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On Representations of Commutative BCK-algebras

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Abstract

We introduce a relative cancellation property for commutative BCK-algebras and we study the problem of embedding of commutative BCK-algebras into Abelian lattice ordered groups. We show that if a BCK-algebra is not upwards directed, then we cannot use the method of Wyler and Baer. Anyway, supposing either union property or unitarily extendibility, we can present the embedding of such a BCK-algebra into the positive cone of an Abelian lattice ordered group with universal property. Finally, some interesting examples are presented.

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1 Introduction

BCK-algebras were introduced by Imai and Iséki [ImIs], [Ise], and they have been intensively studied by several authors ([MeJu] is an interesting monograph about BCK-algebras). A very important class of BCK-algebras is the class of commutative BCK-algebras [RoTr], and in particular, of bounded commutative BCK-algebras which are categorically equivalent to MV-algebras [Mun 1]. For MV-algebras there is a representation theorem by Mundici [Mun] via intervals $[0, u]$ with a strong unit u in an Abelian lattice ordered group. In this case we have also a universal property.

If a commutative BCK-algebra is upwards directed, its representation via Abelian lattice ordered groups with universal property was made in [DvGr].

In the present paper, we shall study a representation of a class of commutative BCK-algebras which are not upwards directed, in general, via lattice ordered groups with universal property. We show that the "words" method of Baer [Bae], and applied by Wyler [Wyl] to clans, does not work in the case of commutative BCK-algebras which are not upwards directed.

More information we obtain if we shall study commutative BCK-algebras with relative cancellation property.

The paper is organized as follows. In section 2, we give some lattice-like properties of commutative BCK-algebras and we introduce relative cancellation property which is a central notion. Some Riesz properties for commutative BCK-algebras known in lattice ordered groups are presented in Section 3. A "words" technique of Baer [Bae] and Wyler [Wyl] is presented in Section 4. Finally, we shall study union BCK-algebras (Section 5) and union BCK-algebras which are unitarily extendible [Hoo] (Section 6). For them we find a lattice ordered group representation with universal property. These results generalize the famous result of Mundici [Mun] for MV-algebras. The paper is accomplished with plenty of illustrating examples.

The relative cancellation property is a necessary condition for a representation of commutative BCK-algebras via Abelian lattice ordered groups. In generally, the problem of representation of commutative BCK-algebras with this property seems to be open, and the authors hope to solve it.

Definition 1.1 A *BCK-algebra* is a non-empty set X with a binary operation $*$ and with a constant element 0 such that the following axioms are satisfied: for all x, y, $z \in X$, (BCK-1) $((x * y) * (x * z)) * (z * y) = 0$;

(BCK-2) $(x * (x * y)) * y = 0$;

(BCK-3) $x * x = 0$;

(BCK-4) $x * y = 0$ and $y * x = 0$ imply $x = y$;

(BCK-5) $0 * x = 0$.

We shall write $X = (X; *, 0)$. In X, there is a binary relation $\leq$ defined by

$$x \leq y \quad \text{iff} \quad x * y = 0.$$

This is a partial order relation on X which is called the BCK-*order*. We recall some elementary properties of BCK-algebras [MeJu, Thms I.1.2, I.1.3, I.1.4]: if $(X; *, 0)$ is a BCK-algebra, then for all x, y, $z \in X$,

(a) $x \leq y$ implies $z * y \leq z * x$;

(b) $(x * y) * z = (x * z) * y$;

(c) $(x * z) * (y * z) \leq x * y$;

(d) $x \leq y$ implies $x * z \leq y * z$;

(e) $x * y \leq x$;

(f) $x * 0 = x$.

If we denote by

$$x \wedge y := y * (y * x), \quad x, y \in X,$$

$x \wedge y$ is a lower bound of x and y, $x \wedge x = x$, $x \wedge 0 = 0 = 0 \wedge x$, and it is true that

$$x * (y \wedge x) = x * y, \quad x, y \in X. \tag{1.1}$$

In general, $x \wedge y \neq y \wedge x$, and a BCK-algebra is said to be *commutative* if it satisfies, for all x, $y \in X$,

$$x * (x * y) = y * (y * x), \quad x, y \in X. \tag{1.2}$$

In this case, $x \wedge y = y \wedge x$ is the greatest lower bound of x and y, and $(X; *, 0)$ is a lower semilattice with respect to the BCK-order. According to [MeJu, Thm I.5.6], a BCK-algebra X is commutative if and only if $x \leq y$ implies $x = y * (y * x)$.

2 Lattice Properties of Commutative BCK-algebras

We recall that it follows from [CST] that any commutative BCK-algebra is a distributive near lattice, and to be self-contained, for reader's convenience we prove the following result.

Theorem 2.1 *Let $(X;*,0)$ be a commutative BCK-algebra, and let $x,y,z,z_1 \in X$.*

(i) *Let $x,y \le z$. Define*

$$x \vee_z y := z * ((z*x) \wedge (z*y)). \tag{2.1}$$

Then $x \le x \vee_z y$, $y \le x \vee_z y$ and, for any $z_1 \in X$ with $z_1 \ge z$, we have

$$x \vee_z y = x \vee_{z_1} y. \tag{2.2}$$

(ii) *$x \vee y$ exists in X if and only if there exists an element $z \in X$ such that $x, y \le z$.*

(iii) *If $x \vee y$ exists in X, then $(z\wedge x) \vee (z \wedge y)$ exists in X for any $z \in X$, and*

$$z \wedge (x \vee y) = (z \wedge x) \vee (z \wedge y). \tag{2.3}$$

(iv) *If $(X;*,0)$ is directed upwards,*[3] *then X is a distributive lattice with respect to the BCK-order.*

Proof. (i) Since X is commutative, then it is a lower semilattice, and $x \wedge y = x*(x*y) = y*(y*x)$, [MeJu, Thm I.5.5]. Let us define

$$x \vee_z y := z * ((z*x) \wedge (z*y))$$

if x, $y \le z$. Since $x \vee_z y = z*((z*x)\wedge(z*y)) \ge z*(z*x) = x$ and similarly $x \vee_z y \ge y$, $x\vee_z y$ is a majorant of x and y. If $x, y \le z_1$ and $z \le z_1$, we have: $z_1 * x \ge z_1 * z$ and $z_1 * y \ge z_1 * z$ so that $z_1 * z \le (z_1 * x) \wedge (z_1 * y)$, and hence $z = z_1 * (z_1 * z) \ge z_1 * ((z_1*x)\wedge(z_1*y)) = x \vee_{z_1} y$.

[3] A poset $(X;\le)$ is said to be *directed upwards* if given $a, b \in X$, there is an element $c \in X$ such that $a \le c$ and $b \le c$.

Since $x \vee_z y \leq z \leq z_1$ and $x, y \leq x \vee_z y$, in the same way we see that $x \vee_z y \geq x \vee_{z_1} y$. From $x, y \leq x \vee_{z_1} y \leq z$ using similar arguments, it follows that $x \vee_{z_1} y \geq x \vee_z y$ and therefore $x \vee_z y = x \vee_{z_1} y$, which proves (2.2).

(ii) One direction is evident. Suppose now $x \leq z$ and $y \leq z$ for some element $z \in X$. In view of (i), $x \vee_z y$ is an upper bound of x and y. Assume that $v \geq x, y$ for some $v \in X$. Then $w := v \wedge z \geq x, y$, and by (i),

$$x, y \leq x \vee_z y = x \vee_w y \leq w = v \wedge z \leq v.$$

This means that $x \vee_z y$ is the least upper bound of x and y in X.

(iii) Let us suppose that $u := x \vee y$ exists in X and let z be an arbitrary element in X. Put $X_u : \{v \in X : v \leq u\}$. Then $(X_u; *, 0)$ is a bounded commutative BCK-subalgebra [4] of X. Define $z_0 := z \wedge u$, then $z_0 \wedge u = z \wedge u$, $z_0 \wedge x = z \wedge u \wedge x = z \wedge x$, $z_0 \wedge y = z \wedge u \wedge y = z \wedge y$, and $x, y, z_0,\ z_0 \wedge x, z_0 \wedge y \in X_u$. Since X_u is a bounded commutative BCK-algebra, the infimum of two elements from X_u taken in it is equal to the infimum taken in the whole X, and, in addition

$$x \vee y = x \vee_u y,$$

where $\vee_u$ is defined via (2.1) for $z = u$. Therefore, $(X_u; *, 0)$ is a distributive lattice, [MeJu, Thm VII.3.4], and

$$z \wedge (x \vee y) = z_0 \wedge (x \vee y) = z_0 \wedge (x \vee_u y) = (z_0 \wedge x) \vee_u (z_0 \wedge y). \tag{2.4}$$

It is evident that $z \wedge (x \vee y) \geq z_0 \wedge x,\ z_0 \wedge y$. Let now $z_0 \wedge x,\ z_0 \wedge y \leq w$, where $w \in X$. Then $z_0 \wedge x,\ z_0 \wedge y \leq w \wedge z_0 \in X_u$ and hence $(z_0 \wedge x) \vee_u (z_0 \wedge y) \leq w \wedge z_0$. This gives in view of (2.4) $z \wedge (x \vee y) \leq w \wedge z \leq w$, so that (2.3) holds. **Q.E.D.**

(iv) This is a consequence of (ii) and (iii).

