this approach with that used in the proof of Example 3.3, we obtain the statements in question.

(ii) follows the same ideas as those in the proof of (ii) in Example 3.2. $\square$

Example 3.5. Let X be a linearly ordered set with the least element 0. Then $X = (X; *, 0)$ is a positively implicative BCK-algebra with the original linear order if and only if $*$ is defined (3.3) for all $x, y \in X$.

If 1 belongs to X, then m with $m(x) = 0$ if $x \in X \setminus \{1\}$ and $m(1) = 1$ is a unique state on X, and any measure on X is a constant multiple of a state. If X is not bounded, then the zero function is a unique measure on X.

Proof. We claim that if $x > y$, then $x * y = x$. Suppose the converse. Then $x > x*y > y$. Let now y_1 be an arbitrary element of X such that $x > y_1 > y$. Then $x > x*y \geq x*y_1 > y_1$. In particular, if $y_1 = x*y$, then $x > x*y \geq x*(x*y) > x*y$ which is a contradiction.

The statement about states and measures follows easily from the proof of Example 3.2. □

4. Representation of Semisimple BCK-algebras

An ideal I is *maximal* if it is proper and there is no proper ideal of X containing I and different of I. We denote by $\mathcal{M} = \mathcal{M}(X)$ the set of all maximal ideals of a bounded BCK-algebra X.[3] The set

$$R(X) := \bigcap \{I : \ I \in \mathcal{M}(X)\}$$

is said to be the *radical* of X. A bounded BCK-algebra is said to be *semisimple* if $R(X) = \{0\}$.

We recall that according to [MeJu, Thm VII.5.6], a bounded positively implicative BCK-algebra X is semisimple iff, for any $x \in X$, we have

$$NNx = x, \tag{4.1}$$

where $Nx = 1 * x$. In addition, by [MeJu, Thm II.5.5], an ideal M of a positively implicative BCK-algebra is maximal iff, for all $x, y \in X$, $x \notin M$ and $y \notin M$ imply $x * y \in M$ and $y * x \in M$.

Example 4.1. Let $\Omega \neq \emptyset$ and denote by $\mathcal{B}$ any Boolean algebra of Ω, i.e., $\mathcal{B} \subseteq 2^\Omega$ such that

(i) $\emptyset \in \mathcal{B}$;
(ii) if $A \in \mathcal{B}$, then $\Omega \setminus A \in \mathcal{B}$;
(iii) if $A, B \in \mathcal{B}$, then $A \cup B \in \mathcal{B}$.

Then $(\mathcal{B}; *, \emptyset)$ is a bounded semisimple positively implicative BCK-algebra, where $*$ is defined via

$$A * B = A \setminus B, \ \ A, B \in \mathcal{B}. \tag{4.2}$$

Example 4.2. Let X be a system of fuzzy sets of Ω, i.e. $X \subseteq [0, 1]^\Omega$ $(\Omega \neq \emptyset)$, such that

(i) $1_\Omega \in X$, where $1_\Omega(\omega) = 1$ for any $\omega \in \Omega$;
(ii) if $f, g \in X$, then $f * g \in X$, where

$$(f * g)(\omega) = \begin{cases} 0 & \text{if } f(\omega) \leq g(\omega), \\ f(\omega) & \text{if } f(\omega) > g(\omega), \end{cases} \quad \omega \in \Omega. \tag{4.3}$$

[3] There is at least one maximal ideal of X.

Then $(X;*,0_\Omega,1_\Omega)$ is a semisimple positively implicative BCK-algebra.

It is worthy to say that (ii) in Example 3.1 is a positively implicative BCK-algebra which is not semisimple whenever K is greater than 1.

Theorem 4.3. *Let $(X;*,0,1)$ be a bounded positively implicative BCK-algebra. Then there are a non-void set Ω, a Boolean algebra $\mathcal{B}\subseteq 2^\Omega$ and a homomorphism h from X onto $\mathcal{B}$.*

Proof. Take $\Omega=\mathcal{M}(X)$ the set of all maximal ideals of X, and, for any $x\in X$, define a mapping $\widehat{x}:\ \mathcal{M}\to\{0,1\}$ via

$$\widehat{x}(M)=\begin{cases}0 & \text{if } x\in M,\\ 1 & \text{if } x\notin M,\end{cases}\quad M\in\mathcal{M}.$$

Put $\widehat{X}:=\{\widehat{x}:\ x\in X\}$ and define $*_1$ on $\widehat{X}$ as follows

$$(\widehat{x}*_1\widehat{y})(M)=\begin{cases}0 & \text{if } \widehat{x}(M)\le\widehat{y}(M),\\ \widehat{x}(M) & \text{if } \widehat{x}(M)>\widehat{y}(M),\end{cases}\quad M\in\mathcal{M}.$$

Then $(\widehat{X},*_1,\widehat{0},\widehat{1})$ is a bounded semisimple BCK-algebra. We have $\widehat{0}=0_\mathcal{M}$ and $\widehat{1}=1_\mathcal{M}$. We claim that $\widehat{x*y}=\widehat{x}*_1\widehat{y}$ for any $x,y\in X$.

