Mathematical Institute
Slovak Academy of Sciences, Bratislava

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Ján Jakubík

Lexicographic products of half linearly ordered groups

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LEXICOGRAPHIC PRODUCTS OF HALF LINEARLY ORDERED GROUPS

Ján Jakubík, Košice

The notion of half linearly ordered group (and, more generally, of half lattice ordered group) was introduced by Giraudet and Lucas [2].

In the present paper we define the lexicographic product of half linearly ordered groups. This definition includes as a particular case the lexicographic product of linearly ordered groups.

We investigate the question on the existence of isomorphic refinements of two lexicographic product decompositions of a half linearly ordered group.

The analogous question for linearly ordered groups was dealt with by Maltsev [5]; his result was generated by Fuchs [1] and the author [3].

The isomorphic refinements of small direct product decompositions of half lattice ordered groups were studied in [4].

1. Preliminaries

Let G be a group and suppose that it is, at the same time, a partially ordered set.

We denote by $G\uparrow$ (or $G\downarrow$) the set of all $x \in G$ such that, whenever $y, z \in G$ and $y \leqq z$, then $xy \leqq xz$ (or $xy \geqq xz$, respectively).

1.1. Definition. (Cf. [2].) G is said to be a half linearly ordered group if the following conditions are satisfied:

1) the partial order $\leqq$ on G is non-trivial;
2) if $x, y, z \in G$ and $y \leqq z$, then $yx \leqq zx$;
3) $G = G\uparrow \cup G\downarrow$;
4) $G\uparrow$ is a linearly ordered set.

The neutral element of G will be denoted by e. In view of 1), $G \neq \{e\}$. It is obvious that the following conditions are equivalent:

(i) $G\downarrow = \emptyset$;
(ii) G is a linearly ordered group with more than one element.

We denote by $\mathcal{HL}$ the class of all half linearly ordered groups. Next, let $\mathcal{HL}_1$ be the class of all elements of $\mathcal{HL}$ which fail to be linearly ordered.

We shall apply the following results (cf. [2]):

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1.2. Proposition. *Let $G \in \mathcal{HL}_1$. Then*

(i) *$G\uparrow$ is a subgroup of the group G having the index 2;*
(ii) *the partially ordered set $G\downarrow$ is isomorphic to $G\uparrow$;*
(iii) *if $x \in G\uparrow$ and $y \in G\downarrow$, then x and y are incomparable.*

1.3. Proposition. *Let $G \in \mathcal{HL}_1$. Then*

(i) *for each $x \in G$ with $x \neq e$ the relation*

$$x^2 = e \Longleftrightarrow x \in G\downarrow$$

is valid;
(ii) *if $x \in G\downarrow$ and $y \in G\uparrow$, then $xyx = y^{-1}$;*
(iii) *the group $G\uparrow$ is abelian.*

2. Lexicographic products

Let I be a nonempty set and for each $i \in I$ let $G_i \in \mathcal{HL}$. We denote by G^1 the cartesian product of the groups G_i $(i \in I)$. The elements of G^1 will be expressed as $g = (\dots, g_i, \dots)_{i\in I}$ or $g = (g_i)_{i\in I}$; g_i is the component of g in G_i. We put

$$I(g) = \{i \in I : g_i \neq e\} .$$

Now let us suppose that I is a linearly ordered set and that either

(i_0) $G_i \in \mathcal{HL}_1$ for each $i \in I$,

or

(ii_0) $G_i \notin \mathcal{HL}_1$ for each $i \in I$.

If (ii_0) is valid then we choose an element $g^{(1)} \in G^1$ such that $g_i^{(1)} \in G_i\downarrow$ for each $i \in I$.

We denote by G^0 the set of all $g \in G^0$ such that either

(i_1) $g_i \in G_i\uparrow$ for each $i \in I$ and the set $I(g)$ is well-ordered,

or

(ii_1) $g_i \in G_i\downarrow$ for each $i \in I$ and the set $I(g^{(1)}g^{-1})$ is well-ordered.

