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Slovak Academy of Sciences, Bratislava

Preprint 12/1999

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July 26, 1999

PREPRINT SERIES

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Abstract.

As an alternative to quantum logics MV algebras have been suggested ([2], [20]). In a quite general class of MV algebras there have been constructed sum of any two observables ([20], [14], [10]). Recently the notion of product MV algebra has beeen introduced ([17], [4], [5], [12]). In any weakly σ-distributive product MV algebra the joint observable of any observables can be constructed and on the base also the sum of any observables. In this paper we show that the two concepts of constructing of the sum are equivalent. [1]

Keywords: MV algebras, states, observables

States and observables on MV algebras

Here we shall present basic definitions. In Section 2 the first construction of sum of observables will be discussed, in Section 3 the second one will be studied. Finally in Section 4 the connection between the concepts will be clarify.

A prototype of an MV algebra is a set $\mathcal{F}$ of real functions $f : \Omega \to [0,1]$ closed under the following binary operations $\oplus$, $\odot$, the unary operation $*$ and two fixed elements 0_Ω, 1_Ω, where

$f \oplus g = \min(f+g, 1)$,

$f \odot g = \max(f+g-1, 0)$,

$f^* = 1 - f$.

Generally an MV algebra ([1]) is an algebraic structure $\mathcal{M} = (M, \oplus, \odot, *, u, 0)$ satisfying some axioms (e.g. [1], [3], [20]). Of course, by the Mundici representation theorem [11] to any MV algebra $\mathcal{M}$ there exists a commutative lattice ordered group $(G, +, \leq)$ with a strong unit u (i.e. to any $a \in G$ there exists $n \in N$ with $a \leq nu$) such that $M = [0, u]$ and

$a \oplus b = (a+b) \wedge u$,

$a \odot b = (a+b-u) \vee 0$,

[1]This research has been supported by grant VEGA 2/6087/99.

$a^* = u - a$,

0 is the neutral element of G and u is the mentioned strong unit of G.

A state is a mapping $m : M \to [0,1]$ satisfying the following conditions:

(i) $m(u)$=1.

(ii) If $a \le b^*$, then $m(a+b) = m(a) + m(b)$.

(iii) If $a_n \nearrow a$, then $m(a_n) \nearrow m(a)$.

A weak observable is a mapping $x : \mathcal{B}(R) \to M$ satisfying the following conditions:

(i) m($x(R)$)=1.

(ii) If $A \cap B = \emptyset$, then $x(A) \le x(B)^*$ and $x(A \cup B) = x(A) + x(B)$.

(iii) If $A_n \nearrow A$, then $x(A_n) \nearrow x(A)$.

A weak observable is observable, if

(i') $x(R) = u$.

If $x : \mathcal{B}(R) \to M$ is a weak observable and $m : M \to [0,1]$ is a state, then the mapping $m_x = m \circ x : \mathcal{B}(R) \to [0,1]$ is a probability measure.

$\odot$-independent observables

Two observables $x, y : \mathcal{B}(R) \to M$ are called to be $\odot$-independent, if

$$m(x(A) \odot y(B)) = m(x(A)).m(y(B))$$

for any $A, B \in \mathcal{B}(R)$.

Of course, the right side of the equality can be rewritten as $m_x(A).m_y(B) = m_x \times m_y(A \times B)$. This formulation leads to the following definition.

Two observables $x, y : \mathcal{B}(R) \to M$ are called to be compatible, if there exists a measure $\mu : \mathcal{B}(R^2) \to [0,1]$ such that

$$m(x(A) \odot y(B)) = \mu(A \times B)$$

for all $A, B \in \mathcal{B}(R)$.

Evidently, if x, y are $\odot$-independent, they are compatible. Therefore we have formulated the following theorem ([10]) for compatible observables, although the proof is a slight modification of the corresponding proof from the book [20]. Of course, it holds in so-called weakly σ-distributive MV algebras ([22]).

Definition 2.1 A weakly σ-distributive MV algebra is a σ-complete MV algebra such that for any bounded sequence (a_{ij}) with $(a_{ij}) \searrow 0 (j \to \infty, i = 1, 2, \ldots)$ we have

$$\bigwedge_{\varphi \in N^N} \bigvee_{i=1}^{\infty} a_{i\varphi(i)} = 0\,.$$

Theorem 2.2. Let M be a weakly σ-distributive MV algebra, $x, y : \mathcal{B}(R) \to M$ be compatible week observables. Then there exists a weak observable $x \oplus y$ such that

$$m((x \oplus y)((-\infty, t))) = \mu(\{(u, v); u + v < t\})$$

for any $t \in R$.

