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1 Introduction

The direct limit construction is a well-known method for building up algebras from families of algebras, e.g. [2], §21. We will handle with monounary algebras and some properties of the direct limit construction will be discussed in this paper.

Let $\mathcal{K}$ be a class of algebras and $\mathbf{L}_{\to}(\mathcal{K})$ be the class of direct limits of algebras of $\mathcal{K}$. The operator $\mathbf{L}_{\to}$ was introduced also by G.Grätzer in [2], §23. If $\mathbf{L}_{\to}(\mathcal{K}) = \mathcal{K}$ is satisfied, then we will say that $\mathcal{K}$ is closed with respect to direct limits. Classes of cyclically ordered groups closed with respect to direct limits have been dealt with by J.Jakubík and G.Pringerová [3].

We will assume classes of monounary algebras be closed with respect to isomorphisms.

In the present paper we investigate classes $\mathcal{K}$ such that $\mathbf{L}_{\to}(\mathcal{K}) = \mathcal{K}$. It will be given several examples of such classes. Further, we will characterize all one-element classes $\mathcal{K}$. (By a one-element class we mean the class of all isomorphic copies of a monounary algebra.) As a consequence we obtain that every class $\mathcal{K} = \mathbf{L}_{\to}(\mathcal{K})$ contains an algebra A such that $\mathbf{L}_{\to}(\{A\}) = \{A\}$.

2 Preliminaries

Let $\mathcal{N}$ be the set of all positive integers, $\mathcal{N}_0 = \mathcal{N} \cup \{0\}$.

The class of all monounary algebras will be denoted by $\mathcal{U}$. We will use the symbol f for the operation in algebras of $\mathcal{U}$.

Let A be a monounary algebra. The algebra A is said to be connected, if for each $x, y \in A$ there are positive integers m, n with $f^m(x) = f^n(y)$. Let $B \subseteq A$ and B be a subalgebra of A. The algebra B is said to be a component of A, if B is a maximal connected part of A.

The class of all connected monounary algebras will be denoted by the symbol $\mathcal{U}^c$.

Let $k \in \mathcal{N}_0$. Let's denote by C_k the following connected monounary algebra:

1. if $k = 0$, then $\parallel C_k \parallel = \aleph_0$; otherwise $\parallel C_k \parallel = k$;

2. operation f of C_k is surjective and injective mapping.

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If $k \neq 0$, then we will say that the algebra C_k is the cycle of length k. The class of all connected monounary algebras having the cycle of length k as its subalgebra will be denoted by $\mathcal{U}^c_k$. For the denotation of the class of all connected monounary algebras without a cycle we will use the symbol $\mathcal{U}^c_0$, more precisely we put $\mathcal{U}^c_0 = \mathcal{U}^c - \bigcup_{k\in\mathcal{N}} \mathcal{U}^c_k$.

Let $A \in \mathcal{U}$. We will say that A has a cycle, if there exists $k \in \mathcal{N}$ such that C_k is the subalgebra of A.

Lemma 1 *Let $A \in \mathcal{U}$ and $i, j \in \mathcal{N}, u \in A$. If $f^i(u) = u$ and $v = f^j(u)$, then $f^i(v) = v$.*

Proof. We have $f^i(v) = f^i(f^j(u)) = f^j(f^i(u)) = f^j(u) = v$. □

Lemma 2 *Let $A \in \mathcal{U}$. If there exist $n \in \mathcal{N}, x \in A$ such that $f^n(x) = x$, then A has a cycle.*

Proof. Put $B = \{f^k(x), k \in \mathcal{N}_0\}$. The algebra B is a subalgebra of A and $B = \{f^k(x), 0 \leq k \leq n-1\}$ according to the previous lemma. If $\| B \| = 1$, then B is a cycle.

Let $\| B \| > 1$. Assume that $a, b \in B$. Then there exist $k, l \in \{0, ..., n-1\}$ such that $a = f^k(x), b = f^l(x)$. Assume that $f(a) = f(b)$. If $k = l$, then $a = b$.