Let A and B be the set of all elements of G^0 which satisfy the condition (i_1) or (ii_1), respectively. Hence G^0 is a disjoint union of A and B.

2.1. Lemma. *G^0 is a subgroup of the group G.*

Proof. If $a, a' \in A$, then clearly $aa' \in A$ and $a^{-1} \in A$. If $b, b' \in B$, then in view of 1.3 (i) we have $bb' \in A$ and $b^{-1} \in B$. For $a \in A$ and $b \in B$ the relations $ab \in B$ and $ba \in B$ are valid. □

We define the binary relation $\leqq$ on G^0 as follows:

for $a, a' \in A$ we put $a \leqq a'$ if either $a = a'$, or $a \neq a'$ and $a_{i(0)} < a'_{i(0)}$, where $i(0)$ is the least element of I $(a'a^{-1})$;

for $b, b' \in B$ we define the relation $b \leqq b'$ analogously;

if $a \in A$ and $b \in B$, then we consider a and b as to be incomparable (i.e., neither $a \leqq b$ nor $b \leqq a$).

It is easy to verify that the relation $\leqq$ is a partial order on G^0.

2.2. Lemma. *G^0 is a half linearly ordered group.*

Proof. In view of 2.1, G^0 is a group. We consider the above defined partial order on G^0. We have to verify that the conditions 1) –– 4) from 1.1 are satisfied.

Choose $i(0) \in I$. There exists $g^{(i(0))} \in G_{i(0)}$ such that $e < g^{(i(0))}$. In view of the definition of G^0 there exists $g \in G^0$ such that $g_{i(0)} = g^{(i(0))}$ and $g_i = e$ for each $i \in I \setminus \{i(0)\}$. Then $e < g$ and hence 1) holds.

The validity of 2) is obvious. Next, $G \uparrow = A$ and $G \downarrow = B$, whence 3) is valid. From the definition of the partial order $\leqq$ on G^0 we conclude that the condition 4) is satisfied as well. □

In the case that (ii$_0$) is valid the half linearly ordered group G^0 is defined by applying the element $g^{(1)} \in G^{(1)}$ with $g_i^{(1)} \in G_i \downarrow$ for each $i \in I$ let us denote $G^0 = G^0(g^{(1)})$.

2.3. Proposition. *Let G^1 be as above and let $g^{(1)}, g^{(2)} \in G^1$ such that $g_i^{(1)}, g_i^{(2)} \in G_i \downarrow$ for each $i \in I$. Then there exists an isomorphism φ of $G^0(g^{(1)})$ onto $G^0(g^{(2)})$ such that $\varphi(g^{(1)}) = g^{(2)}$ and $\varphi(a) = a$ for each $a \in A$.*

Proof. Let A be as above. The set B above was constructed by means of the element $g^{(1)}$; let us now write B^1 instead of B. Analogously we construct the set B^2 by using the element $g^{(2)}$.

Let $b_1 \in B^1$. We have $Ag^{(1)} = B^1$. Hence there exists a uniquely determined element $a \in A$ with $ag^{(1)} = b_1$. We put $\varphi(b_1) = ag^{(2)}$. In particular, we obtain $\varphi(g^{(1)}) = g^{(2)}$. For each $a \in A$ we set $\varphi(a) = a$. Hence φ is a mapping of $B^0(g^{(1)})$ into $B^0(g^{(2)})$.

For each $a_1, a_2 \in A$ we have $\varphi(a_1 a_2) = \varphi(a_1)\varphi(a_2)$. Let $b_1, b_1' \in B^1$. There exist $a, a' \in A$ with $b_1 = ag^{(1)}$, $b_1' = a'g^{(1)}$. Then $\varphi(b_1) = ag^{(2)}$, $\varphi(b_1') = a'g^{(2)}$. Next, $b_1 b_1' \in A$ and hence

$$\varphi(b_1 b_1') = b_1 b_1' = ag^{(1)}a'g^{(1)} = a(g')^{-1}$$

(in the last step we applied 1.3 (ii)). Similarly,

$$\varphi(b_1)\varphi(b_1') = ag^{(2)}a'g^{(2)} = a(a')^{-1} \ ,$$

thus $\varphi(b_1 b_1') = \varphi(b_1)\varphi(b_1')$.

