Mathematical Institute
Slovak Academy of Sciences, Bratislava

Preprint 4/1999

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March 23, 1999

PREPRINT SERIES

# Piron's and Bell's Geometric Lemmas and Gleason's Theorem

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**Abstract**

We study the idea of implantation of Piron's and Bell's geometrical lemmas for proving some results concerning measures on finite as well as infinite-dimensional Hilbert spaces, including also measures with infinite values. In addition, we present parabola based proofs of weak Piron's geometrical and Bell's lemmas. These approaches will not used directly Gleason's theorem, which is a highly non-trivial result.

## 1 Introduction

The Gleason theorem [Gle] is the corner-stone of measurement theory in quantum mechanics. It says, that if the quantum mechanical system can be described by a Hilbert space of dimension at least three, then any state of the physical system corresponds to von Neumann operator. The original proof of this theorem is highly non-trivial, and only after 30 years later a simpler proof using also Piron's geometrical lemma [Pir, pp. 75–78] was present by Cooke, Keane and Moran [CKM].

Today Gleason's theorem is used in quantum measurement as well as in mathematics. Dvurečenskij is the author of a monograph [Dvu], where there are described plenty of applications of Gleason's theorem to different areas of mathematics.

Plenty of authors was trying to simplify the proof of Gleason theorem. One of very perspective methods is Piron's geometrical lemma which gives particular but physically very important results. Such a way was used, e.g. by Calude, Hertling and K. Svozil [CHS]. A different type of geometrical reasoning was used by Specker [Spe], Kochen and Specker [KoSp] and Bell [Bel].

In connection with Kochen-Specker argument, it seems that there are some arguments to prove some particular cases using only Piron's lemma, e.g. the non-existence of two-valued states on the three-dimensional Hilbert space. In the present paper, we give two particular forms of Piron's geometrical lemmas. A different type of geometrical reasoning was used by Bell to prove the non-existence of two-valued states [Bel, pp. 450–451].

From a physical point of view, the non-existence of two-valued measures indicates that there cannot be "elements of physical reality" [EPR] which are independent of the particular measurement context. To state these physical consequences pointedly, let us (wrongly) assume that there indeed exists a "hidden arena" behind the quantum phenomena. Let us further assume that this "hidden arena" behaves classically in the sense that any conceivable logical property is either true or false, formalized by 1 and 0, respectively. A proper formalization of "classical arena" is a Boolean algebra, which possesses a full, separating and unital set of two-valued measures. The connection between the quantum phenomena and the "hidden arena" can then be identified with an injective partial lattice homomorphism (preserving the lattice operations only among commensurable observables).

Classically, it almost goes without saying that any such logical property can be either true or false *independent* of the "measurement context" Redhead [Red]; i.e., independent of which properties are measured alongside with it. Thus a necessary condition for any such classical "hidden arena" to exist is the possibility to define a two-valued measure on it, which should be directly reflected by a two-valued measure on its associated quantum logic via the injective partial lattice homomorphism.

Non-existence of two-valued measures on certain quantum logics indicates a breakdown of the ideas reviewed above of a straightforwardly conceivable "classical arena" together with its connection to the quantum phenomena by injective partial lattice homomorphisms. In particular, it can no longer be stated that physical properties exist independent of the actual way by which they are inferred.

The above results and interpretations do not exclude more general, weaker types of connections [CHS, Svo], as well as the possibility of nonsingular, non two-valued measures.

We apply the geometrical method of Piron to two-valued measures on infinite-dimensional Hilbert space, as well as to the description of all infinite-valued measures on Hilbert-space quantum logic.

## 2 Piron's Geometrical Lemmas

The corner-stone of Gleason's theorem is concentrated on the real three-dimensional case of the Hilbert space $H = \mathbb{R}^3$. First we introduce some terminology of spherical geometry.

Let $\mathcal{S}(\mathbb{R}^3)$ be the unit sphere in $\mathbb{R}^3$. If $p$ and $q$ are two vectors in $\mathcal{S}(\mathbb{R}^3)$, the angle between them is denoted by $\theta(p,q)$. For the point $p$ of the unit sphere $\mathcal{S}(\mathbb{R}^3)$, we define the *northern hemisphere* relative to $p$, $N_p$, as the set of all points $q \in \mathcal{S}(\mathbb{R}^3)$ such that $0 < \theta(p,q) < \pi/2$; the point $p$ is the *north pole* of $N_p$. The *equator* relative to $p$ is the set $E_p$ of all unit vectors orthogonal to $p$.

Let $q \in N_p$, then $q \neq p$. Among the great circles which pass through the point $q$ there is a unique one whose plane cuts the equator $E_p$ at the points which are orthogonal to $q$. We denote this great circle $C(q)$. Then $q$ is the northern-most point on $C(q)$ whose latitude $\alpha$, $0 < \alpha < \pi/2$, is the greatest among all the points in $C(q)$. In addition, any point $\widetilde{q} \in C(q)$ has a latitude from the interval $[-\alpha, \alpha]$.

We say that a vector $p$ in the northern hemisphere *can be reached* from a vector $q$ in the same hemisphere, if there is a finite sequence of vectors $q_0 := q, q_1, \ldots, q_n := p$ in the northern hemisphere such that $q_i \in C(q_{i-1})$ for $i = 1, \ldots, n$.