Let $k > l$. Then $f^{k-l}(f^{l+1}(x)) = f^{k+1}(x) = f(f^k(x)) = f(a) = f(b) = f(f^l(x)) = f^{l+1}(x)$. Substitute in the previous lemma $k-l$ for i, $n-l-1$ for j and $f^{l+1}(x)$ for u. We obtain $v = f^j(u) = f^{n-l-1}(f^{l+1}(x)) = f^n(x) = x$ and $x = v = f^i(v) = f^{k-l}(x)$. Therefore $a = f^k(x) = f^{k-l+l}(x) = f^l(f^{k-l}(x)) = f^l(x) = b$.

We have proved that f is injective on B. The set B is finite and thus the algebra B is a cycle by the definition. □

Lemma 3 *Let A be a homomorphic image of C_0. If A has no cycle, then $A \cong C_0$.*

Proof. Suppose that φ is a homomorphism from C_0 onto A.

Let $x, y \in C_0$ and $\varphi(x) = \varphi(y)$. Assume that $x \neq y$. Then there exists $n \in \mathcal{N}$ such that $x = f^n(y)$. We have $f^n(\varphi(y)) = \varphi(f^n(y)) = \varphi(x) = \varphi(y)$. In view of previous lemma A has a cycle. This is a contradiction. Thus $x = y$ and φ is an isomorphism. □

For the sake of completeness and for fixing the denotations we also recall the notion of direct limit: in fact, we apply it for the case of monounary algebras (cf., e.g. Grätzer [2] for the case of universal algebras).

Let $< P, \leq >$ be a directed partially ordered set, $P \neq \emptyset$. For each $p \in P$ let A_p be a monounary algebra. Suppose that for each pair of elements p and q in P with $p < q$, there is defined a homomorphism φ_{pq} of A_p into A_q such that $p < q < s$ implies that

$$\varphi_{ps} = \varphi_{pq} \circ \varphi_{qs}.$$

For each $p \in P$ let φ_{pp} be the identity on A_p.

Let p and q be the elements of P and let $x \in A_p, y \in A_q$. We put $x \equiv y$ if there exists $s \in P$ with $p \leq s, q \leq s$ such that $\varphi_{ps}(x) = \varphi_{qs}(y)$. For each $z \in \bigcup_{p\in P} A_p$ put $\overline{z} = \{t \in \bigcup_{p\in P} A_p : z \equiv t\}$. Let $\overline{A} = \{\overline{z} : z \in \bigcup_{p\in P} A_p\}$.

If z_1, z_2 are elements of $\bigcup_{p\in P} A_p$ such that $\overline{z}_1 = \overline{z}_2$, then clearly $\overline{f(z_1)} = \overline{f(z_2)}$. Hence if put $f(\overline{z}_1) = \overline{f(z_1)}$, then operation f on $\overline{A}$ is correctly defined and with respect to this operation $\overline{A}$ is a monounary algebra. It is said to be the direct limit of the indexed system $\{A_p\}_{p\in P}$. We express this situation by writing

$$(1) \quad \{A_p\}_{p\in P} \longrightarrow \overline{A}$$

Homomorphisms and endomorphisms of monounary algebras were investigated in [4] - [7].

The definition of direct limit yields following four assertions.

Lemma 4 *Let (1) hold. Let $p \in P$ and φ_p be the mapping from A_p to $\overline{A}$ such that $\varphi_p(x) = \overline{x}$ for every $x \in A_p$. Then φ_p is the homomorphism of A_p to $\overline{A}$.*

Lemma 5 *Let $m \in \mathcal{N}$ and (1) be valid. If $\| A_p \| \leq m$ for every $p \in P$, then $\| \overline{A} \| \leq m$.*

Lemma 6 *Let (1) be valid. If the operation of A_p is 1-1 for every $p \in P$, then operation of $\overline{A}$ is 1-1.*

Lemma 7 *Let (1) be valid and let $p \in P$. If $q \leq p$ for all $q \in P$, then $\overline{A} \cong A_p$.*

Lemma 8 *Let A be an algebra and (1) be valid. If $A_p \cong A$ for all $p \in P$ and φ_{pq} is an isomorphism between A_p and A_q for all $p, q \in P$, then $\overline{A} \cong A$.*

EXAMPLE. Suppose that P is the set of all finite subsets of the interval $(0,1)$. Let $\leq = \subseteq$. For $p \in P, p = \{p_1, ..., p_n\}$,where $n \in \mathcal{N}$, put $A_p = \{x, p_1, ...p_n\}$. Further put $f(x) = f(p_i) = x$ for $i = 1, ..., n$. If $p \subseteq q$, then let $\varphi_{pq} = id_{A_p}$. We get $\overline{A} \cong (0,1) \cup \{\overline{x}\}$ and $f(y) = \overline{x}$ for every $y \in (0,1) \cup \{\overline{x}\}$.

