Mathematical Institute
Slovak Academy of Sciences, Bratislava

Preprint 3/1997

*Anatolij Dvurečenskij*
*Maria Gabriella Graziano*

# Remarks on representations of minimal clans

January 27, 1997

PREPRINT SERIES

# Remarks on Representations of Minimal Clans

Anatolij DVUREČENSKIJ[1], Maria Gabriella GRAZIANO[2]

[1]Mathematical Institute, Slovak Academy of Sciences,
Štefánikova 49, SK-814 73 Bratislava, Slovakia
E-mail: dvurecen@mau.savba.sk
[2]Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università degli Studi di Napoli "Federico II", Complesso Universitario,"Monte S. Angelo", Via Cintia, I-801 26 Napoli, Italia,
E-mail: grazian@matna2.dma.unina.it

**Abstract**

We compare commutative BCK-algebras with commutative minimal clans, and we characterize them using Wyler's representation theorem via (i) lattice ordered groups, (ii) residuable elements of an ordered semigroup, and (iii) fuzzy representation. Finally, we characterize commutative minimal clans with condition (S) via $\Delta$-l-semigroups.

## 1 Introduction

Today there are plenty of algebraic structures which model different phenomena like quantum structures (quantum logics, orthomodular lattices, orthomodular posets, orthoalgebras, effect algebras, D-posets, MV-algebras [Dvu 1], which are models of quantum mechanical events, and which generalize

[1]The paper has been partially supported by grant 2/4033/97 SAV, Bratislava, Slovakia, and by Ministero dell'Università e della Ricerca Scientifica e Tecnologica, and by Gruppo Nazionale per l'Analisi Funzionale e Applicazioni, Italy.

[2]Keywords: Minimal clan, clan, BCK-algebra, MV-algebra, lattice ordered group, posemigroup, fuzzy set, residuable element, condition (S).

AMS classification: 06F35, 03G12.

Kolmogorov's probability model), or BCK-algebras [ImIs], clans [Wyl], minimal clans [Sch 1], which are algebraic abstraction of Boolean rings and algebras.

All these structures are based on partial binary or total binary operations which are analogues of addition or subtraction, respectively. All these structures are nowadays very frequently used, and the aim of the present paper is compare such structures, and give different kinds of their representations.

The paper is organized as follows. Section 2 is devoted to basic models, BCK-algebras, clans, minimal clans, and their representation using basic Wyler's characterization [Wyl] of commutative clans.

Section 3 compares commutative BCK-algebras and positive commutative minimal clans, and it gives a representation of upwards directed commutative BCK-algebras as subsets of a positive cone of lattice ordered groups.

Section 4 describes positive commutative minimal clans as subsets of residuable elements of po-semigroups. It is based on ideas of Fleischer [Fle] for BCK-algebras.

A fuzzy representation of commutative minimal clans having an order determining system of measure-morphisms is in Section 5, and a characterization of minimal clans with condition (S) via Weber's $\Delta$-l-semigroup [Web] is in Section 6. In addition, some examples of such minimal clans as subsets of real numbers in $[0, \infty)$ are given.

## 2 BCK-Algebras, Minimal Clans, and Clans

In the present section, we introduce BCK-algebras, MV-algebras, minimal clans, and clans, and we present a representation theorem of Wyler [Wyl] for commutative clans and its corollary for commutative minimal clans.

**Definition 2.1** *A BCK-algebra is a non-empty set $X$ with a binary operation $*$ and with a constant element $0$ such that the following axioms are satisfied: for all $x$, $y$, $z \in X$,*

(BCK-1) $((x * y) * (x * z)) * (z * y) = 0$;

(BCK-2) $(x * (x * y)) * y = 0$;

(BCK-3) $x * x = 0$;

(BCK-4) $x * y = 0$ *and* $y * x = 0$ *imply* $x = y$;

(BCK-5) $\quad 0 * x = 0.$

We shall write $X = (X; *, 0)$. In $X$, there is a binary relation $\leq$ defined by

$$x \leq y \quad \text{iff} \quad x * y = 0.$$

This is an order relation on $X$ which is called the BCK-*order*. We recall some elementary properties of BCK-algebras [MeJu, Thms I.1.2, I.1.3, I.1.4]: if $(X; *, 0)$ is a BCK-algebra, then for all $x$, $y$, $z \in X$,

(a) $x \leq y$ implies $z * y \leq z * x$;

(b) $(x * y) * z = (x * z) * y$;

(c) $(x * z) * (y * z) \leq x * y$;

(d) $x \leq y$ implies $x * z \leq y * z$;

(e) $x * y \leq x$;

(f) $x * 0 = x$.

If we denote by

$$x \wedge y := y * (y * x), \quad x, y \in X,$$

$x \wedge y$ is a lower bound of $x$ and $y$, $x \wedge x = x$, $x \wedge 0 = 0 = 0 \wedge x$, and it is true that

$$x * (y \wedge x) = x * y, \quad x, y \in X. \tag{2.1}$$

In general, $x \wedge y \neq y \wedge x$, and a BCK-algebra is said to be *commutative* if it satisfies, for all $x$, $y \in X$,

$$x * (x * y) = y * (y * x), \quad x, y \in X. \tag{2.2}$$

In this case, $x \wedge y = y \wedge x$ is the greatest lower bound of $x$ and $y$ and $(X; *, 0)$ is a lower semilattice with respect to the BCK-order. According to [MeJu, Thm I.5.6], a BCK-algebra $X$ is commutative if and only if $x \leq y$ implies $x = y * (y * x)$.

Special types of BCK-algebras are MV-algebras which have been introduced by Chang [Cha]. According to a result of Mundici [Mun 1], MV-algebras are categorically equivalent to bounded commutative BCK-algebras.

We recall that MV-algebras can be characterized as follows [Mun]. An MV-*algebra* is a non-empty set $L$ with two special elements 0 and 1 $(0 \neq 1)$, with a binary operation $\oplus : L \times L \to L$ and with a unary operation ${}^* : L \to L$ such that, for all $a, b, c \in L$, we have:

(MV-1) $a \oplus b = b \oplus a$;

(MV-2) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$;

(MV-3) $a \oplus 0 = a$;

(MV-4) $a \oplus 1 = 1$;

(MV-5) $(a^*)^* = a$;

(MV-6) $a \oplus a^* = 1$;

(MV-7) $0^* = 1$;

(MV-8) $(a^* \oplus b)^* \oplus b = (a \oplus b^*)^* \oplus a$.

Given an MV-algebra $L$, we define, for all $a, b \in L$,

$$\begin{aligned} a \odot b &= (a^* \oplus b^*)^*, \\ a \wedge b &= (a \oplus b^*) \odot b, \\ a \vee b &= (a \odot b^*) \oplus b. \end{aligned}$$

Then $\leq$, defined via $a \leq b$ iff $a = a \wedge b$, is a partial order on $L$, and $(L; \vee, \wedge, 0, 1)$ is a distributive lattice with the least and greatest elements 0 and 1, respectively [Cha].

We recall that two MV-algebras are isomorphic if there is a bijection between them preserving $\oplus$, $^*$, 0 and 1.

**Example 2.2** *Let $(L; +, \leq, 0)$ be a lattice ordered group with a strong unit $u$.*
*Let us define in $[0, u]$*

$$x \oplus y := (x + y) \wedge u, \quad x^* := u - x, \quad x, y \in [0, u]. \tag{2.3}$$

*Then $([0, u]; \oplus, ^*, 0, u)$ is an MV-algebra.*

*Conversely, any MV-algebra $(M; \oplus, ^*, 0, 1)$ is isomorphic with some $([0, u]; \oplus, ^*, 0, 1)$, where $[0, u]$ is an order interval of some lattice ordered group $G$ and $u$ is a strong unit* [Mun].

Minimal clans are a common abstraction of Boolean rings and lattice ordered groups which were introduced by Schmidt [Sch 1], and they have also a closed connection with BCK-algebras as we see below.

**Definition 2.3** *A minimal clan is a set $E$ with a relation $S \subseteq E \times E$, a map $+ : S \to E$ and an order relation $\leq$ such that:*

(MC-1) *there exists an element $0 \in E$ such that $(0, x) \in S$, $(x, 0) \in S$ and $x + 0 = x = 0 + x$, for all $x \in E$;*

(MC-2) *for all $x, y, z \in E, (x, y) \in S$ and $(x + y, z) \in S$ iff $(y, z) \in S$ and $(x, y + z) \in S$ and in this case $(x + y) + z = x + (y + z)$;*

(MC-3) *$x = y$ holds for all $x, y \in E$ such that $u + x + v = u + y + v$ for some $u, v \in E$ such that $(u, x), (u + x, v) \in S$ and $(u, y), (u + y, v) \in S$ (the cancellation property);*

(MC-4) *$u + x + v \leq u + y + v$ holds for all $x, y \in E$ such that $x \leq y$ and for all $u, v \in E$ with $(u, x), (u + x, v) \in S$ and $(u, y), (u + y, v) \in S$;*

(MC-5) *$(E; \leq)$ is a lattice;*

(MC-6) *for all $x, y \in E$ there exist $u, v \in E$ such that $0 \leq u$, $0 \leq v$, $(u, x) \in S$, $(x, v) \in S$, $(u, x \wedge y) \in S$, $(x \wedge y, v) \in S$ and $u + x = x \vee y = x + v$, $u + x \wedge y = y = x \wedge y + v$ (the difference property).*

Every minimal clan is a distributive lattice [Sch 1, Thm 1.1.14], and $u + x + v \leq u + y + v$ implies $x \leq y$ [Sch 1, Thm 1.1.9].