We say that a commutative BCK-algebra $(X; *, 0)$ has a *relative cancellation property* if, for $a, x, y \in X$, $a \leq x, y$ with $x * a = y * a$ imply $x = y$.

Lemma 2.2 *Let $(X; *, 0)$ be any upwards directed commutative BCK-algebra. Then it has the relative cancellation property.*

[4] A nonvoid subset A of a BCK-algebra $(X; *, 0)$ is said to be a BCK-*subalgebra* of X if $x, y \in A$ implies $x * y \in A$.

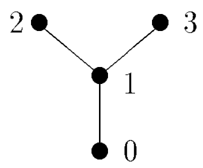
Proof. Suppose that there exist $c, c' \in X$ such that $a \leq c, c'$ and $c * a = c' * a$. We have to prove that $c = c'$. Let $u \in X$ be such that $c, c' \leq u$.

Then $(u * a) * (u * c) = (u * (u * c)) * a = (u \wedge c) * a = c * a$. Since $a \leq c$, we have $u * c \leq u * a$ and then $[u * (c * a)] * a = (u * a) * (c * a) = (u * a) * [(u * a) * (u * c)] = (u * a) \wedge (u * c) = u * c$.

In the same way we show that $[u * (c' * a)] * a = u * c'$. From $c * a = c' * a$, it follows $u * c' = u * c$. Hence $c = u * (u * c) = u * (u * c') = c.'$ **Q.E.D.**

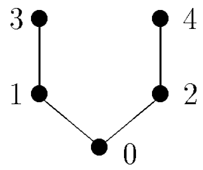
Example 2.3 *The example B_{4-3-3} from* [MeJu], *where $*$ is given by the table (next to it there is its Hasse diagram), gives a commutative BCK-algebra which is not upwards directed, consequently it is not a lattice. It has not the relative cancellation property. Indeed, we have $1 \leq 2, 3$ and $2 * 1 = 1 = 3 * 1$ but $2 \neq 3$, consequently, X cannot be embedded into the positive cone of a lattice ordered group.*

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	1	0	1
3	2	1	1	0



Example 2.4 *The example B_{5-2-7} from* [MeJu], *where $*$ is given by the table, gives a commutative BCK-algebra which has relative cancellation property but which is not directed upwards.*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	2	4	0



Theorem 2.5 *Let $(X;*,0)$ be a commutative BCK-algebra having relative cancellation property. Define a partial binary operation $+$ with the domain $S = \{(a,b) \in X \times X :$ there exists $c \in X$ with $c \geq b$ and $a = c * b\}$, and we define $c = a+b$ if $c \geq b$, and $a = c*b$. Then, for all $a,b,c \in X$, we have*

(I) *$a+0$, $0+a$ exist for any $a \in X$, and $a+0 = a = 0+a$.*

(II) *$a+b$ exists if and only if $b+a$ exists, and then $a+b = b+a$ (the commutativity).*

(III) *$a+b$ and $(a+b)+c$ exist in X if and only if $b+c$ and $a+(b+c)$ exist in X, and in this case $(a+b)+c = a+(b+c)$ (the associativity).*

(IV) *If $a+b$ exists in X and if $a_1 \leq a$, $b_1 \leq b$, then a_1+b_1 exists in X, and $a_1+b_1 \leq a+b$.*

(V) *$a = b$ holds whenever $u+a = u+b$ for some $u \in X$ (the cancellation property).*

(VI) *If $a+b \leq a+b'$ then $b \leq b'$. Conversely, if $b \leq b'$ and $a+b'$ exists in X, then $a+b \leq a+b'$.*

(VII) *If $a+b$ and $a+c$ exist in X, then $(a \wedge b)+c$ exists in X, and*

$$(a+c) \wedge (b+c) = (a \wedge b)+c \tag{2.5}$$

(VIII) *Let $u \geq x,y$. Then $x \vee y$ exists in X, and*

$$(x \vee y) * y = x * (x \wedge y) = x * y, \tag{2.6}$$

and $(x*(x \wedge y))+y$, $(x*y)+y$ exist in X, and

$$x \vee y = (x*(x \wedge y))+y = (x*y)+y. \tag{2.7}$$

Proof. The relative cancellation property entails that the partial binary operation $+$ is defined correctly.

(I) and (II) are evident. To prove prove (III) suppose that $a+b$ and $(a+b)+c$ exist in X. Then there exist $x, y \in X$ such that $a \leq x$ and $x * a = b$, $a + b = x \leq y$ and $y * x = y * (a+b) = c$.

Since $c = y * x \leq y * a$ and $(y * a) * c = (y * a) * (y * x) = [y * (y * x)] * a = x * a = b$, then $b + c \in X$ and $b + c = y * a$.

From $a \leq y$ and $y * a = b + c$, it follows $(a + (b + c)) \in X$ and $a + (b + c) = y = (a + b) + c$.

(IV) There are x, y in X such that $a_1 + x = a$ and $b_1 + y = b$. Using commutativity and associativity, we have $a + b = (a_1 + x) + (b_1 + y) = (a_1 + b_1) + (x + y)$ which entails the assertion.

(V) It follows easily from definition of relative cancellation property.

(VI) There exists $c \in X$ such that $(a + b) + c = a + b'$. Using associativity and commutativity of $+$, we have $a + (b + c) = a + b'$ and $(b + c) + a = b' + a$, so that $b + c = b'$ and, consequently, $b \leq b'$. The second part follows from (IV).

(VII) Since $a + c \geq c$ and $b + c \geq c$, we have $(a + c) \wedge (b + c) \geq c$. Then using property (b), we have

$$\begin{aligned} ((a+c) \wedge (b+c)) * c &= ((a+c) * ((a+c) * (b+c))) * c \\ &= a * ((a+c) * (b+c)). \end{aligned}$$

Calculate

$$a * b = ((a+c) * c) * ((b+c) * c) \leq (a+c) * (b+c),$$

when we have used property (c) of BCK-algebras, so that

$$a * ((a+c) * (b+c)) \leq a * (a * b) = a \wedge b.$$

Hence,

$$((a+c) \wedge (b+c)) * c \leq a \wedge b. \tag{2.8}$$

Since $a \geq a \wedge b$ and $a + c$ exists in X, we have by (IV) that $(a \wedge b) + c$ exists in X. It is clear that $a + c \geq (a \wedge b) + c$ and $b + c \geq (a \wedge b) + c$, so that

$$
\begin{aligned}
(a+c)\wedge(b+c) &\geq (a\wedge b)+c,\\
((a+c)\wedge(b+c))*c &\geq ((a\wedge b)+c)*c,\\
((a+c)\wedge(b+c))*c &\geq a\wedge b,
\end{aligned}
$$

which in view of (2.8) entails $((a+c)\wedge(b+c))*c = a\wedge b$, and, consequently, we have (2.5).

(VIII) Due to (1.1), we have $x*(x\wedge y) = x*y$. Using the basic properties (b), (1.1) and (1.2), we have

$$
\begin{aligned}
(x\vee y)*y &= (u*((u*x)\wedge(u*y))*y\\
&= (u*y)*((u*x)\wedge(u*y))\\
&= (u*y)*(u*x)\\
&= (u*(u*x))*y = x*y,
\end{aligned}
$$

which proves (2.6) and, consequently, (2.7).

Q.E.D.

In the paper [DvGr, Thm 3.4], we have proved that any upwards directed commutative BCK-algebra has relative cancellation property, and it can be converted via Theorem 2.5 into a commutative minimal clan. Motivated by this, we call the structure $(X;+,*,0)$ the BCK-*clan*, where $+$ is derived from Theorem 2.5, supposing that $(X;*,0)$ is a commutative BCK-algebra having relative cancellation property.

Example 2.6 *Let $G=(G;+,\leq,0)$ be a lattice ordered group and let $G^+ := \{x\in G : x\geq 0\}$ be its positive cone. Define a total binary operation $*_G$ on G^+ via*

$$x *_G y := (x-y)\vee 0, \tag{2.9}$$

for all $x,y\in G^+$. Then

$$x *_G y = x-(x\wedge y) = (x\vee y)-y, \quad x,y\in G^+, \tag{2.10}$$

and $(G^+;*_G,0)$ is an upwards directed commutative BCK-algebra having relative cancellation property.