We now present two forms of Piron's geometrical lemmas which we apply in our considerations. Their proofs can be found in [Pir, p. 75–78] (we recall that the first result is used to prove the second one).

**Lemma 2.1 (Weak Piron's Geometrical Lemma)** *Let $p$ be and $q$ be two unit vectors in the northern hemisphere such that $p$ lies in the region between $C(q)$ and the equator. Then there is a vector $\widetilde{q} \in C(q)$ from the northern hemisphere such that $p \in C(\widetilde{q})$.*

**Lemma 2.2 (Piron's Geometrical Lemma)** *If $p$ and $q$ are unit vectors in the northern hemisphere with $\alpha_p < \alpha_q$, where $\alpha_p$ and $\alpha_q$ are the latitudes of $p$ and $q$, then $p$ can be reached from $q$.*

These results have been formulated by C. Piron [Pir, p.75–78] (see also Cooke, Keane, and Moran [CKM], Kalmbach [Kal], Dvurečenskij [Dvu]), and they were applied by Cooke, Keane, and Moran [CKM] as a one to present a more simpler proof of the original Gleason theorem. Weak Piron's Geometrical Lemma has been used by Calude, Hertling and Svozil [CHS] to prove the non-existence of two-valued measure on $\mathcal{L}(\mathbb{R}^3)$ using topological arguments.

In what follows, we present some applications of these lemmas to different situations of two-valued measures and infinite-valued measures.

## 3 Two-valued Measures

Let $H$ be a real, complex, or quaternion Hilbert space with an inner product $(\cdot,\cdot)$. We denote by $\mathcal{L}(H)$ the system of all closed subspaces of $H$. For a given $M \in \mathcal{L}(H)$, $M^\perp := \{x \in H : (x,y) = 0 \text{ for any } y \in M\}$. Then $\mathcal{L}(H)$ is a complete orthomodular lattice. By $\bigoplus_t M_t$ we denote the joint of a system $\{M_t\}_t$ of mutually orthogonal subspaces of $H$. By $sp(\{x_t\}_t)$ we mean the span generated by the system of vectors $\{x_t\}_t$.

A mapping $m : \mathcal{L}(H) \to [0,1]$ such that

$$m(H) = 1, \tag{3.1}$$

$$m(\bigoplus_{t\in T} M_t) = \sum_{t\in T} m(M_t) \tag{3.2}$$

is said to be a *finitely additive, $\sigma$-additive,* or *completely additive measure* on $\mathcal{L}(H)$ if (3.2) holds for any finite, countable or arbitrary index set $T$. If the dimension of $H$ is finite, the $\sigma$-additivity, or the complete additivity are superfluous, and we shall say in this case that $m$ is a measure.

We recall that Gleason [Gle] proved that if $H$ is a separable real or complex Hilbert space of dimension at least three, then there is a one-to-one correspondence between $\sigma$-additive measures $m$ on $\mathcal{L}(H)$ and Hermitian positive operators $T$ on $H$ of trace 1 which is given as follows

$$m(M) = tr(TP_M), \ \ M \in \mathcal{L}(H), \tag{3.3}$$

where $P_M$ denotes the orthogonal projector from $H$ onto $M$.

The first application of Piron's Geometrical Lemmas gives the following result without using Gleason's theorem and which has serious physical consequences. We prove the result using both geometrical lemmas to present them.

**Theorem 3.1** *On $\mathcal{L}(\mathbb{R}^3)$ there is no two-valued measure.*

**Proof 1.** (*Application of Weak Piron's Geometrical Lemma.*) Suppose that $m$ is a two-valued measure. Choose a unit vector $p$ in $\mathbb{R}$ such that $m(sp(p)) = 1$. Consider $p$ as the north pole; the equator is determined by $P := sp(p)^\perp$.

Let $C(p)$ be the great circle passing through $p$ and the vectors $(-1,0,0)$ and $(1,0,0)$, i.e., it lies in the plane "$y = 0$". On this circle, one of the vectors $(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2})$ and $(-\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2})$, say $p_1 = (\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2})$, determines a one-dimensional subspace of measure

0. Applying Weak Piron's Geometrical Lemma, we see that all vectors of the northern part lying in the region between $C(p_1)$ and the equator determine one-dimensional subspaces of measure 0. Therefore, there is a greatest latitude $\alpha$ with $\pi/4 \le \alpha \le \pi/2$ such that all vectors of northern hemisphere lying on $C(p)$ with a positive $x$-th coordinate have a latitude less than $\alpha$ and simultaneously they determine one-dimensional subspaces of measure zero.

Similarly, there is a least latitude $\beta$ with $-\pi/4 \le \beta \le 0$ such that all vectors of $C(q)$ having a positive $x$-th coordinate have a latitude greater than $\beta$ and simultaneously they determine one-dimensional subspaces of measure zero.

Consequently, the circle $C(p)$ is divided into four connective arcs each of length $\pi/2$ such that all vectors of any arc determine subspaces of the same measure.

One of border points has to determine a one-dimensional subspace of measure 1; it is now our new north pole. We can assume that we have now a new orientation of the sphere such that the north pole has a coordinate $(1,0,0)$, and the vector $(1,0,0)$ is a border point among arcs determining a one-dimensional subspace of measure 0.