This example shows that Lemma 5 cannot be generalized to the case when instead of m will be put an infinite cardinal number.

Let $\mathcal{K} \subseteq \mathcal{U}$. The equality $\mathbf{L}_{\rightarrow}(\mathcal{K}) = \mathcal{K}$ is equivalent to the following condition: whenever (1) holds and $A_p \in \mathcal{K}$ for each $p \in P$, then each monounary algebra isomorphic to $\overline{A}$ belongs to $\mathcal{K}$.

Lemma 9 *a) $\mathbf{L}_{\rightarrow}(\emptyset) = \emptyset$, $\mathbf{L}_{\rightarrow}(\mathcal{U}) = \mathcal{U}$.*

b) Let J be a nonempty set and for $j \in J$ let $\mathbf{L}_{\rightarrow}(\mathcal{K}_j) = \mathcal{K}_j$. Then $\mathbf{L}_{\rightarrow}(\bigcap_{j\in J} \mathcal{K}_j) = \bigcap_{j\in J} \mathcal{K}_j$.

c) If $\mathbf{L}_{\rightarrow}(\mathcal{K}_1) = \mathcal{K}_1, \mathbf{L}_{\rightarrow}(\mathcal{K}_2) = \mathcal{K}_2$, then $\mathbf{L}_{\rightarrow}(\mathcal{K}_1 \cup \mathcal{K}_2) = \mathcal{K}_1 \cup \mathcal{K}_2$.

Proof. The assertions a, b follow from definitions.

Suppose that (1) is valid and $A_p \in \mathcal{K}_1 \cup \mathcal{K}_2$ for all $p \in P$. Denote $Q = \{q \in P : A_p \in \mathcal{K}_2\}$.

First let there exist $p \in P$ such that for every $q \in Q$ the relation $p \not\leq q$ is true. Put $R = \{r \in P : p \leq r\}$. The set R is directed. Further if $r \in R$ then $A_r \in \mathcal{K}_1$. If $s \in P$, then we can choose $s' \in P$ such that $s \leq s', p \leq s'$. We have $s' \in R$. That means that R is cofinal with P. Thus $\{A_r\}_{r \in R} \to \overline{A}$ and $\overline{A} \in \mathcal{K}_1$.

Now let for every $p \in P$ there exists $q \in Q$ such that $p \leq q$. Then Q is cofinal with P and $\{A_q\}_{q \in Q} \to \overline{A}$. Because $\mathcal{K}_2$ is closed with respect to direct limits we have $\overline{A} \in \mathcal{K}_2$. $\square$

It is obvious that Lemmas 4, 5, 7, 8, 9 are not specific for monounary algebras, they are valid for any type of algebraic systems.

3 Several closed classes

This section is devoted to the search of some concrete classes of monounary algebras which are closed with respect to direct limits.

We will often use these three properties of direct limits.

Lemma 10 $\mathbf{L}_{\to}(\mathcal{U}^c) = \mathcal{U}^c$.

Proof. Let $\overline{x}, \overline{y} \in \overline{A}$. There exist $p \in P$ and $x_1 \in \overline{x}, y_1 \in \overline{y}$ such that $x_1, y_1 \in A_p$. We can find $m, n \in \mathcal{N}_0$ such that $f^m(x_1) = f^n(y_1)$ by the connectivity of A_p. That means that $f^m(\overline{x}) = \overline{f^m(x_1)} = \overline{f^n(y_1)} = f^n(\overline{y})$. $\square$

Lemma 11 *Let (1) be valid. The family $\{A_p\}_{p \in P}$ contains an algebra with a cycle if and only if $\overline{A}$ has a cycle. More precisely, $\overline{A}$ contains a cycle of length k, where k is the length of the shortest cycle in algebras of $\{A_p\}_{p \in P}$.*

Proof. Assume that $\{A_p\}_{p \in P}$ contains an algebra with a cycle. Let l be the length of the shortest cycle in algebras of $\{A_p\}_{p \in P}$. Every algebra of $\{A_p\}_{p \in P}$ can be homomorphically embedded into $\overline{A}$ according to Lemma 4. This implies that $\overline{A}$ is the algebra with a cycle. Moreover $\overline{A}$ has at least one cycle with length less or equal to l.

