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Anatolij Dvurečenskij

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ON LOOMIS–SIKORSKI'S THEOREM FOR MV-ALGEBRAS AND BCK-ALGEBRAS

Anatolij DVUREČENSKIJ

Mathematical Institute, Slovak Academy of Sciences,
Štefánikova 49, SK-814 73 Bratislava, Slovakia
dvurecen@mat.savba.sk

Abstract. We generalize the Loomis–Sikorski theorem for perfect MV-algebras, Dedekind σ-complete BCK-algebra, and Dedekind σ-complete ℓ-rings showing that there exist special kinds of tribes, BCK-tribes or r-tribes consisting of fuzzy sets, or of bounded functions, defined on a compact Hausdorff space. Tribes are closed under point supremas of bounded sequences and they are a direct generalization of σ-algebras of crisp subsets.

1. Introduction

MV-algebras entered mathematics in 1958 by Chang [Cha] as a non-latticeal generalization of Boolean algebras and they arise from many valued logic of Łukasiewicz in the same manner as Boolean algebras arise from classical two-valued logic. BCK-algebras were introduced in a similar way by Imai and Iséki [ImIs] in 1966, and both structures met interest of plenty mathematicians, logicians, algebraists, experts in fuzzy sets as well as in quantum structures [DvKi].

Many MV-algebras have an isomorphic representation by an MV-algebra of fuzzy sets, called a tribe, more precisely by continuous fuzzy sets on a compact Hausdorff space which is an analogue of the Stone representation of Boolean algebras. The similar situation is with some commutative BCK-algebras.

The famous Loomis–Sikorski theorem [Sik] plays a crucial rôle for analysis of Boolean σ-algebras, and it says that every Boolean σ-algebra is a σ-homomorphic image of a σ-algebra of subsets.

In [Dvu 5] and [Mun 1], it was shown that for σ-complete MV-algebras there exists an analogue of σ-complete algebras of sets, tribes, which are σ-complete systems of fuzzy sets closed under point limits of non-decreasing sequences of fuzzy sets, and these tribes play the corresponding rôle in the Loomis–Sikorski theorem for MV-algebras.

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In the present paper, we show that the Loomis–Sikorski theorem for σ-complete MV-algebras can be extended to perfect MV-algebras which have a representation by bounded functions only on radicals, Section 2. In Section 3, we present the Loomis–Sikorski theorem for Dedekind σ-complete commutative BCK-algebras with the relative cancellation property having a quasi strong unit. Section 4 is devoted to the Loomis–Sikorski theorem for Dedekind σ-complete product BCK-algebras, product MV-algebras and product perfect MV-algebras, respectively.

2. Perfect MV-algebras

In the present section, we recall MV-algebras and perfect MV-algebras, and we generalize for them the Loomis–Sikorski theorem.

We recall that an MV-*algebra* is a non-empty set M with two special elements 0 and 1 ($0 \neq 1$), with a binary operation $\oplus : M \times M \to M$ and with a unary operation $^* : M \to M$ such that, for all $a, b, c \in M$, we have

(MVi) $a \oplus b = b \oplus a$ (commutativity);
(MVii) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ (associativity);
(MViii) $a \oplus 0 = a$;
(MViv) $a \oplus 1 = 1$;
(MVv) $(a^*)^* = a$;
(MVvi) $a \oplus a^* = 1$;
(MVvii) $0^* = 1$;
(MVviii) $(a^* \oplus b)^* \oplus b = (a \oplus b^*)^* \oplus a$.

We define the following binary operations $\odot, \vee, \wedge$ as follows:

$$a \odot b := (a^* \oplus b^*)^*, \;\; a, b \in M, \tag{2.1}$$
$$a \vee b := (a^* \oplus b)^* \oplus b, \quad a \wedge b := (a^* \vee b^*)^*, \;\; a, b \in M.$$

Then $(M; \odot, 1)$ is a semigroup written "multiplicatively" with the neutral element 1.

If, for $a, b \in M$, we define

$$a \leq b \;\Leftrightarrow\; a = a \wedge b,$$

then $\leq$ is a partial order on M, and $(M; \vee, \wedge, 0, 1)$ is a distributive lattice with the least and greatest elements 0 and 1, respectively, [Cha]. We recall that $a \leq b$ iff $b \oplus a^* = 1$.

Without loss of generality we shall write for MV-algebras $M = (M; \oplus, \odot, ^*, 0, 1)$, where $\odot$ is defined by (2.1).

Similarly, we can define a (total) binary operation $*$ on M defined by

$$a * b := a \odot b^*, \;\; a, b \in M. \tag{2.2}$$

Then

$$a * b = a * (a \wedge b),$$

and $1 * a = a^*$ for any $a \in M$.

Let $(G;+,0,\le)$ be an Abelian ℓ-group with a strong unit u, i.e., given $v \in G$, there is an integer $n \ge 1$ such that $-nu \le v \le nu$.

Define

$$\Gamma(G,u) = \{g \in G:\ 0 \le g \le u\} \tag{2.3}$$

and set for all $g_1, g_2, g \in \Gamma(G,u)$

$$\begin{aligned} g_1 \oplus g_2 &= (g_1+g_2) \wedge u,\\ g_1 \odot g_2 &= (g_1+g_2-u) \vee 0,\\ g^* &= u-g. \end{aligned}$$

Then $(\Gamma(G,u);\oplus,\odot,^*,0,u)$ is an MV-algebra. The famous Mundici result [Mun] says that given an MV-algebra M there is an Abelian ℓ-group G with a strong unit u such that M is isomorphic with some $\Gamma(G,u)$, in addition, Γ defines a categorical equivalence between the category of unital ℓ-groups and the category of MV-algebras. We shall denote any representation ℓ-group $(G;+,0,\le)$ with a strong unit u of an MV-algebra as a unital ℓ-group $G=(G;+,0,\le,u)$, or simply $G=(G,u)$.

