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RELATIVELY ORTHOMODULAR LATTICES

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Abstract.
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M.F. Janowitz defined a generalized orthomodular lattice [GOM–lattice, GOML] as a lattice with a smallest element 0 and with an orthogonality relation $\perp$. He proved that any such a lattice is isomorphic to a prime ideal of an orthomodular lattice [OM–lattice, OML]. Equivalently, a GOM–lattice can be defined as a lattice with a smallest element 0, in which every interval containing 0 is an OM–lattice and a natural additional condition is satisfied.

We define a relatively orthomodular lattice [ROM–lattice, ROML] as a lattice with a commutativity relation C, in such a way that every GOM–lattice is an ROM–lattice. Equivalently, an ROM–lattice can be defined as a lattice in which every interval is an OM–lattice and a natural additional condition is satisfied. We prove that every ROM–lattice is isomorphic to a prime dual ideal of a GOM–lattice and that this embedding is in some sense minimal. Consequently, an ROM–lattice can be defined as a sublattice of an OM–lattice closed under the relative orthocomplements.

Introduction

The purpose of this paper is a primary investigation of lattices, which we call relatively orthomodular lattices and which include convex sublattices of orthomodular lattices. The process will be similar to that of M.F. Janowitz who introduced in [4] generalized orthomodular lattices which include ideals of orthomodular lattices. Because we shall frequently use the names of just mentioned classes of lattices, we introduce the following abbreviations:

orthomodular lattice = OM–lattice = OML,
generalized orthomodular lattice = GOM–lattice = GOML,
relatively orthomodular lattice = ROM–lattice = ROML.

Though our aim in this article is to investigate some generalization of an OM–lattice, nevertheless a basic notion remains the notion of an OM–lattice. As in the case of GOM–lattices (see [4]), a primary developing of the theory of ROM–lattices showed, in fact, that we have stayed in the original theory of OM–lattices. Therefore we begin with several observations on OM–lattices, relating to the orthogonality relation, the commutativity relation and the relative orthocomplements, in Section 1. Usually, an OM–lattice is axiomatically defined as a bounded lattice on which a unary operation called an orthocomplementation is given having some additional properties. In Section 1 we also show that an OM–lattice L can alternatively be defined in such a way that instead of the orthocomplementation operation on L the orthogonality relation on L or the commutativity relation on L is used.

In Section 2 we recall some known facts about GOM–lattices. Janowitz's embedding theorem states that every GOM–lattice is isomorphic to a prime ideal of an

OM–lattice. In Section 3 we start with a definition and a very elementary theory of ROM–lattices and we show how OM–lattices and GOM–lattices are connected with ROM–lattices. GOM–lattices are axiomatically defined by using an orthogonality relation $\perp$ on them. For ROM–lattices we use an axiomatic description of the commutativity relation C on them.

Section 4 contains our main result which is that an ROM–lattice is a prime dual ideal of a suitably defined GOM–lattice. Since every GOM–lattice is a prime ideal of a suitably defined OM–lattice, ROM–lattices are exactly the sublattices of OM–lattices, closed under the relative orthocomplements. Therefore we think that the name "relatively orthomodular lattice" is justified.

It is also shown that the GOM–lattice extension of a given ROM–lattice constructed in the main theorem is minimal which means that the extension is contained (in a natural sense) in every GOM–lattice containing the given ROM–lattice.

The theory of OM–lattices is widely developed, we refer to books of G. Kalmbach [6] and L. Beran [1]; several parts of a book of F. Maeda and S. Maeda [8] are also devoted to OM–lattices. Two chapters of a book of L. Beran [1] are treated with GOM–lattices while the beginning of a book of F. Maeda and S. Maeda [8] is treated with lattices which are more general than GOM–lattices (cf. also a previous article of S. Maeda [9]).

For general lattice theory we refer to G. Birkhoff [2].

As usual, when $(L, \wedge, \vee)$ is a lattice and $x, y \in L$ are such that $x \leq y$, an interval $[x, y]$ is the set $\{z \in L | x \leq z \leq y\}$.

1. Orthomodular lattices

In this section we recall some basic properties of orthomodular lattices relating to the orthogonality relation, the commutativity relation and the relative orthocomplements.

A system $(L, \wedge, \vee, 0, 1, ^\sharp)$ is called an *orthomodular lattice*, briefly an *OM–lattice*, an *OML*, if $(L, \wedge, \vee, 0, 1)$ is a bounded lattice and $^\sharp$ is a unary operation on L (called an *orthocomplementation*) having the following properties: $x^\sharp$ is a complement of x, $x^{\sharp\sharp} = x$, $x \leq y$ implies $y^\sharp \leq x^\sharp$, and the *orthomodular law*, $x \leq y$ implies $x \vee (y \wedge x^\sharp) = y$. The element $x^\sharp$ is called the *orthocomplement* of x.

The orthomodular law is equivalent with each of the following three conditions: $x \leq y$ implies $y \wedge (x \vee y^\sharp) = x$; $x \leq y$ and $y \wedge x^\sharp = 0$ implies $x = y$; $x \leq y$ implies $x \leq z^\sharp$ and $x \vee z = y$ for some z. (For the latter condition see [8], Theorem (29.13).)

A system $(L, \wedge, \vee, 0, 1, ^\sharp)$ satisfying all defining properties of an OM–lattice except the orthomodular law is called an *ortholattice* or an *orthocomplemented lattice*.

As known, every interval $[x, y]$, where $x, y \in L$ and $x \leq y$, in an OM–lattice $(L, \wedge, \vee, 0, 1, ^\sharp)$, is an OM–lattice itself under an orthocomplementation $z \mapsto z^{\sharp_{x,y}} = x \vee (y \wedge z^\sharp) = y \wedge (x \vee z^\sharp)$, see [8], [6], [1]. The unary operation $^{\sharp_{x,y}}$ on the set $[x, y]$ is called the *relative orthocomplementation* on $[x, y]$ and the element $z^{\sharp_{x,y}}$ is called the *relative orthocomplement* of z in the interval $[x, y]$. If $x \in L$, the symbol $^{\sharp_x}$ will be used as an abbreviation for the symbol $^{\sharp_{0,x}}$. Thus, if $y \in [0, x]$ then the element $y^{\sharp_x} = x \wedge y^\sharp$ is the relative orthocomplement of y in the interval $[0, x]$.

Concerning the relative orthocomplements in an OM–lattice $(L, \wedge, \vee, 0, 1, ^\sharp)$, the next observation is of interest. If $x, y, z, u, v \in L$ and $u \leq x \leq z \leq y \leq v$ then

$$z^{\sharp_{x,y}} = y \wedge (x \vee z^{\sharp_{u,v}}) = x \vee (y \wedge z^{\sharp_{u,v}}).$$

(See [1], Theorems 2.3, 3.4, 4.2, 3.10 and Lemma 6.2 in Chapter II.)

If $(L, \wedge, \vee, 0, 1, ^\sharp)$ is an OM–lattice and if $x, y \in L$ then it is said that x, y are *orthogonal* (in notation $x \perp y$) if $x \leq y^\sharp$. The binary relation $\perp$ on L is called an *orthogonality relation* of L. As known, the orthogonality $\perp$ has the following properties ($x, y, z \in L$): $x \perp x$ implies $x = 0$; $x \perp y$ implies $y \perp x$; $x \perp y$ and $z \leq x$ implies $z \perp y$; $x \perp y$ and $x \perp z$ implies $x \perp y \vee z$; $x \leq y$ implies there exists z such that $x \perp z$ and $x \vee z = y$. (See [8], Lemma (29.3), Remark (29.5) and Theorem (29.13).) Let us note that in the latter condition, the element z is unique, namely z is just the relative orthocomplement of x in the interval $[0, y]$, i.e., $z = x^{\sharp_y}$ (cf. [8], Remark (29.14)).

If $(L, \wedge, \vee, 0, 1, ^\sharp)$ is an OM–lattice and $x, y \in L$ then it is said that x, y *commute* (in notation xCy) if $x = (x \wedge y) \vee (x \wedge y^\sharp)$. The binary relation C on L is called a *commutativity relation* of L. As known, the commutativity C has the following properties ($x, y, z, t \in L$): $x \leq y$ implies xCy; xCy implies yCx; xCy implies $xCy^\sharp$; xCy and xCz implies $xCy \wedge z$ and $xCy \vee z$; xCy and xCz implies $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$; $x \leq y \leq z$ implies there exists t such that yCt, $y \wedge t = x$ and $y \vee t = z$. (See [1], Theorems 2.3, 3.4, 4.2, 3.10 and Lemma 6.2 in Chapter II.) The element t in the latter condition is unique, t is just the relative orthocomplement of y in the interval $[x, z]$, i.e., $t = y^{\sharp_{x,z}}$. Let us recall that for $x, y \in L$, xCy if and only if $x = (x \vee y) \wedge (x \vee y^\sharp)$ (cf. [1], Theorem 3.6 in Chapter II). Let us also remember the following property. If $x, y, u, v \in L$, $u \leq v$ and $x, y \in [u, v]$ then

$$x, y \text{ commute in } L \quad \text{if and only if} \quad x, y \text{ commute in } [u, v].$$

(See [1], Chapter II, Theorem 6.4.)