We shall write $E = (E; +, S, \leq, 0)$ for minimal clans.

A minimal clan is (i) *commutative* if $S$ is symmetric and the sum commutes, and (ii) *positive* if $x \geq 0$ for any $x \in E$. Commutative minimal clans are also called *Vitali spaces* [Con].

In the sequel, let $E$ be a commutative minimal clan.

**Lemma 2.4** [Sch 1, Lem. 1.1.1, Thm 1.1.13], [Con, Prop. 1.23].

(a) *If $u \leq x$, $v \leq y$ and $(x, y) \in S$, then $(u, v) \in S$;*

(b) *if $(x, z) \in S$ and $(y, z) \in S$, then $(x \wedge y, z) \in S$ and $(x + z) \wedge (y + z) = (x \wedge y) + z$;*

(c) *$(x, z), (y, z) \in S$ if and only if $(x \vee y, z) \in S$ and in this case $(x + z) \vee (y + z) = (x \vee y) + z$.*

In every commutative minimal clan, the difference property can be used to define a (partial) subtraction, $-$, in the following way: the domain of $-$ is

$$S' = \{(x,y) \in E \times E : \exists\, z \in E \text{ for which } (y,z) \in S \text{ and } x = y + z\}.$$

We set $x - y := z$, for all $(x,y) \in S'$. By the cancellation property, $z$ is unique. In the case that $E$ is positive, we have $(x,y) \in S'$ if and only if $y \leq x$.

**Lemma 2.5** *Let $x$, $y$, $z \in E$, then*

(a) *if $x - z$ and $y - z$ are defined, then $x - z \leq y - z$ iff $x \leq y$;*

(b) *if $z - x$ and $z - y$ are defined, then $z - x \leq z - y$ iff $y \leq x$;*

(c) *if $(x,z) \in S'$, then $(y,z) \in S'$ holds for every $y \geq x$;*

(d) *if $(x,z) \in S'$, then $(x,z') \in S'$ holds for every $z' \leq z$.*

**Lemma 2.6** *If $x$, $y$, $z \in E$, $x-y$ and $x-z$ are defined, then $(x-y) \wedge (x-z) = x - (y \vee z)$.*

**Proof.** Since $x - y$ and $x - z$ are defined, there exist $m$, $n \in E$ such that $x = y + m$ and $x = z + n$. By Lemma 2.4 (a), $(y, m \wedge n), (z, m \wedge n) \in S$ and by (MC-4) $y + m \wedge n \leq y + m = x$ and $z + m \wedge n \leq z + n = x$. Then $(y + m \wedge n) \vee (z + m \wedge n) = y \vee z + m \wedge n \leq x$, there exists $p \geq 0$ such that $y \vee z + m \wedge n + p = x$.

Since $m \wedge n + p = x - y \vee z \leq x - y = m$ and similarly $m \wedge n + p \leq n$, we have $m \wedge n + p \leq m \wedge n$ and, consequently, $p = 0$. Hence $x - y \vee z = m \wedge n = (x - y) \wedge (x - z)$. **Q.E.D.**

**Example 2.7** (a) *Every lattice ordered group is a commutative minimal clan.*

(b) *Let $E$ be a Boolean ring. Then $E$ endowed with the set $S = \{(x,y) \in E \times E : x \wedge y = 0\}$ and with the map $\vee : S \to E$ is a positive commutative minimal clan.*

Let $E$ and $F$ be two commutative minimal clans. A mapping $\varphi : E \to F$ is said to be an *isomorphism* if (i) it is a lattice-isomorphism, and if (ii) $a + b$ exists in $E$ if and only if $\varphi(a) + \varphi(b)$ exists in $F$, and in this case $\varphi(a + b) = \varphi(a) + \varphi(b)$. In particular, we have $\varphi(0) = 0$ and $\varphi(a - b) = \varphi(a) - \varphi(b)$ for all $(a,b) \in S'$.

If $F$ is a sublattice of a commutative minimal clan $E$ such that

(i) $0 \in F$;

(ii) $x, y \in F, x \leq y \Rightarrow y - x \in F$;

(iii) $x, y, z, (x+y), (x+y)+z \in F \Rightarrow y + z \in F$;

then $F$ with the set $S_F = \{(x,y) \in F \times F : (x,y) \in S, x+y \in F\}$ and with the map $+ : S_F \to F$ is a commutative minimal clan, and $F$ is called a *subclan* of $E$.

Another interesting structure with a partially defined subtraction is a clan defined by Wyler [Wyl]. We recall that a lattice $(E; \leq)$ is said to be a *clan* if there is a subset $S \subseteq E \times E$ (called the domain of a subtraction $-$) and a mapping $- : S \to E$ (called the *subtraction)*), (if $(a,b) \in S$ we write $-(a,b) := b - a$) such that, for all $a, b, c, p \in E$, we have:

(C1) if $a \leq b$, then $b - a$ is defined;

(C2) if $a \vee b \leq c$, we have $a \leq b$ if and only if $c - b \leq c - a$;

(C3) if $b - a$ and $c - a$ are defined and $b \leq c$, then $b - a \leq c - a$, and $(c-a) - (b-a) = c - b$;

(C4) if $b \leq p$ and $a \leq p - b$, then there is an element $c \in E$ such that $p - c$ is defined, and $p - c = (p-b) - a$.

By Wyler [Wyl, Thm 2.4], the equation $x - x = x$ has a unique solution which we denote by 0, and for it we have $0 = x - x \leq y - x$.

For a clan $E = (E; -, S, \leq)$, define a relation

$$\tilde{S} := \{(x,y) \in E \times E : \ \exists\, z \in E \text{ with } y \leq z, \text{ and } x \leq z - y\}.$$

For each pair $(x,y) \in \tilde{S}$ there is the unique element $u \in Z$ satisfying

$$z - u = (z - y) - x$$

for all $z \in E$ such that $y \leq z$ and $x \leq z - y$. We denote this element

$$z := x + y,$$

and the partial operation $+ : \ \tilde{S} \to E$ is well-defined.

A clan $E = (E; -, S, \leq)$ is (i) *symmetric*, if for all $x, y \in E$ with $x \leq y$, there exists some $v \in E$ satisfying $(x, v) \in \tilde{S}$ and $x + v = y$; (ii) *commutative* if, for all $x, y \in E$, $(x,y) \in \tilde{S}$ if and only if $(y,x) \in \tilde{S}$, and $x + y = y + x$.

Comparing clans and minimal clans (see [Sch 1, Sch]), we can see that every symmetric clan is a minimal clan, and every minimal clan is a symmetric clan having a minimal domain of subtraction (for more details see [Sch , Thms 6.1, 6.2, Cor. 6.3]), which explains the name of minimal clans.

A crucial result of commutative clans is a result of Wyler [Wyl, Thm 7.9]:

**Theorem 2.8** [Wyler] *Any commutative clan $E = (E; -, S, \leq)$ can be embedded into a lattice ordered group preserving all joins, meets in $E$, and all subtractions $b - a$ whenever $a \leq b$ in $E$.*

From this basic result we have immediately the following representation theorem for commutative minimal clans.

**Corollary 2.9** *Every commutative minimal clan $(E; +, S, \leq, 0)$ can be embedded, as a subclan, into a lattice ordered group $(L_E; +, \leq, 0)$. If $\alpha_E : E \to L_E$ is the embedding map, $\alpha_E(E)$ is a subclan of $L_E$ in the previous sense and $\alpha_E$ is an isomorphism of minimal clans from $E$ onto $\alpha_E(E)$. Moreover, $L_E$ is generated, as a group, by $\alpha_E(E)$.*

*In particular, if $(E; +, S, \leq, 0)$ is a positive commutative minimal clan, then it can be embedded into a positive cone of $L_E$.*

The couple $(\alpha_E, L_E)$ has the universal property ([Wyl, Thm 6.7]): if $h : E \to G$ is an additive and order preserving mapping of the (minimal) commutative clan $E$ into an order Abelian group $G$, then $h = h_1 \circ \alpha_E$ for a uniquely determined homomorphism $h_1 : L_E \to G$ of ordered Abelian groups. This group $L_E$ is unique up to isomorphism.

Another corollary is the following result. Let $E$ be a bounded positive commutative minimal clan with the least and greatest elements 0 and 1, respectively. Then $\alpha_E(E) = [0, u]$, where $u = \alpha_E(1)$, is an order interval in $L_E$ and $u$ is a strong unit.

Indeed, for any $x \in L_E$, there exist $a_1, \ldots, a_n, b_1, \ldots, b_m \in E$ such that $x = (\alpha_E(a_1) + \ldots + \alpha_E(a_n)) - (\alpha_E(b_1) + \ldots + \alpha_E(b_m))$. Since $E$ is positive and $\alpha_E$ preserves the order, we have $\alpha_E(b_1) + \ldots + \alpha_E(b_m) \geq 0$ and thus $x \leq \alpha_E(a_1) + \cdots + \alpha_E(a_n) \leq nu$.

**Remark 2.10** *Let $(M; \oplus, ^*, 0, 1)$ be an MV-algebra. Let us define $b \ominus a := (a \oplus b^*)^*$ for $a \leq b$, and $a + b := (a^* \ominus b)^*$ for $b \leq a^*$. Then $M$ with this* *partial addition is a positive commutative minimal clan with 1 as the greatest element* [Pul].