We have two cases:

Case (a). $\widehat{x*y}(M)=0$, then $x*y\in M$ and we have four possibilities:

(1) $x\in M,\ y\in M$, then $\widehat{x}(M)=0=\widehat{y}(M)$ and $(\widehat{x}*_1\widehat{y})(M)=0$.
(2) $x\notin M,\ y\in M$, then $x*(x*y)\le y\in M$ so that $x\in M$, which is a contradiction.
(3) $x\in M,\ y\notin M$, then $\widehat{x}(M)=0,\ \widehat{y}(M)=1$ and $(\widehat{x}*_1\widehat{y})(M)=0$.
(4) $x\notin M,\ y\notin M$, then $\widehat{x}(M)=1,\ \widehat{y}(M)=1$ and $(\widehat{x}*_1\widehat{y})(M)=0$.

Case (b). $\widehat{x*y}(M)=1$, then $x*y\in M$. We have again four possibilities:

(1) $x\in M,\ y\in M$, then by [MeJu, Cor VII.5.4], $1*x\notin M,\ 1*y\notin M$ and by [MeJu, Thm II.5.5] $(1*y)*(x*y)\in M$. But $(1*y)*(x*y)=(1*x)*y\in M$, hence $1*x\in M$ while $y\in M$ and M is an ideal, which gives a contradiction.
(2) $x\notin M,\ y\in M$, then $\widehat{x}(M)=1,\ \widehat{y}(M)=0,\ (\widehat{x}*_1\widehat{y})(M)=1$.
(3) $x\in M,\ y\notin M$, then by [MeJu, Thm II.5.5] $(x*y)*y\in M$, but X is positively implicative, and by [MeJu, Thm I.4.2(b)] $x*y=(x*y)*y\in M$, which is a contradiction.
(4) $x\notin M,\ y\notin M$. Since M is maximal, according to [MeJu, Thm II.5.5], $x*y\in M$ which is again a contradiction.

Consequently, the mapping $h:\ X\to\widehat{X}$ defined via $h(x)=\widehat{x},\ x\in X$, is a homomorphism from X onto $\widehat{X}$. Since $(\widehat{X};*_1,\widehat{0},\widehat{1})$ is a semisimple BCK-algebra which is isomorphic with $\overline{X}:=\{I_{\widehat{x}}:\ x\in X\}$, where $I_{\widehat{x}}$ denotes the indicator of the set $\widehat{x}$, which is a Boolean algebra in question. $\square$

Theorem 4.4. *A positively implicative BCK-algebra $(X;*,0,1)$ is semisimple if and only if the mapping $x\mapsto\widehat{x}$, $x\in X$, from Theorem 4.3 is an isomorphism.*

Proof. Suppose that X is semisimple. We assert that $\widehat{x}=\widehat{y}$. First let $\widehat{x}=\widehat{0}$. This means that $x\in M$ for any $M\in\mathcal{M}$. Since $R(X)=\{0\}$, we have $x=0$. If now

$\widehat{x} = \widehat{y}$, then $\widehat{x} *_1 \widehat{y} = \widehat{0}$ and $\widehat{y} *_1 \widehat{x} = \widehat{0}$ which means that $\widehat{x * y} = \widehat{0} = \widehat{y * x}$ and hence $x * y = 0 = y * x$ and $x = y$.

Let now the mapping $x \mapsto \widehat{x}$ be an isomorphism. If $x \in R(X)$, then $\widehat{x}(M) = 0$ for any $M \in \mathcal{M}$, hence $\widehat{x} = \widehat{0}$ and $x = 0$ which gives $R(X) = \{0\}$. □

Theorem 4.5. *Any positively implicative BCK-algebra $(X; *, 0, 1)$ is semisimple if and only if it is isomorphic to some Boolean algebra of subsets of a set Ω.*

Proof. If X is semisimple, the assertion follows from Theorem 4.4, and the converse statement is evident. □

Theorem 4.6. *A bounded positively implicative BCK-algebra $(X; *, 0, 1)$ is semisimple if and only if X possesses an order-determining system of states on X.*

Proof. Let X be semisimple. According to Theorem 4.5, there is a Boolean algebra $\mathcal{B}$ of subsets of a set $\Omega \neq \emptyset$ which is isomorphic with X. It is clear that the mapping $m_\omega : \mathcal{B} \to [0, 1]$ defined via $m_\omega(A) = I_A(\omega)$, $A \in \mathcal{B}$, where I_A is the indicator of A, is a state on $\mathcal{B}$, and the set of all these m_ω's is an order-determining system of states. Using given isomorphism between X and $\mathcal{B}$, we see that X possesses an order-determining system of states.

Let now $\mathcal{S}$ be an order-determining system of states on X. Since for any state m on X we have $m(1 * x) = 1 - m(x)$, $x \in X$, we see that $m(NNx) = m(x)$ for any $m \in \mathcal{M}$. Hence $NNx = x$ for any $x \in X$, which, by [MeJu, Thm VII.5.6], implies that X is semisimple. □

Theorem 4.7. *Let $(X; *, 0, 1)$ be a bounded positively implicative BCK-algebra. The following statements are equivalent:*

(i) *X is semisimple.*
(ii) *X is isomorphic to a Boolean algebra of subsets of a set.*
(iii) *X is implicative.*
(iv) *X is commutative.*
(v) *X possesses an order-determining system of states.*

Proof. According to Theorems 4.5 and 4.6, (i), (ii) and (v) are equivalent, and in view of [MeJu, Thm I.6.2], for positively implicative BCK-algebras, (iii) and (iv) are equivalent. Using [MeJu, Thm VII.2.7], any implicative BCK-algebra is isomorphic to a Boolean algebra of subsets of a set, so that (ii) and (iii) are equivalent. □

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