We choose in the northern hemisphere two open parts, the right half $S_1 := \{q \in \mathcal{S}(\mathbb{R}^3) : q_x > 0,\, q_z > 0\}$ and the left half $S_2 := \{q \in \mathcal{S}(\mathbb{R}^3) : q_x < 0,\, q_z > 0\}$. Applying Weak Piron's Geometrical Lemma, we can see that all vectors from $S_1$ determine one-dimensional subspaces of measure 0.

On the other hand, any vector $q = (q_x, q_y, q_z) \in S_2$ determines a one-dimensional subspace of measure 1. Indeed, the vector $q$ together with the vector $(q_y, -q_x, 0)/\sqrt{q_x^2 + q_y^2}$ in the equator and with the vector $(-q_x, -q_y, \frac{q_x^2+q_y^2}{q_z}//||(-q_x, -q_y, \frac{q_x^2+q_y^2}{q_z})||$ in $S_1$ forms an orthonormal basis. Consequently, $q$ determines a subspace of measure 1.

Choose now three orthonormal vectors $(-\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2})$, $(-\frac{1}{2}, -\frac{1}{\sqrt{2}}, \frac{1}{2})$, and $(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$, we see that the first two lie in the left half and the third one in the right half, which gives a contradiction.

**Proof 2.** (*Application of Piron's Geometrical Lemma.*) Suppose the converse. Let $m$ be a two-valued measure on $\mathcal{L}(\mathbb{R}^3)$. Then there is a unit vector $p$ in $\mathbb{R}$ such that $m(sp(p)) = 1$ and $m(P) = 0$, where $P = sp(p)^\perp$. Consider $p$ as the north pole; then $P$ passes through the equator $E_p$.

Take two mutually orthogonal vectors in the northern hemisphere whose latitude is $\alpha = \pi/4$. Then one of them, denote it $q$, determines a one-dimensional subspace $Q$ of measure 0. Let $C(q)$ be the great circle containing $q$ and vectors from $Q^\perp \cap P$. Therefore any point $\tilde{q} \in C(q)$ determines a subspace of measure 0.

If now $\tilde{q}$ is a unit vector from the northern hemisphere belonging to $C(q)$, similarly as for $C(q)$, we can show that any vector from $C(\tilde{q})$ determines a one-dimensional subspace of measure zero. Applying Piron's Geometrical Lemma, we can show that any unit vector of the northern hemisphere whose latitude is less than $\pi/4$ determines a one-dimensional subspace of $\mathbb{R}^3$ of measure zero. Consequently, that holds for any unit vector in $\mathbb{R}^3$ whose latitude is from the interval $(-\pi/4, \pi/4)$.

Without loss of generality we can suppose that the vector $q$ lies in the plane "$y = 0$" with a positive $x$-th coordinate, i.e., $q = (\sqrt{2}/2, 0, \sqrt{2}/2)$, and take a unit vector $q_1$ from the northern hemisphere with the coordinate $(-\sqrt{3}/2, 0, 1/2)$, and let $C(q_1)$ be the great circle passing through $q_1$ and $P \cap sp(q_1)^\perp$. It determines a new two-dimensional space $P'$ of measure zero, and suppose that $C(q_1)$ is now a new equator with the vector $q$ in its new northern hemisphere. The latitude of $q$ with respect to $P'$ is now $\pi/4 + \pi/6 = 5\pi/12$.

Repeating the previous process with $q$ and $P'$, we obtain that all vectors whose latitude is from the interval $(-5\pi/12, 5\pi/12)$ determine one-dimensional subspaces of finite measure which gives that $\mathbb{R}^3$ has measure 0 which is a contradiction. **Q.E.D.**

It is easy to see that if $\dim H = 2$, there is plenty of two-valued measures on $\mathcal{L}(H)$.

**Corollary 3.2** *Let $H$ be a complex, or quaternion three-dimensional Hilbert space. Then on $\mathcal{L}(H)$ there is no two-valued measure.*

**Proof.** Choose a unit vector $q$ and three mutually orthogonal vectors $p_0, p_1, p_2$ such that $q \not\perp p_0, p_1, p_2$. Define $\widetilde{p}_i = |(q, p_i)|^{-1}(q, p_i)p_i$, $i = 0, 1, 2$. Then $q = \sum_{i=0}^{2}(q, \widetilde{p}_i)\widetilde{p}_i$ and $(q, \widetilde{p}_i) \in \mathbb{R}$ for any $i = 0, 1, 2$. Define the three-dimensional real vector space $M$ generated by $\widetilde{p}_0, \widetilde{p}_1, \widetilde{p}_2$. Then $q \in M$, and the restriction of $(\cdot, \cdot)$ onto $M \times M$ attains only real values. The measure $m$ on $\mathcal{L}(H)$ determines a unique measure $\widetilde{m}$ on $\mathcal{L}(M)$ in such a way that $\widetilde{m}(sp_{\mathbb{R}}(r)) = m(sp(r))$ for any unit vector $r$ in $M$, where $sp_{\mathbb{R}}$ denotes the span over the real field.