Now let $\overline{A}$ has a cycle of length $n, n \in \mathcal{N}$. Assume that $p \in P$ and $x \in A_p$ are such that $f^n(\overline{x}) = \overline{x}$. We have $f^n(x) \in \overline{x}$ because $f^n(\overline{x}) = \overline{f^n(x)}$. That means that there exists $q \in P$ such that $\varphi_{pq}(x) = \varphi_{pq}(f^n(x))$. We obtain $\varphi_{pq}(x) = f^n(\varphi_{pq}(x))$. Thus A_q has a cycle with length less or equal to n. $\square$

Lemma 12 *Suppose that (1) is valid. Let $\overline{A}$ has no cycle and let the family $\{A_p\}_{p \in P}$ contain an algebra with a subalgebra isomorphic to C_0. Then C_0 is a subalgebra of $\overline{A}$.*

Proof. Let $p \in P$ is such that $C_0 \in S(A_p)$. Consider homomorphism φ_p from Lemma 4. Then $\varphi_p(C_0)$ is the homomorphic image of C_0. The algebra $\overline{A}$ hasn't any cycle by the assumption and that is why $\varphi_p(C_0) \cong C_0$ according to Lemma 3 . Further $\varphi_p(C_0) \in S(\overline{A})$. Therefore $C_0 \in S(\overline{A})$. $\square$

Theorem 1 *Let $k \in \mathcal{N}_0$. Then*
a) $\mathbf{L}_{\rightarrow}(\mathcal{U}^c_k) = \mathcal{U}^c_k$.
b) $\mathbf{L}_{\rightarrow}(\{C_k\}) = \{C_k\}$.

Proof. Let (1) is valid.

Suppose that $k \neq 0$. Let $A_p \in \mathcal{U}^c_k$ for all $p \in P$. We have $\overline{A} \in \mathcal{U}^c$ according to Lemma 10 and $\overline{A}$ has a cycle of length k by Lemma 11.

Now let $A_p \cong C_k$ for all $p \in P$. The algebra $\overline{A}$ is connected by Lemma 10 and $C_k \in S(\overline{A})$ according to Lemma 11. The operation f of $\overline{A}$ is injective according to Lemma 6. We have $\overline{A} \cong C_k$.

Assume that $A_p \in \mathcal{U}^c_0$ for all $p \in P$. We have $\overline{A} \in \mathcal{U}^c_0$ according to Lemmas 10 and 11.

Finally let $A_p \cong C_0$ for all $p \in P$. The algebra $\overline{A}$ is connected by Lemma 10 , $\overline{A}$ has no cycle by Lemma 11 and $C_0 \in S(\overline{A})$ by Lemma 12 . In view of Lemma 6 we have $\overline{A} \cong C_0$. $\square$

Theorem 2 *Let $\mathcal{K}_k \subseteq \mathcal{U}^c_k$ and $\mathbf{L}_{\rightarrow}(\mathcal{K}_k) = \mathcal{K}_k$ for all $k \in \mathcal{N}_0$. If $\mathcal{K} = \bigcup_{k \in \mathcal{N}_0} \mathcal{K}_k$, then $\mathbf{L}_{\rightarrow}(\mathcal{K}) = \mathcal{K}$.*

Proof. Let (1) is valid, $A_p \in \mathcal{K}$ for all $p \in P$.