*In particular, by the previous observations, there exists a lattice ordered group* $(L_M; +, \leq, 0)$ *with a strong unit* $u$ *and an isomorphism of minimal clans* $\alpha_M : M \to [0, u]$.

*Let* ${}^{*_u}$ *and* $\oplus_u$ *be defined in* $[0, u]$ *via* (2.3). *It is easy to see that* $\alpha_M(x^*) = \alpha_M(x)^{*_u}$, *for all* $x \in M$. *It follows from* $a \oplus b = a + (a^* \wedge b)$ *holding for all* $a, b \in M$, *that* $\alpha_M(a \oplus b) = \alpha_M(a) \oplus_u \alpha_M(b)$, *for all* $a, b \in M$.

*Then* $\alpha_M$ *is an isomorphism between the MV-algebras* $(M; \oplus, {}^*, 0, 1)$ *and* $([0, u]; \oplus_u, {}^{*_u}, 0, u)$ *and we obtain in this way the famous result of Mundici, or, in other words, the Mundici representation theorem of MV-algebras follows easily from Wyler's result* [Wyl, Thm 7.9].

# 3 Relations between BCK-Algebras and Minimal Clans

The aim of this section is to study connections among positive commutative minimal clans and upwards directed commutative BCK-algebras. In particular, we show that any upwards directed commutative BCK-algebra can be embedded onto a commutative BCK-subalgebra of a positive cone of some lattice ordered group.

A poset $(E, \leq)$ is *directed upwards* if given $a, b \in E$ there is an element $c \in E$ such that $a \leq c$ and $b \leq c$. Of course any lattice is directed upwards.

We notify that in every positive commutative minimal clan, we can define a new total operation $\ominus$ as follows:

$$x \ominus y := x - x \wedge y, \quad x, y \in E. \tag{3.1}$$

Using this operation, we obtain the following result.

**Theorem 3.1** *Let* $(E; +, S, \leq, 0)$ *be a positive commutative minimal clan. Then* $(E; \ominus, 0)$, *where* $\ominus$ *is defined via* (3.1), *is a commutative BCK-algebra directed upwards.*

**Proof.** We have to verify conditions (BCK-1)–(BCK-5).

The conditions (BCK-5) and (BCK-3) are obviously true. (BCK-4): if $x \ominus y = 0 = y \ominus x$, then $x = x \wedge y$ and $y = x \wedge y$ and so $x = y$.

(BCK-2): from $x = x \wedge y + x \ominus y \geq x \ominus y$ we have $x \ominus (x \ominus y) = x - x \ominus y = x \wedge y$ and then $(x \ominus (x \ominus y)) = x \wedge y - x \wedge y = 0$.

(BCK-1): we have to show that $((x \ominus y) \ominus (x \ominus z)) \ominus (z \ominus y) = 0$. Let us write $(x \ominus y) \ominus (x \ominus z) = m$ and $z \ominus y = c$. We claim that $m \leq c$.

We have the following: $m = (x - x \wedge y) \ominus (x - x \wedge z) = (x - x \wedge y) - [(x - x \wedge y) \wedge (x - x \wedge z)] = (x - x \wedge y) - [x - (x \wedge y) \vee (x \wedge z)] = (x - x \wedge y) - [x - x \wedge (y \vee z)] = (x \ominus y) - (x \ominus (y \vee z)) = (x \ominus y) - (x \ominus (y + c))$.

The last identity follows from the fact that $z \ominus y = c$ implies $z = z \wedge y + c$ and then, by (MC-6), $y \vee z = y + c$. We have now $m = a - b$ with $a$ and $b$ such that $x = x \wedge y + a$ and $x = x \wedge (y + c) + b = x \wedge (x \wedge y + c) + b$.

Since $x \wedge y + a = x \wedge (x \wedge y + c) + b$ and $x \wedge (x \wedge y + c) \geq x \wedge y$, we have $m = a - b = x \wedge (x \wedge y + c) - x \wedge y \leq (x \wedge y + c) - x \wedge y = c$.

Finally, since $x \ominus (x \ominus y) = x \wedge y$ and $y \ominus (y \ominus x) = y \wedge x$, we obtain that $(E; \ominus, 0)$ is a commutative BCK-algebra directed upwards. **Q.E.D.**

We show now the converse statement to Theorem 3.1 for commutative BCK-algebras.

**Lemma 3.2** *Let $(X; *, 0)$ be a commutative BCK-algebra directed upwards with respect to BCK-order. Then $(X; \leq)$ is a lattice with the least element $0$.*

**Proof.** Since $X$ is commutative, then it is a lower semilattice and $x \wedge y = x * (x * y) = y * (y * x)$, [MeJu, Thm I.5.5]. Let us define

$$x \vee_z y := z * ((z * x) \wedge (z * y))$$

if $x,\, y \leq z$. Since $x \vee_z y = z * ((z * x) \wedge (z * y)) \geq z * (z * x) = x$ and similarly $x \vee_z y \geq y$, $x \vee_z y$ is a majorant of $x$ and $y$. If $x, y \leq z_1$ and $z \leq z_1$, we have: $z_1 * x \geq z_1 * z$ and $z_1 * y \geq z_1 * z$ so that $z_1 * z \leq (z_1 * x) \wedge (z_1 * y)$, and hence $z = z_1 * (z_1 * z) \geq z_1 * ((z_1 * x) \wedge (z_1 * y)) = x \vee_{z_1} y$.

Since $x \vee_z y \leq z \leq z_1$ and $x, y \leq x \vee_z y$, in the same way we see that $x \vee_z y \geq x \vee_{z_1} y$. From $x, y \leq x \vee_{z_1} y \leq z$ using similar arguments, it follows that $x \vee_{z_1} y \geq x \vee_z y$ and therefore $x \vee_z y = x \vee_{z_1} y$.

If $x,\, y \leq z, z_1$, there exists $z_2$ such that $z, z_1 \leq z_2$. Since $x \vee_z y = x \vee_{z_2} y$ and $x \vee_{z_1} y = x \vee_{z_2} y$, we have that $x \vee_z y = x \vee_{z_1} y$. Hence, $x \vee_z y$ is independent from the majorant $z$.

We prove that $x \vee_z y$ is the supremum of $x$ and $y$. Let us consider $u \in E$ such that $x, y \leq u$. Then $x \vee_z y = x \vee_u y = u * ((u * x) \wedge (u * y)) \leq u$ and, consequently, $x \vee y := x \vee_z y$ for all $z \in E$ with $x, y \leq z$. **Q.E.D.**

Let now $(X; *, 0)$ be a commutative BCK-algebra directed upwards. Then $X$ can be organized into a positive commutative minimal clan. Define the domain $S$ of a partial addition in the following way:

$$(a, b) \in S \quad \text{and} \quad a + b := c \quad \text{iff} \quad a \leq c \quad \text{and} \quad c * a = b. \tag{3.2}$$

**Lemma 3.3** *Let $(X; *, 0)$ be an upwards directed commutative BCK-algebra. Then the partial addition $+$ defined via (3.2) is well-defined.*

**Proof**. Suppose that there exist $c, c' \in X$ such that $a \leq c, c'$ and $c * a = c' * a$. We have to prove that $c = c'$. Let $u \in X$ be such that $c, c' \leq u$.

Then $(u * a) * (u * c) = (u * (u * c)) * a = (u \wedge c) * a = c * a$. Since $a \leq c$, we have $u * c \leq u * a$ and then $[u * (c * a)] * a = (u * a) * (c * a) = (u * a) * [(u * a) * (u * c)] = (u * a) \wedge (u * c) = u * c$.

In the same way we show that $[u * (c' * a)] * a = u * c'$. From $c * a = c' * a$, it follows $u * c' = u * c$. Hence $c = u * (u * c) = u * (u * c') = c.'$ **Q.E.D.**

**Theorem 3.4** *Let $(X; *, 0)$ be an upwards directed commutative BCK-algebra. Then $(X; +, S, \leq, 0)$, where $+$ and $S$ are defined via (3.2) and $\leq$ is the BCK-order, is a positive commutative minimal clan.*

**Proof**. Let us start observing that the partial addition is commutative. In fact, if $(a, b) \in S$ and $a + b = x$, then $a \leq x$ and $x * a = b$. Since $b = x * a \leq x$ and $x * b = x * (x * a) = a$, we have $(b, a) \in S$ and $b + a = x = a + b$.

The axiom (MC-1) is obvious. (MC-3) follows from (3.2) and (MC-5) from Lemma 3.2.

To prove (MC-2), let us consider $(a, b), (a + b, c) \in S$. Then by (3.2) there exist $x, y \in X$ such that $a \leq x$ and $x * a = b$, $a + b = x \leq y$ and $y * x = y * (a + b) = c$.

Since $c = y * x \leq y * a$ and $(y * a) * c = (y * a) * (y * x) = [y * (y * x)] * a = x * a = b$, then $(b, c) \in S$ and $b + c = y * a$.
From $a \leq y$ and $y * a = b + c$, it follows $(a, b + c) \in S$ and $a + (b + c) = y = (a + b) + c$.

To prove (MC-4), let us suppose that $(a,b),(a,b') \in S$ and $b \le b'$. By definition (3.2), there exist $c, c' \in X$ such that $a \le c$ and $c * a = b$, $a \le c'$ and $c' * a = b'$. Since $b' * b \le b'$ and $b' * (b' * b) = b \wedge b' = b$, then $(b' * b, b) \in S$ and $(b' * b) + b = b'$. Using associativity and commutativity, we have $c' = a + b' = a + [(b' * b) + b] = a + [b + (b' * b)] = (a + b) + (b' * b) \ge a + b = c$.