Then $\widetilde{m}$ is a two-valued measure on the three-dimensional real Hilbert space $M$ which contradicts Theorem 3.1. **Q.E.D.**

The non-existence of any two-valued measure in the finite-dimensional case entails that there must exists a finite number of subspaces for which it is impossible to define a non-trivial two-valued measure. This follows from the compactness of the space of all functions having the values 0 or 1 on subspaces of the space $\mathbb{R}^3$. Kochen and Specker [KoSp] have found such a construction for 117 vectors and Alda [Ald1] for 90 vectors. Recently Peres [Per] produced an elegant proof of 33 vectors in $\mathbb{R}^3$.

Let $S$ be now a real, complex, or quaternion Hilbert space with an inner product $(\cdot, \cdot)$. For any $M \subseteq S$, we put $M^\perp = \{x \in S : (x, y) = 0 \text{ for any } y \in M\}$.

We denote by $F(S)$ the set of all *orthogonally closed subspaces* of $S$, i.e., of all subspaces $M \subseteq S$ such that $M^{\perp\perp} = M$, and by $E(S)$ the set of all *splitting subspaces* of $S$, i.e., of all $M \subseteq S$ such that $M + M^\perp = S$. Then $F(S)$ is a complete orthocomplemented lattice with respect to the set-theoretic inclusion which is not necessary orthomodular, i.e., if $M \subseteq N$, then $N = M \vee (N \cap M^\perp)$. On the other hand, $E(S)$ is an orthomodular poset which is not necessary a complete lattice. We have $E(S) \subseteq F(S)$, and the equality $E(S) = F(S)$ holds iff $S$ is complete, i.e., iff $S$ is a Hilbert space (see [Dvu, Sec 4.1]).

A *finitely additive measure* on $E(S)$, respectively on $F(S)$, is a mapping $m : E(S) \to [0, 1]$ such that $m(S) = 1$, and $m(M \vee N) = m(M) + m(N)$ whenever $M$ and $N$ are mutually orthogonal. On $E(S)$ there is plenty of finitely additive measures, e.g., given a unit vector $x$ in $S$, the mapping $m_x : M \mapsto ||x_M||^2$, where $M \in E(S)$ and $x = x_M + x_{M^\perp}$, $x_M \in M$ and $x_{M^\perp} \in M^\perp$, is a finitely additive measure. On the other hand, on $E(S)$ there is a completely additive measure iff $S$ is complete [Dvu, Thm 4.2.8]. We recall that there is an open problem whether does exist at least one finitely additive measure on $F(S)$.

Another application of Piron's Geometrical Lemma is to the space $F(S)$ which generalizes a result of Alda [Ald 2]. Alda proved originally this result using Gleason's theorem. Below we present this result using Piron's Geometrical Lemma.

**Corollary 3.3** *Let $S$ be a real, complex, or quaternion inner product space of dimension at least three. Then there is no two-valued finitely additive measure on $F(S)$. In particular, there is no two-valued finitely additive measure on the lattice of all closed subspaces of a Hilbert space over $\mathbb{R}$, $\mathbb{C}$ or the quaternion field.*

**Proof.** If $S$ is finite-dimensional, the assertion follows from Corollary 3.2.

Let $\dim S = \infty$. Choose a maximal orthonormal system $\{x_i\}_{i\in I}$ in $S$. We express the index set $I$ in the form of a union of mutually disjoint three-element sets $I_\beta$, i.e., $I = \bigcup_\beta I_\beta$. Let $H_3$ be the three-dimensional Hilbert space over the same field as $S$. Given $\beta$, choose a unitary operator $U_\beta : H_3 \to S_\beta = sp(\{x_i : i \in I_\beta\})$ and define the mapping $U : \mathcal{L}(H_3) \to \mathcal{L}(S)$ as follows

$$U(M) := \bigoplus_\beta U_\beta(M),\ M \in \mathcal{L}(H_3).$$

We recall that $\bigoplus$ denotes here the joint of mutually orthogonal elements taken in $F(S)$. Then $M \perp N$ iff $U(M) \perp U(N)$, and then $U(M+N) = U(M) + U(N)$, $U(H_3) = S$.

Suppose that $m$ is a two-valued measure on $F(S)$. Then $m_U : \mathcal{L}(H_3) \to \{0,1\}$, defined via $m_U(M) = m(U(M))$, $M \in \mathcal{L}(H_3)$, is a two-valued measure on $\mathcal{L}(H_3)$ which is by Corollary 3.2 impossible. **Q.E.D.**

Recently Pták and Weber [PtWe] have constructed an inner product space $S$ such that $E(S)$ consists of all finite- and cofinite-dimensional subspaces of $S$. Such $E(S)$ is then a lattice and of course there exists a two-valued measure $m$ on this $E(S)$ defined by $m(X) = 1$ if and only if $X$ is a cofinite-dimensional subspace. On the other hand, we can show that on $E(S)$, $\dim S \neq 2$, there is no two-valued completely additive measure, because the existence of a completely additive measure on $E(S)$ entails the completeness of $S$ which would be imply that on the Hilbert space $S$ there exists a two-valued measure [Dvu, Thm 4.2.3] which is absurd.

## 4 Measures with Infinite Values

Besides finite measures, in Hilbert space quantum mechanics we can meet measures attaining infinite values. For example,

$$m(M) := \dim(M),\ M \in \mathcal{L}(H),$$

is one of such measures. Integrating quantum mechanical observables through finite measures we can obtain measures which attain improper value, e.g.,

$$m(M) := tr(T(P_M A + AP_M))/2,\ M \in \mathcal{L}(H),$$

can be regarded as the integral of the form $\int_M x_A dm_T$, where $A$ and $x_A$ are observables corresponding to a Hermitian operator (in general a self-adjoint operator) $A$ in $H$, and $m_T$ is a $\sigma$-additive measure on $\mathcal{L}(H)$ determined by $T$ via (3.3).