Assume that $\{A_p\}_{p \in P}$ contains the algebra with a cycle. Let i be the length of the shortest cycle in algebras of $\{A_p\}_{p \in P}$. Put $Q = \{q \in P : A_q \in \mathcal{U}^c_i\}$. Homomorphic image of the cycle of length i can be only the cycle of length less or equal than i, in our case thus the cycle of length i. If $q \in Q$ and $p \in P$ is such that $q \leq p$, then $p \in Q$. Thus Q is directed and cofinal with P. We have $\{A_q\}_{q \in Q} \rightarrow \overline{A}$ and $\overline{A} \in \mathcal{U}^c_i$ by Lemma 11 . Because $\mathcal{K}_i$ is closed with respect to direct limits, we have $\overline{A} \in \mathcal{K}_i$ and $\overline{A} \in \mathcal{K}$.

Now assume that $\{A_p\}_{p \in P}$ doesn't contain the algebra with a cycle. Then $A_p \in \mathcal{K}_0$ for every $p \in P$. Because $\mathcal{K}_0$ is closed with respect to direct limits we obtain $\overline{A} \in \mathcal{K}_0 \subseteq \mathcal{K}$. $\square$

Let A be a monounary algebra. Let A satisfy the following condition: If $C \subseteq A$ and C is the cycle of A, then $C \cong C_1$. Then we will say that A is a cycle-free algebra. Cycle-free algebras have been dealt with by G.Bordalo [1]. Denote by $\mathcal{F}$ the class of all cycle-free algebras.

Theorem 3 $\mathbf{L}_{\rightarrow}(\mathcal{F}) = \mathcal{F}$.

Proof. Follows from Lemma 11. □

Let us denote by N the monounary algebra with universe $\mathcal{N}$ and operation f defined as $f(i) = i + 1$ for each $i \in \mathcal{N}$.

Theorem 4 $\mathbf{L}_{\rightarrow}(\{N, C_0\}) = \{N, C_0\}$.

Proof. Let (1) is valid and $A_p \cong N$ or $A_p \cong C_0$ for every $p \in P$. Put $Q = \{q \in P : A_q \cong C_0\}$. If Q is cofinal with P, then $\{A_q\}_{q \in Q} \rightarrow \overline{A}$. The set $\{C_0\}$ is closed with respect to direct limits by Theorem 1 and that is why $\overline{A} \cong C_0$.

Suppose that Q isn't cofinal with P. Then $P - Q$ is cofinal with P and $\{A_p\}_{p \in P-Q} \rightarrow \overline{A}$. The algebra $\overline{A}$ is connected by Lemma 10.

Let C_0 be a subalgebra of $\overline{A}$. The operation of $\overline{A}$ is injective by Lemma 6. Therefore $\overline{A} \cong C_0$.

Now let C_0 isn't any subalgebra of $\overline{A}$. In view of Lemma 11 $\overline{A}$ has no cycle and thus N is a subalgebra of $\overline{A}$. According to the injectivity of the operation of $\overline{A}$ we obtain $\overline{A} \cong N$. □

4 The algebras of type τ

Let A be a monounary algebra and $\{B_j\}_{j \in J}$ be the component partition of A.

We shall say that A is of type τ if following two conditions are valid:

1. if $j \in J$, then there exists $k \in \mathcal{N}$ such that $A_j \cong C_k$;
2. if $i, j \in J, i \neq j$ and $k, l \in \mathcal{N}$ are such that $A_i \cong C_k, A_j \cong C_l$, then k doesn't divide l.

Denote by $\mathcal{T}$ the class of all algebras of type τ.

We will prove that $\mathcal{T}$ is closed with respect to direct limits and some special subclasses of $\mathcal{T}$ are closed with respect to direct limits.

The definition of algebras of type τ yields that $C_k \in \mathcal{T}$ for every $k \in \mathcal{N}$. Further if A is of type τ and C_1 is a subalgebra of A, then $A \cong C_1$. Finally if $A \in \mathcal{T}$ and B is a subalgebra of A, then $B \in \mathcal{T}$.

Lemma 13 *Let A is of type τ. Then every endomorphism of A is the isomorphism.*

Proof. Monounary algebras have the following property: If $k \in \mathcal{N}$ and φ is a homomorphism of C_k, then $\varphi(C_k)$ is a cycle of length l, where l is a divisor of k. From this follows the lemma. □

Theorem 5 *If $A \in \mathcal{T}$, then* $\mathbf{L}_{\rightarrow}(\{A\}) = \{A\}$.