Finally we have to prove the difference property. Let us consider $u$ such that $(a \vee b) * a = u$. Since $a \le a \vee b$, by (3.2) $(a, u) \in S$ and $a + u = a \vee b$. Since $X$ is directed upwards, there exists $c \in X$ such that $a, b \le c$.

By Lemma 3.2 we have $a \vee b = c * [(c*a) \wedge (c*b)]$. Then $(a \vee b) * a = [c * ((c*a) \wedge (c*b))] * a = (c*a) * [(c*a) \wedge (c*b)] = (c*a) * (c*b) = (c * (c*b)) * a = b * a$.

Since $a \wedge b \le b$ and $b * (b \wedge a) = b * a = (a \vee b) * a$, by (3.2) $(a \wedge b, u) \in S$ and $a \wedge b + u = b$. **Q.E.D.**

As an easy corollary of Theorem 3.4 is a fact that any upwards directed commutative BCK-algebra is a distributive lattice. This fact was known only for bounded commutative BCK-algebras (see [MeJu, Thm VII.3.4]).

Let us start with a positive commutative minimal clan $(E; +, S, \le, 0)$. Then, by Theorem 3.1, $(E; \ominus, 0)$, where $\ominus$ is defined by (3.1), is an upwards directed commutative BCK-algebra and $+_\ominus$, induced by $\ominus$ via (3.2), coincides with original addition $+$.

A similar consideration holds if we start with an upwards directed commutative BCK-algebra $(X; *, 0)$. Then $(X; +, S, \le, 0)$ is a positive commutative minimal clan, where $+$ is induced by $*$ via (3.2). The total binary operation $*_+$, induced by $+$ via (3.1), coincides with the original binary operation $*$ on $X$.

Let $X = (X; *, 0)$ be a BCK-algebra. A subset $X_0$ of $X$ is a BCK-*subalgebra* of $X$ if (i) $0 \in X_0$, and (ii) $x * y \in X_0$ whenever $x, y \in X_0$.

Finally, we present the following representation of upwards directed commutative BCK-algebras.

**Theorem 3.5** *Let $G = (G; +, \le, 0)$ be a lattice ordered group and let $G_+ := \{x \in G : x \ge 0\}$ be its positive cone. Define a total binary operation $*_G$ on $G_+$ via*

$$x *_G y := (x - y) \vee 0, \tag{3.3}$$

*for all $x, y \in G_+$. Then*

$$x *_G y = x - (x \wedge y) = (x \vee y) - y, \quad x, y \in G_+, \tag{3.4}$$

*and* $(G_+; *_G, 0)$ *is an upwards directed commutative BCK-algebra.*

*Let* $G_0$ *be a non-void sublattice of* $G_+$ *such that* $x, y \in G_0$ *entails* $x *_G y \in G_0$. *Then* $(G_0; *_G, 0)$ *is an upwards directed commutative BCK-subalgebra of* $(G_+; *_G, 0)$.

*Conversely, let* $X = (X; *, 0)$ *be an upwards directed commutative BCK-algebra. Then there is a lattice ordered group* $(G; +, \leq, 0)$ *and a sublattice* $G_0$ *of* $G_+$ *such that* $(X; *, 0)$ *is isomorphic with* $(G_0; *_G, 0)$.

**Proof.** From (3.3) and basic properties of lattice ordered groups, [Bir], we have (3.4), and Theorem 3.1 entails the first part of Theorem.

The second part follows from Corollary 2.9. **Q.E.D.**

We recall that the authors do not know whether any commutative BCK-algebras can be also embedded into some lattice ordered group like upwards directed ones. The following example shows such a possibility (see also a remark to Theorem 6.7, where is another partial positive answer).

**Example 3.6** *Let* $\Omega \neq \emptyset$ *and let* $A, B$ *be two non-void disjoint subsets of* $\Omega$. *Define* $E := \{X \subseteq \Omega : X \subseteq A \text{ or } X \subseteq B\}$. *Then* $(E; *, \emptyset)$, *where* $*$ *is the set-theoretical difference, is a commutative BCK-algebra which is not upwards directed. Hence, it cannot be converted into a positive minimal clan with the set-theoretical inclusion via Theorem* 3.4. *On the other hand,* $E$ *can be embedded into* $\mathcal{A}(E)$, *the minimal algebra of subsets of* $\Omega$ *generated by* $E$, *which defines a positive commutative BCK-algebra, and which can be embedded into a lattice ordered group via Theorem* 3.5.

# 4 Representation of Commutative Minimal Clans via Residuable Elements

The aim of this section is to obtain a representation theorem for positive commutative minimal clans using residuable elements of a partially ordered semigroup. The basic ideas are due to Fleischer [Fle], who represented BCK-algebras, and for the reader's convenience we give them in details.

Let $(G; +, \leq, 0)$ be a commutative, positive and partially ordered semigroup written additively (a po-semigroup in abbreviation), that is, $\leq$ is a partial order on $G$, and $+$ is a binary operation on $G$ such that, for all $x, y, z \in G$, we have

(P1) $x + 0 = x = 0 + x$;

(P2) $x + y = y + x$;

(P3) $x + (y + z) = (x + y) + z$;

(P4) $x \leq y$ implies $x + z \leq y + z$;

(P5) $0 \leq x$.

Let $a, b \in G$. An element $a \div b$ of $G$ is said to be a *residuation* of $a$ by $b$ if it satisfies: (i) $b + a \div b \geq a$; (ii) if $b + x \geq a$, then $x \geq a \div b$. These conditions mean that $a \div b$ is the least element of the set $\{x \in G : b + x \geq a\}$. If $a \div b$ exists, then it is unique. If, for any $b \in G$, $a \div b$ exists, then $a$ is said to be a *residuable element* of $G$.

The following statements are evident.

**Lemma 4.1** *Let $R(G)$ denote the set of all residuable elements of $G$. Then* (i) $0 \in R(G)$, *and* (ii) *if $a \in R(G)$ and $b \in G$, then $a \div b \in R(G)$, and, for any $c \in G$, $(a \div b) \div c = a \div (b + c)$.*

**Lemma 4.2** *For any $a, b, c \in R(G)$, we have* (i) *$a \leq b$ if and only if $a \div b = 0$.*
(ii) *$a \leq b$ implies $a \div c \leq b \div c$ and $c \div b \leq c \div a$.*

**Theorem 4.3** *Let $(G; \leq, +, 0)$ be a positive and commutative po-semigroup. Let $M$ be a non empty subset of $R(G)$ such that*

(i) *if $a, b \in M$ then $a \div b \in M$;*

(ii) *$a \div (a \div b) = b \div (b \div a)$, for all $a, b \in M$;*

(iii) *for any $a, b \in M$ there exists $z \in M$ such that $a \leq z$ and $b \leq z$.*

*Then $0 \in M$.*

*Define a partial operation $+_M$ on $M$ with domain $S_M$ in such a way*

$$(a, b) \in S_M \quad \text{and} \quad a +_M b = c \quad \text{iff} \quad a \leq c \quad \text{and} \quad b = c \div a. \tag{4.1}$$

*Then $(M; +_M, S_M, \leq, 0)$ is a positive commutative minimal clan.*

**Proof.** It is clear that if $a \in M$, then $0 = a \div a \in M$. We define a partial binary operation $*$ on $M$ in such a way $a * b := a \div b$.

By a direct calculation or using [MeJu, Th. VIII.2.5], we see that $(M; *, 0)$ is a BCK-algebra, where $a \leq b$ if and only if $b \div a = 0$.

Condition (ii) says that $(M; *, 0)$ is a commutative BCK-algebra and, in view of (iii), it is directed upwards.

By Theorem 3.6, $(M; +_M, S_M, \leq, 0)$, where $+_M$ is defined via (4.1), is a positive commutative minimal clan. **Q.E.D.**

The converse of Theorem 4.3 is also true.

**Theorem 4.4** *Let $(E; +, S, \leq, 0)$ be a positive commutative minimal clan. Then there exists a positive commutative po-semigroup $G$ and a non empty subset $M$ of $R(G)$ such that:*

(i) *$a, b \in M$ implies $a \div b \in M$;*

(ii) *$a \div (a \div b) = b \div (b \div a)$, for all $a, b \in M$;*

(iii) *for all $a, b \in M$ there exists $c \in M$ such that $a \leq c, b \leq c$.*

*Moreover, $(M; +_M, S_M, \leq, 0)$, where $S_M$ and $+_M$ are defined as in (4.1), is a positive commutative minimal clan which is isomorphic with $(E; +, S, \leq, 0)$.*

**Proof.** The proof will consists of several steps.

1) Using Theorem 3.1, we can convert the minimal clan $(E; +, S, \leq, 0)$ into a commutative BCK-algebra $(E; *, 0)$ where $a * b := a - a \wedge b$.

2) For any $a \in E$, we define a mapping $\varphi_a : E \to E$ such that $\varphi_a(x) := x*a$, for all $x \in E$. Denote by $G(E) := \{\varphi_{a_1} \circ \ldots \circ \varphi_{a_n} : a_1, \ldots, a_n \in E, n \geq 1\}$. Then $(G(E); \circ, \varphi_0)$ is a semigroup and, using property (b) of BCK-algebras, it is not difficult to show that $G(E)$ is commutative.