Let G_0 be a non-void subset of G^+ such that $x,y\in G_0$ entail $x*_G y\in G_0$. Then $(G_0;*_G,0)$ is a commutative BCK-subalgebra of $(G^+;*_G,0)$ having relative cancellation property, which is a lower semilattice.

Our aim is to prove that some commutative BCK-algebras having relative cancellation property can be embedded onto some $(G_0; *_G, 0)$ of some lattice ordered group $(G; +, \leq, 0)$ with $G_0 \subseteq G^+$.

3 BCK-algebras and Riesz Decomposition Property

A poset $(X; \leq)$ has the *interpolation property* (or simply has *interpolation*), iff $x, y, p, q \in X$ with $x, y \leq p, q$ imply that there exists $z \in X$ with $x, y \leq z \leq p, q$. A BCK-clan $(X; +, *, 0)$ has the *Riesz decomposition property* iff $a, b, x \in X$ with $x \leq a+b$ imply that there exist $a_1, b_1 \in X$ with $a_1 \leq a$, $b_1 \leq b$ and $x = a_1 + b_1$.

It is evident that any commutative BCK-algebra $(X; *, 0)$ has interpolation. Indeed, let $x, y \leq p, q$ then $x, y \leq p \wedge q \leq p, q$.

Example 3.1 *Let $\Omega \neq \emptyset$ and let A, B be two non-void disjoint subsets of Ω. Define $E := \{X \subseteq \Omega : X \subseteq A$ or $X \subseteq B\}$. Then $(E; *, \emptyset)$, where $*$ is the set-theoretical difference, is a commutative BCK-algebra having relative cancellation property which is not upwards directed. The BCK-clan $(E; +, \setminus, \emptyset)$ is not a minimal clan but it has the Riesz decomposition property.*

Similarly, it is possible to show, (see e.g. [Wyl, Thm 5.9]) that any commutative clan has the Riesz decomposition property, consequently, any BCK-clan $(X; +, *, 0)$ has it whenever X is upwards directed; this follows also from a more general proposition:

Theorem 3.2 *Let $(X; *, 0)$ be a commutative BCK-algebra having relative cancellation property. Then the BCK-clan $(X; +, *, 0)$ has the Riesz decomposition property.*

Proof. Suppose that $c \leq a + b$, and define $u := a + b$ and $X_u := \{x \in X : x \leq u\}$. Put now $v = c * (b \wedge c)$ and $a_1 = a \wedge v$, so that $a_1 \leq a$. On the other hand, $c * v = b \wedge c$ and if we put $b_1 = c * a_1$,, then a_1, b_1 are defined in X_u and, moreover, $a_1 + b_1 = c$.

We must show that $b_1 \leq b$. Applying (2.7) and (2.5), we obtain

$$\begin{aligned} a_1 + b_1 = c &\leq (a+b) \wedge (b \vee c) \\ &= (a+b) \wedge [(c * (b \wedge c)) + b] \\ &= (a+b) \wedge (v+b) \\ &= (a \wedge v) + b = a_1 + b, \end{aligned}$$

and thus by (VI), $b_1 \leq b$. **Q.E.D.**

Lemma 3.3 *Let, for $a_1, a_2, b_1, b_2 \in X$, we have $a_1 + a_2 = b_1 + b_2$, where $(X; +, *, 0)$ is a BCK-clan. Then there are $c_{11}, c_{12}, c_{21}, c_{22} \in X$ such that*

$$\begin{aligned} a_1 &= c_{11} + c_{12}, & b_1 &= c_{11} + c_{21}, \\ a_2 &= c_{21} + c_{22}, & b_2 &= c_{12} + c_{22}. \end{aligned}$$

Proof. Since $a_1 \leq b_1 + b_2$, due to Theorem 3.2, we have that $a_1 = c_{11} + c_{12}$, where $c_{11} \leq b_1$ and $c_{12} \leq b_2$. Then there are $c_{21}, c_{22} \in X$ such that $c_{11} + c_{21} = b_1$ and $c_{12} + c_{22} = b_2$.

Calculate

$$(c_{11} + c_{12}) + a_2 = (c_{11} + c_{21}) + (c_{12} + c_{22}).$$

Using associativity and cancellation (V), we have $a_2 = c_{21} + c_{22}$. **Q.E.D.**

4 Construction of a Derived Semigroup

Throughout this section we shall suppose that $X = (X; +, *, 0)$ is a BCK-clan derived from a commutative BCK-algebra $(X; *, 0)$ which has relative cancellation property. The aim of this section is a construction of an ordered Abelian semigroup in that X can be embedded preserving $*$ and $+$; that will be possible, for instance, if X is directed upwards.

We shall follow ideas of Wyler [Wyl] who used Baer's approach [Bae].

We define $[X] := \bigcup_{n=1}^{\infty} X^n$. An element $A = (x_1, \ldots, x_n) \in [X]$ is said to be a *word* with entries, (letters), $x_1, \ldots, x_n$ from the alphabet X, and n is the length of the word A. Two words $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_m)$ are the same if $n = m$ and $x_i = y_i$ for any i with $1 \leq i \leq n$.

$[X]$ can be organized into an additive semigroup via

$$(x_1,\dots,x_n)+(y_1,\dots,y_m) := (x_1,\dots,x_n,y_1,\dots,y_m),$$

where $(x_1,\dots,x_n),\ (y_1,\dots,y_m)\in[X]$.

We call two words $A+(a,b)+B$ and $A+(c)+B$, where $A,B\in[X]$ and $a,b,c\in X$, *directly similar* if $a+b=c$. Here A or B or both may be the empty words. For $A,B\in[X]$, we write $A\sim B$ iff A and B are directly similar. Then (i) $A\sim A$, and (ii) if $A\sim B$, then $B\sim A$. Let $\simeq$ be the transitive closure of $\sim$, i.e., $A\simeq B$ iff there is a finitely many words $A_1,\dots,A_n\in[X]$ such that $A=A_1\sim A_2\sim\cdots\sim A_n=B$.

Then $\simeq$ is a congruence relation. Indeed, suppose that $A\simeq B$ and $C\simeq D$. Without loss of generality we can assume that there are two sequences of words $A_1,\dots,A_n$ and $C_1,\dots,C_m$ with $n=m$ such that $A=A_1\sim A_2\sim\cdots\sim A_n=B$ and $C=C_1\sim C_2\sim\cdots\sim C_n=D$. Then $A+C=A+C_1\sim A+C_2\sim\cdots\sim A+C_n=A+D\sim A_1+D\sim A_2+D\sim\cdots\sim A_n+D=B+D$.

Denote by $S(X)$ the quotient semigroup $(X)/\simeq$; we call it a *derived semigroup* of a BCK-algebra $(X;*,0)$. Given a word $A\in[X]$, we denote by $[A]$ the corresponding equivalence class in $S(X)$ given by A. Hence $[(0)]$ is the neutral element of $S(X)$, and if $(a_1,\dots,a_n)\in[X]$, we have

$$(a_1,\dots,a_n)=(a_1)+\cdots+(a_n),$$

so that

$$[(a_1,\dots,a_n)]=[(a_1)]+\cdots+[(a_n)]. \tag{4.1}$$

According to Baer [Bae], given a word $(a_1,\dots,a_n)$, inductively define a subset of X, $\mathcal{S}(a_1,\dots,a_n)$, as follows: (i) if $n=1$, define $\mathcal{S}(a_1)=\{a_1\}$, (ii) if $n>1$, then $a\in\mathcal{S}(a_1,\dots,a_n)$ iff there exist an integer i and $b,c\in X$ with $1\le i\le n-1$, $b\in\mathcal{S}(a_1,\dots,a_i)$, $c\in\mathcal{S}(a_{i+1},\dots,a_n)$ such that $a=b+c$.

Lemma 4.1 *Let $(X;+,*,0)$ be a BCK-clan.*

(i) *If $a\in\mathcal{S}(a_1,\dots,a_n)$, where $n>1$, then there exists an integer i with $1\le i<n$ such that a_i+a_{i+1} exists in X and $a\in\mathcal{S}(a_1,\dots,a_{i-1},a_i+a_{i+1},a_{i+2},\dots,a_n)$.*

(ii) *If $1\le i<n$, $n>1$, and if a_i+a_{i+1} exists in X, then $\mathcal{S}(a_1,\dots,a_{i-1},a_i+a_{i+1},a_{i+2},\dots,a_n)\subseteq\mathcal{S}(a_1,\dots,a_n)$.*

(iii) *If $a_1+\cdots+a_n=a$, then $a\in\mathcal{S}(a_1,\dots,a_n)$.*

(iv) *If* $\mathcal{S}(a_1,\dots,a_n)\neq\emptyset$, *then* $a := a_1+\cdots+a_n$ *exists in* X, *and* $\mathcal{S}(a_1,\dots,a_n) = \{a\}$.