Proof. Suppose that (1) is valid. Put $A_p = A$ for each $p \in P$. Let $p, q \in P$. The homomorphism φ_{pq} is an endomorphism of A. The algebra A is of type τ and thus φ_{pq} is an isomorphism between A_p and A_q. That implies $\overline{A} \cong A$ according to Lemma 8. $\square$

Lemma 14 *Let $A \in \mathcal{U}$. If $C_k \in S(A)$, then there exists $x \in A$ such that $f^k(x) = x$ and $\| \{x, f(x), ..., f^{k-1}(x)\} \| = k$.*

Proof. Follows from the definition of a cycle. $\square$

Lemma 15 *Let (1) be valid and $k \in \mathcal{N}$. If $C_k \in S(\overline{A})$, then there exists $p \in P$ such that for each $q \in P$ with $p \leq q$ holds $C_k \in S(A_q)$.*

Proof. We prove this assertion indirectly. Suppose that fer each $p \in P$ there exists $q \in P, p \leq q$ such that $C_k \notin S(A_q)$.

We shall show that for every $\overline{x} \in \overline{A}$ there is $f^k(\overline{x}) \neq \overline{x}$ or

$$\| \{\overline{x}, f(\overline{x}), ..., f^k(\overline{x})\} \| < k.$$

Then $C_k \notin S(\overline{A})$.

Assume that $\overline{x} \in \overline{A}$ and $f^k(\overline{x}) = \overline{x}$. Let $p \in P$ be such that $x \in A_p$. In view of $\overline{f^k(x)} = \overline{x}$, there exists $q \in P, p \leq q$ such that $\varphi_{pq}(x) = \varphi_{pq}(f^k(x))$. We obtain $f^k(\varphi_{pq}(x)) = \varphi_{pq}(x)$. Thus A_q has a cycle of length m, where $m \leq k$.

Let $m < k$. The equality

$$\{\overline{x}, f(\overline{x}), ..., f^k(\overline{x})\} = \{\overline{\varphi_{pq}(x)}, f(\overline{\varphi_{pq}(x)}), ..., f^k(\overline{\varphi_{pq}(x)})\}$$

is valid. Therefore $\| \{\overline{x}, f(\overline{x}), ..., f^k(\overline{x})\} \| < k$.

Let $m = k$. Choose $s \in P, q \leq s$ such that A_s doesn't contain a cycle of length k. Then the element $\varphi_{qs}(\varphi_{pq}(x))$ belongs to a cycle of A_s which has length $n, n < k$. Analogously as in the previous case we obtain $\| \{\overline{x}, f(\overline{x}), ..., f^k(\overline{x}) \| \leq n < k$. $\square$

Theorem 6 $\mathbf{L}_{\rightarrow}(\mathcal{T}) = \mathcal{T}$.

Proof. Let (1) is valid and $A_p \in \mathcal{T}$ for all $p \in P$. According to Lemma 6 and Lemma 12 every component of $\overline{A}$ is isomorphic to C_k for some $k \in \mathcal{N}$.

Assume that $\overline{B}, \overline{C}$ are components of $\overline{A}$ such that $\overline{B} \cong C_k, \overline{C} \cong C_l, k, l \in \mathcal{N}$. In view of Lemma 15 there exist $p, r \in P$ such that for each $q \in P, p \leq q$ the algebra A_q contains a cycle of length k and for each $s \in P, r \leq s$ the algebra A_s contains a cycle of length l. Choose $t \in P$ such that $r \leq t$ and $p \leq t$. We obtain that A_t has cycles of lengths k, l. The algebra A_t is of type τ. Therefore if $k \neq l$, then k doesn't divide l. And if $k = l$, then $\overline{B} = \overline{C}$. $\square$

Theorem 7 *Let $\mathcal{K} \subseteq \mathcal{T}$ and $n \in \mathcal{N}$. If every element of $\mathcal{K}$ has less than n components, then $\mathbf{L}_{\rightarrow}(\mathcal{K}) = \mathcal{K}$.*

Proof. Let (1) is valid and $A_p \in \mathcal{K}$ for all $p \in P$. We have $\overline{A} \in \mathcal{T}$ by the previous theorem.