The relations $\perp$ and C are connected by the following principle:

$$x \perp y \quad \text{if and only if} \quad xCy \quad \text{and} \quad x \wedge y = 0.$$

For our purposes it is useful to remember alternative definitions of $\perp$ and C:

$$x \perp y \quad \text{if and only if} \quad x = y^{\sharp_{x \vee y}},$$

$$xCy \quad \text{if and only if} \quad x = y^{\sharp_{x \wedge y, x \vee y}}.$$

We will use these facts in the next two sections.

The following two results show that an OM–lattice $(L, \wedge, \vee, 0, 1, ^\sharp)$ can alternatively be defined in such a way that instead of the unary operation $^\sharp$ on L, the binary relation $\perp$ on L or the binary relation C on L is used.

1.1. Theorem. *The orthogonality relation $\perp$ of an OM–lattice $(L, \wedge, \vee, 0, 1, ^\sharp)$ satisfies the following conditions ($x, y, z \in L$):*

(P1) $x \perp x$ *implies* $x = 0$,
(P2) $x \perp y$ *and* $z \leq x$ *implies* $y \perp z$,
(P3) $x \perp y$ *and* $x \perp z$ *implies* $x \perp y \vee z$,
(P4) $x \leq y$ *implies there exists* z *such that* $x \perp z$ *and* $x \vee z = y$.

Moreover, for every $x \in L$, the element $x^\sharp$ is a unique element of L such that $x \perp x^\sharp$ and $x \vee x^\sharp = 1$.

Conversely, let $(L, \wedge, \vee, 0, 1, \perp)$ be a bounded lattice with a binary relation $\perp$ on L satisfying conditions (P1)–(P4). Then for every $x \in L$ there is a unique $x^\sharp \in L$ such that $x \perp x^\sharp$ and $x \vee x^\sharp = 1$. Moreover, $(L, \wedge, \vee, 0, 1, ^\sharp)$ is an OM–lattice and for all $x, y \in L$,

$$x \perp y \quad \textit{if and only if} \qquad x \leq y^\sharp .$$

Remark. Instead of a single condition (P2) the following two conditions are usually used in the literature:

(i) $x \perp y$ implies $y \perp x$,
(ii) $x \perp y$ and $z \leq x$ implies $z \perp y$.

Evidently, condition (P2) is equivalent with the system $\{(i), (ii)\}$ even in any system $(L, \leq, \perp)$, where L is a set, $\leq$ is a reflexive binary relation on L and $\perp$ is a binary relation on L. So, we shall use the symmetry (i) of the binary relation $\perp$ without any reference in any such a system $(L, \leq, \perp)$ satisfying condition (P2).

Proof of Theorem. The first part of Theorem is successively proved, for instance, in [8], Lemma (29.3), Remark (29.5), Theorem (29.13) and Remark (29.14).

The second part of Theorem follows, for example, by [9], Theorem 1 and Lemma 1. □

1.2. Theorem. *The commutativity relation C of an OM–lattice $(L, \wedge, \vee, 0, 1, ^\sharp)$ satisfies the following conditions ($x, y, z, t \in L$):*

(C1) xCy *and* xCz *implies* $y \vee zCx$ *and* $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,
(C2) $x \leq y \leq z$ *implies there exists* t *such that* yCt, $y \wedge t = x$ *and* $y \vee t = z$.

Moreover, for every $x \in L$, the element $x^\sharp$ is a unique element of L such that $xCx^\sharp$, $x \wedge x^\sharp = 0$ and $x \vee x^\sharp = 1$.

Conversely, let $(L, \wedge, \vee, 0, 1, C)$ be a bounded lattice with a binary relation C on L satisfying conditions (C1) and (C2). Then for every $x \in L$ there exists a unique $x^\sharp \in L$ such that $xCx^\sharp$, $x \wedge x^\sharp = 0$ and $x \vee x^\sharp = 1$. Moreover, $(L, \wedge, \vee, 0, 1, ^\sharp)$ is an OM–lattice and for all $x, y \in L$,

$$xCy \quad \textit{if and only if} \quad x = (x \wedge y) \vee (x \wedge y^\sharp).$$

Proof. The first part of Theorem is well known (see e.g., [1], Chapter II, Theorems 3.4, 4.2, Lemma 3.8, Theorem 2.3 (iii), Lemma 6.2 and Proposition 5.3).

The converse part of Theorem follows by (iv) of Lemma 3.2 and by Proposition 3.4. □

The next Theorem is analogous using another axiom system for C.

1.3. Theorem. *The commutativity relation C of an OM–lattice $(L, \wedge, \vee, 0, 1, ^\sharp)$ satisfies the following conditions ($x, y, z, t \in L$):*

(D1) xCy *implies* yCx,

(D2) xCy *and* xCz *implies* $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ *and* $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$,

(D3) $x \le y \le z$ *implies there exists* t *such that* yCt, $y \wedge t = x$ *and* $y \vee t = z$.

Moreover, for every $x \in L$, the element $x^\sharp$ is a unique element of L such that $xCx^\sharp$, $x \wedge x^\sharp = 0$ and $x \vee x^\sharp = 1$.

Conversely, let $(L, \wedge, \vee, 0, 1, C)$ be a bounded lattice with a binary relation C on L satisfying conditions (D1)–(D3). Then for every $x \in L$ there exists a unique $x^\sharp \in L$ such that $xCx^\sharp$, $x \wedge x^\sharp = 0$ and $x \vee x^\sharp = 1$. Moreover, $(L, \wedge, \vee, 0, 1, ^\sharp)$ is an OM–lattice and for all $x, y \in L$,

$$xCy \quad \text{if and only if} \quad x = (x \wedge y) \vee (x \wedge y^\sharp).$$

Proof. The first part of Theorem is well known (see e.g., [1], Chapter II, Theorems 3.4, 3.10, Lemma 6.2, Theorem 2.3 (iii) and Proposition 5.3).

The converse part of Theorem follows by Theorems 3.6 and 1.2. Further proofs of this part of Theorem can be obtained by combining corresponding results in Section 3. □

2. Generalized orthomodular lattices

Generalized orthomodular lattices were introduced by M.F. Janowitz in [4] and it was shown that their study can be reduced to the study of ideals of orthomodular lattices.

2.1. Definition. Let $(L, \wedge, \vee, 0)$ be a lattice with a smallest element and let $\perp$ be a binary relation on L such that conditions (P1) – (P4) are satisfied for all $x, y, z \in L$. The system $(L, \wedge, \vee, 0, \perp)$ is called a *generalized orthomodular lattice* (abbreviated, a *GOM–lattice*, a *GOML*) and $\perp$ is called an *orthogonality relation* on $(L, \wedge, \vee, 0)$. If $x, y \in L$ and $x \perp y$ we say that x, y are *orthogonal*.

Remark. Conditions (P1)–(P4) are introduced in Theorem 1.1.

In the above definition of a GOM–lattice we used a slightly simpler system of axioms than in [4]. That this system is equivalent with the original one follows from the following observations.

Let $(L, \wedge, \vee, 0, \perp)$ be a GOM–lattice. By (P4), there exists $x \in L$ such that $0 \perp x$ and $0 \vee x = 0$, hence $x = 0$ and thus $0 \perp 0$. If $x, y \in L$ and $x \perp y$ then by (P2), $y \perp x$. (A GOM–lattice was defined in [4] as a lattice L with 0 on which there is given a binary relation $\perp$ satisfying the following axioms for all $x, y, z \in L$: $x \perp x$ iff $x = 0$; $x \perp y$ implies $y \perp x$; $x \perp y$, $z \le x$ implies $z \perp y$; $x \perp y$, $x \perp z$ implies $x \perp y \vee z$; if $x \le y$ then there exists z such that $x \perp z$ and $x \vee z = y$.)

Every OM–lattice is a GOM–lattice, as is observed in the previous Section.

As known, if $(L, \wedge, \vee, 0, \perp)$ is a GOML and $x, y \in L$ then $x \leq y$ implies there exists a unique $z \in L$ such that $x \perp z$ and $x \vee z = y$. (An immediate proof of this is as follows: If there is another $t \in L$ with $x \perp t$ and $x \vee t = y$ then by (P4) and (P3), $z \perp u$ and $x \perp z \vee t = z \vee u \geq u$ for some $u \in L$, hence by (P2) and the symmetry of $\perp$, $u \perp x$ which with $u \perp z$ gives $u \perp x \vee z = y \geq u$ by (P3). By (P2), $u \perp u$, hence $u = 0$ by (P1). This means that $z = z \vee t$, i.e., $t \leq z$. Similarly we get $z \leq t$ and thus $z = t$.) The element z will be denoted by the symbol $x^{\sharp_y}$. The element $x^{\sharp_y}$ is called the relative orthocomplement of x in the interval $[0, y]$. For every $x \in L$, ${}^{\sharp_x}$ denotes the unary operation $y \mapsto y^{\sharp_x}$ on the interval $[0, x]$.