Corollary 2.3. If M is a weakly σ-distributive MV algebra and $x, y : \mathcal{B}(R) \to M$ are $\odot$ - independent weak observables, then there exists a weak observable $x \oplus y$ such that

$$m((x \oplus y)((-\infty, t))) = m_x \times m_y(\{(u, v); u + v < t\}).$$

Of course, the last equality can be reformulated as follows. Put $g(u, v) = u + v$. Then $g^{-1}((-\infty, t)) = \{(u, v); u + v < t\}$, hence the mentioned equality can be rewritten by

$$m((x \oplus y)((-\infty, t)) = m_x \times m_y(g^{-1}((-\infty, t)))$$

(for $t \in R$), or by the equality

$$m \circ (x \oplus y) = m_x \times m_y \circ g^{-1}.$$

Although the construction of $x \oplus y$ is a little complicated, it is interesting that the strong law of large numbers has been proved in this concept (see [15]).

Product MV algebras

If $\mathcal{F}$ is a σ-algebra of fuzzy sets and $x, y : \mathcal{B}(R) \to \mathcal{F}$ are observables, then their joint observable is a morphism $h : \mathcal{B}(R^2) \to \mathcal{F}$ such that

$$h(A \times B) = x(A).y(B)$$

for any $A, B \in \mathcal{B}(R)$. Since $x(A).y(B)$ are real functions from $\mathcal{F}$, $x(A).y(B)$ is their product. Of course, in general MV algebra we need not have such a product. Therefore we introduce the following notion.

Definition 3.1 A product MV algebra is an algebraic system $\mathcal{M} = (M, \oplus, \odot, ., *, 0, u)$, where $(M, \oplus, \odot, *, 0, u)$ is an MV algebra and . is a binary operation on M satisfying the following properties:

(i) $u.u = u$.

(ii) . is commutative and associative.

(iii) If $a \leq b^*$, then $(a + b).c = a.c + b.c$.

(iv) If $a_n \downarrow 0, b_n \downarrow 0$, then $a_n \cdot b_n \downarrow 0$.

A product MV algebra is called strong product MV algebra, if

$$u.a = a$$

for any $a \in M$.

In a weaker form the notion of a product MV algebra is discussed in [4]. In [5] a class of MV algebras is studied (weakly divisible MV algebras) in which a product can be introduced. From an algebraic point of view the product MV algebras are studied in [6].

Of course, any MV algebra of fuzzy sets is an MV algebra. Foundations of probability theory on such MV algebras have been developped in [20]. Although in [20] some results have been presented concerning general MV algebras, the absence of the existence of the joint observable did not permit to generalize the previous results. It was one of reasons for introducing the notion of the product MV algebra.

First of all the notion of the joint observable can be defined in the product MV algebras.

Definition 3.2. If $x, y : \mathcal{B}(R) \to M$ are observables, then their joint observable is a mapping $h : \mathcal{B}(R^2) \to M$ satisfying the following conditions:

(i) $h(R^2) = u$.

(ii) If $A \cap B = \emptyset$, then $h(A) \leq h(B)^*$ and $h(A \cup B) = h(A) + h(B)$.

(iii) If $A_n \nearrow A$, then $h(A_n) \nearrow h(A)$.

(iv) $h(A \times B) = x(A).y(B)$ for any $A, B \in \mathcal{B}(R)$.

Theorem 3.3. Let $\mathcal{M}$ be a weakly σ-distributive product MV algebra, $x, y : \mathcal{B}(R) \to M$ be any observables. Then there exists the joint observable of x, y.

Proof. See [17].

Definition 3.4. If M is a weakly σ-distributive product MV algebra, $x, y : \mathcal{B}(R) \to M$ are observables and h is their joint observable, then we define an observable $x + y$ by the formula

$$x + y = h \circ g^{-1},$$

where $g : R^2 \to R, g(u, v) = u + v$.

It is not difficult to see that Theorem 3.3 holds also for weak observables, hence $x + y$ can be defined also for weak observables.

Definition 3.5. Two weak observables are p-independent, if

$$m(x(A).y(B)) = m(x(A)).m(y(B))$$

for any $A, B \in \mathcal{B}(R)$.