Let us define a binary relation $\ll$ on $G(E)$ by

$$\varphi_{b_1} \circ \cdots \circ \varphi_{b_m} \ll \varphi_{a_1} \circ \cdots \circ \varphi_{a_n} \text{ iff } (\varphi_{a_1} \circ \cdots \circ \varphi_{a_n})(x) \leq (\varphi_{b_1} \circ \cdots \circ \varphi_{b_m})(x), \quad x \in E.$$

The fact that $\ll$ is a partial order on $G(E)$ follows from one that $\leq$ is a partial order on $E$. $(G(E); \circ, \ll, \varphi_0)$ is now a positive po-semigroup. Indeed, for $a_1, \ldots, a_n \in E$, we have $(\cdots (x * a_n) * \cdots) * a_1 \leq x = \varphi_0(x)$, $x \in E$, and then $\varphi_0$ is the smallest element of $G(E)$.

To prove that $G(E)$ is a po-semigroup, suppose that $\varphi_{a_1} \circ \cdots \circ \varphi_{a_n} \ll \varphi_{b_1} \circ \cdots \circ \varphi_{bm}$. Then for all $x \in E$

$$(\cdots (x * a_n) * \cdots) * a_1 \geq (\cdots (x * b_m) * \cdots) * b_1.$$

This is also true for the element $(\cdots (x * u_k) * \cdots) * u_1$, for any $u_1, \ldots, u_k \in E$. Using property (b) of BCK-algebras, it is easy to see that (P4) is satisfied for any $\varphi_{u_1} \circ \cdots \circ \varphi_{u_k} \in G(E)$.

We now show that, for any $a \in E$, $\varphi_a$ is a residuable element of $G(E)$ and, in addition, $\varphi_a \div \varphi_b = \varphi_{a*b}$ holds for any $b \in E$. Indeed, $\varphi_b \circ \varphi_{a*b} \gg \varphi_a$, since $(x * (a * b)) * b = (x * b) * (a * b) \leq x * a$ for all $x \in E$.

If $\varphi_b \circ (\varphi_{a_1} \circ \cdots \circ \varphi_{a_n}) \gg \varphi_a$, then $((\cdots (x * a_n) * \cdots) * a_1) * b \leq x * a$ for any $x \in X$, and $((\cdots (a * a_n) * \cdots) * a_1) * b \leq 0$. Hence, $((\cdots (a * a_n) \cdots) * a_1) * b = 0$ and, using property (b), $(\cdots (a * b) * \cdots) * a_1 = 0$. Since

$$\begin{aligned}
&((\cdots (x * a_n) * \cdots) * a_1) * (x * (a * b)) = \\
&(\cdots ((x * (x * (a * b))) * a_n) * \cdots) * a_1 = \\
&(\cdots ((a * b) * a_n) * \cdots) * a_1 = 0
\end{aligned}$$

for all $x \in E$, then $\varphi_{a_1} \circ \cdots \varphi_{a_n} \gg \varphi_{a*b}$.

Hence, $\varphi_a$ is a residuable element, and $\varphi_a \div \varphi_b = \varphi_{a*b}$.

3) Define $M = \{\varphi_a : a \in E\}$. It is clear that $M$ satisfies the property (i)–(iii). Then $(M; \div, \varphi_0)$ is a commutative BCK-algebra directed upwards. We can convert it into a positive commutative minimal clan $(M; +_M, S_M, \leq, \varphi_0)$ via Theorem 4.3 and (4.1).

Finally, define a mapping $\varphi : E \to M$ such that $\varphi(a) = \varphi_a$ for any $a \in E$.

From the equivalencies $a = b$ iff $x * a = x * b$ for all $x \in E$ iff $\varphi_a = \varphi_b$, it follows that $\varphi$ is a bijection.

$\varphi$ preserves the order: $\varphi_a \gg \varphi_b$ implies $z * a \leq z * b$ for any $z \in E$. If $a, b \leq z$, then $a = z * (z * a) \leq z * (z * b) = b$.

Using (4.1) and the fact that $\varphi_a \div \varphi_b = \varphi_{a*b}$, it is easy to see that $\varphi$ is an isomorphism. **Q.E.D.**

# 5 Fuzzy Representation of Minimal Clans

In the present section, we give another kind of representation of minimal clans using fuzzy set approach of clans having an order determining system of measure-morphisms.

**Definition 5.1** *Let $(E; S, +, \leq, 0)$ be a positive commutative minimal clan. A mapping $m : E \to [0,1]$ is said to be* (i) *a measure on $E$ if $m(x+y) = m(x) + m(y)$ for all $(x,y) \in S$,* (ii) *a state if $E$ has the greatest element 1 and $m$ is is a measure with $m(1) = 1$, and* (iii) *a measure-morphism on $E$ if $m(x - x \wedge y) = \max\{0, m(x) - m(y)\}$, for all $x, y \in E$.*

It is clear that every measure-morphism on $E$ is a measure. We recall that a non-empty family $\mathcal{S}$ of measures (states, or morphisms) is an *order determining* if $m(a) \leq m(b)$ for all $m \in \mathcal{S}$ implies $a \leq b$.

**Theorem 5.2** *Let $\Omega$ be a non-empty set. Let $X \subseteq [0,1]^\Omega$ a system of fuzzy sets of $\Omega$ such that:*

(i) $X \neq \emptyset$;

(ii) *if $f, g \in X$, then $f * g \in X$, where $(f * g)(\omega) = \max\{0, f(\omega) - g(\omega)\}$ for all $\omega \in \Omega$;*

(iii) *for all $f, g \in X$ there exists $h \in X$ such that $f, g \leq h$.*

*Then the null function belongs to $X$.*

*We define a partial addition $+_X$ on $X$ with domain $S_X$ via*

$$(f,g) \in S_X \quad \textit{and} \quad f +_X g = h \quad \textit{iff} \quad f(\omega) + g(\omega) = h(\omega),\ \omega \in \Omega. \tag{5.1}$$

*Let $\leq$ be the pointwise order on $X$. Then $(X; +_X, S_X, \leq, 0)$ is a positive commutative minimal clan.*

*The mapping $m_\omega : X \to [0,1]$, defined via $m_\omega(f) := f(\omega)$, $f \in X$, is a measure-morphism on $X$, and the set $\mathcal{S} = \{m_\omega : \omega \in \Omega\}$ is an order determining system of measure-morphisms on $X$.*

**Proof**. It is clear that if $f \in X$, then $0 = f * f \in X$. By a direct calculation we can show that $(X; *, 0)$ is a commutative BCK-algebra directed upwards. According to Theorem 3.4 and formula (3.2), we convert it into a positive commutative minimal clan. Then it is easy to see that $S = S_X$ and $+ = +_X$.

Since $m_\omega(f - f \wedge g) = m_\omega(f * g) = (f * g)(\omega) = \max\{0, f(\omega) - g(\omega)\} = \max\{0, m_\omega(f) - m_\omega(g)\}$, $m_\omega$ is a measure-morphism on $X$ and obviously $\mathcal{S}$ is an order determining system. **Q.E.D.**

We prove now the converse statement of Theorem 5.2.

**Theorem 5.3** *Let* $(E;+,S,\leq,0)$ *be a positive commutative minimal clan with an order determining system of measure-morphisms* $\mathcal{S}$*. Let* $X \subseteq [0,1]^{\mathcal{S}}$ *be the system of fuzzy sets* $X = \{\bar{x} \in [0,1]^{\mathcal{S}} : \bar{x}(m) := m(x), m \in \mathcal{S}, x \in E\}$*. Then* $X$ *satisfies* (i), (ii), (iii) *of Theorem* 5.2 *and* $(E;+,S,\leq,0)$ *is isomorphic with* $(X;+_X,S_X,\leq,0)$*, where* $S_X$ *and* $+_X$ *are defined via* (5.1).

**Proof.** The property (i) of Theorem 5.3 is obviously true.

Since $(\bar{x} * \bar{y})(m) = \max\{0, m(x) - m(y)\} = m(x - x \wedge y) = \overline{x - x \wedge y}(m)$ for all $m \in \mathcal{S}$, then $\bar{x} * \bar{y} = \overline{x - x \wedge y}$ and (ii) is satisfied.

The property (iii) follows from $\bar{x}, \bar{y} \leq \overline{x \vee y}$. Using Theorem 5.1, we have that $(X;+_X,S_X,\leq,\bar{0})$ is a positive commutative minimal clan.

Define a mapping $\varphi : E \to X$ via $\varphi(x) := \bar{x}$, $x \in E$. If $(x,y) \in S$ and $x+y=z$, we have, since $m$ is also a measure, $\bar{x}(m)+\bar{y}(m) = m(x)+m(y) = m(z) = \bar{z}(m)$.

Then $(\bar{x},\bar{y}) \in S_X$ and $\varphi(x+y) = \varphi(x) + \varphi(y)$.

Conversely, if $(\bar{x},\bar{y}) \in S_X$, there exists $\bar{z} \in X$ with $\bar{x}(m) + \bar{y}(m) = \bar{z}(m)$ for all $m \in \mathcal{S}$.

Then $m(x) + m(y) = m(z)$ implies $m(x) \leq m(z)$ for all $m \in \mathcal{S}$ and then $x \leq z$.