Let a, b_1 be as above and let $a_1 \in A$. Then

$$\varphi(a_1 b_1) = \varphi(a_1 a g^{(1)}) = a_1 a g^{(2)} \ ,$$

$$\varphi(a_1)\varphi(b_1) = a_1 a g^{(2)} = \varphi(a_1 b_1) \ .$$

Consider the element $g^{(1)}a_1$. Clearly $g^{(1)}a_1 \in B^1$. There exists $x \in G^0(g^{(1)})$ with $g^{(1)}a_1 = xg^{(1)}$. If $x \in B^1$, then $xg^{(1)} \in A$, which is impossible. Hence $x \in A$ and thus

$$\varphi(g^{(1)}a_1) = \varphi(xg^{(1)}) = xg^{(2)} \ .$$

Moreover, in view of 1.2 and 1.3 we have

$$x = g^{(1)}a_1 g^{(1)} = a_1^{-1} \ ,$$

hence

$$\varphi(g^{(1)}a_1) = a_1^{-1}g^{(2)} \ .$$

Next, in a similar way we obtain

$$\varphi(g^{(1)})\varphi(a_1) = g^{(2)}a_1 = a_1^{-1}g^{(2)} = \varphi(g^{(1)}a_1) \ .$$

It is obvious that φ is a monomorphism. If $y \in G^0(g^{(2)})$, then either $y \in A$ and hence $\varphi(y) = y$, or there is $a \in A$ with $y = ag^{(2)}$ and in this case $\varphi(ag^{(1)}) = y$. Thus φ is a bijection. By summarizing, φ is an isomorphism of the group $G^0(g^{(1)})$ onto the group $G^0(g^{(2)})$.

Let $x_1, x_2 \in G^0(g^{(1)})$, $y_i = \varphi(x_i)$ $(i = 1, 2)$. Suppose that x_1 and x_2 are comparable. Then either (i) $x_1, x_2 \in A$, or (ii) $x_1, x_2 \in B^1$. If (i) holds, then we have trivially

$$x_1 \leqq x_2 \Longleftrightarrow y_1 \leqq y_2 \ .$$

Let (ii) be valid. There are $a_1, a_2 \in A$ with $x_i = a_i g^{(1)}$ $(i = 1, 2)$. We have

$$x_1 \leqq x_2 \Longleftrightarrow a_1 \leqq a_2 \Longleftrightarrow y_1 \leqq y_2 \ ,$$

completing the proof. □

Proposition 2.3 shows that if we consider our construction up to isomorphism, then the choosing of $g^{(1)}$ is not essential. Let us write again B^0 instead of $B^0(g^{(1)})$.

2.4. Definition. Let G_i $(i \in I), g^{(1)}$ and B^0 be as above. Then B^0 is said to be the lexicographic product of half linearly ordered groups G_i and we express this fact by writing

$$G^0 = \Gamma_{i \in I} G_i \ ; \tag{1}$$

G_i are called lexicographic factors of G^0.

It is clear that if G_i are linearly ordered groups, then the above definition coincides with the usual notion of lexicographic product of linearly ordered groups (cf., e.g., [1], [5]).

Let I, G_i and G^0 be as above. Suppose that for each $i \in I$ we have

$$G_i = \Gamma_{j \in J(i)} G_{ij} \ . \tag{2}$$

Then without loss of generality we can suppose that $J(i(1)) \cap J(i(2)) = \emptyset$ whenever $i(1)$ and $i(2)$ are distinct elements of I.

Let K be the set of all pairs (i, j) where $i \in I$ and $j \in J(i)$. The set K is linearly ordered by applying the lexicographic linear order (i.e., $(i_1, j_1) \leqq (i_2, j_2)$ if either $i_1 < i_2$, or $i_1 = i_2$ and $j_1 \leqq j_2$). Then from (1) and (2) we obtain

$$G^0 = \Gamma_{(i,j) \in K} G_{ij} \ . \tag{3}$$

The relation (3) is said to be a refinement of (1).

