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MOMENT PROBLEM IN SOME ORDERED VECTOR SPACES

Camille Debiève - Miloslav Duchoň - Beloslav Riečan

Abstract. We present a moment problem in the context of (boundedly) complete vector lattices. A generalization of the Hausdorff moment theorem is formulated and proved.

1. Introduction

If g is a real-valued function of bounded variation on the unit interval I of the real line, the numbers

$$a_k = \int_0^1 t^k dg(t), \qquad k \in N.$$

are called the (power) moments of g.

For every sequence of real numbers $a_k, k \in N$ and every pair of non-negative integers n, k, set

$$\Delta^n a_k := \sum_{j=0}^{n} (-1)^j \binom{n}{j} a_{k+j}$$

The sequence $(a_k, k \in N)$ is called completely monotone if $\Delta^n a_k \geq 0$ for all integers $n, k \geq 0$. For a sequence (a_k) of real numbers to be the moment sequence of some non-decreasing function g (this case being of particular interest), it is necessary and sufficient that (a_k) be completely monotone ([3],[5]). The result permits a generalization to the situation where (a_k) is a completely monotone sequence of elements of a (Dedekind) complete vector lattice the definition of completely monotone sequence being the same.

In the paper [2] we considered the same question in the case of a (boundedly) σ-complete, weakly σ-distributive vector lattice V satisfying the following condition: Any interval $[a, b] \subset V$ is sequentially o-compact, i.e. to every sequence $(a_i) \subset [a, b]$ there is an o-convergent subsequence (a_{n_i}). In this paper we do not use these assumptions. On the other side we use here the representation of an order bounded linear operator on the space of continuous functions on the interval $[a, b]$ of real line into a (Dedekind) complete vector lattice Y ([1]).

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2. Moment problem theorem

Let Y be a (Dedekind) complete vector lattice. Denote by $L^o(X,Y)$ the vector space of all $o-$bounded operators on the normed space X into Y, that is of those U that, if $U \in L^o(X,Y)$, then $\{U(x); \|x\| \leq 1\}$ is an $o-$bounded subset of Y. For $U \in L^o(X,Y)$ we put

$$\bigl| U \bigr| = \sup\{|U(x)|; \|x\| \leq 1\}$$

If we take into consideration a general form of an order bounded linear operator on the space $C[a,b]$ into Y we can formulate a task in the considered case as follows: Decide under which conditions there exists such a non-decreasing function $g(t) : [a,b] \to Y$ such that

$$\int_a^b t^n dg(t) = y_n, \quad n = 0, 1, \dots$$

Recall that a function g, defined on $T = [a,b]$ and taking values in Y, is said to be of (o)-bounded variation, if the set of all elements of the form

$$\sum_j |g(t_{j+1}) - g(t_j)|,$$

corresponding to all finite partitions of the interval T, is o-bounded. We shall denote by $(o) - var_{t\in T} g(t)$ the least upper bound of this set.

We shall need the following result ([1], VII.1.5).

The general form of the (o)-bounded linear operator $U : C(T) \to Y$ is given by the formula

$$U(f) = \int_T f(t) dg(t)$$

where $g : T \to Y$ is a function of $(o)-$bounded variation.

We shall derive a concrete result relating to the power moment problem in the interval $[0,1]$.

Theorem. *In order that there exists a non-decreasing function $g(t) : [0,1] \to Y$ such that*

$$\int_0^1 t^n dg(t) = y_n, \quad y_n \in Y, \quad n = 0, 1, \dots \tag{1}$$

it is necessary and sufficient that

$$\Delta^n y_k \geq 0, \quad n, k = 0, 1, \dots, \tag{2}$$

where $\Delta^m y_k$ denotes the m-th differences for the sequence (y_k) defined inductively by equalities

$$\Delta^{m+1} y_k = \Delta^m y_k - \Delta^m y_{k+1}, \quad \Delta^0 y_k = y_k,$$

$$m = 0, 1, \dots; k = 0, 1, \dots. \tag{3}$$

Proof. The necessity.