Theorem 3.6 If M is a weakly σ-distributive product MV algebra and $x, y : \mathcal{B}(R) \to M$ are p-independent weak observables, then

$$m_{x+y} = (m_x \times m_y) \circ g^{-1}.$$

Proof. By the definition

$$m_{x+y}((-\infty, t)) = m(h(g^{-1}((-\infty, t)))) =$$

$$= m(h(\{(u,v); u+v<t\})).$$

Put $A_i^n = [(i-1)/2^n, i/2^n), \alpha_n(t) = \{(i,j); i+j < 2^n t\}$. Then

$$m(h(\{(u,v); u+v<t\})) =$$

$$= m(h(\bigcup_{n=1}^{\infty} \bigcup_{(i,j)\in\alpha_n(t)} A_i^n \times A_j^n)) =$$

$$= \lim_{n\to\infty} \sum_{(i,j)} m(x(A_i^n).y(A_j^n)) =$$

$$= m_x \times m_y(\bigcup_{n=1}^{\infty} \bigcup_{(i,j)\in\alpha_n(t)} A_i^n \times A_j^n) =$$

$$= m_x \times m_y(\{(u,v); u+v<t\}) =$$

$$= m_x \times m_y(g^{-1}((-\infty,t))).$$

Corollary 3.7. If M is a weakly σ-distributive MV algebra and x, y are $\odot$-independent and p-independent observables, then

$$m_{x\oplus y} = m_{x+y}.$$

Proof. It follows by Corollary 2.3 and Theorem 3.6.

Two kinds of independency

We have seen that $x \oplus y$ and $x + y$ have the same probability distribution provided they are p-independent as well as $\odot$-independent. We shall show that $\odot$-independency in a strong product MV algebra implies p-independency.

Theorem 4.1. If M is a strong product MV algebra and $x, y : \mathcal{B}(R) \to M$ are $\odot$ independent weak observables, then x, y are p-independent.

Proof. Put $\mu = m_x \times m_y : \mathcal{B}(R^2) \to [0,1]$. Put $a = x(A), b = x(B), c = y(C)$, where $A, B, C \in \mathcal{B}(R), A \cap B = \emptyset$. Then

$$m((a+b)\odot c) = m(x(A\cup B)\odot y(C))$$

$$= \mu((A\cup B)\times C) = \mu(A\times C) + \mu(A\times C)$$

$$= m(a\odot c) + m(b\odot c)$$

$$= m(a\odot c + b\odot c),$$

hence

$$m((a+b)\odot c - (a\odot c + b\odot c)) = 0.$$

But

$$a\odot c + b\odot c =$$

$$= (a+c-u)\vee 0 + (b+c-u)\vee 0 =$$

$$= (a+c-u+(b+c-u)\vee 0)\vee(b+c-u)\vee 0 =$$

$$= (a+b+c-u+c-u)\vee(a+c-u)\vee(b+c-u)\vee 0,$$

hence

$$0 = m((a+b)\odot c - (a\odot c + b\odot c)) =$$

$$= m((a+b+c-u)\vee 0-$$

$$-((a+b+c-u+c-u)\vee(a+c-u)\vee(b+c-u)\vee 0)) =$$

$$= m((a+b+c-u)\vee 0$$

$$+(u-a-b-c+u-c)\wedge(u-a-c)\wedge(u-b-c)\wedge 0) =$$

$$= m(((u-c)\wedge b\wedge a\wedge(a+b+c-u))\vee d),$$

where

$$d = (u-a-b-c+u-c)\wedge(u-a-c)\wedge(u-b-c)\wedge 0.$$

Evidently $m(d) = 0$. Put $B = A^c$. Then $m(a)+m(b)-1 = m(x(A)) + m(x(A^c)) - 1 = 0$, hence

$$0 = m(y(C^c)\wedge x(A^c)\wedge x(A)\wedge y(C)).(\star)$$

On the other hand $a.b \leq a\wedge b, a\odot b \leq a.b$. Therefore

$$m(x(A).y(C)) - m(x(A)\odot y(C)) =$$

$$= m(x(A).y(C) - (x(A)+y(C)-u)\vee 0) =$$

$$= m(((u-x(A)).(u-y(C)))\wedge(x(A).y(C))) =$$

$$= m((x(A^c).y(C^c))\wedge(x(A).y(C))) \leq$$

$$\leq m(y(C^c)\wedge x(A^c)\wedge x(A)\wedge y(C)) = 0$$

by ($\star$). Therefore

$$m(x(A).y(C)) = m(x(A)\odot y(C)).$$

We see that x, y are p-idependent. Moreover $m_{x+y} = m_{x\oplus y}$ by Corollary 3.7.

Corollary 4.2 Let M be a weakly σ-distributive product MV algebra. Let $x, y : \mathcal{B}(R) :\rightarrow M$ be $\odot$-independent observables. Then

$$m_{x+y} = m_{x\odot y}.$$

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