(v) *For any word* $A\in[X]$ *we have* $|\mathcal{S}(A)|\leq 1$; *and if* $(a_1,\dots,a_n)\sim(b_1,\dots,b_{n+1})$ *and* $\mathcal{S}(a_1,\dots,a_n)\neq\emptyset$, *then* $\mathcal{S}(b_1,\dots,b_{n+1})\neq\emptyset$.

(vi) *If* $(a_1,\dots,a_n)\sim(b_1,\dots,b_{n+1})$ *and* $\mathcal{S}(a_1,\dots,a_n)\neq\emptyset$, *then* $\mathcal{S}(a_1,\dots,a_n) = \mathcal{S}(b_1,\dots,b_{n+1})$.

Proof. (i) Let $n=2$. If $a\in\mathcal{S}(a_1,a_2)$, then $a=a_1'+a_2'$, where $a_1'\in\mathcal{S}(a_1)$, $a_2'\in\mathcal{S}(a_2)$, and by definition, $a_1=a_1'$, $a_2=a_2'$, consequently $a\in\mathcal{S}(a_1+a_2)$. Let now $n=3$ and $a\in\mathcal{S}(a_1,a_2,a_3)$. Then there are $b,c\in X$ with $a=b+c$ such that either $b\in\mathcal{S}(a_1)$ and $c\in\mathcal{S}(a_2,a_3)$ or $b\in\mathcal{S}(a_1,a_2)$ and $c\in\mathcal{S}(a_3)$. In the first case, by induction step for $n=2$, we have $c\in\mathcal{S}(a_2+a_3)$, consequently, $a\in\mathcal{S}(a_1,a_2+a_3)$. Similarly for the second case. The general case of n follows from complete induction.

(ii) Let $n=2$ and let a_1+a_2 exist in X. Then $a_1+a_2\in\mathcal{S}(a_1,a_2)$. Since $\mathcal{S}(a_1+a_2)=\{a_1+a_2\}$, we conclude $\mathcal{S}(a_1+a_2)\subseteq\mathcal{S}(a_1,a_2)$. Let now $n=3$. Suppose, for example, a_1+a_2 exists in X and let $a\in\mathcal{S}(a_1+a_2,a_3)$. Then $a=b+c$, where $b\in\mathcal{S}(a_1+a_2)\subseteq\mathcal{S}(a_1,a_2)$ (in view of induction step for $n=2$) and $c\in\mathcal{S}(a_3)$. Hence $a=(a_1+a_2)+a_3\in\mathcal{S}(a_1,a_2,a_3)$. Similarly we exhibit the second possibility. The rest of this proof uses the complete induction and the associativity of $+$.

(iii) Let $A=(a_1,\dots,a_n)$. If $n=1$, we have $a=a_1$ and $a_1\in\mathcal{S}(A)=\{a_1\}$, so that $a\in\mathcal{S}(a_1)$. If $n=2$, then $a=a_1'+a_2'$, where $a_1'\in\mathcal{S}(a_1)=\{a_1\}$ and $a_2'\in\mathcal{S}(a_2)=\{a_2\}$. Therefore, $a=a_1+a_2\in\mathcal{S}(a_1,a_2)$. The general case of n can be proved by complete induction.

(iv) If $n=1$, the statement holds. Let now $n=2$, then there exists $a\in\mathcal{S}(a_1,a_2)$ such that $a=a_1+a_2$, which proves $\mathcal{S}(a_1,a_2)=\{a_1+a_2\}$. Let now $n=3$ and suppose $a\in\mathcal{S}(a_1,a_2,a_3)$. Then $a=b_1+b_2$, where either $b_1\in\mathcal{S}(a_1)$ and $b_2\in\mathcal{S}(a_2,a_3)$, or $b_1\in\mathcal{S}(a_1,a_2)$ and $b_2\in\mathcal{S}(a_3)$. In the first case we have $\mathcal{S}(a_1)=\{a_1\}=\{b_1\}$ and $b_2=a_2+a_3$. Using associativity of $+$, we have $a=a_1+a_2+a_3$ and $\mathcal{S}(a_1,a_2,a_3)=\{a\}$. In analogical way we proceed in the second case.

Similarly we obtain the general case of n by complete induction.

(v) Due to (iii) and (iv) either $\mathcal{S}(A)=\emptyset$ or $|\mathcal{S}(A)|=1$.

Let $n=1$, then $a=a_1=b_1+b_2$, so that $a\in\mathcal{S}(b_1,b_2)$. Let $n=2$. By (iv), $a=a_1+a_2$ exists in X, and either $a_1=b_1+b_2$ and $a_2=b_3$ or $a_1=b_1$

and $a_2 = b_2 + b_3$. For the first case we have $a_1 \in \mathcal{S}(b_1, b_2)$ and $a_2 \in \mathcal{S}(b_3)$, so that $a = a_1 + a_2 \in \mathcal{S}(b_1, b_2, b_3) \neq \emptyset$.

For the second case we have $a_1 \in \mathcal{S}(b_1)$ and $a_2 \in \mathcal{S}(b_2, b_3)$. Using again (iv), we have $a = a_1 + a_2 = b_1 + b_2 + b_3 \in \mathcal{S}(b_1, b_2, b_3)$. The general case of n follows from complete induction.

(vi) It follows from (v) and (iii), (iv). **Q.E.D.**

Lemma 4.2 *Let* $(X; +, *, 0)$ *be a BCK-clan. Then* (i) $(a_1, \ldots, a_n) \simeq (b)$ *if and only if* $a_1 + \cdots + a_n$ *exists in* X *and equals* b*;* (ii) *if* $[(a)] = [(b)]$ *then* $a = b$.

Proof. It follows from (v) Lemma 4.1 and the results of Baer [Bae, (iv), (v) Thm 1.1] (compare also [Wyl]). **Q.E.D.**

Lemma 4.3 *Let* a *and* b *from* X *have an upper bound in* X*. Then*

$$[(a)] + [(b)] = [(b)] + [(a)]. \tag{4.2}$$

In particular, if X *is upwards directed, then* $S(X)$ *is a commutative semi-group.*

Proof. Let $u \in X$ be an upper bound of a and b. Then due to (ii) Theorem 2.1, $a \vee b$ exists in X. Using (2.6) and (2.7), we have

$$\begin{aligned} (a, b) &\sim (a \wedge b, a * (a \wedge b), b) = (a \wedge b, (a \vee b) * b, b) \\ &\sim (a \wedge b, a \vee b) \sim (a \wedge b, (a \vee b) * a, a) \\ &= (a \wedge b, b * (a \wedge b), a) \sim (b, a). \end{aligned}$$

The second assertion follows from (4.1) and (4.2). **Q.E.D.**

Lemma 4.4 *Let, for* $A, B \in [X]$*, we have* $[A] + [B] = [(0)]$*. Then* $[A] = [(0)] = [B]$.

Proof. It follows easily from Lemma 4.2 (i). **Q.E.D.**

We note that if X is not upwards directed, then (4.2) can fail. Indeed, take X from Example 2.4. Then the partial binary operation $+$ is given by the table below, and here we have $[(1)] + [(2)] = [(1, 2)] \neq [(2, 1)] = [(2)] + [(1)]$.

+	0	1	2	3	4
0	0	1	2	3	4
1	1	3			
2	2		4		
3	3				
4	4				

The following result was proved originally by Wyler [Wyl, Thm 5.4] for commutative clans. His proof works also in our case because it does not depend on the existence of suprema in X, and, for reader's convenience, we present it in the whole completeness (see also Ravindran [Rav] for a special case of X).

Lemma 4.5 *Let $A, B, W \in [X]$ such that $[A] + [B] = [W]$. Then there are elements $c_i', c_i'' \in X$, for $i = 1, \ldots, k$, such that $W = (c_1' + c_1'', \ldots, c_k' + c_k'')$, and $A \simeq (c_1', \ldots, c_k')$, $B \simeq (c_1'', \ldots, c_k'')$.*

Proof. Since $[A] + [B] = [W]$, there is a finite sequence of words $W_1, \ldots, W_n \in [X]$ such that $A + B = W_1 \sim W_2 \sim \cdots \sim W_n = W$. The proof will follow complete induction on n.