If $x + y' = z$, from $m(x) + m(y) = m(z)$ and $m(x) + m(y') = m(z)$, it follows $m(y) = m(y')$ for all $m \in S$ and then $y = y'$. Hence, $\varphi$ is an isomorphism in question. **Q.E.D.**

# 6 Minimal Clans with Condition (S) and Δ-L-Semigroups.

In the present section, we show that if a positive commutative minimal clan is with condition (S), then the original partial addition $+$ can be extended to a complete operation $\oplus$ such that $(E;\oplus,\leq,0)$ is a special kind of ordered semigroups, a $\Delta$-l-semigroup, introduced by Weber [Web].

The condition (S) for BCK-algebras was introduced by Iséki [Ise] and studied by several authors. For minimal clans, according to Theorem 3.1, we define the condition (S) as follows.

**Definition 6.1** *A positive commutative minimal clan* $(E;+,S,\leq,0)$ *is with condition* (S) *if, for all* $x,y \in E$*, there exists an element* $x \oplus y \in E$ *such that*

(i) $(x \oplus y) \ominus x \leq y$;

(ii) $z \ominus x \leq y$ *implies* $z \leq x \oplus y$.

It follows immediately that $x \leq x \oplus y$, $y \leq x \oplus y$ and $x \oplus 0 = x = 0 \oplus x$ for any $x, y \in E$.

Every positive commutative minimal clan with a greatest element 1 is with condition (S) and

$$x \oplus y = 1 \ominus ((1 \ominus x) \ominus y)$$

for any $x, y \in E$, where $\ominus$ is defined via (3.1) [MeJu, Thm I.7.16], and $(E; \oplus, 0)$ is never a group under $\oplus$ if $E$ is nontrivial.

It is possible to prove that if $E$ is with condition (S), then $(E; \oplus, \leq)$ is a positive, commutative ordered semigroup with zero 0 [MeJu, Thm I.7.7].

Not any positive commutative minimal clan is with condition (S):

**Example 6.2** *Let* $E = ([0,1); \tilde{+}, S, \leq, 0)$, *where* $\leq$ *is the usual order of real numbers, and* $a \tilde{+} b$ *is defined if and only if the usual addition of real numbers* $a + b < 1$, *then* $a \tilde{+} b := a + b$. *Then* $E$ *is a positive commutative minimal clan without condition* (S).

**Proof.** To verify that, it is easy to see that, e.g. $1/2 \oplus 1/2$ does not exist in $E$. **Q.E.D.**

The following three examples follow from relations between commutative minimal clans and BCK-algebras, Section 3, and from the paper [Dvu], and, therefore their proofs are omitted.

**Example 6.3** *Let the interval of real numbers* $E = [0,\, 1]$ *be ordered by the natural way.*

(i) *Let* $g$ *be a continuous, increasing function* $g : [0,\, 1] \to [0,\, 1]$ *with* $g(0) = 0$ *and* $g(1) = 1$ *(called a generator). Put* $S_g = \{(x, y) \in E \times E : g(x) + g(y) \leq 1\}$, *and define a partial binary operation* $+_g$ *via*

$$x +_g y := g^{-1}(g(x) + g(y)),\ (x, y) \in S_g. \tag{6.1}$$

*Then* $([0,1]; +_g, S_g, \leq, 0)$ *is a positive minimal commutative clan.*

(ii) *A system* $([0,1,];+,S,\leq,0)$ *is a positive commutative minimal clan if and only if there is a continuous, increasing function* $g:\ [0,\,1]\to[0,\,1]$ *with* $g(0)=0$ *and* $g(1)=1$ *such that* $S=S_g$ *and* $a+b=a+_g b$.

*In addition, all such minimal clans are isomorphic and with condition* (S).

**Example 6.4** *Let* $E=[0,\infty)$ *be ordered by the natural way.*

(i) *If* $g:\ [0,\,\infty)\to[0,\,\infty)$ *is an increasing and continuous function with* $g(0)=0$, *then* $([0,\,\infty);+_g,S_g,\leq,0)$ *is a commutative minimal clan, where* $S_g=\{(x,y)\in E\times E:\ g(x)+g(y)\leq\sup\{g(x):\,x\in E\}\}$, *and* $+_g$ *is defined via* (6.1).

*If* $g_0:=id_{[0,\,\infty)}$, *then any* $([0,\,\infty);+_g,S_g,\leq,0)$ *is isomorphic with* $([0,\,\infty);+_{g_0},S_g,\leq,0)$ *if and only if* $g$ *is unbounded. In this case* $S_g=E\times E$, *and it is with condition* (S).

*If* $g$ *is bounded, then* $([0,\,\infty);+_g,S_g,\leq,0)$ *is isomorphic with* $([0,\,1);\tilde{+},S,\leq,0)$, *from Example* 6.2. *In this case it is without condition* (S).

(ii) *If* $([0,\,\infty);+,S,\leq,0)$ *is a positive commutative minimal clan, then there is an increasing and continuous function* $g:\ [0,\infty)\to[0,\,\infty)$ *with* $g(0)=0$ *such that* $+=+_g$, *where* $+_g$ *is defined via* (6.1), *and* $S=S_g$.

**Example 6.5** *Let the interval* $E=[0,\,1)$ *be ordered by the natural way.*

(i) *If* $g:\ [0,\,1)\to[0,\,\infty)$ *is an increasing and continuous function with* $g(0)=0$, *then* $([0,\,\infty);+_g,S_g,\leq,0)$ *is a commutative minimal clan, where* $S_g=\{(x,y)\in E\times E:\ g(x)+g(y)<\sup\{g(x):\,x\in E\}\}$, *and* $+_g$ *is defined via* (6.1).

*If* $g$ *is bounded, then* $([0,\,1);+_g,S_g,\leq,0)$ *is isomorphic with Example* (6.2), *and it is without condition* (S). *If* $g$ *is unbounded, then* $([0,\,1);+_g,S_g,\leq,0)$ *is isomorphic with* $([0,\,\infty);+,E\times E,\leq,0)$, *and it is with condition* (S).

(ii) *Let* $([0,\,1);+,S,\leq,0)$ *be a positive commutative minimal clan. Then there is an increasing and continuous function* $g:\ [0,1)\to[0,\,\infty)$ *with* $g(0)=0$ *such that* $+=+_g$, *where* $+_g$ *is defined via* (6.1), *and* $S=S_g$.

**Proposition 6.6** *Let* $E=(E;+,S,\leq,0)$ *be a positive commutative minimal clan. If there is a total binary operation* $\tilde{\oplus}$ *on* $E$ *such that*

$$(x\ominus y)\ominus z=x\ominus(y\,\tilde{\oplus}\,z),\quad x,y,z\in X,\tag{6.3}$$

*then* $E$ *is with condition* (S), *and*

$$x\oplus y=x\,\tilde{\oplus}\,y,\quad x,y\in X.$$

**Proof.** Due to the property (6.3), we have $((x \tilde{\oplus} y) \ominus x) \ominus y = (x \tilde{\oplus} y) \ominus (x \tilde{\oplus} y) = 0$, so that $(x \tilde{\oplus} y) \ominus x \leq y$. If now $z \ominus x \leq y$, then $z \ominus (x \tilde{\oplus} y) = (z \ominus x) \ominus y = 0$, which shows that $z \leq x \tilde{\oplus} y$ and, consequently, $E$ is with condition (S). **Q.E.D.**

A non-empty family $E \subseteq 2^{\Omega}$, $\Omega \neq \emptyset$, is said to a *difference algebra* if $A, B \in E$ implies $A \setminus B \in E$. A difference algebra is closed under the set-theoretical intersection, and it is called a $\cup$-*algebra*, if $A \cup B \in E$, whenever $A \cap B = \emptyset$, $A, B \in E$. Then $A \cup B \in E$ for all $A, B \in E$.

It is clear that $(E; \setminus, \emptyset)$ is a commutative BCK-algebra, and $(E; \cup, S, \subseteq, \emptyset)$, where $S = \{(A, B) \in E \times E : A \cap B = \emptyset\}$, is a positive commutative minimal clan. Any Boolean ring is isomorphic with a $\cup$-algebra of subsets, and it is with condition (S). In the case

$$A \oplus B = A \cup B, \quad A, B \in E.$$

**Theorem 6.7** *Let $E = (E; +, S, \leq, 0)$ be a positive commutative minimal clan with condition* (S)*, and let $x, y, z \in E$. The following assertions are equivalent:*

(i) $x \leq y$ *implies* $x \oplus y = y$.

(ii) $(x \oplus y) \oplus y = x \oplus y$.

(iii) $x \oplus x = x$.

(iv) $(x \oplus y) \ominus z = (x \ominus z) \oplus (y \ominus z)$.

(v) $x \oplus y = x \vee y$.

(vi) $x \oplus y = x \oplus (y \ominus x)$.

(vii) $(x \ominus y) \ominus y = x \ominus y$.

(viii) *$E$ is isomorphic to some $\cup$-algebra of subsets of some $\Omega \neq \emptyset$.*

**Proof.** Using relations between the upwards directed commutative BCK-algebra $(E; \ominus, 0)$ and the positive commutative minimal clan $(E; +, S, \leq, 0)$, Section 3, we have that (i)–(vii) are equivalent [MeJu, Thm I.7.12], and they give an isomorphism of $(E; \ominus, 0)$ with some difference algebra [MeJu, Thm VII.2.7]. Consequently, we have the equivalency of (viii) with (i)–(vii). **Q.E.D.**

We recall that similarly as in Example 3.6, any difference algebra $E$ can be embedded into the algebra $\mathcal{A}(E)$, and, consequently, into some lattice ordered group, which gives a partial positive answer to the problem posed in the end of Section 3.