Let the moment problem (1) be solved. Denote by L an order bounded linear operator on the space $C[0,1] \to Y$ generated by a non-decreasing function (hence of order bounded variation, [1]), $g(t) : [0,1] \to Y$. Put

$$x_k^{(m)}(t) = t^k(1-t)^m, \quad m,k = 0,1,\dots. \tag{4}$$

Since

$$x_k^{(m+1)}(t) = t^k(1-t)^{m+1} = t^k(1-t)^m - t^{k+1}(1-t)^m =$$
$$= x_k^{(m)}(t) - x_{k+1}^{(m)}(t),$$

we have

$$L(x_k^{(m+1)}) = L(x_k^{(m)}) - L(x_{k+1}^{(m)}), \quad m,k = 0,1,\dots.$$

Further

$$L(x_k^{(0)}) = \int_0^1 t^k dg(t) = y_k, \quad k = 0,1,\dots.$$

If we take into the consideration (3), we can easily see (by induction) that

$$L(x_k^{(m)}) = \int_0^1 t^k(1-t)^m dg(t) = \Delta^m y_k, \quad m,k = 0,1,\dots.$$

From this we deduce that $\Delta^m y_k \geq 0, \quad m,k = 0,1,\dots$, i. e. (2) is satisfied.

The sufficiency.

Let L_0 denote the operator defined on the set of functions (x_n), $x_n(t) = t^n$, $n = 0,1,\dots$ into Y by the formula

$$L_0(x_n) = y_n, \quad n = 0,1,\dots.$$

Extend L to the linear hull of the set (x_n), i. e. to the set of all polynomials. Namely if $x(t) = c_0 + c_1 t + \cdots + c_n t^n$, we put

$$L(x) = c_0 y_0 + c_1 a_1 = \cdots + c_n y_n.$$

Since the functions x_n are linearly independent the definition of L will be unique.

The operator L defined above will be clearly additive and homogeneous.

We shall see that the condition (2) implies that L is positive and order bounded on the vector space of all polynomials.

Let us note that (even without the condition (2)) the operator L is order bounded on the set P_m of all polynomials whose degree is $\leq m$, because P_m is finite-dimensional space (as coordinates we take coefficients of polynomial), hence the convergence (boudedness) in P_m is coordinatewise.

We have

$$L(x_k^{(s)}) = \Delta^s y_k, \quad k,s = 0,1,\dots.$$

Take any polynomial $p(t)$. Let the degree of $p(t)$ be equal to m. Let us take the sequence of corresponding Berstein polynomials (of $p(t)$),

$$p_n(t) = B_n(p;t) = \sum_{k=0}^n \binom{n}{k} p(\frac{k}{n}) t^k (1-t)^{n-k}.$$

It is well-known that the degree of the polynomial $p_n(t)$ for any $n = 1, 2, \ldots$ is not greater than m, and since $p_n(t)$ uniformly converge to $p(t)$ for $n \to \infty$, we have (according to remarks above) $L(p_n) \to L(p)$ for $n \to \infty$. But

$$|L(p_n)| \leq \sum_{k=0}^{n} \binom{n}{k} |p(\frac{k}{n})||L(x_n^{(n-k)})| \leq$$

$$||p|| \sum_{k=0}^{n} \binom{n}{k} \Delta^{n-k} y_k = ||p|| y_0.$$

If we take the limit on the left side for $n \to \infty$, we obtain

$$|L(p)| \leq ||p|| y_0.$$

Since p is an arbitrary polynomial, we have proved that L is order bounded on the space of all polynomials. Now we may extend L to the whole space $C[0,1]$, since polynomials are uniformly dense in $C[0,1]$ ([1],VI.3.3). We may now use for this extension the theorem concerning general form of the order bounded linear operator on the space $C[0,1]$ ([1], VII.1.5) according to which

$$L(f) = \int_0^1 f(t) dg(t), \quad f \in C[0,1],$$

where $g(t) : [0,1] \to Y$ is a function of order bounded variation. In particular

$$y_n = L(x_n) = \int_0^1 t^n dg(t), \quad n = 0, 1, \ldots$$

We may further extend L to characteristic functions of all intervals in $[0,1]$. Since by assumption $0 \leq \Delta^k y_n, \quad k, n = 0, 1, \ldots,$ we can see that L is positive for positive f. By definition $g(t) = L(c_{[0,t]})$, hence we can see that g is non-decreasing function.

We may reformulate the preceeding theorem in the form

Theorem bis. *A sequence $(a_k) \subset Y$ is the moment sequence of a non-decreasing function g on $[0,1]$ into Y if and only if (a_k) is completely monotone.*

Remark. We would like to mention that moment theorem can be applied to obtain a new proof of spectral theorem for Hermitian operators. Cf. [4], where moment theorem is considered in ordered locally convex spaces with some properties.

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