Consider that $\{\overline{B}_i\}_{i \in I}$ is the component partition of$\overline{A}$. If I is finite, put $m = \| I \|$, else put $m = n$. Let $i(1), ..., i(m)$ be different elements of I.

Assume that $j \in \{1, ..., m\}$ and $k(j) \in \mathcal{N}$ is such that $\overline{B}_{i(j)} \cong C_{k(j)}$. According to Lemma 15 in P exists $p(j)$ such that for each $q \in P, p(j) \leq q$ holds that the algebra A_q has a cycle of the length $k(j)$.

Now let $s \in P$ be such that $p(1) \leq s, ..., p(m) \leq s$. The algebra A_s contains cycles of length $k(1), ..., k(m)$. Numbers $k(1), ..., k(m)$ are different and thus $\overline{A}$ is a subalgebra of A_s and $m < n$.

Assume that the algebra A_s has a component B such that $B \ncong C_{k(j)}$ for $j = 1, ..., m$. Then B is the cycle of length k and $k(j)$ doesn't divide k for $j = 1, ...m$. According to Lemma 4 can be A_s homomorphically embedded into $\overline{A}$, this is a contradiction. Thus $\overline{A} \cong A_s$ and $\overline{A} \in \mathcal{K}$. □

5 One-element $\mathcal{D}$-classes

In this section we will characterize all one-element classes of monounary algebras which are closed with respect to direct limits.

Let A be a monounary algebra.

The notion of degree $s(x)$ of an element $x \in A$ was introduced by M.Novotný [7]. Put $M = \{m \in \mathcal{N}$: there exists $y \in A$ such that $f^m(y) = x\}$. If M is not upperbounded, then $s(x) = \infty$. If M has an upper bound, then $s(x)$ is the maximum of M.

Let B be a subalgebra of A. Assume that there exists a homomorphism φ of A onto B such that $\varphi(b) = b$ for each $b \in B$. Then B is said to be a retract of A and φ is called a retract mapping corresponding to B.

Retracts of monounary algebras was thoroughly studied by D. Studenovská. In view of [8] Theorem 1.3 we have

Lemma 16 *Let $A \in \mathcal{U}, B \subseteq A$ and $B \in S(A)$. Then B is a retract of A if and only if the following conditions are satisfied:*

(a) If $y \in A$ is such that $f(y) \in B$, then there is $z \in M$ such that $s(y) \leq s(z)$.

(b) For any component D of A with $D \cap B = \emptyset$, the following conditions are satisfied:

  (b1) If D contains a cycle with d elements, then there is a component D' of A with $D' \cap B \neq \emptyset$ and there is $n \in \mathcal{N}$ such that n divides d and D' has a cycle with n elements.

(b2) *If D contains no cycle and $x \in D$, then there is $y \in B$ such that $s(f^k(x)) \leq s(f^k(y))$ for every $k \in \mathcal{N}_0$.*

In view of [3], Lemma 1.1 we have

Lemma 17 *Let $\mathbf{L}_{\rightarrow}(\mathcal{K}) = \mathcal{K}$ and $A \in \mathcal{K}$. Let B be a retract of A. Then $B \in \mathcal{K}$.*

Lemma 18 *Let $\mathbf{L}_{\rightarrow}(\mathcal{K}) = \mathcal{K}$ and $\mathcal{K} \neq \emptyset$. Then $\mathcal{K}$ contains an algebra B such that each component of B is isomorphic to C_k for any $k \in \mathcal{N}_0$.*

Proof Choose $A \in \mathcal{K}$. Put $P = \mathcal{N}, A_p = A$ for all $p \in P, \varphi_{p,p+1} = f$ for all $p \in P$. So we have defined direct family of monounary algebras. We will show that the operation f on $\overline{A}$ is injective and surjective mapping. Then the proof will be ready.

Assume that $p, q \in P, a \in A_p, b \in A_q$ and $f(\overline{a}) = f(\overline{b})$. Then $\overline{f(a)} = \overline{f(b)}$ and that is why there exists $s \in P, p \leq s, q \leq s$ such that $\varphi_{ps}(f(a)) = \varphi_{qs}(f(b))$. ¿From this follows $\varphi_{p,s+1}(a) = f^{s+1-p}(a) = f^{s-p}(f(a)) = \varphi_{ps}(f(a)) = \varphi_{qs}(f(b)) = f^{s-q+1}(b) = \varphi_{q,s+1}(b)$. That means that $\overline{a} = \overline{b}$.