The notion of $\Delta$-l-semigroups is due to Weber. According to [Web], a structure $(P;+,\Delta,\leq\ )$ is called a $\Delta$-l-*semigroup* if $(P;+)$ is a commutative semigroup, $(P;\leq)$ is a lattice, and $\Delta:(x,y)\to y\Delta x$ is a map from $\{(y,x)\in P\times P: x\leq y\}$ into $P$ such that for all $x,y,z\in P$ the following properties hold:

(P1) $x\leq x+y$;

(P2) $x\leq y\leq z+x$ implies $y\Delta x\leq z$;

(P3) $y=(y\Delta x)+x$ if $x\leq y$.

Examples of $\Delta$-l-semigroups are Boolean rings, the positive cone of lattice ordered groups, MV-algebras.

The following properties of $\Delta$-l-semigroups are proved in [Web]:

If $P$ is a $\Delta$-l-semigroup, $P$ has a smallest element 0, and, for all $x,y,z,t\in P$, we have

(a) $x\Delta x=0$ and $0+x=x=x+0$;

(b) if $x\leq y$, then $x+z\leq y+z$;

(c) if $t\leq x\leq y\leq z$, then $y\Delta x\leq z\Delta t$;

(d) if $y\leq z$ and $x\leq z\Delta y$, then $x+y\leq z$ and $z\Delta(x+y)=(z\Delta y)\Delta x$;

(e) $(x\vee y)\Delta x=y\Delta(x\wedge y)$.

Using $\Delta$-l-semigroups, we obtain a representation theorem for minimal clans with condition (S).

**Theorem 6.8** *Let* $(E;+,S,\leq,0)$ *be a positive commutative minimal clan with condition* (S) *and define* $\oplus$ *via condition* (S)*. Then* $\oplus$ *is an extension of* $+$*, i.e., for* $x,y\in E$ *if* $x+y$ *exists in* $E$*, then*

$$x+y=x\oplus y,$$

*and* $(E;\oplus,\leq,0)$ *is a positive po-semigroup, in which any element is residuable, and*

$$x \div y = x \ominus y, \quad x,y \in E, \tag{6.2}$$

*where* $\ominus$ *is defined via* (3.1). *Moreover,* $(E;\oplus,\leq,\Delta)$, *where* $y\Delta x = y - x$ *for* $x \leq y$, *is a* $\Delta$-*l-semigroup.*

**Proof.** Suppose that $x+y$ exists and $z\ominus x \leq y$ for some $z \in E$. Then $x\wedge y + z$ exists and, by (MC-4), $z = x\wedge z + z\ominus x \leq x\wedge z + y \leq x+y$. From $(x+y)\ominus x = y$, it follows $x\oplus y = x+y$. Due to Theorem 3.1 and (BCK-2), we have $x\ominus(x\ominus y) \leq y$ so that $x \leq y\oplus(x\ominus y)$. Let now $z\oplus y \geq x$ for some $z \in E$, then $z \geq (z\oplus y)\ominus y \geq x\ominus y$ which yields $x\div y = x\ominus y$.

We know that $(E;\leq)$ is a lattice and $(E;\oplus)$ is a commutative semigroup (Theorem 3.4 and [MeJu, Thm I.7.7]).

(P1) follows from definition of $\oplus$. To prove (P2), let $x \leq y \leq z\oplus x$. By (a) of Lemma 2.5, $y - x \leq z\oplus x - x \leq z$, that is $y\Delta x \leq z$.

To show (P3), let $x \leq y$. Then $y = (y-x)+y = y\Delta x \oplus x$. **Q.E.D.**

The converse statement is also true.

**Theorem 6.9** *Let* $(E;+,S,\leq,0)$ *be a positive and commutative minimal clan and let* $+$ *have an extension* $\oplus$ *such that* $(E;\oplus,\leq,\Delta)$, *where* $y\Delta x = y-x$ *for* $x \leq y$, *is a* $\Delta$-*l-semigroup. Then* $(E;+,S,\leq,0)$ *is with condition* (S).

**Proof.** It follows from (P1) that $x\oplus y \geq x$ for every $x,y \in E$. Then, by (P2), $(x\oplus y)\ominus x = (x\oplus y)\Delta x \leq y$. If, for $z \in E$, $z\ominus x \leq y$, by property (b), $z = x\wedge z + z\ominus x = x\wedge z \oplus z\ominus x \leq x\wedge z\oplus y \leq x\oplus y$ and hence $E$ has properties (i) and (ii) of Definition 6.1. **Q.E.D.**

It is worth to recall that the condition (S) in Theorem 6.8 allows us to use the same set $E$ for representation the positive minimal minimal clan $E$ in the semigroup $(E;\oplus,\leq,0)$ via Theorem 4.4 because any element in $E$ is residuable.

**Proposition 6.10** *Let* $E = (E;+,S,\leq,0)$ *be a positive commutative minimal clan with condition* (S).

(i) *If* $\bigvee_i a_i$ *exists in* $E$, *then, for any* $b \in E$, $\bigvee_i (a_i\ominus b)$ *also exists, and*

$$\bigvee_i (a_i\ominus b) = (\bigvee_i a_i)\ominus b.$$

(ii) *If* $\bigwedge_i b_i$ *exists in* $E$*, then so does* $\bigvee_i(a \ominus a_i)$ *for any* $a \in E$*, and*

$$\bigvee_i (a \ominus a_i) = a \ominus (\bigwedge_i b_i).$$

(iii) *If* $\bigwedge_i a_i$ *exists in* $E$*, so does* $\bigwedge_i(a_i \oplus b)$ *for any* $b \in E$*, and*

$$(\bigwedge_i a_i) \oplus b = \bigwedge_i (a_i \oplus b),$$

*and* $(E; \oplus, \leq, 0)$ *is a lattice ordered semigroup.*

(iv) $$(a \vee b) \oplus (a \wedge b) = a \oplus b, \quad a, b \in E.$$

**Proof.** (i) Let $c = (\bigvee_i a_i) \ominus b$. Then $c \oplus b \geq \bigvee_i a_i \geq a_i$ and $c \oplus b \geq a_i$ for any $i$. Hence, $c \geq a_i \ominus b$.

If now $x \geq a_i \ominus b$, then $x \oplus b \geq a_i$ and $x \oplus b \geq \bigvee_i a_i$, so that $x \geq (\bigvee_i a_i) \ominus b$.

Similarly we prove (ii).

(iii) It is clear that $(\bigwedge_i a_i) \oplus b \leq a_i \oplus b$ for any $i$. Let $x \leq a_i \oplus b$ for any $i$. Then $x \ominus b \leq a_i$, $x \ominus b \leq \bigwedge_i a_i$ and $x - x \wedge b \leq \bigwedge_i a_i$ which gives

$$\begin{aligned} x \ominus b &\leq a_i, \\ x \ominus b &\leq \bigwedge_i a_i, \\ x - x \wedge b &\leq \bigwedge_i a_i, \\ (x - x \wedge b) + (x \wedge b) &\leq (\bigwedge_i a_i) \oplus (x \wedge b) \leq (\bigwedge_i a_i) \oplus b, \\ x &\leq (\bigwedge_i a_i) \oplus b. \end{aligned}$$

(iv) According to (MC-6), we have

$$\begin{aligned} (a \vee b) - a &= b - (a \wedge b), \\ ((a \vee b) - a) \oplus a &= (b - (a \wedge b)) \oplus a, \\ ((a \vee b) - a) + a &= (b - (a \wedge b)) \oplus a, \\ a \vee b &= (b - (a \wedge b)) \oplus a, \\ (a \vee b) \oplus (a \wedge b) &= (b - (a \wedge b)) \oplus (a \wedge b) \oplus a, \\ (a \vee b) \oplus (a \wedge b) &= a \oplus b. \end{aligned}$$

**Q.E.D.**

**Theorem 6.11** *Let* $(P; +, \leq, \Delta)$ *be a* $\Delta$*-l-semigroup such that if* $x \leq y$*, then* $y\Delta(y\Delta x) = x$*. Let* $0$ *be the smallest element in* $P$*. Let us define a partial binary operation* $\oplus$ *in* $P$ *in such a way that* $a \oplus b$ *exists and* $a \oplus b = c$ *if and only if* $b \leq c$ *and* $c\Delta b = a$*, and let* $S_P$ *be the set of all such pairs* $(a, b)$*.*

*Then* $\oplus$ *is well-defined and* $(P; \oplus, 0, \leq)$ *is a positive commutative minimal clan with condition* (S)*.*

**Proof.** To prove that $\oplus$ is well-defined, let us suppose that $a \oplus b = c$ and $a \oplus b = c'$. By definition of $\oplus$, $b \leq c, b \leq c'$ and $c\Delta b = a = c'\Delta b$. Then by (P3), $c = (c\Delta b) + b = (c'\Delta b) + b = c'$.

The operation $+$ is a total extension of $\oplus$. Indeed, if $a \oplus b$ exists, $a = (a \oplus b)\Delta b$ and, by property (P3), $a \oplus b = [(a \oplus b)\Delta b] + b = a + b$.

The partial addition is commutative. In fact, if $a \oplus b = c$, it is $b \leq c$ and $c\Delta b = a$. By (P1), $c + b \geq c \geq b$ and, by (P2), $a = c\Delta b \leq c$. Since, by assumptions on $P$, $c\Delta a = c\Delta(c\Delta b) = b$, $b \oplus a$ exists and $b \oplus a = a \oplus b$.