Let $n = 1$. Then $A + B = W$ and let $A = (a_1, \ldots, a_r)$, $B = (b_1, \ldots, b_s)$, $W = (c_1, \ldots, c_{r+s})$. We define

$$c_i' = \begin{cases} a_i & \text{if } 1 \le i \le r, \\ 0 & \text{if } r+1 \le i \le r+s, \end{cases} \qquad c_i'' = \begin{cases} 0 & \text{if } 1 \le i \le r, \\ b_i & \text{if } r+1 \le i \le r+s. \end{cases}$$

Then we have $(a_1, \ldots, a_r) \simeq (c_1', \ldots, c_{r+s}')$, $(b_1, \ldots, b_s) \simeq (c_1'', \ldots, c_{r+s}'')$, and $W = (c_1' + c_1'', \ldots, c_{r+s}' + c_{r+s}'')$.

Suppose now that the statement holds for any finite sequence of at most n words. We prove the statement for a sequence of $n+1$ words. Assume thus

$$A + B = W_1 \sim W_2 \sim \cdots \sim W_n \sim W_{n+1} = W.$$

There are two cases.

Case (a). $W = (c_1, \ldots, c_k)$, there is i with $1 \le i < k$ such that $c_i + c_{i+1}$ exists in X, and $W_n = (c_1, \ldots, c_{i-1}, c_i + c_{i+1}, c_{i+2}, \ldots, c_k)$. By induction hypothesis, there are elements

$$\begin{gathered} c_1', \ldots, c_{i-1}', c', c_{i+2}', \ldots, c_k' \in X, \\ c_1'', \ldots, c_{i-1}'', c'', c_{i+2}'', \ldots, c_k'' \in X, \end{gathered}$$

such that $c_j = c'_j + c''_j$ for $j = 1, \ldots, i-1$ and for $j = i+2, \ldots, k$, $c' + c'' = c_i + c_{i+1}$, and

$$\begin{array}{rcl} A & \simeq & (c'_1, \ldots, c'_{i-1}, c', c'_{i+2}, \ldots, c'_k), \\ B & \simeq & (c''_1, \ldots, c''_{i-1}, c'', c''_{i+2}, \ldots, c''_k). \end{array}$$

According to Lemma 3.3, there are four elements $x_{11}, x_{12}, x_{21}, x_{22} \in X$ such that $c' = x_{11} + x_{12}$, $c'' = x_{21} + x_{22}$, $c_i = x_{11} + x_{21}$, and $c_{i+1} = x_{12} + x_{22}$. Put $c'_i = x_{11}$, $c''_i = x_{21}$, $c'_{i+1} = x_{12}$, $c''_{i+1} = x_{22}$. Then $c_i = c'_i + c''_i$, $c_{i+1} = c'_{i+1} + c''_{i+1}$, $c' = c'_i + c'_{i+1}$, $c'' = c''_i + c''_{i+1}$, and $W = (c_1, \ldots, c_k) = (c'_1 + c''_1, \ldots, c'_k + c''_k)$. On the other side,

$$\begin{array}{rcl} A & \simeq & (c'_1, \ldots, c'_{i-1}, c', c'_{i+2}, \ldots, c'_k) \\ & = & (c'_1, \ldots, c'_{i-1}, c'_i + c'_{i+1}, c'_{i+2}, \ldots, c'_k) \\ & \simeq & (c'_1, \ldots, c'_{i-1}, c'_i, c'_{i+1}, c'_{i+2}, \ldots, c'_k) \end{array}$$

and

$$\begin{array}{rcl} B & \simeq & (c''_1, \ldots, c''_{i-1}, c'', c''_{i+2}, \ldots, c''_k) \\ & = & (c''_1, \ldots, c''_{i-1}, c''_i + c''_{i+1}, c''_{i+2}, \ldots, c''_k) \\ & \simeq & (c''_1, \ldots, c''_{i-1}, c''_i, c''_{i+1}, c''_{i+2}, \ldots, c''_k). \end{array}$$

Case (b). $W_n = (c_1, \ldots, c_k)$, $c_i + c_{i+1}$ exists in X for some i with $1 \leq i < k$, and $W_{n+1} = (c_1, \ldots, c_{i-1}, c_i + c_{i+1}, c_{i+2}, \ldots, c_k)$. By induction hypothesis on W_n, there are elements

$$c'_1, \ldots, c'_k \in X \text{ and } c''_1, \ldots, c''_k \in X$$

such that $c_i = c'_i + c''_{i+1}$, $i = 1, \ldots, k$, $A \simeq (c'_1, \ldots, c'_k)$ and $B \simeq (c''_1, \ldots, c''_k)$. Since $c_i + c_{i+1}$ exists in X, we have due to associativity and commutativity of $+$

$$\begin{array}{rcl} c_i + c_{i+1} & = & (c'_i + c''_i) + (c'_{i+1} + c''_{i+1}) \\ & = & (c'_i + c'_{i+1}) + (c''_i + c''_{i+1}) \\ & = & c' + c'', \end{array}$$

where $c' = c'_i + c'_{i+1}$, $c'' = c''_i + c''_{i+1}$. Then

$$
\begin{aligned}
A &\simeq (c_1', \ldots, c_{i-1}', c_i', c_{i+1}', c_{i+2}', \ldots, c_k') \\
&\simeq (c_1, \ldots, c_{i-1}', c', c_{i+2}', \ldots, c_k'), \\
B &\simeq (c_1'', \ldots, c_{i-1}'', c_i'', c_{i+1}'', c_{i+2}'', \ldots, c_k'') \\
&\simeq (c_1'', \ldots, c_{i-1}'', c'', c_{i+2}'', \ldots, c_k''),
\end{aligned}
$$

and $W = (c_1' + c_1'', \ldots, c_{i-1}' + c_{i-1}'', c' + c'', c_{i+2}' + c_{i+2}'', \ldots, c_k' + c_k'')$, which makes the proof finished. **Q.E.D.**

If we assume that X is upwards directed, then we can say more about the derived semigroup $S(X)$.

Theorem 4.6 *Let $(X; +; *, 0)$ be an upwards directed BCK-clan. Then, for all $A, B, C \in S(X)$, we have*

(i) $[A] + [B] = [B] + [A]$.

(ii) *If $[A] + [B] = [A] + [C]$, then $[B] = [C]$.*

(iii) *$\leq_X$, where $[A] \leq_X [B]$ if and only if there is an $C \in [X]$ such that $[A] + [C] = [B]$, is a partial order on $S(X)$.*

(iv) $[(a)] \wedge [(b)] = [(a \wedge b)]$, $a, b \in X$.

(v) *$(S(X); +, \leq_X, [(0)])$ is an Abelian partially ordered semigroup which is a lower semilattice and it is the generating positive cone of a lattice ordered group $G(X)$.*

(vi) *A mapping $h : X \to G(X)$ defined via*

$$h(a) := [(a)], \; a \in X,$$

is injective and preserving $\wedge$, and $+$.

(vii) *The couple $(G(X), h)$ has a universal property, i.e., if $g : X \to G_1$ is an order and $+$ preserving mapping into a partially ordered Abelian group G_1, then $g = g' \circ h$ for a uniquely determined group homomorphism $g' : G(X) \to G_1$.*

(viii) *If X is bounded with the greatest element $1 \in X$, then $h(1)$ is a strong unite in $G(X)$ and h maps X onto the order interval* $[[(0)],[(1)]] := \{g \in G(X) : [(0)] \leq_X g \leq_X [(1)]\}$.[5]

(ix) *If $g \in G(X)$, $0 \leq g \leq h(x)$ for some $x \in X$, then there exists a unique $y \in X$ such that $y \leq x$ and $h(y) = g$.*

Proof. This result has been proved in [DvGr, Thm 3.5] using Wyler's ideas [Wyl], or it can follow ideas from this section. **Q.E.D.**

Remark 4.7 We recall that if, for the derived group $S(X)$, (4.2) holds, then $S(X)$ is the positive generating cone of a lattice ordered group $G(X)$ as it is indicated in Theorem 4.6. Unfortunately, that is not a general case, and we do not know other conditions as upwards directness guaranteeing (4.2). Hence, we have to look for other ways for representing commutative BCK-algebras via lattice ordered groups. Such possibilities will be done in Sections 5 and 6.

The lattice ordered group $G(X)$ from the proof of the last theorem is said to be the *derived group* of $(X; +, *, 0)$.

5 Union BCK-algebras

In this section, we introduce a family of commutative BCK-algebras which can be embedded into the positive cones of lattice ordered groups.