Let (1) be valid and let φ be an isomorphism of a half linearly ordered group G onto G^0. Then φ is called a lexicographic product decomposition of G.

3. Congruence relations

In this section some auxiliary results on congruence relations will be obtained. Next we prove that to each lexicographic product decomposition of $G\uparrow$ there corresponds a lexicographic product decomposition of G.

Congruence relations on half lattice ordered groups were investigated in [4]. In the particular case of half linearly ordered groups stronger results than those in [4] can be proved.

Let $G \in \mathcal{HL}$ and $a, b \in G, a \leqq b$. Then we write $a \vee b = b$ and $a \wedge b = a$. Hence $\vee$ and $\wedge$ are partial binary operations on G.

Let ρ be an equivalence relation on G. Consider the following conditions for this relation:

(i) ρ is a congruence relation with respect to the group operation.

(ii) If $\circ \in \{\vee, \wedge\}, x, y, z \in G, x\rho y$ and if $x \circ z$ exists in G, then $y \circ z$ exists in G and $(x \circ z)\rho(y \circ z)$.

3.1. Definition. An equivalence relation ρ on G is said to be a congruence relation on G if it satisfies the conditions (i) and (ii).

The set of all congruence relations on G will be denoted by $\operatorname{Con} G$. The symbol $\operatorname{con} G$ denotes the set of all equivalence relations on G which satisfy the condition (i). Both the sets $\operatorname{Con} G$ and $\operatorname{con} G$ are partially ordered in the usual way; then they are complete lattices.

The symbols $\operatorname{Con} G\uparrow$ and $\operatorname{con} G\uparrow$ have analogous meanings.

For $x \in G$ and $\rho \in \operatorname{con} G$ we put $\overline{x}(\rho) = \{y \in G : y\rho x\}$. For $x \in G\uparrow$ and $\tau \in \operatorname{con} G\uparrow$ the meaning of $\overline{x}(\tau)$ is analogous.

3.2. Lemma. *Let $G \in \mathcal{HL}_1$ and let X be a subgroup of $G\uparrow$. Then X is normal in G.*

Proof. Let $x \in X$ and $g \in G$. If $g \in G\uparrow$, then in view of 1.3 (iii) we have $g^{-1}xg = x$. Next let $g \in G\downarrow$. Then according to 1.3 (i), $g^{-1} = g$. Thus 1.3 (ii) yields that $g^{-1}xg = x^{-1}$ and hence $g^{-1}xg \in X$. □

For $\rho \in \operatorname{con} G$ and $x, y \in G\uparrow$ we put $x\rho^1 y$ iff $x\rho y$. Next, for $\tau \in \operatorname{con} G$ and $u, v \in G$ we set $u\tau' v$ iff $(u^{-1}v)\tau e$.

3.3. Lemma. *Let $\rho \in \operatorname{Con} G$ and $\tau \in \operatorname{con} G$. Then*

(i) *$\rho^1 \in \operatorname{con} G\uparrow$; moreover, if $\rho \in \operatorname{Con} G$, then $\rho^1 \in ConG\uparrow$;*

(ii) *$\tau' \in \operatorname{con} G$; if $\tau \in \operatorname{Con} G\uparrow$, then $\tau' \in \operatorname{Con} G$.*

Proof. The assertion (i) is obvious. By applying 3.2 and by using the same method as in Section 3 of [4] we conclude that $\tau' \in \operatorname{con} G$. If, moreover, $\tau \in \operatorname{Con} G\uparrow$, then the results of [4] yield that τ' belongs to $\operatorname{Con} G$. □

For $\rho \in \operatorname{Conv} G$ the symbol G/ρ has the obvious meaning. If we assume only that $\rho \in \operatorname{con} G$, then (G/ρ) denotes the corresponding factor group. For $\tau \in \operatorname{Conv} G\uparrow$ or $\tau \in \operatorname{conv} G\uparrow$ the symbols $G\uparrow/\tau$ or $(G\uparrow/\tau)$ have analogous meanings.