For our aims we do not limit ourselves only to positive measures on $\mathcal{L}(H)$. A mapping $m : \mathcal{L}(H) \to \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ such that

$$m(O) = 0, \tag{4.1}$$

$$m(\bigoplus_{t\in T} M_t) = \sum_{t\in T} m(M_t), \tag{4.2}$$

where $O$ is the null-space of $H$, is said to be a *finitely additive, $\sigma$-additive,* or *completely additive measure* on $\mathcal{L}(H)$ if (4.2) holds for any finite, countable or arbitrary index set $T$. It is possible to show that $m$ attains from the improper value $\pm\infty$ at most one.

The first description of measures with infinite values on $\mathcal{L}(H)$ is given by Lugovaja and Sherstnev [LuSh]. The following lemma is of similar importance as that for finite measures in Gleason's proof. Here we present it proving it in a different way as in the original (see [LuSh], or [Dvu, Lem 3.4.2]), namely, we use Piron's Geometrical Lemma.

**Lemma 4.1 (Lugovaja-Sherstnev)** *Let $m : \mathcal{L}(\mathbb{R}^3) \to (-\infty, +\infty]$ be a finitely additive measure with $m(\mathbb{R}^3) = \infty$, and let $Q$ and $P$ be one- and two-dimensional subspaces of $\mathbb{R}^3$ of finite measure. Then $P \subseteq Q$.*

**Proof.** If $Q \subseteq P$, we are ready. Suppose thus that $Q \not\subseteq P$. It is clear that $Q \not\perp P$. Denote by $\mathcal{S}(\mathbb{R}^3)$ the unit sphere in $\mathbb{R}^3$ and suppose that $P$ determines the equator. Choose a unit vector $q$ in $Q$ supposing that $q$ belongs to the northern hemisphere, i.e., its latitude $\alpha$ with respect to $P$ satisfies $0 < \alpha < \pi/2$, and let $C(q)$ be the great circle determined by $q$ and by an orthogonal vector $q_1$ from $P \cap Q^\perp$. Then $sp(q, q_1)$ is of finite measure, so that any vector $\widetilde{q} \in C(q)$ lying in the northern hemisphere determines a subspace of finite measure. By the same way as for $C(q)$ we can show that any point from the northern hemisphere belonging to $C(\widetilde{q})$ determines a one-dimensional subspace of finite measure.

Using Piron's Geometrical Lemma, we see that any unit vector of the northern hemisphere whose latitude is less than $\alpha$ determines a one-dimensional subspace of finite measure.

Consequently, all unit vectors whose latitude is from the interval $(-\alpha, \alpha)$ determines a one-dimensional subspace of finite measure.

If $\alpha > \pi/4$, we obtain the contradiction. Otherwise, we continue as follows. Without loss of generality we can assume that vector $q$ lies in the plane "$y = 0$", i.e., $q = (q_x, 0, q_z)$, where $q_x > 0, q_z > 0$, and $\sin\alpha = q_z$. Choose now a unit vector $p$ in the northern hemisphere such that $p = (p_x, 0, p_z)$ and $p_x < 0, p_z = \sin\frac{4}{5}\alpha$. Then the great circle $C(p)$ determines a new two-dimensional space $P_1$ of finite measure, and we shall suppose that $P_1$ determines a new equator with the vector $q$ in its northern hemisphere. The new latitude of $q$ with respect to $P_1$ is $\alpha' = \alpha + \frac{4}{5}\alpha = \frac{9}{5}\alpha$. Repeating all above considerations, we can show that all unit vectors whose latitude is from the interval $(-\alpha', \alpha) = (-\frac{9}{5}\alpha, \frac{9}{5}\alpha)$ determine a one-dimensional subspace of $\mathbb{R}^3$ of finite measure.

Repeating this process sufficiently many times , we can found the least integer $n$ such that $(\frac{18}{5})^n\alpha > \pi/4$, and, in addition, we can found a two-dimensional subspace $P_n$ of $\mathbb{R}^3$ such that the latitude of $q$ relative to $P_n$ is greater than $\pi/4$. Therefore, all vectors in $\mathcal{S}(\mathbb{R}^3)$ determine one-dimensional subspaces of finite measure which is a contradiction with the hypothesis that $m(\mathbb{R}^3) = \infty$. **Q.E.D.**

**Corollary 4.2** *Let $H$ be a complex, or quaternion three-dimensional Hilbert space. Let $m : \mathcal{L}(H) \to (-\infty, +\infty]$ be a finitely additive measure with $m(H) = \infty$, and let $Q$ and $P$ be one- and two-dimensional subspaces of $H$ of finite measure. Then $P \subseteq Q$.*