Let $\{(X_i; *_i, 0_i)\}_{i\in I}$ be a system of BCK-algebras such that (i) $0_i = 0$ for any $i \in I$; and (ii) $X_i \cap X_j = \{0\}$ for $i \neq j$, $i, j \in I$. Put $X = \bigcup_{i\in I} X_i$ and define a total binary operation $*$ on X via

$$x * y = \begin{cases} x *_i y & \text{if } x, y \in X_i,\ i \in I, \\ x & \text{if } x \in X_i,\ y \in X_j,\ i \neq j,\ i, j \in I. \end{cases}$$

Then $(X; *, 0)$ is a BCK-algebra called a *union BCK-algebra*, and we denote $X = \bigoplus_{i\in I} X_i$. A union BCK-algebra is commutative iff each X_i is commutative. It is clear that any BCK-algebra $X = \bigoplus_{i\in I} X_i$, where $X_i = X$ and

[5] In this case, a commutative BCK-algebra X can be converted by a natural way into an MV-algebra and conversely, and (viii) gives Mundici's famous representation theorem of MV-algebras [Mun].

$|I| = 1$. Moreover, any X_i is a BCK-subalgebra of $X = \bigoplus_{i\in I} X_i$, and if $u \in X_i$, then

$$X_u := \{x \in X : x \leq u\} \subseteq X_i.$$

Conversely, let $\{X_i\}_{i\in I}$ be a family of subsets of a BCK-algebra $(X; *, 0)$ such that

(a) $X = \bigcup_{i\in I} X_i$,

(b) $X_i \cap X_j = \{0\}$,

(c) $u \in X_i$ implies $X_u \subseteq X_i$,

then each X_i is a BCK-subalgebra of X, and X is the union BCK-algebra of $\{X_i\}_{i\in I}$.

More about union BCK-algebras is in [MeJu].

We recall that if $X = \bigoplus_{i\in I} X_i$ and there are two different subalgebras X_i and X_j having at least two elements, then X does not satisfy the condition (S). [6] Indeed, if $a \in X_i$ and $b \in X_j$ are non-zero elements, then $a * a \leq b$ and $b * a \leq b$ but there is no element $c \in X$ such that $a \leq c$ and $b \leq c$.

We say that a BCK-algebra $(X; *, 0)$ has a *u.d. union property* if X can be expressed as $X = \bigoplus_{i\in I} X_i$, where each X_i is directed upwards. A BCK-clan $(X; +, *, 0)$ has a *u.d. union property* if $(X; *, 0)$ has it.

Since $\{0\}$ is trivially an upwards directed BCK-subalgebra of X, using Zorn's lemma we conclude that in any BCK-algebra X there is a maximal upwards directed BCK-subalgebra of X.

Any BCK-algebra with u.d. union property has a unique decomposition $X = \bigoplus_{i\in I} X_i$, where all X_i are upwards directed:

Lemma 5.1 *Let $X = \bigoplus_{i\in I} X_i$, where each X_i is upwards directed. Then, if $X = \bigoplus_{j\in J} Y_j$, where each Y_j is upwards directed, for any $i \in I$ there is a unique $j \in J$ such that $X_i = Y_j$ and conversely.*

---

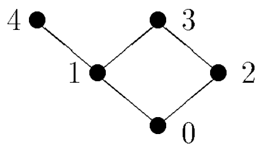
[6] A BCK-algebra $(X; *, 0)$ satisfies *condition* (S) if given elements $a, b \in X$ there is an element $a \circ b \in X$ such that (i) $(a \circ b) * a \leq b$, and (ii) if $x * a \leq b$ then $x \leq a \circ b$. Then $(X; \circ, 0, \leq)$ is a commutative po-semigroup [MeJu, ThmI.7.7]. Since $a \leq a \circ b$ and $b \leq a \circ b$, $(X; *, 0)$ is directed upwards and by [DvGr], X can be embedded into the positive cone of some lattice ordered group.

Proof. Let $u \in X_i$ be non-zero. Then there is a unique Y_j such that $u \in Y_j$. Hence, $X_u \subseteq X_i$ and $X_u \subseteq Y_j$. Let now v be an arbitrary element of X_i. There is an element $w \in X_i$ which as an upper bound of u and v, i.e., $u, v \leq w$. Then $v \in X_w \subseteq X_i$ and since $u \in Y_j$, we conclude that $X_w \subseteq Y_j$, so that $X_i \subseteq Y_j$. By symmetrical reasonings we can show that $Y_j \subseteq X_i$. **Q.E.D.**

We recall that Example 2.4 gives a BCK-algebra and a BCK-clan which have u.d. union property, while the following example not.

Example 5.2 *The example* B_{5-2-8} *from* [MeJu], *where* $*$ *is given by the table below, gives a commutative BCK-algebra which has not u.d. union property because* $1 \leq 3, 4$ *while* 3 *and* 4 *has no upper bound in it.*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	2
3	3	2	1	0	2
4	4	1	4	1	0



Lemma 5.3 (i) *Any commutative BCK-algebra with u.d. union property has relative cancellation property.*

(ii) *If* $X = \bigoplus_{i \in I} X_i$ *is a commutative BCK-algebra, then, for* $x \in X_i$ *and* $y \in X_j$, *where* $x \neq 0 \neq y$ *and* $i \neq j$, $x + y$ *does not exist in* X. *In general, if* $u + v$ *exists in* X, *then* u *and* v *belong to the same subalgebra* X_i.

Proof. (i) Suppose that $a \leq x, y$ and $x * a = y * a$. If $a = 0$, then trivially $x = y$. If $a \neq 0$, then $a \in X_x \cap X_y$, so that there is a unique X_i in $X = \bigoplus_{i \in I} X_i$ such that $x, y \in X_i$. Applying Lemma 2.2, we see that $x = y$.

(ii) This part follows from an observation that if $x + y$ exists in X, then $x, y \leq x + y$ so that $x, y \in X_{x+y}$, which means that x and y have to belong to the same subalgebra. **Q.E.D.**

Lemma 5.4 *Let $X = \bigoplus_{i\in I} X_i$ and let $(X;+,*,0)$ be a BCK-algebra having u.d. union property. Let all elements of $(a_1,\dots,a_n)$ belong to a fixed X_s, $s \in I$. If $(a_1,\dots,a_n) \sim (b_1,\dots,b_m)$, then all elements b_j's belong to X_s.*

If $[(a_1,\dots,a_n)] = [(b_1,\dots,b_m)]$, then all $b_j \in X_s$, for $j = 1,\dots,m$.

Proof. Let $(a_1,\dots,a_n)$ be directly similar with $(b_1,\dots,b_m)$. Then either some $a_i = a' + a''$ and $b_i = a'$, $b_{i+1} = a''$ and $a_k = b_k$ for $1 \le k < i$ and $a_k = b_{k+1}$ for $i < k \le n$, or $b_j = a_j + a_{j+1}$ and other elements coordinately coincide. In the first case $b_i, b_{i+1} \le a_i$ so that $b_i, b_{i+1} \in X_s$ and similarly in the second one.

The rest of Lemma is now evident. **Q.E.D.**

We now present the main result, a representation theorem for commutative BCK-algebras having u.d. union property, which converts in some sense Example 2.6.

Theorem 5.5 *Let $(X;*,0)$ be a commutative BCK-algebra with u.d. union property. Then there is a lattice ordered group $(G;+,\le,0)$ with the positive generating cone G^+ and a subset G_0 of G^+ generating G^+, which is a lower semilattice closed under $*_G$, such that X can be injectively embedded onto G_0 with an embedding h satisfying*

$$h(a+b) = h(a) + h(b) \tag{5.1}$$

whenever $a+b$ exists in X, and

$$h(x*y) = h(x) *_G h(y) := h(x) - h(x) \wedge h(y),\ \ x,y \in X. \tag{5.2}$$

Proof. (i) Suppose $X = \bigoplus_i X_i$, where each X_i is an upwards directed BCK-subalgebra. Since this decomposition is due to Lemma 5.1 unique and by Lemma 5.4, $(a_1,\dots,a_n) \simeq (b_1,\dots,b_m)$ with entries $a_k \in X_i$ for any $1 \le k \le n$, we can construct the derived semigroups $S(X_i)$ for each X_i such that $S(X_i) \subseteq S(X)$ for any i with $S(X_i) \cap S(X_j) = \{[0]\}$ for $i \ne j$. According to Theorem 4.6, for each i, the derived group $G_i := G(X_i)$ is a lattice derived group and a mapping $h_i : X_i \to G_i$ is an embedding preserving $+$ and the order in X_i.

Define the direct sum G of the system of groups $\{G(X_i)\}_i$, $G = G(X) := \bigoplus_i G(X_i)$. That is, $G(X)$ is the subset of the product $\prod_i G(X_i)$ consisting of all elements $(w_i)_i$, where $w_i \in X_i$, with finitely many non-zero w_i's, and the addition and the ordering are by coordinates. In addition, $G(X)$ is a lattice ordered group.