Suppose that

$$\varphi : G\uparrow \longrightarrow \Gamma_{i\in I} A_i \tag{1}$$

is a lexicographic product decomposition of the linearly ordered group $G\uparrow$ and that $G\downarrow\neq\emptyset$. For $i\in I$ and $x,y\in G\uparrow$ we put $x\tau^i y$ if

$$\varphi(x)_i=\varphi(y)_i\ .$$

3.4. Lemma. *For each $i\in I$, τ^i belongs to* $\operatorname{con} G\uparrow$.

Proof. This is an immediate consequence of (1). □

Let us remark that, in general, τ^i need not belong to $\operatorname{Conv} G\uparrow$.

In what follows we suppose that $A_i\neq\{e\}$ for each $i\in I$. For $i\in I$ and $a_i\in A_i$ we denote by a_i^0 the element of $G\uparrow$ such that

$$(\varphi(a_i^0))_i=a_i\ ,\quad (\varphi(a_i^0))_{i(1)}=e\quad\text{for each}\quad i(1)\in I\setminus\{i\}\ ;$$

in view of the definition of the lexicographic product, such an element a_i^0 does exist in $G\uparrow$. Next we put

$$A_i^0=\{a_i^0:a_i\in A_i\}\ .$$

For $x\in G\uparrow$ we set

$$\chi_i(\overline{x}(\tau^i))=\varphi(x)_i^0\ .$$

Then χ_i is correctly defined (i.e., the result of applying χ_i does not depend on the choice of the element $x\in\overline{x}(\tau^i)$).

For $x,y\in G\uparrow$ we define $\overline{x}(\tau^i)\leqq\overline{y}(\tau^i)$ to be valid in $(G\uparrow/\tau^i)$ if and only if

$$\varphi(x)_i^0\leqq\varphi(y)_i^0\ .$$

Also, the relation $\leqq$ on $(G\uparrow/\tau^i)$ is correctly defined. We have obviously

3.5. Lemma. *Under the relation $\leqq$, $(G\uparrow/\tau^i)$ is a linearly ordered group and χ_i is an isomorphism of this linearly ordered group onto A_i.*

Put $H_i^{(1)}=(G\uparrow/\tau^i)$ under the linear order defined above. For $x\in G\uparrow$ let

$$\varphi^{(1)}(x)=(\overline{x}(\tau^i))_{i\in I}\ .$$

Then 3.5 and (1) yields that we have a lexicographic product decomposition

$$\varphi^{(1)}:G\uparrow\longrightarrow\Gamma_{i\in I}H_i^{(1)}\ . \tag{2}$$

Let us have a fixed element i of the set I. We construct $\tau^i\in\operatorname{conv} G\uparrow$ and $(\tau^i)'\in\operatorname{conv} G$ as above. Put

$$H_i^{(2)}=\{\overline{y}((\tau^i)'):y\in G\downarrow\}\ ,$$

$$G_i=H^{(1)}\cup H^{(2)}\ .$$

Then G_i is a group, namely, $G_i=(G/(\tau^i)')$.

Choose a fixed element $g^{(1)}$ of $G\downarrow$. By means of $g^{(1)}$ we define a relation $\leqq$ on $H_i^{(2)}$ as follows.

Let $h^{(1)}, h^{(2)} \in H_i^{(2)}$. There are $y_i, y_2 \in G\downarrow$ such that

$$(*) \qquad h^{(j)} = \overline{y}_j((\tau^i)') \quad (j = 1, 2) \ .$$

Then $y_1 y_2^{-1} \in G\uparrow$. We put

$$h^{(1)} \leqq h^{(2)}$$

if

$$\overline{y_1 g^{(1)}}(\tau^i) \leqq \overline{y_2 g^{(2)}}(\tau_i) \ .$$

The relation $\leqq$ is corretly defined on $H_i^{(2)}$ (i.e., it does not depend from the choice of y_1, y_2 satisfying $(*)$). It is a routine to verify that this relation is reflexive, transitive and antisymmetric. Finally, we have either $h^{(1)} \leqq h^{(2)}$ or $h^{(2)} \leqq h^{(1)}$. Hence $\leqq$ is a linear order on $H_i^{(2)}$. Also, $H_i^{(2)}$ is isomorphic to $H_i^{(1)}$.