We recall the following known result since we shall prove an analogous theorem for ROM–lattices. This result enables to define a GOM–lattice in another way, using just mentioned unary operations.

2.2. Theorem. *If $(L, \wedge, \vee, 0, \perp)$ is a GOM–lattice then the following conditions are satisfied for all $x, y, z \in L$:*

(G1) *The interval $[0, x]$ is an OM–lattice with an orthocomplementation ${}^{\sharp_x}$.*
(G2) *If $x \leq y \leq z$ then $x^{\sharp_y} = y \wedge x^{\sharp_z}$.*

Moreover, for all $x, y \in L$,

$(*)$ *$x \perp y$ if and only if $x = y^{\sharp_{x \vee y}}$.*

Conversely, let $(L, \wedge, \vee, 0)$ be a lattice with a smallest element and for every $x \in L$ let ${}^{\sharp_x}$ be a unary operation on the interval $[0, x]$ given by $y \mapsto y^{\sharp_x}$, such that conditions $(G1), (G2)$ are satisfied. If $\perp$ is a binary relation on L given by condition $(*)$ then conditions $(P1) - (P4)$ are satisfied and for all $x, y \in L$, $x \leq y$ implies $x \perp x^{\sharp_y}$ and $x \vee x^{\sharp_y} = y$.

For Theorem 2.2 see [9],[4],[8].

If $(L, \wedge, \vee, 0, \perp)$ is a GOM–lattice, $x, y, z \in L$ and $x \leq y \leq z$, let us denote $y^{\sharp_{x,z}} = x \vee y^{\sharp_z}$, this element is called the relative orthocomplement of y in the interval $[x, z]$. For every $x, y \in L$ with $x \leq y$, by the symbol ${}^{\sharp_{x,y}}$ we denote the unary operation $z \mapsto z^{\sharp_{x,y}}$ on the interval $[x, y]$.

Using properties (G1) and (G2) of a GOM–lattice, one can easily obtain the following conditions for all $x, y, z, t \in L$:

(1) If $x \leq y$ then the interval $[x, y]$ is an OM–lattice with an orthocomplementation ${}^{\sharp_{x,y}}$.
(2) $x \leq y \leq z \leq t$ implies $y^{\sharp_{x,z}} = z \wedge y^{\sharp_{x,t}}$ and $z^{\sharp_{y,t}} = y \vee z^{\sharp_{x,t}}$.

M.F. Janowitz in [4] has shown that every GOM–lattice is a prime ideal of an OM–lattice. We recall Janowitz's result in a slightly modified form. The result can be found also in the book [1] of L. Beran which repeates Janowitz's proof.

2.3. Theorem (M.F. Janowitz [4]). *Every GOM–lattice is a prime ideal of an OM–lattice.*

Proof. Let $(L, \wedge, \vee, 0, \perp)$ be a GOM–lattice. Let M be a set such that $L \bigcap M = \varnothing$ and let $L \to M, x \mapsto x^\sharp$ be a bijection. Let us denote $0^\sharp = 1$ and $x^{\sharp\sharp} = x$ for every

$x \in L$. Thus $^\sharp$ can be considered as a unary operation on the set $K = L \bigcup M$. Let us define an extension of the partial order $\leq$ of L by the following rules ($x, y \in L$):

$$x \leq y^\sharp \quad \text{if and only if} \quad x \perp y,$$

$$x^\sharp \leq y^\sharp \quad \text{if and only if} \quad y \leq x.$$

Then $\leq$ is a partial order on K, 0 is the smallest element and 1 is the greatest element in $(K, \leq)$.
Moreover, $(K, \leq)$ is a lattice, the lattice operations are extensions of $\wedge$ and $\vee$ and are as follows ($x, y \in L$):

$$x^\sharp \wedge y^\sharp = (x \vee y)^\sharp,$$

$$x^\sharp \vee y^\sharp = (x \wedge y)^\sharp,$$

$$x \wedge y^\sharp = x \wedge y^{\sharp_{x \vee y}},$$

$$x \vee y^\sharp = (x^{\sharp_{x \vee y}} \wedge y)^\sharp.$$

Observe that for every $x \in K$, $x^\sharp$ is a complement of x. The system $(K, \wedge, \vee, 0, 1,^\sharp)$ is an OM–lattice and L is a prime ideal in the lattice $(K, \wedge, \vee)$. $\square$

Looking at the proof of the preceding Theorem, we see that if the GOM–lattice $(L, \wedge, \vee, 0, \perp)$ has no greatest element, then the construction presented here is, in fact, an adding to the set L exactly what is necessary, namely just the missing orthocomplements of the elements of L.

3. Relatively orthomodular lattices

In this section a relatively orthomodular lattice is introduced by means of a commutativity relation. We prove several basic facts which are used to show a connection with orthomodular lattices and generalized orthomodular lattices.

3.1. Definition. Let $(L, \wedge, \vee)$ be a lattice and let C be a binary relation on L such that conditions (C1) and (C2) are satisfied for all $x, y, z, t \in L$. The system $(L, \wedge, \vee, C)$ is called a *relatively orthomodular lattice* (in abbreviation, an *ROM–lattice*, an *ROML*) and C is called a *commutativity relation* on $(L, \wedge, \vee)$. When $x, y \in L$ and xCy, we say that x and y *commute*.

Remark. Conditions (C1) and (C2) are introduced in Theorem 1.2.

3.2. Lemma. *If $(L, \wedge, \vee, C)$ is an ROM–lattice then the following conditions are satisfied for all $a, b, x, y, z, t, u, v \in L$:*

(i) *xCy implies yCx,*
(ii) *$x \leq y$ implies xCy,*
(iii) *$u \leq x \leq z \leq y \leq v$, zCa, zCb, $z \wedge a = x$, $z \wedge b = u$, $z \vee a = y$ and $z \vee b = v$ implies $a = y \wedge (x \vee b) = x \vee (y \wedge b)$,*
(iv) *$x \leq y \leq z$ implies there exists a unique t such that yCt, $y \wedge t = x$ and $y \vee t = z$.*

Proof. (i) If xCy then by (C1), $y \vee yCx$ which means that yCx.

(ii) If $x \leq y$ then by (C2) it follows that there is $z \in L$ with xCz, $x \wedge z = x$ and $x \vee z = y$. Hence $x \leq z$ and then $z = x \vee z = y$, thus xCy.

(iii) Denote $c = a \vee b$. From $b \leq c$ it follows bCc by (ii), hence cCb by (i). According to (C1), cCz and

$$c \wedge z = z \wedge (a \vee b) = (z \wedge a) \vee (z \wedge b) = x \vee u = x.$$

Again by (C1), cCz and cCb implies

$$x \vee b = (c \wedge z) \vee (c \wedge b) = c \wedge (z \vee b) = c \wedge v = c,$$

and cCz and cCa implies

$$a = x \vee a = (c \wedge z) \vee (c \wedge a) = c \wedge (z \vee a) = c \wedge y = y \wedge (x \vee b).$$

Let $d \in L$ be such that zCd, $z \wedge d = u$ and $z \vee d = y$. Using the preceding part of the proof we get $d = y \wedge (u \vee b) = y \wedge b$. By (C1) we obtain

$$a = c \wedge y = c \wedge (z \vee d) = (c \wedge z) \vee (c \wedge d) = x \vee d = x \vee (y \wedge b).$$

(iv) The existence of t is guaranteed by condition (C2). The uniqueness of t follows by (iii). $\square$

Let $(L, \wedge, \vee, C)$ be an ROM–lattice. If $x, y, z \in L$ and $x \leq y \leq z$, let us denote the unique element $t \in L$ with yCt, $y \wedge t = x$ and $y \vee t = z$ by the symbol $y^{\sharp_{x,z}}$. For every $x, y \in L$, $x \leq y$, the symbol $^{\sharp_{x,y}}$ denotes the unary operation $z \mapsto z^{\sharp_{x,y}}$ on the interval $[x, y]$.

3.3. Corollary. *If $(L, \wedge, \vee, C)$ is an ROM–lattice then the following conditions are satisfied for all $x, y, z, u, v \in L$:*

(i) $u \leq x \leq z \leq y \leq v$ *implies* $z^{\sharp_{x,y}} = y \wedge (x \vee z^{\sharp_{u,v}}) = x \vee (y \wedge z^{\sharp_{u,v}})$,

(ii) xCy *if and only if* $x = y^{\sharp_{x \wedge y, x \vee y}}$.

Proof. (i) follows from (iii) of Lemma 3.2. (ii) is clear from the definition of the element $y^{\sharp_{x \wedge y, x \vee y}}$ and from (i) of Lemma 3.2. $\square$

3.4. Proposition. *If $(L, \wedge, \vee, C)$ is an ROM–lattice then every interval $[u, v]$, where $u, v \in L$ and $u \leq v$, is an OM–lattice under an orthocomplementation $^{\sharp_{u,v}}$. Moreover, for all $x, y \in [u, v]$ the following is true:*

$$xCy \quad \textit{if and only if} \quad x, y \quad \textit{commute in} \quad [u, v].$$

Proof. Let $u, v \in L$ be such that $u \leq v$.