**Proof.** It is clear that $Q \not\perp P$.

Choose a unit vector $q \in Q$ and three mutually orthogonal vectors $p_0, p_1, p_2$ such that $p_0 \perp P$, $p_1, p_2 \in P$ and $q \not\perp p_0, p_1, p_2$. Define $\widetilde{p}_i = |(q, p_i)|^{-1}(q, p_i)p_i$, $i = 0, 1, 2$. Then $q = \sum_{i=0}^{2}(q, \widetilde{p}_i)\widetilde{p}_i$ and $(q, \widetilde{p}_i) \in \mathbb{R}$ for any $i = 0, 1, 2$. Define the three-dimensional real vector space $M$ generated by $\widetilde{p}_0, \widetilde{p}_1, \widetilde{p}_2$. Then $q \in M$, and the restriction of $(\cdot, \cdot)$ onto $M \times M$ attains only real values. The measure $m$ on $\mathcal{L}(H)$ determines a unique measure

$\widetilde{m}$ on $\mathcal{L}(M)$ in such a way that $\widetilde{m}(sp_{\mathbb{R}}(r)) = m(sp(r))$ for any unit vector $r$ in $M$, where $sp_{\mathbb{R}}$ denotes the span over the real field.

Then $\widetilde{m}(M) = \infty$, and $\widetilde{m}(\widetilde{Q})$ and $\widetilde{m}(\widetilde{P})$ are finite, where $\widetilde{Q}$ and $\widetilde{P}$ are real subspaces of $M$ generated by $q$ and $\widetilde{p}_1, \widetilde{p}_2$, respectively. Applying Lemma 4.1, we have $q \in \widetilde{Q} \subseteq \widetilde{P}$ such that $q \in P$, and $Q \subseteq P$. **Q.E.D.**

**Corollary 4.3** *Let $H$ be a real, complex, or quaternion Hilbert space, $3 \leq \dim H = n < \infty$, and let $m$ be a finitely additive measure on $\mathcal{L}(H)$ with $m(H) = \infty$. If $P$ and $Q$ are two subspaces of $H$ of dimension $n-1$ and $1$, respectively, and of finite measure, then $Q \subseteq P$.*

**Proof.** If $\dim H = 3$, the assertion follows from Corollary 4.2. Suppose thus that $\dim H > 3$. Then $\dim(Q^{\perp} \cap P) \geq 2$ and there exists a unit vector $y_1 \in Q^{\perp} \cap P$, i.e., $y_1 \perp Q$, $y_1 \perp P^{\perp}$, and $\dim(Q \vee P^{\perp} \vee sp(y_1)) = 3$. Hence

$$m(Q \vee P^{\perp} \vee sp(y_1)) = \infty.$$

In addition, $\dim((Q \vee P^{\perp} \vee sp(y_1)) \cap P) = 2$.

We assert that $Q \subseteq P$. If not, then there exists a unit vector $y_2 \perp y_1$ belonging to $(Q \vee P^{\perp} \vee sp(y_1)) \cap P$ such that $Q \vee (sp(y_1, y_2)) = Q \vee P^{\perp} \vee sp(y_1)$. Using Corollary 4.2, we have $m(Q \vee P^{\perp} \vee sp(y_1)) < \infty$. **Q.E.D.**

**Corollary 4.4** *Let $H$ be a real, complex, or quaternion finite-dimensional Hilbert space. Let $m$ be a finitely additive measure on $\mathcal{L}(H)$, and let $M$ be a two-dimensional subspace of finite measure. Denote by*

$$D(m) := \{x \in H : m(sp(x)) < \infty\}. \tag{4.3}$$

*Then $D(m)$ is a linear submanifold of $H$.*

**Proof.** It is clear that $M \subseteq D(m)$. Take two linearly independent vectors $x$ and $y$ from $D(m)$. If $x \notin M$, then $\dim(M \vee sp(x)) = 3$ and $m(M \vee sp(x)) < \infty$ by Corollary 4.3. If $y \in M \vee sp(x)$, then $x + y \in D(m)$. Otherwise, let $y \notin M \vee sp(x)$. Then $3 \leq \dim(M \vee sp(x, y)) \leq 4$, and by Corollary 4.3, $m(M \vee sp(x, y)) < \infty$, which proves $x + y \in D(m)$. **Q.E.D.**

For the reader's convenience we present the following example to illustrate Corollary 4.4

Let $\Delta$ be a set of one-dimensional subspaces in a real, complex, or quaternion finite-dimensional Hilbert space $H$. If two elements of $\Delta$ are never orthogonal, we can define a measure $m$ on $\mathcal{L}(H)$ by :

$$m(P) = \begin{cases} 0 & \text{if } P \in \Delta \text{ or } P = \{0\}, \\ \infty & \text{otherwise.} \end{cases}$$

In this case $D(m) = \bigcup_{D \in \Delta} D$. Thus $D(m)$ is not a subspace and $m$ is not defined as in the previous Corollary 4.4.

