Mathematical Institute
Slovak Academy of Sciences, Bratislava

Preprint 2/1999

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Some remarks on the moment problem

March 2, 1999

PREPRINT SERIES

SOME REMARKS ON THE MOMENT PROBLEM

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Abstract. We solve the moment problem for the polar margins of a physically relevant class of phase space observables.

1. Introduction and notations

The representation of an observable as a positive operator measure in the Hilbert space formulation of quantum mechanics arises the question of determining the moment operators of an observable and of the uniqueness of the operator measure in terms of its moment operators. Though the classical moment problem has been thoroughly investigated, see, e.g., [1], [2], [3], [4], much less has been published on the operator measure counterpart of the problem. In this paper we investigate the operator measure formulation of the moment problem, and we solve the problem for the unsharp phase and number observables which are obtained as the polar coordinate angle and radial margins of the phase space observables.

Let $\mathcal{H}$ be a complex Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the set of bounded operators on $\mathcal{H}$. Let Ω be a nonempty set and $\mathcal{A}$ a σ-algebra of subsets of Ω. Let $E : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ be an operator measure, and let $E_{\psi,\varphi}$, $\psi, \varphi \in \mathcal{H}$, denote the complex measure defined by $E_{\psi,\varphi}(X) = \langle\, \psi \mid E(X)\varphi \,\rangle$. In this paper we always assume that the operator measures are positive, that is, $E(X) \geq O$ for all $X \in \mathcal{A}$, and normalized, that is, $E(\Omega) = I$. Let $f : \Omega \to \mathbb{C}$ be an $\mathcal{A}$-measurable function and let $\mathcal{D}(L^E(f))$ denote the set of those vectors $\varphi \in \mathcal{H}$ for which f is $E_{\psi,\varphi}$-integrable for each $\psi \in \mathcal{H}$. The integral of f with respect to E is the unique linear operator $L^E(f)$, with the domain $\mathcal{D}(L^E(f))$, such that $\langle\, \psi \mid L^E(f)\varphi \,\rangle = \int_\Omega f\, dE_{\psi,\varphi}$ for all $\varphi \in \mathcal{D}(L^E(f)), \psi \in \mathcal{H}$. We denote this operator also by $\int f\, dE$. Further details and properties of this notion of an operator integral are given in [5] and [6].

Assume now that $\Omega = \mathbb{R}$, with $\mathcal{A}$ being the σ-algebra $\mathcal{B}(\mathbb{R})$ of the Borel subsets of the real line $\mathbb{R}$. Let id denote the identity function on $\mathbb{R}$. Let $\mathbb{N}$ denote the set of natural numbers including 0. For any $k \in \mathbb{N}$, we call the linear operator $L^E(id^k)$ the *k-th moment operator* of the operator measure E. We also write $L^E(id^k) = \int_\mathbb{R} x^k\, dE(x)$. In general, $L^E(id^k)$ need not be densely defined; it is even possible that $\mathcal{D}(L^E(id^k)) = \{0\}$. We call the number $\langle\, \psi \mid L^E(id^k)\varphi \,\rangle = \int_{\mathbb{R}} id^k \, dE_{\psi,\varphi}$, $\psi \in \mathcal{H}, \varphi \in \mathcal{D}(L^E(id^k))$, the *k-th moment* of the complex measure $E_{\psi,\varphi}$.

2. Operator measures concentrated on bounded sets

Consider an operator measure $E : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ which is concentrated on a closed interval $[a,b]$ of the real line $\mathbb{R}$. Its moment operators $L^E(id^k)$ are bounded self-adjoint operators. It is a well-known consequence of the Weierstrass approximation theorem and the uniqueness part of the Riesz representation theorem that the moment operators $L^E(id^k)$ determine E uniquely. We ask under what conditions the sequence of bounded self-adjoint operators may appear as a sequence of the moment operators of an operator measure concentrated on $[a,b]$. A characterization of such operator sequences is given, for instance, in [7, Theorem II, p. 463]. The multitude of the polynomial conditions of that result may, however, be reduced by using the notion of completely monotone operator sequences.