To prove the first part of Proposition let us denote $^\sharp = {}^{\sharp_{u,v}}$. Clearly, $x \wedge x^\sharp = u$, $x \vee x^\sharp = v$ and $x^{\sharp\sharp} = x$ for every $x \in [u, v]$. Let $x, y \in [u, v]$ be such that $x \leq y$. Denote $a = x^\sharp$ and $b = y^\sharp$. Using (i) of Corollary 3.3 we obtain

$$x = y^{\sharp_{x,y}} = y \wedge (x \vee b) = u \vee ((x \vee b) \wedge y) = b^{\sharp_{u, x \vee b}},$$

hence the orthomodular law in $[u,v]$ and xCb. By (C1), from xCa and xCb it follows that

$$x \wedge (a \vee b) = (x \wedge a) \vee (x \wedge b) = u \vee u = u,$$

hence by (i) of Corollary 3.3 we get

$$a \vee b = a^{\sharp_{a,a\vee b}} = a \vee ((a \vee b) \wedge x) = a \vee u = a,$$

which means that $b \leq a$.

Therefore the interval $[u,v]$ is an OM–lattice.

To prove the second part of Proposition let us observe that the following is true for all $x, y \in [u,v]$:

$$x, y \text{ commute in } [u,v] \text{ if and only if}$$
$$x = (x \wedge y) \vee ((x \vee y) \wedge y^{\sharp_{u,v}}) \text{ if and only if}$$
$$x = y^{\sharp_{x\wedge y, x\vee y}} \text{ if and only if } xCy,$$

by (ii) of Corollary 3.3. □

The following Theorem demonstrates that the name relatively orthomodular lattice is justified.

3.5. Theorem. *If $(L, \wedge, \vee, C)$ is an ROM–lattice then the following conditions are satisfied for all $x, y, z, u, v \in L$ with $x \leq y$ and $u \leq v$:*

(R1) *The interval $[x,y]$ is an OM–lattice with an orthocomplementation ${}^{\sharp_{x,y}}$.*
(R2) *If $z \in [x,y] \subseteq [u,v]$ then $z^{\sharp_{x,y}} = x \vee (y \wedge z^{\sharp_{u,v}})$.*

Moreover, for all $x, y \in L$,

($*$) *xCy if and only if $x = y^{\sharp_{x\wedge y, x\vee y}}$.*

Conversely, let $(L, \wedge, \vee)$ be a lattice and for every $x, y \in L$ with $x \leq y$ let ${}^{\sharp_{x,y}}$ be a unary operation $z \mapsto z^{\sharp_{x,y}}$ on the interval $[x,y]$ such that conditions (R1), (R2) are satisfied. If C is a binary relation on L given by condition ($*$) then conditions (C1), (C2) are satisfied and for all $x, y, z \in L$, $x \leq y \leq z$ implies $yCy^{\sharp_{x,z}}$, $y \wedge y^{\sharp_{x,z}} = x$ and $y \vee y^{\sharp_{x,z}} = z$.

Proof. The necessity of conditions (R1), (R2) and ($*$) is proved in Proposition 3.4 and Corollary 3.3.

The sufficiency follows from the fact that every interval $[u,v]$, where $u, v \in L$ and $u \leq v$, is an OM–lattice with an orthocomplementation ${}^{\sharp_{u,v}}$ (condition (R1)) and that the relation C on L given by condition ($*$) restricts to its commutativity relation: if $x, y \in [u,v]$ then

$$x, y \text{ commute in } [u,v] \text{ if and only if (by (R1))}$$
$$x = (x \wedge y) \vee ((x \vee y) \wedge y^{\sharp_{u,v}}) \text{ if and only if (by (R2))}$$
$$x = y^{\sharp_{x\wedge y, x\vee y}} \text{ if and only if (by } (*)) \ xCy. \qquad \square$$

Remark. Every OM–lattice is an ROM–lattice. Using (1) and (2) of the preceding section we see that every GOM–lattice is an ROM–lattice. Let us note that the dual of an ROM–lattice is an ROM–lattice, that an ROM–lattice with a smallest element is a GOM–lattice and that a bounded ROM–lattice is an OM–lattice.

A dual of a GOM–lattice is an ROM–lattice. A dual of a GOM–lattice which has no greatest element is an example of an ROM–lattice which has no smallest element, hence which is not a GOM–lattice.

Every principal dual ideal of an ROM–lattice is a GOM–lattice. Every convex sublattice of an ROM–lattice is again an ROM–lattice. More generally, every sublattice of an ROM–lattice closed under the relative orthocomplements is again an ROM–lattice.

Condition (1) of Section 2 is identical with condition (R1) above. Condition (R2) can be replaced by condition (2) of Section 2 in the preceding Theorem. Thus these conditions (1) and (2) are also characteristic for ROM–lattices.

The following Theorem shows that we can use another system of axioms for C in the definition of an ROM–lattice. (Conditions (D1)–(D3) used in the following Theorem are introduced in Theorem 3.1.)

3.6. Theorem. *Let $(L, \wedge, \vee)$ be a lattice and let C be a binary relation on L. Then $(L, \wedge, \vee, C)$ is an ROM–lattice if and only if conditions (D1)–(D3) are satisfied for all $x, y, z, t \in L$.*

Proof. If $(L, \wedge, \vee, C)$ is an ROM–lattice and $x, y, z, t \in L$ then (D1) follows by (i) of Lemma 3.2, (D2) follows by Proposition 3.4 and (D3) is identical with (C2).

Conversely, assume that conditions (D1)–(D3) are satisfied for all $x, y, z, t \in L$. Since (C2) = (D3), we have (ii) of Lemma 3.2. We prove (iii) of Lemma 3.2. By (D2), $z \wedge (a \vee b) = (z \wedge a) \vee (z \wedge b) = x \vee u = x$, hence by (D1), Lemma 3.2 (ii) and (D2) we obtain

$$x \vee b = b \vee (z \wedge (a \vee b)) = (b \vee z) \wedge (a \vee b) = v \wedge (a \vee b) = a \vee b,$$

hence again by (D1), Lemma 3.2 (ii) and (D2) we get

$$a = a \vee x = a \vee (z \wedge (a \vee b)) = (a \vee z) \wedge (a \vee b) = y \wedge (a \vee b) = y \wedge (x \vee b).$$

Analogously, using dual arguments, it can be shown that $a = x \vee (y \wedge b)$.

We have also (iv) of Lemma 3.2 because (C2) = (D3) and we have shown (iii) of Lemma 3.2.

If $x, y, z \in L$ and $x \leq y \leq z$ let us denote the unique element $t \in L$ with yCt, $y \wedge t = x$ and $y \vee t = z$ by the symbol $y^{\sharp_{x,z}}$. The symbol $^{\sharp_{x,z}}$ denotes the unary operation $y \mapsto y^{\sharp_{x,z}}$ on the interval $[x, z]$.

Conditions (i) and (ii) of Corollary 3.3 are now evident. Similarly as in Proposition 3.4 we can show that every interval $[u, v]$, where $u, v \in L$ and $u \leq v$, is an OM–lattice under an orthocomplementation $^{\sharp_{u,v}}$ and that C restricts to its commutativity relation. From this it follows that condition (C1) is satisfied. Thus $(L, \wedge, \vee, C)$ is an ROM–lattice.

The proof is complete. $\square$

Remark. We tried to prove conditions (C1) and (C2) directly from conditions (D1)–(D3) but we were not successful. Therefore we, in fact, mimicked the proof which led to Proposition 3.4.

We were not successful also in proving conditions (D1)–(D3) directly from conditions (C1) and (C2), the above proof goes through Proposition 3.4, too.

Further considerations provide an alternative proof of Proposition 3.4. The proof is not direct as the previous one using C. In the following proof $\perp$ is used.

3.7. Lemma. *If $(L, \wedge, \vee, C)$ is an ROM–lattice then the following conditions are satisfied for all $x, y, z, t \in L$:*

(i) *xCy implies yCx,*
(ii) *$x \leq y$ implies xCy,*
(iii) *$x \leq y \leq z$ implies there exists a unique t such that yCt, $y \wedge t = x$ and $y \vee t = z$,*
(iv) *xCy and $x \vee (y \wedge z) = z$ implies $xCy \wedge z$.*

Proof. (i) is (i) of Lemma 3.2.

(ii) is (ii) of Lemma 3.2.