The axiom (MC-5) is obviously true. (MC-1) follows from property (a) of $\Delta$-l-semigroups. (MC-6) follows immediately from (e).

To prove (MC-2), let us suppose that $a \oplus b$ and $(a \oplus b) \oplus c$ exist. Then there exist $z, z' \in P$ such that $a \leq z$ and $z\Delta a = b$, $z = a \oplus b \leq z'$ and $z'\Delta(a \oplus b) = c$.

Since $a \leq z \leq z'$, by property (c) $b = z\Delta a \leq z'\Delta a$. By property (d), $(z'\Delta a)\Delta b = z'\Delta(a + b) = z'\Delta(a \oplus b) = c$, then $b \oplus c$ exists and $b \oplus c = z'\Delta a$. From the last identity we have that $a \oplus (b \oplus c)$ exists and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

To show (MC-3), let $z \oplus b = z \oplus a = c$. Then $b \leq c$ and $c\Delta b = z, a \leq c$ and $c\Delta a = z$. Hence $b = c\Delta(c\Delta b) = c\Delta(c\Delta a) = a$.

For (MC-4), let us suppose that $a \leq b$ and $z \oplus a, z \oplus b$ exist for some $z$ in $P$. By definition of $\oplus$, $(z \oplus a)\Delta a = z = (z \oplus b)\Delta b$. Then by (P3) and property (b) $z \oplus a = [(z \oplus a)\Delta a] + a \leq [(z \oplus a)\Delta a] + b = [(z \oplus b)\Delta b] + b = z \oplus b$.

Finally, we have to prove the condition (S). Let us observe that in the positive commutative minimal clan $(P; \oplus, \leq, 0)$, by definition of subtraction, $x - y$ is defined and $z = x - y$ if and only if $y \leq x$ and $x\Delta y = z$. In particular, $x \ominus y = x - x \wedge y = x\Delta(x \wedge y)$. Let now $z \in P$ such that $z \ominus x \leq y$. Then, by (b), $z = z \wedge x \oplus (z \ominus x) = z \wedge x + (z \ominus x) \leq z \wedge x + y \leq x + y$.

Moreover, since $x \leq x + y$ by (P1), it follows from (P2) that $y \geq (x + y)\Delta x = (x + y) \ominus x$. **Q.E.D.**

**Acknowledgement.** One of the authors, A.D., is grateful to Mathematical Department of the University “Federico II” of Naples for their hospitality

during his stay in Oct.–Nov. 1996, when the present paper has been initiated.

# References


[Bir] G. Birkhoff, *Lattice Theory*, Amer. Math. Soc. Coll. Publ, Vol. **25**, Providence, Rhode Island, 1967.

[Cha] C.C. Chang, Algebraic analysis of many valued logics, Trans. Amer. Math. Soc. **88** (1958), 467–490.

[Con] C. Constantinescu, *Some Properties of Spaces of Measures*, Suppl. Atti Sem. Mat. Fis. Univ. Modena **35** (1989), 1–289.

[Dvu] A. Dvurečenskij, Measures and states on BCK-algebras, Atti Sem. Mat. Fis. Univ. Modena (1997).

[Dvu 1] A. Dvurečenskij, Fuzzy set representation of some quantum structures, Fuzzy Sets and Systems, (1997).

[Fle] I. Fleischer, Every BCK-algebra is a set of residuables in an integral pomonoid, J. Algebras **119** (1988), 360–365.

[ImIs] Y. Imai, K. Iséki, On axiom systems of propositional calculi, Proc. Japan Acad. **42** (1966), 19–22.

[Ise] K. Iséki, A special class of BCK-algebras, Math. Seminar Notes **5** (1979), 107–119.

[MeJu] J. Meng, Y.B. Jun, *BCK-algebras,* Kyung Moon Sa Co., Seoul, 1994.

[Mun] D. Mundici, Interpretation of AF $C^*$-algebras in Lukasiewicz sentential calculus, J. Funct. Anal. **65** (1986), 15–63.

[Mun 1] D. Mundici, MV-algebras are categorically equivalent to bounded commutative BCK-algebras, Math. Japonica, **6** (1986), 889–894.

[Pul] S. Pulmannová, On connection among some orthomodular structures, Preprint MÚ SAV, Bratislava (1996).

[Fle] K.D. Schmidt, Minimal clans: A class of ordered partial semigroups including Boolean rings and lattice-ordered groups. In: Semigroups – Theory and Applications (Oberwolfach 1986). LNM **1320**, Springer-Verlag, Berlin, Heidleberg, New York, 1988, pp. 300–341.

[Fle 1] K.D. Schmidt, *Jordan Decompositions of Generalized Vector Measures*, Longman Scientific & Technical, Longman group UK Limited, Essex, England, 1989.

[Web] H. Weber, An abstraction of clans of fuzzy sets, preprint Univ. Udine (1996).

[Wyl] O. Wyler, Clans, Compos. Math. **17** (1966/67), 172–189.

This preprint series was founded in 1996. Its purpose is to present manuscripts of submitted or unpublished papers and reports of fellows of the Mathematical Institute of the Slovak Academy of Sciences in Bratislava and its Košice branch. The authors are fully responsible for the content of the preprints. Postal addresses:

*Mathematical Institute*
*Slovak Academy of Sciences*
*Štefánikova 49*
*SK–814 73 Bratislava*
*SLOVAKIA*

Tel./Fax (+421 7) 5249 7316

*Mathematical Institute*
*Slovak Academy of Sciences*
*Grešákova 6*
*SK–040 01 Košice*
*SLOVAKIA*

Tel./Fax (+421 95) 6228291

PostScript files of preprints are available at WWW address:
`http://www.mat.savba.sk/preprints`

## Latest preprints of the series:

**3/1997** *Anatolij Dvurečenskij, Maria Gabriella Graziano*: Remarks on representations of minimal clins

**2/1997** *Ivan Korec*: Elementary definability from Pascal's triangle modulo $p$ and the set of $e$-th powers

**1/1997** *Ján Jakubík*: Lexicographic products of half linearly ordered groups

**34/1996** *Emília Halušková*: Direct limits of monounary algebras

**33/1996** *Ivan Korec*: A list of arithmetical structures strongest with respect to the first order definability

**32/1996** *Ivan Korec*: Definability of Pascal's triangles modulo 4 and 6 and some other binary operations from their associated equivalence relations

**31/1996** *Anatolij Dvurečenskij*: Measures and states on BCK-algebras

**30/1996** *Ján Haluška*: Uncertainty and tuning in music

**29/1996** *Ján Haluška*: On numbers 256/243, 25/24, 16/15

**28/1996** *Ján Haluška*: COMMA 32 805/32 768

**27/1996** *Beloslav Riečan*: On the sum of observables in MV algebras of fuzzy sets

**26/1996** *Beloslav Riečan*: On the strong law of large numbers for weak observables in MV algebras

**25/1996** *Ivan Korec*: Algebraical constructions of reversible generalized Pascal triangles and one–dimensional cellular automata

**24/1996** *Peter Vojtáš*: Uncertain reasoning with floating connectives

**23/1996** *Ivan Korec*: Definability of addition from multiplication and neighbourhood relation and some related results

**22/1996** *Ivan Korec*: Open problems more or less related to generalized Pascal triangles

**21/1996** *Sylvia Pulmannová*: On connections among some orthomodular structures

**20/1996** *Pekka Lahti, Sylvia Pulmannová*: Coexistent observables and effects in quantum mechanics

**19/1996** *Ivan Korec*: Definability of arithmetical operations from Pascal's triangle modulo $n$ and arithmetical functions

**18/1996** *Ivan Korec*: Undecidability and uniform definability in classes of structures related to Pascal's triangles modulo $n$

**17/1996** *Beloslav Riečan*: Probability theory on some ordered structures

**16/1996** *Beloslav Riečan*: Weak observables in MV algebras

**15/1996** *Anatolij Dvurečenskij, Hee Sik Kim*: On connections between BCK-algebras and difference posets

*(continued on inside back cover)*

**14/1996** *Anatolij Dvurečenskij, Karl Svozil*: Product of partition logics, orthoalgebras and automata

**13/1996** *Robin L. Hudson, Sylvia Pulmannová*: Chaotic expansion of elements of the universal enveloping algebra of a Lie algebra associated with a quantum stochastic calculus

**12/1996** *Sylvia Pulmannová*: Congruences in partial abelian semigroups

**11/1996** *Karol Nemoga, Štefan Schwarz*: An explicit description of the set of all normal bases generators of a finite field

**10/1996** *Ján Jakubík*: Subdirect product decompositions of $MV$-algebras

**9/1996** *Ivan Korec*: Reversibility in generalized Pascal triangles and binary reversibility in one-dimensional cellular automata

**8/1996** *Jozef Bobok, Milan Kuchta*: X-minimal patterns and generalization of Sharkovskiĭ's theorem

**7/1996** *Jarmila Hedlíková, Sylvia Pulmannová*: Generalized difference posets and orthoalgebras

**6/1996** *Ján Borsík, Roman Frič*: Pointwise convergence fails to be strict

**5/1996** *Miroslav Ploščica*: Affine completions of distributive lattices

**4/1996** *Ján Haluška*: Equal temperament and Pythagorean tuning: a geometrical interpretation in the plane

**3/1996** *Winfried Just, Peter Vojtáš*: On matrix rapid filters

**2/1996** *Anatolij Dvurerečenskij*: Fuzzy set representations of some quantum structures

**1/1996** *Miloslav Duchoň, Beloslav Riečan*: On the product of semigroup valued measures