We say that a sequence of bounded self-adjoint operators $(S_k)_{k=0}^{\infty} \subset \mathcal{L}(\mathcal{H})$ is *completely monotone* with respect to an interval $[a,b]$ if, for all $m, n \in \mathbb{N}$,

$$\sum_{j=0}^{n} \sum_{i=0}^{m+j} \frac{(-1)^j}{(b-a)^j} (-a)^i \binom{n}{j} \binom{m+j}{i} S_{m+j-i} \geq O.$$

Using a simple transformation, the result [8, Theorem 4.2] can easily be adopted to characterize such operator sequences.

Theorem 2.1. *A sequence of bounded self-adjoint operators $(S_k)_{k=0}^{\infty} \subset \mathcal{L}(\mathcal{H})$, with $S_0 = I$, is a sequence of the moment operators of an operator measure E concentrated on the interval $[a,b]$ if and only if $(S_k)_{k=0}^{\infty}$ is completely monotone with respect to the interval $[a,b]$.*

3. Operator measures with unbounded support

Consider now an operator measure $E : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ which is not necessarily concentrated on a bounded set. Its moment operators $L^E(id^k)$, $k \in \mathbb{N}$, are symmetric, that is, $\langle\, \psi \mid L^E(id^k)\varphi \,\rangle = \langle\, L^E(id^k)\psi \mid \varphi \,\rangle$ for all $\psi, \varphi \in \mathcal{D}(L^E(id^k))$, but they need not be densely defined. Assuming that $\mathcal{D} \subseteq \cap_{k=0}^{\infty} \mathcal{D}(L^E(id^k))$, one may ask whether for a given $\varphi \in \mathcal{D}$ the positive measure $E_{\varphi,\varphi}$ is determinate, that is, if the moment sequence $(\langle\, \varphi \mid L^E(id^k)\varphi \,\rangle)_{k\geq 0}$ determines the measure $E_{\varphi,\varphi}$. One may also ask if the totality of the moment sequences $(\langle\, \varphi \mid L^E(id^k)\varphi \,\rangle)_{k\geq 0}, \varphi \in$

$\mathcal{D}$, determines the moment operators $L^E(id^k)$, and, finally, if the moment operators $L^E(id^k)$ determine the operator measure E.

It is well known that in the unbounded case a moment sequence $(\langle\, \varphi \mid L^E(id^k)\varphi\,\rangle)_{k=0}^{\infty}$, $\varphi \in \mathcal{D}$, need not determine the measure $E_{\varphi,\varphi}$. Indeed, consider the operator measures E and F which are concentrated on $[0,\infty)$ and defined as follows:

$$\begin{aligned} E(X) &= \frac{1}{24} I \int_X e^{-u^{1/4}}\, du \\ F(X) &= \frac{1}{24} I \int_X e^{-u^{1/4}} [1 - \sin(u^{1/4})]\, du \end{aligned}$$

for all $X \in \mathcal{B}([0,\infty))$. Then (see, e.g., [9, p. 126])

$$\int_0^\infty x^k\, dE_{\varphi,\varphi}(x) = \frac{1}{4}(4k+3)! = \int_0^\infty x^k\, dF_{\varphi,\varphi}(x)$$

for all $\varphi \in \mathcal{H}$, $k \in \mathbb{N}$. Thus $L^E(id^k) = L^F(id^k)$ for all $k \in \mathbb{N}$, though $E \neq F$; in fact, $E_{\varphi,\varphi} \neq F_{\varphi,\varphi}$ for each $\varphi \in \mathcal{H} \setminus \{0\}$.

The classical results concerning the Hamburger and the Stieltjes moment problems (see, e.g., [3], or [4]) lead immediately to necessary conditions for a sequence of (unbounded) self-adjoint operators $(S_k)_{k=0}^{\infty}$ to appear as the sequence of the moment operators of an operator measure E concentrated on $\mathbb{R}$ and $\mathbb{R}^+$, respectively. In view of the applications to be discussed, we give the Stieltjes version of such conditions: a necessary condition that a sequence of (unbounded) self-adjoint operators $(S_k)_{k=0}^{\infty}$ is the sequence of the moment operators of an operator measure $E : \mathcal{B}([0,\infty)) \to \mathcal{L}(\mathcal{H})$ is that

$$\sum_{l,m=0}^{n} \overline{\alpha}_l \alpha_m S_{l+m} \geq O$$

and

$$\sum_{l,m=0}^{n} \overline{\alpha}_l \alpha_m S_{l+m+1} \geq O$$

for every finite sequence $\alpha_0, \cdots, \alpha_n$ of complex numbers, with the understanding that the operator inequalities are defined on the respective domains $\cap_{k=0}^{2n} \mathcal{D}(S_k)$ and $\cap_{k=0}^{2n+1} \mathcal{D}(S_k)$.