(iii) is (iv) of Lemma 3.2 and we need use another proof. The existence of t follows by (C2), to show the uniqueness let $x \leq y \leq z$, yCt, yCu, $y \wedge t = y \wedge u = x$, $y \vee t = y \vee u = z$ and let us denote $v = t \vee u$. According to (C1), vCy and $y \wedge v = (y \wedge t) \vee (y \wedge u) = x$, hence using (ii), (i) and (C1) we obtain

$$t = t \vee x = (v \wedge t) \vee (v \wedge y) = v \wedge (t \vee y) = v \wedge (u \vee y) = (v \wedge u) \vee (v \wedge y) = u \vee x = u.$$

(iv) If xCy and $x \vee (y \wedge z) = z$ then $x \wedge y \leq x \leq z$, hence by (C2), xCt, $x \wedge t = x \wedge y$ and $x \vee t = z$ for some $t \in L$. From xCy and xCt by (C1) it follows $y \vee tCx$ and $x \wedge (y \vee t) = (x \wedge y) \vee (x \wedge t) = x \wedge y$. Moreover, $x \vee (y \vee t) = y \vee z = x \vee y$. Using (iii) we get $y = y \vee t$ which means that $t \leq y$. Then

$$y \wedge z = y \wedge (x \vee t) = (y \wedge x) \vee (y \wedge t) = (y \wedge x) \vee t = t$$

and therefore $xCy \wedge z$. $\square$

The proofs of the following two Theorems depend only on Lemma 3.7 and on axioms (C1) and (C2).

3.8. Theorem. *Assume that an ROM–lattice $(L, \wedge, \vee, C)$ has a smallest element 0 and let $\perp$ be a binary relation on L defined by*

$$x \perp y \quad \text{if and only if} \quad xCy \quad \text{and} \quad x \wedge y = 0.$$

Then conditions (P1)–(P4) are satisfied for all $x, y, z \in L$.

Proof. (P1): If $x \perp x$ then $x \wedge x = 0$, hence $x = 0$.
(P2): If $x \perp y$ and $z \leq x$ then xCz by (ii) and (i) of Lemma 3.7 and $z \wedge y = 0$ since $z \wedge y \leq x \wedge y = 0$. From xCy and xCz by (C1) it follows $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) = 0 \vee z = z$, hence $y \vee (x \wedge (y \vee z)) = y \vee z$ which with yCx by (iv) of Lemma 3.7 gives $yCx \wedge (y \vee z) = z$, i.e., yCz and thus $y \perp z$.
(P3): From $x \perp y$ and $x \perp z$ it follows xCy and xCz, hence $xCy \vee z$ and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) = 0 \vee 0 = 0$ which means that $x \perp y \vee z$.
(P4): This is clear by (C2). $\square$

3.9. Theorem. *Assume that an ROM–lattice $(L, \wedge, \vee, C)$ has 0 and 1. For every $x \in L$ let $x^\sharp \in L$ be the unique element for which $xCx^\sharp$, $x \wedge x^\sharp = 0$ and $x \vee x^\sharp = 1$. Then for all $x, y \in L$,*

$$x \perp y \quad \text{if and only if} \quad x \leq y^\sharp.$$

Moreover, $(L, \wedge, \vee, 0, 1, ^\sharp)$ is an OM–lattice and for all $x, y \in L$,

$$xCy \quad \text{if and only if} \quad x = (x \wedge y) \vee (x \wedge y^\sharp).$$

Proof. If $x, y \in L$ and $x \perp y$, let $z \in L$ be such that $x \vee y \perp z$ and $x \vee y \vee z = 1$. From $y \perp x$ and $y \perp z$ it follows that $y \perp x \vee z$, hence $y^\sharp = x \vee z \geq x$. Conversely, if $x, y \in L$ and $x \leq y^\sharp$ then from $y^\sharp \perp y$ it follows that $x \perp y$.

Clearly, for every $x \in L$, $x^{\sharp\sharp} = x$. If $x, y \in L$ and $x \leq y$ then $x \leq y^{\sharp\sharp}$ which means that $x \perp y^\sharp$, hence $y^\sharp \perp x$ and therefore $y^\sharp \leq x^\sharp$.

If $x, y \in L$ and $x \leq y$ then from yCx and $yCy^\sharp$ it follows that $y \wedge (x \vee y^\sharp) = (y \wedge x) \vee (y \wedge y^\sharp) = x \vee 0 = x$ which gives the orthomodular law. Thus L is an OM–lattice.

Let $x, y \in L$ and xCy. From yCx and $yCy^\sharp$ it follows that $x \vee y^\sharp Cy$ and $y \wedge (x \vee y^\sharp) = (y \wedge x) \vee (y \wedge y^\sharp) = (y \wedge x) \vee 0 = y \wedge x$. From $x \vee y^\sharp Cx$ and $x \vee y^\sharp Cy$ it follows that

$$(x \vee y^\sharp) \wedge (x \vee y) = ((x \vee y^\sharp) \wedge x) \vee ((x \vee y^\sharp) \wedge y) = x \vee (x \wedge y) = x,$$

hence $x = (x \wedge y) \vee (x \wedge y^\sharp)$ since L is an OM–lattice.

Conversely, let $x, y \in L$ and $x = (x \wedge y) \vee (x \wedge y^\sharp)$, hence $x \vee y^\sharp = (x \wedge y) \vee y^\sharp$. Since L is an OM–lattice, we have $x = (x \vee y) \wedge (x \vee y^\sharp)$. From $yCx \wedge y$ and $yCy^\sharp$ it follows $yC(x \wedge y) \vee y^\sharp$, hence $yCx \vee y^\sharp$. By (iv) of Lemma 3.7, from $yCx \vee y^\sharp$ and $y \vee ((x \vee y^\sharp) \wedge (x \vee y)) = x \vee y$ it follows $yC(x \vee y^\sharp) \wedge (x \vee y) = x$, hence yCx and thus xCy. □

4. Embedding theorem

This section contains our main result: Every relatively orthomodular lattice (ROM–lattice) is a prime dual ideal of a generalized orthomodular lattice (GOM–lattice). To construct, for a given ROM–lattice L, an extension K of L such that K is a GOM–lattice and L constitutes a prime dual ideal of K, we shall need a sequence of technical results which pass through the whole Section 4 and culminate in a final theorem.

4.1. Lemma. *Let L be an ROM–lattice and let $x, y, z \in L$ be such that $x \leq y$. Then the following three conditions are equivalent:*

(i) $x^{\sharp_{x \wedge z, y}} = y \wedge z$,
(ii) $y \wedge x^\sharp \leq z$ *in some interval of L containing* x, y, z,
(iii) $y \wedge x^\sharp \leq z$ *in all intervals of L containing* x, y, z.

Moreover, the following three conditions are equivalent:

(i)* $y^{\sharp_{x, y \vee z}} = x \vee z$,
(ii)* $y \wedge x^\sharp \leq z^\sharp$ *in some interval of L containing* x, y, z,
(iii)* $y \wedge x^\sharp \leq z^\sharp$ *in all intervals of L containing* x, y, z.

Proof. First observe that $x^{\sharp_{x\wedge z,y}} = (x \wedge z) \vee (y \wedge x^\sharp)$ in every interval of L containing x, y, z. Hence, in the series (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) of implications, only the implication (iii) $\Rightarrow$ (i) needs a proof.

(iii) $\Rightarrow$ (i). Let us denote $t = x^{\sharp_{x\wedge z,y}}$ and $^\sharp = {}^{\sharp_{x\wedge z,y\vee z}}$. Then $t \le y \wedge z$, hence by the orthomodular law in $[x \wedge z, y \vee z]$,

$$y \wedge z = t \vee (y \wedge z \wedge t^\sharp) = t \vee (z \wedge t^{\sharp_{x\wedge z,y}}) = t \vee (x \wedge z) = t.$$

Using duality, it can be shown that the remaining three conditions are equivalent, too. $\square$

Let $(L, \wedge, \vee)$ be an ROM–lattice. Let us denote

$$Q(L) = \{(x, y) \in L \times L \mid x \le y\}.$$

On the set $Q(L)$ define a binary relation, denoted by $\equiv$, by the following rule:

$$(x, y) \equiv (z, t) \quad \text{if and only if}$$

$$y \wedge x^\sharp = t \wedge z^\sharp \quad \text{in some interval of } L \text{ containing} \quad x, y, z, t.$$

By Lemma 4.1, this expression is equivalent with the following one:

$$y \wedge x^\sharp = t \wedge z^\sharp \quad \text{in all intervals of } L \text{ containing} \quad x, y, z, t.$$

Then $\equiv$ is an equivalence relation on the set $Q(L)$. By the symbol $(x, y)^\equiv$ we denote the equivalence class of $\equiv$ corresponding to the element $(x, y) \in Q(L)$, i.e.,

$$(x, y)^\equiv = \{(z, t) \in Q(L) \mid (x, y) \equiv (z, t)\}.$$

The quotient set of $Q(L)$ under $\equiv$ will be denoted by $M(L)$, i.e.

$$M(L) = Q(L)/_\equiv = \{(x, y)^\equiv \mid (x, y) \in Q(L)\}.$$

We can assume that L and $M(L)$ are disjoint sets. Let us denote

$$K(L) = L \cup M(L).$$

Remark. If the ROM–lattice $(L, \wedge, \vee)$ has a smallest element 0 and $(x, y), (z, t) \in Q(L)$ then

$$(x, y) \equiv (z, t) \quad \text{if and only if} \quad x^{\sharp_{0,y}} = z^{\sharp_{0,t}}.$$

This means that different elements $x, z \in L$ have the same orthocomplement in different intervals containing 0.