The mapping $h : X \to G$ defined via $h(a) := (b_i)_i$, where $b_i = [(a)]$ if $a \in X_i$ and $b_i = [(0)]$ if $a \notin X_i$, is an injection.

(ii) It is possible to show that $h(a) \le h(b)$ in G iff $a \le b$ in X, and $h(0) = [(0)]$. Indeed, let $a \le b$ in X. Since $a \in X_b := \{x \in X : x \le b\}$, both a and b belong to the same subalgebra X_i of X, so that $h(a) \le h(b)$. Conversely, let $h(a) \le h(b)$ in G. By definition of h, both a and b belong to the same subalgebra X_i, and using (iv) Theorem 4.6, we conclude that $h_i(a) \le h_i(b)$, so that, $a \le b$.

(iii) Let now $c = a \wedge b$, $a, b \in X$. Then $c \in X_a \cap X_b$. If a and b belong to the same subalgebra, say X_i, then by (iv) Theorem 4.6, $h_i(c) = h_i(a) \wedge h_i(b)$, so that $h(c) = h(a) \wedge h(b)$. If a and b belong to different subalgebras, say X_i and X_j, then $c = 0$ and by definition of h, $h(a) \wedge h(b) = [(0)] = h(c)$.

(iv) Equation (5.1) is a consequence of (ii) Lemma 5.3.

Let now x and y be arbitrary elements of X. Due to (5.1), we have

$$h(x * (x \wedge y)) = h(x) - h(x \wedge y) = h(x) - h(x) \wedge h(y).$$

On the other hand, we have $x * y = x * (y \wedge x) = x * (x \wedge y)$ which proves (5.2) and finishes the proof. **Q.E.D.**

We say that a partially ordered Abelian group $(G; +, \le, 0)$ with a mapping $h : X \to G^+$ is a *universal group* for a BCK-clan $(X; +, *, 0)$ if (i) the positive cone G^+ is generating for G; (ii) $h(X)$ generates G^+; (iii) $h(x + y) = h(x) + h(y)$ whenever $x + y$ exists in X, $x, y \in X$, and (iv) for any partially ordered Abelian group G_1 and any order and $+$ preserving mapping $g : X \to G_1$ there is a group homomorphism $g' : G \to G_1$ such that $g = g' \circ h$. The universal group, if it exists, is unique up to isomorphism, and g' from (iv) is a unique group homomorphism with that property. We denote the universal group $G = (G, h)$.

Theorem 5.6 *Under the conditions of Theorem* 5.5*, the group* $G(X)$ *and the embedding* h *from the proof of Theorem* 5.5 *has a universal property for* $(X; +, *, 0)$*, and* $(G(X), h)$ *is a universal group for* $(X; +, *)$*.*

Proof. Let g be a mapping from X into a partially ordered Abelian group G_1 preserving $+$ and the order in X. Then the restriction $g_i : X_i \to G_1$ of

g onto X_i preserves $+$ and the order in X_i, and due to (vii) of Theorem 4.6, there is a unique group homomorphism $g'_i : G_i(X) \to G_1$ such that $g_i = g'_i \circ h_i$. Define $g' : G \to G_1$ via

$$g'((w_i)_i) := \sum_i g'_i(w_i), \quad (w_i)_i \in G(X). \tag{5.3}$$

Since in any $(w_i)_i$ there are only finitely many non-zero elements $w_i \in G(X_i)$, the sum on the right-hand side of (5.3) is defined well, and g' is a group homomorphism from $G = G(X)$ into G_1.

Take $a \in X$. If $a = 0$, then

$$g'(h(0)) = g'([(0)]) = 0 = g(0).$$

If $a \neq 0$, then there is a unique subalgebra X_i containing a. Then

$$g'(h(a)) = g_i(h_i(a)) = g_i(a) = g(a).$$

The uniqueness of g' is now clear because if $g' : G(X) \to G_1$ is a group homomorphism such that $g''(h(a)) = g(a)$, $a \in X$, then for $a \in X_i$, $g_i(a) = g(a) = g''(h(a)) = g''(\cdots, [(0)], \cdots, [(0)], h_i(a), [(0)], \cdots [(0)], \cdots) = g'(h_i(a))$. The definition of $G(X) = \bigoplus_i G(X_i)$ gives $g' = g''$.

The last statement follows from the construction of derived semigroups and derived groups. **Q.E.D.**

The following example shows that there is a commutative BCK-algebra which can be embedded into a lattice ordered group but not via Theorem 5.5.

Example 5.7 *Let* $\Omega = \{1, 2, 3, 4\}$, *and* $X = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}\}$. *Then* $(X; \setminus, \emptyset)$, *where* $\setminus$ *is the set-theoretical difference, is a commutative BCK-algebra having relative cancellation property but not u.d. union property. Therefore, for its embedding into a lattice ordered group we cannot use Theorem* 5.5, *but it can be embedded into a lattice ordered group, because it can be embedded into the Boolean algebra* 2^Ω, *and* 2^Ω *into a lattice ordered group.*

We recall that Example 5.7 is a special case of a positively implicative commutative BCK-algebra, i.e., X satisfies $x * y = (x * y) * y$ for all $x, y \in$

X. Such algebras are so-called implicative BCK-algebras. In this case (see [DvGr, Thm 6.7] and a discussion to this theorem, and [MeJu, Thm VII.2.7]), it was proved that X can be embedded into a Boolean algebra of subsets, and consequently, into a lattice ordered group.

Remark 5.8 Let a BCK-clan $(X;+,*,0)$ be isomorphic under an isomorphism h with a BCK-subclan $(G_0;+,*_G,0)$ of the BCK-clan $(G^+;+,*_G,0)$ of some lattice ordered group G such that (G,h) is a universal group for X. Hence, if (G_1,h_1) is any universal group for X, then h_1 is injective.

Remark 5.9 Let Z be the group of all integers with the usual addition and the order. It is worth to recall that

(i) Example 2.3 cannot be embedded into any lattice ordered group.
(ii) Example 2.4 has a universal group (Z^2,h), where $h(0)=(0,0)$, $h(1)=(1,0)$, $h(2)=(0,1)$, $h(3)=(2,0)$, $h(4)=(0,2)$.
(iii) Example 5.2 has a universal group (Z^2,h), where $h(0)=(0,0)$, $h(1)=(0,1)$, $h(2)=(1,0)$, $h(3)=(1,1)$, $h(4)=(0,2)$.
(iv) Example 5.7 has a universal group (Z^3,h), where $h(\emptyset)=(0,0,0)$, $h(\{1\})=(1,0,0)$,h($\{2\}$) = (0,1,0),h($\{3\}$) = (0,0,1),h($\{1,2\}$) = (1,1,0), $h(\{2,3\})=(0,1,1)$.

6 Unitarily Extendible BCK-algebras

In the present section, we introduce another class of commutative BCK-algebras, which can be embedded into lattice ordered groups.

According to Hoo, [Hoo], a commutative BCK-algebra $(X;*,0)$ is said to be *unitarily extendible* if, for $e\notin X$, we can extend the commutative BCK-algebra structure on X to the commutative BCK-algebra structure on $X\cup\{e\}$ with e as the greatest element in $X\cup\{e\}$. It is clear that any non-zero BCK-algebra is unitarily extendible. Hoo proved the following two important statements:

Theorem 6.1 (i) *A non-zero commutative BCK-algebra $(X;*,0)$ is unitarily extendible if and only if there is a function $f: X\setminus\{0\}\to X\setminus\{0\}$ such that $f(f(x))=x$ and $x*f(y)=y*f(x)$ for all $x,y\in X\setminus\{0\}$.*[7]

[7]We recall that in this case $e*x=f(x)$ for each $x\in X\setminus\{0\}$.

(ii) *A bounded commutative BCK-algebra* $(X; *, 0)$ *is unitarily extendible if and only if there exists an element* $x_0 \in X \setminus \{0\}$ *such that* $x_0 \leq x$ *for all* $x \in X \setminus \{0\}$.

We note that Example 2.3 is a case of a commutative BCK-algebra which is not unitarily extendible; Example 2.4 is a case of a BCK-algebra having u.d. union property but it is not unitarily extendible. Similarly, the set of all nonnegative integers $N = \{0, 1, 2, \ldots\}$ ordered by the natural way can be converted into a commutative BCK-algebra only in one way: $n * m := \max\{0, n - m\}$, $n, m \in N$, in this case it has a linear structure and it can be embedded into the lattice ordered group of all integers, but N is not unitarily extendible.