If $h^{(1)} \in H_i^{(1)}$ and $h^{(2)} \in H_i^{(2)}$, then we consider $h^{(1)}$ and $h^{(2)}$ is to be incomparable. Thus $\leqq$ turns out to be a partial order on G_i.

Now let us verify that G_i is a half linearly ordered group; we have to consider the conditions 1) $-$ 4) from 1.1.

Since $H_i^{(1)}$ is isomorphic to $A_i \neq \{0\}$ and A_i is linearly ordered we conclude that the partial order on G_i is non-trivial, thus 1) holds. The condition 2) is obviously valid. Clearly $G_i\uparrow = H_i^{(1)}$ and $G_i\downarrow = H_i^{(2)}$. Thus 3) and 4) are also satisfied.

For $g_1, g_2 \in G$ and $i \in I$ we put $g_1 \rho^i g_2$ if either

$$g_1, g_2 \in G\uparrow \quad \text{and} \quad g_1 \tau_i g_2 \ ,$$

or

$$g_1, g_2 \in G\downarrow \quad \text{and} \quad g_1 \tau_i' g_2 \ .$$

Then $\rho^i \in \operatorname{con} G$. For each $g \in G$ we put

$$\varphi_1(g) = (\overline{g}(\rho^i))_{i \in I} \ .$$

Hence φ_1 is a mapping of G into the cartesian product of the half linearly ordered groups G_i $(i \in I)$.

3.6. Lemma. *φ_1 is an isomorphism of the group G into the cartesian product of the groups G_i $(i \in I)$.*

Proof. For each $i \in I$, the mapping

$$g \longrightarrow \overline{g}(\rho^i) \quad (g \in G)$$

is a homomorphism of the group G into the group G_i. Hence φ_1 is a homomorphism of the group G into the cartesian product of the groups G_i $(i \in I)$.

Let $g, g' \in G$ and suppose that $\varphi_1(g) = \varphi_1(g')$. If $g \in G\uparrow$ and $g' \in G\downarrow$, then $\overline{g}(\rho^i) \neq \overline{g'}(\rho^i)$ for each $i \in I$, whence $\varphi_1(g) \neq \varphi_1(g')$. Thus either (i) $g, g' \in G\uparrow$, or (ii) $g, g' \in G\downarrow$.

Let (i) be valid. Then in view of (2) we obtain that $g = g'$. Next suppose that (ii) holds and let $g^{(1)}$ be as above. Then $gg^{(1)}, g'g^{(1)} \in G\uparrow$ and $\varphi_1(gg^{(1)}) = \varphi_1(g'g^{(1)})$. Thus $gg^{(1)} = g'g^{(1)}$ yielding that $g = g'$. Therefore φ_1 is a monomorphism. $\square$

3.7. Lemma. *The set $\varphi_1(G)$ coincides with the underlying set of the lexicographic product $\Gamma_{i\in I}G_i$ (constructed with respect to the element $\varphi_1(g^{(1)})$).*

Proof. In view of 2.4 we have to verify that the conditions (i_1) and (ii_1) from Section 2 are satisfied. The relation (2) yields that (i_1) is valid. Let $g \in G$ and suppose that $(\varphi_1(g))_i \in G_i \downarrow = H_i^{(2)}$ for each $i \in I$. Then $g^{(1)}g^{-1} \in G \uparrow$ and hence the condition (i_1) holds for $\varphi_1(g^{(1)}g^{-1})$; thus (ii_1) is satisfied. □

3.8. Lemma. *Let $g, g' \in G$. Then $g \leqq g'$ if and only if $\varphi_1(g) \leqq \varphi_1(g')$.*

Proof. Let $g \leqq g'$. Then either (i) $g, g' \in G \uparrow$, or (ii) $g', g' \in G \downarrow$. Suppose that (i) holds. Then in view of (2), $\varphi_1(g) \leqq \varphi_1(g')$. Next let (ii) be valid. Thus $gg^{(1)}, g'g^{(1)} \in G \uparrow$ and $gg^{(1)} \leqq g'g^{(1)}$. Hence $\varphi_1(gg^{(1)}) \leqq \varphi_1(g'g^{(1)})$ and then $\varphi_1(g)\varphi_1(g^{(1)}) \leqq \varphi_1(g')\varphi_1(g^{(1)})$ yielding that $\varphi(g) \leqq \varphi(g')$.