In the sequel we shall restrict our attention to the case of exponentially bounded positive Borel measures, which appear in the applications to be considered, and which are known to have the following powerful properties (see, for instance, [10, Theorems II.4.3 and II.5.2])

Lemma 3.1. *Let $\mu : \mathcal{B}(\mathbb{R}) \to [0,\infty]$ be a positive Borel measure such that for some $a > 0$*

$$\int_{\mathbb{R}} e^{a|t|}\, d\mu(t) < \infty.$$

a) All the moments of μ are finite, that is, the integrals

$$s_k = \int_{\mathbb{R}} t^k\, d\mu(t), \quad k \in \mathbb{N},$$

exist.
b) The monomials t^k, $k \in \mathbb{N}$, are complete in $L^2(\mathbb{R},\mu)$, that is, if $f \in L^2(\mathbb{R},\mu)$ is such that

$$\int_{\mathbb{R}} f(t) t^k\, d\mu(t) = 0$$

for all $k \in \mathbb{N}$, then $f(t) = 0$ μ-a.e..
c) The measure μ is determinate, that is, it is uniquely determined by its moment sequence $(s_k)_{k=0}^{\infty}$.

4. The phase space observables $A^{|s\rangle}$

The polar margins of the phase space observables constitute a class of physically interesting examples of bounded and unbounded operator measures for which the moment problems can be solved using the above results. To do that we shall first introduce the relevant phase space observables. In the following we assume that the Hilbert space is separable.

For any two vectors $\varphi, \psi \in \mathcal{H}$, we let $|\varphi\rangle\langle\psi|$ denote the operator $|\varphi\rangle\langle\psi|(\eta) = \langle\psi\,|\,\eta\rangle\,\varphi$. Let $(|\,n\rangle)_{n=0}^{\infty}$ be a fixed orthonormal basis of $\mathcal{H}$. We call it the number basis. Let

$$\begin{aligned} N &= \sum_{n\geq 0} n|n\rangle\langle n|, \\ a &= \sum_{n\geq 0} \sqrt{n+1}|n\rangle\langle n+1|, \\ a^* &= \sum_{n\geq 0} \sqrt{n+1}|n+1\rangle\langle n|, \end{aligned}$$

be the associated number, lowering, and raising operators, with the appropriate domains. Let $\lambda : \mathcal{B}(\mathbb{C}) \to [0,\infty]$ be the two-dimensional Lebesgue measure. Let $D_z = e^{za^* - \bar{z}a}$, $z \in \mathbb{C}$, be the (unitary) shift operator, and let T be any positive trace one operator, that is, a state. The set function

$$\mathcal{B}(\mathbb{C}) \ni Z \mapsto A^T(Z) := \frac{1}{\pi}\int_Z D_z T D_z^*\, d\lambda(z) \in \mathcal{L}(\mathcal{H}),$$

is then a (positive normalized) operator measure, the phase space observable associated with the state T [11],[12]. It is known [13] that the only phase space observables A^T whose polar coordinate marginal measures define number and phase observables are those which arise from the number states $T = |s\rangle\langle s|$, or from their mixtures $T = \sum_{s=0}^{\infty} \lambda_s |s\rangle\langle s|$, $\lambda_s \geq 0$, $\sum_{s=0}^{\infty} \lambda_s = 1$. Therefore, from now on we only consider the phase space observables which are defined by the number states $|s\rangle\langle s|$. We denote them by $A^{|s\rangle}$.