Example 6.2 *Let* $X = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}\}$. *Then* $(X; \setminus, \emptyset)$, *where* $\setminus$ *is the set-theoretical difference, is a commutative BCK-algebra which has no u.d. union property, but* X *is unitarily extendible.*

The proofs of the following three examples follow from [Dvu, Exs 3.2–3.4], Theorem 6.1, and the definition of unitarily extendible BCK-algebras.

Example 6.3 *Let* $X = [0, 1)$ *be the interval of real numbers ordered by the natural way. Let* $g : [0, 1) \to [0, \infty)$ *be an increasing, continuous function with* $g(0) = 0$. *Define a total binary operation* $*_g$ *on* X *via*

$$x *_g y := g^{-1}(\max\{0,\, g(x) - g(y)\}), \quad x, y \in X. \tag{6.1}$$

Then $([0, 1); *_g, 0)$ *is a commutative BCK-algebra with the natural order. Conversely, if* $([0, 1); *, 0)$ *is a commutative BCK-algebra with the natural order, then there is an increasing, continuous function* $g : [0, 1) \to [0, \infty)$ *with* $g(0) = 0$ *such that* $* = *_g$, *where* $*_g$ *is defined via* (6.1). *Moreover,* $([0, 1); *_g, 0)$ *is unitarily extendible if and only if* g *is bounded; in this case the function* f *from Theorem* 6.1 *is defined via* $f(x) = g(1^-) - g(x)$, $x \in (0, 1)$, *and* $([0, 1); *_g, 0)$ *is isomorphic with* $([0, 1); *_{id_{[0,1)}}, 0)$.

Example 6.4 *Let* $X = [0, \infty)$ *be the interval of real numbers ordered by the natural way. Then* $([0, \infty); *, 0)$ *is a commutative BCK-algebra with the natural order if and only if there exists an increasing, continuous function* $g : [0, \infty) \to [0, \infty)$ *with* $g(0) = 0$ *such that* $* = *_g$, *where* $*_g$ *is defined via*

(6.1). $([0,\infty);*_g,0)$ *is unitarily extendible if and only if* g *is bounded. In this case,* $([0,\infty);*_g,0)$ *is isomorphic with* $([0,1);*_{id_{[0,\infty)}},0)$*, and the function* f *from Theorem* 6.1 *is defined via* $f(x)=g(+\infty)-g(x),\ x\in(0,\infty)$.

Example 6.5 *Let* $X=[0,1]$ *be the interval of real numbers ordered by the natural way. Then* $([0,1];*,0)$ *is a commutative BCK-algebra with the natural order if and only if there exists an increasing, continuous function* $g:[0,1]\to[0,1]$ *with* $g(0)=0$ *and* $g(1)=1$ *such that* $*=*_g$*, where* $*_g$ *is defined via* (6.1). *Any* $([0,1];*_g,0)$ *is bounded and isomorphic with* $([0,1];*_{id_X},0)$*. Moreover,* $([0,1];*_g,0)$ *is not unitarily extendible.*

Theorem 6.6 *Let* $X=\bigoplus_{i\in I}X_i$ *be a union BCK-algebra, where each* X_i *is either directed upwards or a non-zero and unitarily extendible commutative BCK-subalgebra of* X*. Then* X *can be converted into a BCK-clan* $(X;+,*,0)$*, and there is a lattice ordered group* $(G;+,\leq,0)$ *with the positive generating cone* G^+ *and a lower semilattice* G_0 *of* G^+*, which generates* G^+*, such that* X *can be embedded onto* G_0 *with an injective embedding* h *satisfying* (5.1) *and* (5.2).

Proof. Let $X=\bigoplus_{i\in I}X_i$ satisfy the conditions of Theorem. Then X has a relative cancellation property, and $(X;+,*,0)$ is a well-defined BCK-clan.

For any $i\in I$, let $\tilde{X}_i:=X_i$ if X_i is upwards directed and $\tilde{X}_i:=X_i\cup\{e_i\}$ when X_i is only unitarily extendible, where e_i is the greatest element in $X_i\cup\{e_i\}$. Without loss of generality we can assume that all e_i are diverse. Define the commutative union BCK-algebra $\tilde{X}:=\bigoplus_{i\in I}\tilde{X}_i$. According to Theorems 5.5 and 5.6, there exists a universal group (G,h) for $(\tilde{X};+,*,0)$ satisfying (5.1) and (5.2), so that h is an embedding of $\tilde{X}$ preserving $+$ and the order in $\tilde{X}$. Consequently, the restriction h_0 of h onto X also satisfies (5.1) and (5.2), respectively.

We know that $h(\tilde{X})$ generates the positive cone G^+ of G. We show that also $h(X)$ generates G^+. Take $g\in G^+$. Then $g=\sum_i h(x_i)+\sum_j h(e_j)$, $x_i\in X_i$, where the sums $\sum_i$ and $\sum_j$ are both over finite (also empty) sets of indices. For any $e_j\in\tilde{X}_j$, there are two non-zero elements $x_j',x_j''\in X_j$ such that $e_j=x_j'+x_j''$, where the sum is in $\tilde{X}_j$, hence $h(e_j)=h(x_j')+h(x_j'')$. This means that any element $g\in G^+$ can be expressed as a finite sum of elements from $h(X)$. **Q.E.D.**

Theorem 6.7 *Under the conditions of Theorem* 6.6, $(X;+,*,0)$ *possesses a lattice ordered universal group.*

Proof. Take the lattice ordered group G and the embedding h from $\tilde{X}$ into G which have been used in the proof of Theorem 6.6. Let h_0 be the restriction of h onto X. We claim that $(G;h_0)$ is a universal group for $(X;+,*,0)$ in question. For that it is necessary to verify only the condition (iv) of universal groups.

So let G_1 be an ordered Abelian group and let $g: X \to G_1$ be a mapping preserving $+$ and the order in X. We extend g to $\hat{g}: \tilde{X} \to G_1$ which will preserve order and all existing sums in $\tilde{X}$. Suppose that $\tilde{X}_i = X_i \cup \{e_i\}$, where $e_i \notin X_i$ and e_i is the greatest element in $\tilde{X}_i$. Take two non-zero vectors $x_1, y_1 \in X_i$. Then $x_2 := e_i * x_1,\ y_1 := e_i * y_1 \in X_i \setminus \{0,\}$ and $x_1 + x_2 = e_i = y_1 + y_2$ in $\tilde{X}_i$. We assert that

$$g(x_1) + g(x_2) = g(y_1) + g(y_2).$$

Indeed, according to Lemma 3.3, there are four elements $c_{11}, c_{12}, c_{21}, c_{22} \in \tilde{X}_i$ such that $x_1 = c_{11} + c_{12}$, $x_2 = c_{21} + c_{22}$, $y_1 = c_{11} + c_{21}$, $y_2 = c_{12} + c_{22}$. It is clear that $c_{11}, c_{12}, c_{21}, c_{22} \in X_i$ and all sums in the last four equalities exist also in X_i.

Calculate

$$\begin{aligned} g(x_1) + g(x_2) &= g(c_{11} + c_{12}) + g(c_{21} + c_{22}) \\ &= g(c_{11}) + g(c_{12}) + g(c_{21}) + g(c_{22}) \\ &= g(c_{11}) + g(c_{21}) + g(c_{12}) + g(c_{22}) \\ &= g(c_{11} + c_{21}) + g(c_{12} + c_{22}) \\ &= g(y_1) + g(y_2). \end{aligned}$$

Then $\hat{g}: \tilde{X} \to G_1$ defined via $\hat{g}(x) := g(x)$ if $x \in X$, and $\hat{g}(e_i) := g(x_1) + g(x_2)$, if $x_1 + x_2 = e_i$, $x_1, x_2 \in X_i \setminus \{0\}$, preserves order and sums in $\tilde{X}$. Due to Theorem 5.6, (G,h) is a universal group for $(\tilde{X};+,*,0)$, so that there exists a unique group homomorphism $g': G \to G_1$ such that $\hat{g} = g' \circ h$. Consequently, $g = g' \circ h_0$, and g' is a unique group homomorphism satisfying the equation $g = g' \circ h_0$. **Q.E.D.**

Remark 6.8 We recall that Example 5.2 gives a unitarily extendible BCK-algebra which has not u.d. union property. Indeed, the function $f: X \setminus$

$\{0\} \to X \setminus \{0\}$ in question can be chosen as follows: $f(4) = 2$, $f(3) = 1$, $f(2) = 4$, $f(1) = 3$. According to Theorem 6.7, it has a universal group which is a lattice ordered one, namely Z^2, as it has been shown in (iii) of Remark 5.9.

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