If $(L, \wedge, \vee)$ has no smallest element then the set $M(L)$ represents "the missing orthocomplements of the elements of L", the elements of $M(L)$ are thus candidates for completing the ROM–lattice $(L, \wedge, \vee)$ to a GOM–lattice. The equivalence classes of $\equiv$ collects those elements of L which will have the same relative orthocomplements in corresponding intervals containing 0 and thus such a collection can represent a corresponding missing relative orthocomplement. For example, $\{(x,x) \mid x \in L\}$ is an equivalence class of $\equiv$ which represents the missing smallest element.

So, we shall try to equip the set $K(L)$ with a GOM–lattice structure which will extend the ROM–lattice structure on L and L will constitute a prime dual ideal of $K(L)$.

We define an extension of $\leq$ (where $\leq$ is the partial order on L) from L to $K(L)$ in the following way:

(1) $(x,y)^{\equiv} \leq z$ if and only if
$y \wedge x^{\sharp} \leq z$ in some interval of L containing x,y,z.

(2) $(x,y)^{\equiv} \leq (z,t)^{\equiv}$ if and only if
$y \wedge x^{\sharp} \leq t \wedge z^{\sharp}$ in some interval of L containing x,y,z,t.

By Lemma 4.1, the word some in (1) and (2) can be replaced by the word all. The extension $\leq$ is well defined and $\leq$ is a partial order on the set $K(L)$. Let us note that for every $x \in L$ it holds $(x,x)^{\equiv} = \{(y,y) \mid y \in L\}$, this element is a smallest element in $(K(L),\leq)$ and will be denoted by the symbol 0.

In $(K(L),\leq)$, the meet and the join of elements $x,y \in L$ exist and equal to $x \wedge y$ and $x \vee y$, respectively. In the following series of assertions we will show that $(K(L),\leq)$ is a GOM–lattice which contains L as a prime dual ideal. The next two observations will be useful:

(a) If $x,y,z \in L$ are such that $x \leq y,z$ then
$(x,y)^{\equiv} \leq (x,z)^{\equiv}$ if and only if $y \leq z$.

(b) If $x,y,z \in L$ are such that $x,y \leq z$ then
$(x,z)^{\equiv} \leq (y,z)^{\equiv}$ if and only if $y \leq x$.

4.2. Proposition. *$(K(L),\leq)$ is a lattice, rules for computation of meets and joins in $(K(L),\leq)$ are as follows:*

(i) $(x,y)^{\equiv} \wedge z = (x \vee (y \wedge z)^{\sharp_{x\wedge z,y}}, y)^{\equiv}$,
(ii) $(x,y)^{\equiv} \vee z = x^{\sharp_{x\wedge z,y}} \vee z$,
(iii) $(x,y)^{\equiv} \wedge (z,t)^{\equiv} = (x \vee z \vee (y \wedge t)^{\sharp_{x\wedge z,y\vee t}}, y \vee t)^{\equiv}$,
(iv) $(x,y)^{\equiv} \vee (z,t)^{\equiv} = (x \wedge z, x^{\sharp_{x\wedge z,y}} \vee z^{\sharp_{x\wedge z,t}})^{\equiv}$.

Proof. First observe that (ii) is evident. We prove (i), (iii) and (iv) simultaneously. Let us denote $u = x \vee (y \wedge z)^{\sharp_{x\wedge z,y}}$, $v = x \vee z \vee (y \wedge t)^{\sharp_{x\wedge z,y\vee t}}$ and $w = x^{\sharp_{x\wedge z,y}} \vee z^{\sharp_{x\wedge z,t}}$. Let $(p,q) \in Q(L)$ and let J be an interval containing the elements x,y,z,t,p,q. Let $^{\sharp}$ denote the orthocomplementation in J. Then we obtain $y \wedge u^{\sharp} = y \wedge x^{\sharp} \wedge z$ and

$$(y \vee t) \wedge v^{\sharp} = (x \vee z)^{\sharp} \wedge y \wedge t = y \wedge x^{\sharp} \wedge t \wedge z^{\sharp}.$$

If we denote $s = (y \wedge x^{\sharp}) \vee (t \wedge z^{\sharp})$ then $w = s \vee (x \wedge z)$, hence from $s \leq (x \wedge z)^{\sharp}$ by orthomodularity we get

$$w \wedge (x \wedge z)^{\sharp} = (s \vee (x \wedge z)) \wedge (x \wedge z)^{\sharp} = s.$$

The following relations are now evident:

$$(u, y)^{\equiv} \text{ is a lower bound of } (x, y)^{\equiv} \text{ and } z,$$

$$(v, y \vee t)^{\equiv} \text{is a lower bound and} (x \wedge z, w)^{\equiv} \text{is an upper bound}$$
$$\text{of} (x, y)^{\equiv} \text{ and} (z, t)^{\equiv}.$$

If $(p, q)^{\equiv}$ is a lower bound of $(x, y)^{\equiv}$ and z then $q \wedge p^{\sharp}$ is a lower bound of $y \wedge x^{\sharp}$ and z, hence $(p, q)^{\equiv} \leq (u, y)^{\equiv}$. If $(p, q)^{\equiv}$ is a lower bound of $(x, y)^{\equiv}$ and $(z, t)^{\equiv}$ then $q \wedge p^{\sharp}$ is a lower bound of $y \wedge x^{\sharp}$ and $t \wedge z^{\sharp}$ which means that $(p, q)^{\equiv} \leq (v, y \vee t)^{\equiv}$. If $(p, q)^{\equiv}$ is an upper bound of $(x, y)^{\equiv}$ and $(z, t)^{\equiv}$, i.e. $q \wedge p^{\sharp}$ is an upper bound of $y \wedge x^{\sharp}$ and $t \wedge z^{\sharp}$, then $w \wedge (x \wedge z)^{\sharp} = s \leq q \wedge p^{\sharp}$ which means that $(x \wedge z, w)^{\equiv} \leq (p, q)^{\equiv}$.

The proof is complete. □

4.3. Lemma. *Let $x, y, z, t, p, q \in L$ be such that $x \leq z \leq y$ and $p \leq t \leq q$. Denote $u = z^{\sharp_{x,y}}$ and $v = t^{\sharp_{p,q}}$. If $(x, y) \equiv (p, q)$ and $(z, y) \equiv (t, q)$ then $(u, y) \equiv (v, q)$.*

Proof. Let J be an interval containing x, y, p, q and let $\sharp$ denote the orthocomplementation of J. $(x, y) \equiv (p, q)$ means that $y \wedge x^{\sharp} = q \wedge p^{\sharp}$ and $(z, y) \equiv (t, q)$ means that $y \wedge z^{\sharp} = q \wedge t^{\sharp}$. By Lemma 4.1, $y \wedge z^{\sharp} \leq t^{\sharp}$ means that $y^{\sharp_{z, y \vee t}} = z \vee t$, hence

$$t \wedge p^{\sharp} = t \wedge q \wedge p^{\sharp} \leq (z \vee t) \wedge y \wedge x^{\sharp} = z \wedge x^{\sharp}.$$

Analogously it can be shown that $z \wedge x^{\sharp} \leq t \wedge p^{\sharp}$ and thus $z \wedge x^{\sharp} = t \wedge p^{\sharp}$. Compute:

$$\begin{aligned} y \wedge u^{\sharp} &= y \wedge (x \vee (y \wedge z^{\sharp}))^{\sharp} = y \wedge x^{\sharp} \wedge (z \vee y^{\sharp}) = z \wedge x^{\sharp} \\ &= t \wedge p^{\sharp} = q \wedge (t \vee q^{\sharp}) \wedge p^{\sharp} = q \wedge (p \vee (q \wedge t^{\sharp}))^{\sharp} \\ &= q \wedge v^{\sharp}. \end{aligned}$$

This means that $(u, y) \equiv (v, q)$. □

4.4. Lemma. *If $x \in L$ then in $(K(L), \leq)$, the interval $[0, x]$ is the set*

$$\{y \in L \mid y \leq x\} \cup \{(y, x)^{\equiv} \mid y \in L, \quad y \leq x\}.$$

If $y \in L$, $\quad y \leq x$, put $y^{\sharp} = (y, x)^{\equiv}$ and $(y, x)^{\equiv\sharp} = y$. Then $[0, x]$ is an OM–lattice with an orthocomplementation $\sharp$.

If $(x, y) \in Q(L)$ then in $(K(L), \leq)$, the interval $[0, (x, y)^{\equiv}]$ is the set

$$\{(z, y)^{\equiv} \mid z \in L, \quad x \leq z \leq y\}.$$

If $z \in L$, $\quad x \leq z \leq y$, put $(z, y)^{\equiv\sharp} = (z^{\sharp_{x,y}}, y)^{\equiv}$. Then $[0, (x, y)^{\equiv}]$ is an OM–lattice with an orthocomplementation $\sharp$.