Further let $p \in P$ and $a \in A_p$. Choose $a' \in A_{p+1}$ such that $a = a'$ in the algebra A. Because $\varphi_{p,p+1}(a) = f(a')$, we have $\overline{a} = \overline{f(a')} = f(\overline{a'})$. □

Lemma 19 *Let $\mathbf{L}_{\rightarrow}(\mathcal{K}) = \mathcal{K}$. If $\mathcal{K}$ contains an algebra with a cycle, then $\mathcal{K}$ contains an algebra B such that each component of B is isomorphic to C_k for any $k \in \mathcal{N}$.*

Proof. Let $\{B_j\}_{j \in J}$ be the component partition of A. Put $J' = \{j \in J : B_j$ has a cycle $\}$. Further for each $j \in J'$ denote D_j the cycle of B_j and denote by B the algebra which has the component partition $\{D_j\}_{j \in J'}$.

We have $J' \neq \emptyset$ and B is a retract of A according to Lemma 16. Thus $B \in \mathcal{K}$ according to Lemma 17. □

Lemma 20 *Let $\mathbf{L}_{\rightarrow}(\mathcal{K}) = \mathcal{K}$. If $\mathcal{K}$ contains an algebra with a cycle, then $\mathcal{K}$ contains an algebra of type τ.*

Proof. Let $A \in \mathcal{K}$ and all components of A are cycles. It is easy to see that there exists $B \subseteq A$ such that the algebra $B \in S(A)$ and $B \in \mathcal{T}$. Moreover B is a retract of A and thus $B \in \mathcal{K}$. □

Theorem 8 *Let $A \in \mathcal{U}$. Then $\mathbf{L}_{\rightarrow}(\{A\}) = \{A\}$ if and only if $A \cong C_0$ or A is of type τ.*

Proof. In view of Theorem 1 we have that $\{C_0\}$ is closed with respect to direct limits. If A is an algebra of type τ, then $\{A\}$ is closed with respect to direct limits by Theorem 5 .

The opposite direction we shall prove indirectly. Assume that $A \not\cong C_0$ and A isn't of type τ. Let $\mathcal{K}$ be closed with respect to direct limits and $A \in \mathcal{K}$.

If A has a cycle, then $\mathcal{K}$ contains an algebra of type τ by Lemma 20. Thus $\mathcal{K}$ has more than one element.

If A has no cycle, then $\mathcal{K}$ contains an algebra B such that each component of B is isomorphic to C_k for any $k \in \mathcal{N}_0$ according to Lemma 18 . If $A \not\cong B$, then we are ready. If $A \cong B$, then A isn't connected and each component of A is isomorphic to C_0. Thus C_0 is a retract of A and $C_0 \in \mathcal{K}$ by Lemma 17. We conclude $\mathcal{K}$ has more than one element. $\square$

Theorem 9 *Let* $\mathbf{L}_{\rightarrow}(\mathcal{K}) = \mathcal{K}$. *Then there exists an algebra* $A \in \mathcal{K}$ *such that* $\mathbf{L}_{\rightarrow}(\{A\}) = \{A\}$.

Proof. Choose $B \in \mathcal{K}$. If B is of type τ or $B \cong C_0$, we can put $A = B$ by Theorem 5 and Theorem 1.

Suppose that B isn't of type τ and $A \not\cong C_0$. If B has a cycle, then $\mathcal{K}$ contains an algebra A of type τ according to Lemma 20. We have $\{A\}$ is closed with respect to direct limits by Theorem 5.

If B has no cycle, then $\mathcal{K}$ contains an algebra B' such that each component of B' is isomorphic to C_k for any $k \in \mathcal{N}_0$ according to Lemma 18. If B' is of type τ, then put $A = B'$.

Assume that B' isn't of type τ. If B' has a cycle, then $\mathcal{K}$ contains an algebra A of type τ. If B' has no cycle, then each component of B' is isomorphic to C_0. Thus $C_0 \in \mathcal{K}$ in view of Lemma 17 and we can put $A = C_0$. $\square$

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