**Corollary 4.5** *Let $H$ be a real, complex, or quaternion finite-dimensional Hilbert space. Let $m$ be a finitely additive measure on $\mathcal{L}(H)$, and let there be a two-dimensional subspace of finite measure. Then there is a finite measure $m_0$ on $\mathcal{L}(D(m))$ such that*

$$m(M) = \begin{cases} \infty & \text{if } M \not\subseteq D(m), \\ m_0(M) & \text{if } M \subseteq D(m). \end{cases}$$

*Conversely, if $m_0$ is a finite measure on $\mathcal{L}(M_0)$, where $M_0$ is a subspace of $H$, then the mapping $m$ defined on $\mathcal{L}(H)$ via*

$$m(M) = \begin{cases} \infty & \text{if } M \not\subseteq M_0, \\ m_0(M) & \text{if } M \subseteq M_0, \end{cases}$$

*defines a measure on $\mathcal{L}(H)$. In addition, if $M_0$ is at least a two-dimensional subspace, then $D(m) = M_0$.*

**Proof.** It follows from Corollary 4.4. **Q.E.D.**

**Corollary 4.6** *Let $m$ be a finitely additive measure on $\mathcal{L}(H), \dim H < \infty$, such that there is a three-dimensional subspace of finite measure. Then there is a unique Hermitian trace operator $T_m$ on $D(m)$ such that*

$$m(M) = \begin{cases} \infty & \text{if } M \not\subseteq D(m), \\ tr(T_m P_M) & \text{if } M \subseteq D(m). \end{cases}$$

**Proof.** It follows from Corollary 4.5 and the classical Gleason theorem. **Q.E.D.**

We say that a measure $m$ is *$\sigma$-finite* if there is a sequence of mutually orthogonal subspaces $\{M_n\}_n$ with $\bigoplus_n M_n = H$ such that $m(M_n) < \infty$ for any $n \geq 1$.

Lugovaja and Sherstnev [LuSh] (see also [Dvu, Thm 3.4.8]) presented the generalization of Gleason's theorem for $\sigma$-finite measures with infinite values. They argued as follows: $D(m)$ is a linear submanifold dense in $H$. There is a Hermitian bilinear form $t$ with $D(t) = D(m)$ which is defined via $t(x,x) := ||x||^2 m(sp(x))$, $x \in D(m)$. Given $M \in \mathcal{L}(H)$, we define $t \circ P_M$ as a bilinear form such that $D(t \circ P_M) = \{x \in H : P_M x \in D(m)\}$, and $(t \circ P_M)(x,x) = t(P_M x, P_M x)$. If $t \circ P_M$ is a bounded bilinear form determined by some trace operator $T_M$ on $H$, we say that $t \circ P_M \in Tr(H)$, where $Tr(H)$ is the system of all Hermitian trace operators on $H$, and write $tr(t \circ P_M) := tr(T_M)$. This, roughly speaking, proves the following result (for details see [LuSh], or [Dvu, Thm 3.4.8]; in the second paper there are also some generalizations of this result).

**Theorem 4.7** *Let $H$ be a separable infinite-dimensional Hilbert space. Let $m$ be a $\sigma$-finite $\sigma$-additive measure on $\mathcal{L}(H)$. Then there exists a unique symmetric bilinear form $t$ with domain $D(t)$ dense in $H$ such that*

$$m(M) = \begin{cases} \infty & \text{if } t \circ P_M \notin Tr(H), \\ tr(T_m P_M) & \text{if } t \circ P_M \in Tr(H). \end{cases}$$

## 5 Bell's Geometrical Lemma

J. Bell, the author of the well-known Bell inequalities, in his paper [Bel] on the problem of hidden variables in quantum mechanics, was interesting in Gleason's theorem which in the particular case of $H = \mathbb{R}^3$ entails that on $\mathcal{L}(\mathbb{R}^3)$ there is no two-valued measure. He was looking for a simpler proof of Gleason's theorem, and about his effort the following anecdote [CKM0], [Dvu, p.130] is told: When J. Bell became familiar with the Gleason result, he said that either he would find a relatively simpler proof, or he would leave from this area. Fortunately, he found a relatively simple proof of the partial result [Bel] that there is no two-valued measure on $\mathcal{L}(\mathbb{R}^3)$. Below we present a modification of his proof, and we apply it also for measures with infinite values. It is worthy to recall that it holds for any finitely additive measure on any $\mathcal{L}(H)$, $\dim H \geq 3$.

**Lemma 5.1 (Bell's Geometrical Lemma)** *Let $m$ be a finite finitely additive measure on $\mathcal{L}(\mathbb{R}^3)$. Let $P$ and $Q$ be two one-dimensional subspaces of measure* 1 *and* 0*, respectively. Then, for the angle $\rho$ between $P$ and $Q$, we have $\rho > \arctan 1/2$.*

**Proof.** If $P$ and $Q$ are orthogonal, the statement is evident. Let now $P \not\perp Q$. Choose a non-zero vector $q \in Q$ and let $p$ and $p_1$ be the projections of $q$ onto $P$ and $P^\perp$. Then $p_1 \neq 0 \neq p$, and dividing $p, q, p_1$ by $||p||$, we can assume that $q$ is chosen in such a way that $||p|| = 1$, and that $p$ and $q$ lies in the same hemisphere determined by $P^\perp$. Then $q = p + p_1$, and let us express $p_1$ in the form $p_1 = \epsilon p'$, where $p'$ is a unit vector and $\epsilon > 0$. Then $q = p + \epsilon p'$ and $\tan \rho = \epsilon$.