Proof. Let $x \in L$ and let $(z, t) \in Q(L)$ be such that $(z, t)^{\equiv} \leq x$. Denote $y = (x \wedge t)^{\sharp_{x \wedge z, x}}$. We have $y \leq x$ and by Lemma 4.1, $y^{\sharp_{x \wedge z, x}} = x \wedge t = z^{\sharp_{x \wedge z, t}}$ which means that $(y, x) \equiv (z, t)$, i.e. $(y, x)^{\equiv} = (z, t)^{\equiv}$. Now it is easy to see that the interval $[0, x]$ has the above form.

Let $(x,y),\quad (z,t)\in Q(L)$ be such that $(z,t)^{\equiv}\le (x,y)^{\equiv}$. Then $(z,t)^{\equiv}\le y$ and by the preceding discussion, $(z,t)^{\equiv}=(p,y)^{\equiv}$ for some $p\in L,\quad p\le y$. Clearly, $\quad x\le p$. We see that the interval $[0,(x,y)^{\equiv}]$ has the desired form. Let us observe that the unary operation $^\sharp$ on $[0,(x,y)^{\equiv}]$ is well defined. This is warranted by Lemma 4.3.

It is easy to show that the intervals $[0,x]$ and $[0,(x,y)^{\equiv}]$ with the above defined orthocomplementations are ortholattices. It remains to prove the orthomodularity.

Let $x,y,z\in L$ be such that $y,z\le x$. If $y\le z$ and $z\wedge y^\sharp=0$ then $0=(y,x)^{\equiv}\wedge z=(z^{\sharp_{y,x}},x)^{\equiv}$, hence $z^{\sharp_{y,x}}=x$ which means that $y=z$. If $(y,x)^{\equiv}\le(z,x)^{\equiv}$ and $(z,x)^{\equiv}\wedge(y,x)^{\equiv\sharp}=0$ then $z\le y$ and $y\wedge z^\sharp=0$, hence by the preceding, $\quad y=z$.

Let $(x,y)\in Q(L)$ and $z,t\in L$ be such that $x\le z,t$ and $y\ge z,t$. If $(z,y)^{\equiv}\le (t,y)^{\equiv}$ and $(t,y)^{\equiv}\wedge(z,y)^{\equiv\sharp}=0$ then $t\le z$ and

$$0=(t,y)^{\equiv}\wedge(z^{\sharp_{x,y}},y)^{\equiv}=(t\vee z^{\sharp_{x,y}},y)^{\equiv}=(z^{\sharp_{t,y}},y)^{\equiv}$$

which means that $z^{\sharp_{t,y}}=y$, hence $z=t$. Thus $(z,y)^{\equiv}=(t,y)^{\equiv}$. $\square$

Let u be an arbitrary element of $K(L)$. Consider the interval $[0,u]$ in $(K(L),\le)$. Then $[0,u]$ is an OM–lattice with the orthocomplementation $^\sharp$ determined in Lemma 4.4. Let us denote this unary operation on $[0,u]$ by the symbol $^{\sharp_u}$ which is an abbreviation for the symbol $^{\sharp_{0,u}}$. The following observation shows that the notation is justified:

$$\textit{if}\quad x,y,z\in L\quad \textit{and}\quad x\le y\le z\quad \textit{then}\quad y^{\sharp_{x,z}}=x\vee y^{\sharp_z}.$$

This can be proved by an easy computation: $x\vee y^{\sharp_z}=x\vee(y,z)^{\equiv}=x\vee y^{\sharp_{x,z}}=y^{\sharp_{x,z}}$.

4.5. Lemma. *If $x,y,z\in K(L)$ and $x\le y\le z$, then $x^{\sharp_y}=y\wedge x^{\sharp_z}$.*

Proof. If $x,y,z\in L$ and $x\le y\le z$ then

$$y\wedge x^{\sharp_z}=y\wedge(x,z)^{\equiv}=(y^{\sharp_{x,z}},z)^{\equiv}=(x,y)^{\equiv}=x^{\sharp_y}.$$

If $x\in M(L),\quad y,z\in L$ and $x\le y\le z$ then $x=(t,y)^{\equiv}=(u,z)^{\equiv}$ and hence $t=y\wedge(t\vee u)$ and $u=z\wedge(t\vee u)$. Compute:

$$y\wedge x^{\sharp_z}=y\wedge u=y\wedge z\wedge(t\vee u)=y\wedge(t\vee u)=t=x^{\sharp_y}.$$

If $x,y\in M(L),\quad z\in L$ and $x\le y\le z$ then $x=(t,z)^{\equiv}$ and $y=(u,z)^{\equiv}$, hence

$$y\wedge x^{\sharp_z}=(u,z)^{\equiv}\wedge t=(t^{\sharp_{u,z}},z)^{\equiv}=(t,z)^{\equiv\sharp_{(u,z)^{\equiv}}}=x^{\sharp_y}.$$

If $x,y,z\in M(L)$ and $x\le y\le z$ then $x=(t,s)^{\equiv},\quad y=(u,s)^{\equiv}$ and $z=(v,s)^{\equiv}$, from which it follows

$$y\wedge x^{\sharp_z}=(u,s)^{\equiv}\wedge(t^{\sharp_{v,s}},s)^{\equiv}=(t^{\sharp_{u,s}},s)^{\equiv}=(t,s)^{\equiv\sharp_{(u,s)^{\equiv}}}=x^{\sharp_y}.\quad\square$$

4.6. Theorem. *$(K(L),\wedge,\vee,0)$ is a GOM–lattice and L is a prime dual ideal in the lattice $K(L)$.*

Proof. $K(L)$ is a lattice with a smallest element by Proposition 4.2. Condition (G1) is proved in Lemma 4.4 and condition (G2) in Lemma 4.5. Thus $K(L)$ is a GOM–lattice.

The second part of the theorem follows from the fact that $M(L)$ is a prime ideal in the lattice $K(L)$. For, $M(L)$ is an ideal since for all $x,y\in M(L)$ also $x\vee y\in M(L)$ and for all $x\in M(L)$ and $y\in K(L),\quad y\le x$ implies $y\in M(L)$. $M(L)$ is prime because for all $x,y\in K(L),\quad$ if $x\wedge y\in M(L)$ then $x\in M(L)$ or $y\in M(L)$. $\square$

5. Examples

Example (a). Let $(L, \wedge, \vee)$ be a relatively complemented distributive lattice and let C be a binary relation on L. Then $(L, \wedge, \vee, C)$ is an ROM–lattice if and only if xCy for all $x, y \in L$.

The sufficiency is evident. To prove the necessity let $(L, \wedge, \vee, C)$ be an ROM–lattice which is distributive. If $x, y \in L$ and $z = y^{\sharp_{x\wedge y, x\vee y}}$ then from $z \wedge y = x \wedge y$ and $z \vee y = x \vee y$ by distributivity it follows that $z = x$, hence xCy must hold.

Let $(L, \wedge, \vee)$ be a relatively complemented distributive lattice and let C be a binary relation on L such that xCy for all $x, y \in L$. We call the corresponding ROM–lattice $(L, \wedge, \vee, C)$ by the name *relatively Boolean algebra*.

Example (b). This example was motivated by [7].

As a concrete example of Example (a) let us consider an associative ring R. By $I(R)$ denote the set of idempotents of R, i.e.,

$$I(R) = \{x \in R \,|\, x^2 = x\}.$$

Let L be a nonempty subset of I(R) such that for all $x, y, z \in L$ the following is satisfied:

(i) $xy = yx \in L$,
(ii) $x + y - xy \in L$,
(iii) $xz = x$, $zy = z$ implies $x + y - z \in L$.

Let us observe that for all $x, y, z \in I(R)$ it holds:

$$xz = x,\ zy = z \Rightarrow xy = x,$$

$$zx = x,\ yz = z \Rightarrow yx = x.$$

The set L is well defined since for all $x, y, z \in I(R)$ the following is true:

$$xy = yx \Rightarrow xy \in I(R),$$

$$xy = yx \Rightarrow x + y - xy \in I(R),$$

$$xz = zx = x,\ zy = yz = z \Rightarrow x + y - z \in I(R).$$

Define a binary relation $\leq$ on L by:

$$x \leq y \quad \text{if and only if} \quad xy = x.$$

It is easy to see that $\leq$ is a partial order on the set L. (Let us observe that if $0 \in L$ then $0 \leq x$ for all $x \in L$ and if R has a unit 1 and $1 \in L$ then $x \leq 1$ for all $x \in L$. But we rather assume that $0 \notin L$ and that R is without a unit or if R has a unit 1 then $1 \notin L$.) Moreover, $(L, \leq)$ is a lattice, the lattice operations on L are as follows $(x, y \in L)$:

$$x \wedge y = xy \quad \text{and} \quad x \vee y = x + y - xy.$$

It is easy to see that $(L, \wedge, \vee)$ is a distributive lattice. If $x, y, z \in L$ and $x \leq z \leq y$, let us denote

$$z^{\sharp_{x,y}} = x + y - z.$$

By (iii), $z^{\sharp_{x,y}} \in L$. Clearly, $x \leq z^{\sharp_{x,y}} \leq y$. The element $z^{\sharp_{x,y}}$ is a relative complement of z in the interval $[x, y]$. Let C be a binary relation on L given by: $xCy \Leftrightarrow x = y^{\sharp_{x\wedge y, x\vee y}}$. If $x, y \in L$ then $y^{\sharp_{x\wedge y, x\vee y}} = x\wedge y + x\vee y - y = xy + x + y - xy - y = x$, hence xCy for all $x, y \in L$.