5. The polar margins of the phase space observable $A^{|s\rangle}$

Writing $\mathbb{C} \ni z = |z|e^{i\arg z} = re^{i\theta}, (r,\theta) \in [0,\infty) \times [0,2\pi)$ we may define the polar coordinate marginal measures of A^T:

$$\begin{aligned}
\mathcal{B}([0,\infty)) \ni R \mapsto A^{|s\rangle}((R \times [0,2\pi)) &=: \tilde{E}^{|s\rangle}(R) \in \mathcal{L}(\mathcal{H}),\\
\mathcal{B}([0,2\pi)) \ni X \mapsto A^{|s\rangle}([0,\infty) \times X) &=: F^{|s\rangle}(X) \in \mathcal{L}(\mathcal{H}).
\end{aligned}$$

The operator measures $\tilde{E}^{|s\rangle}$ and $F^{|s\rangle}$, $s \in \mathbb{N}$, are known to represent an unsharp number observable and the conjugate unsharp phase observable, respectively [14],[15]. The unsharp phase observable $F^{|s\rangle}$ is an operator measure concentrated on a bounded interval of the real line, so that its moment problem is completely solved by the discussion in Section 2. On the other hand, the unsharp number observable $\tilde{E}^{|s\rangle}$ is unbounded, so that it could well happen that its moment operators do not suffice to determine the operator measure $\tilde{E}^{|s\rangle}$. We shall turn to investigate the moment problem for the unsharp number observables. For notational simplicity, we consider from now on instead of $\tilde{E}^{|s\rangle}$ the operator measures $E^{|s\rangle}$ defined by

$$E^{|s\rangle}(X) = \tilde{E}^{|s\rangle}(\sqrt{X}),\ X \in \mathcal{B}([0,\infty)),$$

so that, in particular,

$$\int_0^{\infty} x^k\, dE^{|s\rangle}(x) = \int_0^{\infty} (x^2)^k\, d\tilde{E}^{|s\rangle}(x),$$

for all $k \geq 0$.

6. The moment problem for the operator measure $E^{|s\rangle}$

The moment operators of the operator measures $E^{|s\rangle}$, $s \in \mathbb{N}$, have already been determined [5]. They are:

$$\begin{aligned}
L^{E^{|s\rangle}}(id^k) &= \sum_{n=0}^{\infty} \sigma(s,k,n)|n\rangle\langle n| =: \sigma(s,k,N)\\
\mathcal{D}(L^{E^{|s\rangle}}(id^k)) &= \mathcal{D}(N^k),
\end{aligned}$$

where

$$\begin{aligned}\sigma(s,k,n) &= \sum_{l,l'=0}^{[n,s]}(-1)^{l+l'}\binom{s}{l}\binom{n}{l'}\binom{n+k+s-l-l'}{n-l}\binom{k+s-l'}{k}k!\\ &= \sum_{i,j=0}^{\infty}a_{ij}s^{k-j}n^i,\quad a_{i,j}\in\mathbb{N},\end{aligned}$$

with $[n,s] := \min(n,s)$. We now have $(\mid n\rangle)_{n=0}^{\infty}\subset\cap_{k\geq 0}\mathcal{D}(L^{E^{|s\rangle}}(id^k))$, and the complex measures $E^{|s\rangle}_{|n\rangle,|m\rangle}$ associated with the number basis $(\mid n\rangle)_{n=0}^{\infty}$ are easily seen to be the following:

$$\begin{aligned}E^{|s\rangle}_{|n\rangle,|m\rangle}(X) &= \langle\, n\mid E^{|s\rangle}(X)|n\,\rangle\\ &= \Big\langle\, n\mid A^{|s\rangle}(\sqrt{X}\times[0,2\pi))|m\,\Big\rangle = 0,\quad\text{for } n\neq m,\\ E^{|s\rangle}_{|n\rangle,|n\rangle}(X) &= \langle\, n\mid E^{|s\rangle}(X)|n\,\rangle = \Big\langle\, n\mid A^{|s\rangle}(\sqrt{X}\times[0,2\pi))|n\,\Big\rangle\\ &= \int_X g_n^{(s)}(x)\,dx,\end{aligned}$$

where

$$g_n^{(s)}(x) = \frac{n!}{s!}\sum_{\ell,\ell'=0}^{[n,s]}(-1)^{l+l'}\binom{s}{\ell}\binom{s}{\ell'}\frac{(x)^{s+n-\ell-\ell'}}{(n-\ell)!(n-\ell')!}\,e^{-x}.$$