Conversely, suppose that $\varphi_1(g) \leqq \varphi_1(g')$. From this we infer that we have either (i) or (ii). This shows that g, g' are comparable and that $g > g'$ cannot hold. □

3.9. Theorem. *Let $G \in \mathcal{HL}_1$ and suppose that for $G \uparrow$ the relation (1) is valid. Then φ_1 is a lexicographic product decomposition of G.*

Proof. This is a consequence of 3.6, 3.7 and 3.8. □

Consider a lexicographic product decomposition

$$\varphi : G \longrightarrow \Gamma_{i\in I}T_i \ . \tag{3}$$

We denote by φ the mapping ψ reduced to the subset $G \uparrow$ ov G; next we put $T_i \uparrow = A_i$ for each $i \in I$. Then we obtain that (1) holds.

For $g, g' \in G$ and $i \in I$ we put $g\rho_i^* g'$ if $\psi(g)_i = \psi(g')_i$. Thus if $g\rho_i^* g'$ then either (i) $g, g' \in G \uparrow$, or (ii) $g, g' \in G \downarrow$.

We apply the symbols τ_i and ρ_i as above. If (i) is valid, then

$$g\rho_i^* g' \iff g\tau_i g' \iff g\rho_i g' \ .$$

If (ii) holds, then we obtain

$$g\rho_i^* g' \iff (gg^{(1)})\rho_i^*(g', g^{(1)}) \iff (gg^{(1)})\rho_i(g'g^{(1)}) \iff g\rho_i g^{(1)} \ .$$

Thus $\rho_i = \rho_i^*$ for each $i \in I$.

In view of (3), the group (G/ρ_i^*) is isomorphic to T_i. Hence we have

3.10. Lemma. *For each $i \in I$, the groups T_i and G_i are isomorphic.*

In more detail, the isomorphism under consideration is constructed as follows. Let $i \in I$ and $t^i \in T_i$. We denote by X the class of all $g \in G$ such that $\psi(g)_i = t^i$. Then $X \in G/\rho_i^* = G/\rho_i = G_i$ and we assigne the element X of G_i to the element t^i.

Let $(t^i)'$ be another element of T and let $X' \in G_i$ be assigned to $(t^i)'$. Then according to the above defined partial order on G/ρ_i we have $X < X'$ if and only if $t < t'$. Hence the mapping of T_i onto G_i under consideration turns out to be also an isomorphism with respect to the partial order. By summarizing, we get

3.11. Proposition. *Let us apply the notation as above and let $i \in I$. Then the half linearly ordered groups T_i and G_i are isomorphic.*

4. Isomorphic refinements

Again, let $G \in \mathcal{HL}$ and let us have two lexicographic product decompositions

$$\alpha : G \longrightarrow \Gamma_{i\in I} G_i \ ,$$

$$\beta : G \longrightarrow \Gamma_{k\in K} T_k \ .$$

These lexicographic product decompositions are said to be isomorphic if there exists a monotone bijection $b : I \longrightarrow K$ such that for each $i \in I$, G_i is isomorphic to $T_{b(i)}$.

The following definition is a generalization of that given in Section 2.

4.1. Definition. The lexicographic product decomposition β is said to be a refinement of α if for each $i \in I$ there exists a subset $K(i)$ of K and a lexicographic product decomposition

$$\alpha_i : G_i \longrightarrow \Gamma_{k\in K(i)} T_k$$

such that, whenever $g \in G$, $i \in I$ and $k \in K(i)$, then

$$\beta(g)_k = \alpha_i(\alpha(g)_i)_k \ .$$

It is easy to verify that this definition is equivalent with the notion of refinement as applied in [1],[5] (though we use a different notation).