We have shown that the present example is a special case of the preceding Example (a).

Example (c). As another concrete example of Example (a) consider the following example. Let $(L, \wedge, \vee, C)$ be an ROM–lattice and let A be a subset of L such that xCy for all $x, y \in A$. Let B be a subset of L maximal with respect to the following properties: (i) $A \subseteq B$, (ii) xCy for all $x, y \in B$. Then B is a distributive sublattice of $(L, \wedge, \vee)$ closed under the relative orthocomplements. We only show that if $x, y, z, t \in B$ and $x \leq y \leq z$ then $tCy^{\sharp_{x,z}}$. Let $^\sharp$ denote the relative orthocomplementation of the interval $[x \wedge t, z \vee t]$. From tCy it follows $tCy^\sharp$. The rest is clear from the fact that $y^{\sharp_{x,z}} = x \vee (z \wedge y^\sharp)$, tCx and tCz.

Example (d). Let $(L, \wedge, \vee, C)$ be an ROM–lattice. Let A be a nonempty subset of L and let $C(A) = \{x \in L \mid xCy \quad \text{for all} \quad y \in A\}$. Then $C(A)$ is a sublattice of $(L, \wedge, \vee)$ closed under the relative orthocomplements. If $x \in L$ we write simply $C(x)$ for $C(\{x\})$. As usual, the sublattice $C(L)$ of $(L, \wedge, \vee)$ is called the *center* of L. Clearly, $C(L)$ is a distributive lattice.

Example (e). Let $L = \{0, 1, a, a^\sharp\}$ be a four element Boolean algebra. Consider the following binary relation D on L: $D = L \times L \setminus \{(a, 0), (a^\sharp, 0)\}$. For every $x \in L$ denote $D(x) = \{y \in L \mid yDx\}$ (for the notation, cf. [6], p. 23, Chapter I, § 3). Then $D(a) = D(a^\sharp) = D(1) = L$ and $D(0) = \{0, 1\}$. Thus the following two conditions are satisfied for all $x, y \in L$: (i) $x \leq y$ implies xDy, (ii) $D(x)$ is a subalgebra of L. But the relation D is smaller than C, the commutativity of L. Therefore this example of an OML L provides a counterexample for Exercise 10 on p. 233 in the book [6], Chapter 4, § 14.

Just mentioned Exercise 10 can be improved, for instance, by adding a new condition (iii): aCb implies bCa.

We present the following improvement which is simultaneously a simplification of Exercise 10. If L is an OML then the commutativity C of L is the smallest binary relation on L satisfying the following three conditions for all $x, y, z \in L$: (i) $x \leq y$ implies xCy, (ii) xCy implies $x^\sharp Cy$, (iii) xCy and xCz implies $xCy \wedge z$. To show this, it suffices to have a look at the solution of Exercise 10 in [6].
Another such a result is the following one. If L is an OML then the commutativity C of L is the smallest binary relation on L satisfying the following four conditions for all $x, y, z \in L$: (i) $x \leq y$ implies xCy, (ii) $xCx^\sharp$, (iii) xCz and yCz implies $x \vee yCz$, (iv) xCy and xCz implies $xCy \wedge z$. To prove this, let D be the smallest relation on L satisfying (i) – (iv). If $x, y \in L$ and xCy then $x = (x \wedge y) \vee (x \wedge y^\sharp)$ and $y = (x \vee y) \wedge (x^\sharp \vee y)$. By (i), $x \wedge yDx^\sharp \vee y$ and by (ii), $x \wedge y^\sharp Dx^\sharp \vee y$, which by (iii) gives $(x \wedge y) \vee (x \wedge y^\sharp)Dx^\sharp \vee y$, i.e., $xDx^\sharp \vee y$. By (i), $xDx \vee y$ and this with the latter condition gives $xD(x \vee y) \wedge (x^\sharp \vee y)$ by (iv). This means that xDy.

We also obtain a similar result for ROM–lattices. If L is an ROML then the commutativity C of L is the smallest binary relation on L satisfying the following condition for all $x, y, z \in L$: $x \leq y \leq z$ implies $yCy^{\sharp_{x,z}}$. This is clear from the fact that for all $x, y \in L$, xCy if and only if $x = y^{\sharp_{x\wedge y, x\vee y}}$.

Example (f). Let I be an infinite set and let $\mathcal{L}_i = (L_i, \wedge_i, \vee_i, 0_i, 1_i, ^{\sharp_i})$ be an OML such that L_i has more than three elements, for every index $i \in I$. By $\mathcal{L} = (L, \wedge, \vee, 0, 1, ^{\sharp})$ we denote the direct product of the system $(L_i, i \in I)$ of orthomodular lattices. Then L consists of all sequences $(x_i, i \in I)$ where $x_i \in L_i$ for every $i \in I$. If $x = (x_i, i \in I)$ and $y = (y_i, i \in I)$ are in L, then $x \wedge y = (x_i \wedge_i y_i, i \in I)$, $x \vee y = (x_i \vee_i y_i, i \in I)$, $x^{\sharp} = (x_i^{\sharp_i}, i \in I)$ and moreover, $0 = (0_i, i \in I)$ and $1 = (1_i, i \in I)$. As known, $\mathcal{L}$ is again an OM–lattice. For every index $i \in I$, let $a_i \in L_i \setminus \{0_i, 1_i\}$ be an arbitrary but fixed element. By M we denote the set of all sequences $(x_i, i \in I)$ from L for which x_i is different from a_i only for finitely many indices $i \in I$. It is easy to see that the set M forms a convex sublattice of the lattice $(L, \wedge, \vee)$ and thus M is an ROM–lattice. Let us observe that M has no smallest element and no greatest element (hence M can not be an OM–lattice). Therefore M is an ROM–lattice which is not a GOM–lattice (also M is not a dual of a GOM–lattice).

This example was inspired by Example 1.1 (D) in Chapter V of [1].

6. Concluding remarks

Let $(L, \wedge, \vee, C)$ be an ROM–lattice and let $(K, \wedge, \vee, 0, \perp)$ be the GOM–lattice extension of $(L, \wedge, \vee, C)$ constructed in Theorem 4.6. If the ROM–lattice has no smallest element then the minimality of the extension is transparent from the proof of Theorem 4.6: We added to the set L just the missing relative orthocomplements of the elements of L.

With respect to the minimality of the GOM–lattice extension of a given ROM–lattice constructed in our main Theorem, it can be expected that it will have many good preserving and hereditary properties. Similarly, the OM–lattice extension of a given ROM–lattice constructed by combining Theorems 4.6 and 2.3 is expected to be well-behaved.

Let us look, for a moment, at the embedding of a general lattice L into an OM–lattice constructed by G. Kalmbach in [5] (cf. the Bundle Lemma in [6]). If L is an ROM–lattice then it seems that the embedding need not respect possible relative orthocomplements of the elements of L and so the embedding is far from to be minimal in our sense.

M.F. Janowitz [4] defined a GOM–lattice as a lattice with a smallest element 0 and with an orthogonality relation $\perp$. A. Mayet–Ippolito [10] extended Janowitz's definition (and result) in one direction: she defined a weak generalized orthomodular poset (WGOM–poset, WGOMP) as a poset with a smallest element 0 and with relative orthocomplementations on the intervals containing 0. In the present paper we provide an extension of Janowitz's definition (and result) in another direction: we define an ROM–lattice as a lattice with a commutativity relation C (the relation C is an equivalent substitution for the relation $\perp$ not depending on the existence of a smallest element).

As it was observed in [3], a further generalization of the results in the present paper and in [10] is possible by a suitable extension of the definition of a WGOM–poset, which need not have a smallest element. The mentioned paper [3] contains an extension of Mayet–Ippolito's definition (and result) in such a direction for which a smallest element remains: a generalized difference poset (GD–poset, GDP) is a poset with a smallest element 0 having a difference (which is an operation generalizing the operation of relative orthocomplements in the intervals containing 0) and satisfying one additional condition.

Even, it seems to be possible to obtain a common generalization of results in the present paper and in [3] by a suitable modification of the definition of a poset with a difference (cf. [3], where the whole situation is pictured). Questions about these further generalizations will be discussed in a subsequent paper.

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