We denote $E^{|s\rangle}_{|n\rangle,|n\rangle} =: \mu_n^{(s)}$. The probability measure $\mu_n^{(s)}$ is absolutely continuous with respect to the Lebesque measure μ_L, and its Radon-Nikodym derivative $d\mu_n^{(s)}/d\mu_L$ is $g_n^{(s)}$. In fact, the density $g_n^{(s)}$ has the form

$$g_n^{(s)}(x) = h_n^{(s)}e^{-x},$$

where

$$h_n^{(s)}(x) = \frac{n!}{s!}\sum_{\ell,\ell'=0}^{[n,s]}(-1)^{l+l'}\binom{s}{\ell}\binom{s}{\ell'}\frac{x^{s+n-\ell-\ell'}}{(n-\ell)!(n-\ell')!}.$$

This shows immediately that $\int_0^\infty e^{ax}\,d\mu_n^{(s)}(x) < \infty$ for any $0\leq a<1$. By Lemma 3.1 we may thus conclude:

Corollary 6.1. *For all integers $s,n\geq 0$, the monomials x_k, $k\in\mathbb{N}$, are complete in $L^2(\mu_n^{(s)},[0,\infty))$, and the measure $\mu_n^{(s)}$ is determinate.*

Corollary 6.2. *For any $s\geq 0$, the operator measure $E^{|s\rangle}$ is uniquely determined by the moment operator sequence $(\sigma(s,k,N))_{k=0}^{\infty}$.*

Proof. For given $s \geq 0$, assume that there is an operator measure $E : \mathcal{B}([0,\infty)) \to \mathcal{L}(\mathcal{H})$ for which $L^E(id^k) = \sigma(s,k,N)$, for all $k \geq 0$. We recall that the operator measure E is uniquely determined by the complex measures $E_{|n\rangle,|m\rangle}$, $n,m \geq 0$. For any $n \geq 0$, the sequence $(\sigma(s,k,n))_{k=0}^{\infty}$ determines uniquely the probability measure $E^{|s\rangle}_{|n\rangle,|n\rangle}$ so that $E_{|n\rangle,|n\rangle} = E^{|s\rangle}_{|n\rangle,|n\rangle}$. Assume now that $n \neq m$. Let $\mu_{m+i^l n}$ and $\mu^s_{m+i^l n}$ denote the positive measures $E_{|m\rangle+i^l|n\rangle,|m\rangle+i^l|n\rangle}$ and $E^{|s\rangle}_{|m\rangle+i^l|n\rangle,|m\rangle+i^l|n\rangle}$, respectively. Using the polar decomposition we may write

$$E_{|n\rangle,|m\rangle} = \frac{1}{4}\sum_{l=0}^{3} i^l \mu_{m+i^l n}.$$

By assumption,

$$\begin{aligned} \int_0^\infty x^k \, d\mu_{m+i^l n} &= \int_0^\infty x^k \, d\mu^{(s)}_{m+i^l n} \\ &= \int_0^\infty x^k (h_n^{(s)} + h_m^{(s)}) e^{-x} \, dx. \end{aligned}$$

The moment sequence $(\int_0^\infty x^k (h_n^{(s)} + h_m^{(s)}) e^{-x} \, dx)_{k=0}^{\infty}$ determines again uniquely the measure $\mu^{(s)}_{m+i^l n}$, so that $\mu^{(s)}_{m+i^l n} = \mu_{m+i^l n}$. But then

$$\begin{aligned} E_{|n\rangle,|m\rangle} &= \frac{1}{4}\sum_{l=0}^{3} i^l \mu_{m+i^l n} \\ &= \frac{1}{4}\sum_{l=0}^{3} i^l \mu^{(s)}_{m+i^l n} = E^{|s\rangle}_{|n\rangle,|m\rangle}, \end{aligned}$$

which concludes the proof.

Acknowledgement A.D. is very grateful to the Finish Academy for organizing his stay at the Institute of Theoretical Physics, University of Turku in Oct. 1998, when the present paper was initiated.

The paper was partially supported by the grant VEGA No. 4033/98 of the Slovak Academy of Sciences.

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