We have obviously

4.2. Lemma. *Let α and β be isomorphic lexicographic product decompositions of G and let α' be a refinement of α. Then there exists a refinement β' of β such that α' and β' are isomorphic.*

Suppose that the relation (1) from Section 3 is valid. Next suppose that we have a lexicographic product decomposition

(1') $$\chi : G \uparrow\longrightarrow \Gamma_{j\in J} B_j$$

such that (1') is a refinement of (1).

By applying the lexicographic product decomposition φ we construct $\varphi^{(1)}$ as in Section 3; there we proved that (2) holds.

Analogously, by applying χ we construct a lexicographic product decomposition

(2') $$\chi^{(1)} : G \uparrow\longrightarrow \Gamma_{j\in J} K_j^{(1)} \ .$$

Since χ is a refinement of φ, from the construction of $\varphi^{(1)}$ and $\chi^{(1)}$ we obtain

4.3. Lemma. *$\chi^{(1)}$ is a refinement of $\varphi^{(1)}$.*

Again, let φ_1 be as in Section 3. In view of 3.9 we have a lexicographic product decompositions

$$\varphi_1 : G \longrightarrow \Gamma_{i\in I} G_i \ .$$

By using $\chi^{(1)}$ we obtain analogously

$$\chi_1 : G \longrightarrow \Gamma_{j\in J} K_j \ ,$$

where, under a similar notation as in Section 3, $K_j = K_j^{(1)} \cup K_j^{(2)}$. By a routine verification we get

4.4. Lemma. *χ_1 is a refinement of φ_1.*

4.5. Theorem. *Any two lexicographic product decompositions of a half linearly ordered group G have isomorphic refinements.*

Proof. If G is a linearly ordered group, then the assertion is valid in view of [5].

Suppose that $G \in \mathcal{HL}_1$ and that α, β are lexicographic product decompositions of G. Let us denote by α_0 and β_0 the mappings α and β, respectively, reduced to the subset $G\uparrow$ of G. Hence α_0 and β_0 are lexicographic product decompositions of $G\uparrow$. (Cf. Fig. 1.)

$$\begin{tikzcd}
& \alpha \arrow[r] \arrow[d, dash, dashed] & \alpha_0 \arrow[d] & \beta_0 \arrow[d] & \beta \arrow[l] \arrow[d, dash, dashed] & \\
& \alpha_{01} \arrow[d] & \alpha_0^1 \arrow[l] \arrow[d] & \beta_0^1 \arrow[r] \arrow[d] & \beta_{01} \arrow[d] & \\
\mu_2 \arrow[uur] \arrow[r, dash, dashed] & \mu_1 \arrow[r, dash, dashed] & \mu \arrow[r, dash, dashed] & \nu \arrow[r, dash, dashed] & \nu_1 \arrow[r, dash, dashed] & \nu_2 \arrow[uul]
\end{tikzcd}$$

Fig. 1

If we construct $\alpha_0^{(1)}$ and α_{01} (similarly as we constructed $\varphi^{(1)}$ and φ_1 above) then in view of 3.11 the lexicographic product decompositions α and α_{01} are isomorphic.

Under analogous notation, the lexicographic product decompositions β and β_{01} are isomorphic.

Since $G\uparrow$ is a linearly ordered group, according to [5] there exist lexicographic product decompositions μ and ν of $G\uparrow$ such that

μ is a refinement of $\alpha_0^{(1)}$,

ν is a refinement of $\beta_0^{(1)}$,

μ and ν are isomorphic.

Now we construct the lexicographic product decompositions μ_1 and ν_1 of G in the same way as we did for φ_1. In view of 4.4, μ_1 is a refinement of α_{01}, and ν_1 is a refinement of β_{01}.

Hence according to 4.2 there exist lexicographic product decompositions μ_2 and ν_2 such that

μ_2 is a refinement of α,

ν_2 is a refinement of β,

μ_2 and ν_2 are isomorphic.

This completes the proof. □

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