Let $p''$ be a unit vector orthogonal to both $p$ and $p'$, and so to $q$. Therefore,

$$m(sp(p')) = 0, \quad m(sp(p'')) = 0.$$

Since $q \perp p''$, we have

$$m(sp(q + \gamma^{-1}\epsilon p'')) = 0,$$

where $\gamma$ is a real number. Therefore,

$$m(sp(-\epsilon p' + \gamma\epsilon p'')) = 0.$$

The vectors $q + \gamma^{-1}\epsilon p''$ and $-\epsilon p' + \gamma \epsilon p''$ are mutually orthogonal, so we may add them, and we have

$$m(sp(p + \epsilon(\gamma + \gamma^{-1})p'')) = 0.$$

Now if $\epsilon \leq 1/2$, there are real numbers $\gamma$ such that

$$\epsilon(\gamma + \gamma^{-1}) = \pm 1.$$

Therefore,

$$m(sp(p + p'')) = 0 = m(sp(p - p'')).$$

The vectors $p + p''$ and $p - p''$ are orthogonal. Adding them, we obtain $m(sp(p)) = 0$ which is a contradiction. **Q.E.D.**

Using Piron's Geometrical Lemma, we can strengthen Lemma 5.1.

**Lemma 5.2** *Let $m$ be a finite finitely additive measure on $\mathcal{L}(H)$, $\dim H \geq 3$. If $P$ and $Q$ are one-dimensional subspaces of measure one and zero, then $P \perp Q$.*

**Proof.** Suppose that $H = \mathbb{R}^3$. We claim that $Q \subseteq P^\perp$. When we use the proof of Lemma 4.1, changing measure of infinite or finite values to measure one or zero, respectively, we can show that $Q \subseteq P^\perp$.

If $H$ is arbitrary, we define a three-dimensional subspace containing $P, Q$ and a one-dimensional subspace orthogonal to both $P$ and $Q$, which according to the first step proves that $Q \subseteq P^\perp$. **Q.E.D.**

We have seen that if $3 \leq \dim H < \infty$, then on $\mathcal{L}(H)$ there is no two-valued measure; in any rate, there is a measure with $n+1$ values, namely $m(M) := \dim M/n$, $M \in \mathcal{L}(H)$.

Applying Lemma 5.1, we have another proof of Theorem 3.1.

**Corollary 5.3** *On $\mathcal{L}(\mathbb{R}^3)$ there is no two-valued measure.*

**Proof.** It is an easy consequence of Bell's Geometrical Lemma. **Q.E.D.**

The ideas of Bell's Geometrical Lemma can be applied also for measures with infinite values to obtain a weaker form of Lemma 4.1

**Lemma 5.4** *Let $m$ be a measure on $\mathcal{L}(\mathbb{R}^3)$. Let $P$ be a one-dimensional subspace of infinite measure such that $m(P^\perp)$ is finite. Then, for any one-dimensional subspace $Q$ of finite measure, we have $\rho >$ arctan $1/2$, where $\rho$ is the angle between $P$ and $Q$.*

**Proof.** If $P \perp Q$, the statement is evident. Otherwise, suppose $P \not\perp Q$, and let $p \in P, p' \in P^\perp$, and $q \in Q$ be chosen in the same way as in the proof of Lemma 5.1, i.e., $q = p + \epsilon p'$, where $\tan \rho = \epsilon$.

Let $p''$ be a unit vector orthogonal to both $p$ and $p'$, and so to $q$. Then $p'' \in P^\perp$, so that $m(sp(p'))$ and $m(sp(p''))$ are finite. Similarly $m(sp(q + \gamma^{-1}\epsilon p''))$, where $\gamma$ is a real number, and $m(sp(-\epsilon p' + \gamma\epsilon p''))$ are also finite. Consequently, $m(sp(p + \epsilon(\gamma + \gamma^{-1})p''))$ is finite.

Supposing $\epsilon \leq 1/2$, there are real numbers $\gamma$ such that $\epsilon(\gamma + \gamma^{-1}) = \pm 1$. Therefore, $m(sp(p + p''))$ and $m(sp(p - p''))$ are both finite. Adding orthogonal vectors $p + p''$ and $p - p''$, we obtain that $m(sp(p))$ is of finite measure which is absurd. Therefore, $\tan \rho > 1/2$. **Q.E.D.**

We recall that the assumption $m(P^\perp)$ is finite is essential in our considerations, because there are measures $m$ on $\mathcal{L}(\mathbb{R}^3)$ with $m(\mathbb{R}^3) = \infty$ which have not this property (compare with Lemma 4.1 and Corollary 4.5).

## 6 Parabola Based Geometrical Lemma

In the present Section, we show a parabola based proof of a new geometrical lemma which entails both Weak Piron's Geometrical Lemma and Bell's Geometrical lemmas.

Let $D$ be a one-dimensional subspace of $\mathbb{R}^3$ and let $Oxyz$ be Cartesian coordinates[1] such that $D = Oz$ and $D^\perp = xOy$. Let $\Pi$ be the plane $z = 1$, $N$ be the point $(0, 0, 1)$ and $\Phi = \{(0, y, z) \in \mathbb{R}^3 : yz > 0 \text{ or } (y, z) = (0, 0)\}$. Notice that any line contained in $\Phi$ is a one-dimensional subspace of $\mathbb{R}^3$. For the line $\Delta \subset \Phi$, $\alpha$ is the angle between $Oy$ and $\Delta$, $V = \Delta \cap \Pi$ and $T$ is the intersection of $\Pi$ with the plane defined by the lines $Ox$ and $\Delta$. Notice that $T$ is parallel to $Ox$ and so $NV \perp T$.

[1] We recall that the orientation of the coordinatization is different of that used in previous parts.

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