Given an MV-algebra M, we can introduce a partial binary operation $+$ via $a+b$ is defined iff $a \le b^*$, and in this case we put

$$a+b := a \oplus b. \tag{2.4}$$

It is easy to see that $a+0=0+a=a$ for any $a \in M$, and $+$ is commutative, i.e., if $a+b$ is defined in M, so is $b+a$, and $a+b=b+a$; $+$ is associative, i.e., if $(a+b),(a+b)+c$ are defined in M so are defined $b+c$ and $a+(b+c)$, and $(a+b)+c=a+(b+c)$. Identifying the MV-algebra M with $\Gamma(G,u)$ via (2.3), we can see that our partial operation $+$ coincides with the group addition $+$.

A non-void subset I of M is said to be an *ideal* of M if

(i) $x,y \in I$ imply $x \oplus y \in I$,
(ii) $x \in I$, $y \le x$ imply $y \in I$.

A proper ideal A of M is said to be *maximal* if there is no proper ideal of M containing A as a proper subset. Let $\mathcal{M}(M)$ denote the set of all maximal ideals of M. Then $\mathcal{M}(M) \ne \emptyset$, and denote by

$$Rad(M) := \bigcap\{A:\ A \in \mathcal{M}(M)\};$$

we call $Rad(M)$ the *radical* of M. A non-zero element a of M is said to be *infinitesimal* if na exists in M for any integer $n \ge 1$. The set of all infinitesimals in M is denoted by $Infinit(M)$. According to [CDM, Prop. 3.6.4], we have

$$Rad(M) = Infinit(M) \cup \{0\}.$$

The following result has been proved in [DiDv].

Proposition 2.1. *Let M be an MV-algebra. If $a, b \in Rad(M)$, then $a+b$ exists in M, and $a+b \in Rad(M)$.*

An MV-algebra M is said to be (i) *semisimple* if $Rad(M) = \{0\}$, (ii) *perfect* if $M = Rad(M) \cup Rad(M)^*$, where $Rad(M)^* := \{a^* : \ a \in Rad(M)\}$. If M is a perfect MV-algebra, then $Rad(M) \cap Rad(M)^* = \emptyset$.

For example, let G be an Abelian ℓ-group and define the lexicographical product $G(\mathbb{Z}) := \mathbb{Z} \times_{lex} G$, where $\mathbb{Z}$ is the group of all integers. Then the element $(1,0)$ is a strong unit in $G(\mathbb{Z})$ and $\Gamma(G(\mathbb{Z}),(1,0))$ is a perfect MV-algebra, and $Rad(M) = \{(0,g) : \ g \in G^+\}$ and $Rad(M)^* = \{(1,g) : \ g \in G^-\}$. The operation $\oplus$ and * are defined as follows: $(a,b)\oplus(c,d) = (1,0)\wedge(a+c, \ b+d)$ and $(a,b)^* = (1,0)-(a,b) = (1-a,-b)$.

The famous result of Di Nola and Lettieri, [DiLe], says, for any perfect MV-algebra M, there is a (unique up to an ℓ-group isomorphism) ℓ-group G such that $M \cong \Gamma(G(\mathbb{Z}),(1,0))$.

The following result is due to [GeLe].

Proposition 2.2. *Let $\{x_i\}$ be a family of elements of $Rad(M)$ of a perfect MV-algebra M. If $x = \bigvee_i x_i$ exists in M, then $x \in Rad(M)$.*

Proof. If $x \in Rad(M)^*$, then due to $x_i \le y$ for any $y \in Rad(M)^*$, we have $x \le y$. In particular, $x \le x_i^*$ for any i, so that $x_i \le x^*$ which gives the contradiction $x \le x^*$. □

Proposition 2.3. *If h is an MV-homomorphism from an MV-algebra M_1 into second one M_2, then h maps $Rad(M_1)$ into $Rad(M_2)$.*

Proof. Let $a \in Rad(M_1)$. Then, for any $n \ge 1$, $n\,a \in Rad(M_1)$ and $a+b \in Rad(M_1)$ for each $b \in Rad(M_1)$. Hence, for any integer $n \ge 1$, $n\,h(a) \le h(b^*) = h(b)^*$ which implies $h(a) \in Rad(M_2)$.

□

Proposition 2.4. *Let M_1 and M_2 be two perfect MV-algebras and let h be a mapping from $Rad(M_1)$ into $Rad(M_2)$ preserving $+$ and $\wedge$ and $\vee$. Then h can be uniquely extended to an MV-homomorphism $\hat{h}$ from M_1 into M_2.*

Proof. Claim 1. Let G_1^+ and G_2^+ be positive cones of Abelian ℓ-groups G_1 and G_2, respectively, and let $f : \ G_1^+ \to G_2^+$ preserves $+$, $\wedge$ and $\vee$. Then f can be uniquely extended to an ℓ-group homomorphism $\hat{f} : \ G_1 \to G_2$.

Indeed, extend f to $\hat{f}$ via $\hat{f}(a_1 - a_2) = f(a_1) - f(a_2)$, whenever $a_1, a_2 \in G_1^+$; then $\hat{f}$ is a group-homomorphism. We show that it preserves meets. Let $a = a' - a''$ and $b = b' - b''$, where $a', a'', b', b'' \in G_1^+$. Then

$$\hat{f}((a'+b'')\wedge(b'+a'')) = f((a'+b'')\wedge(b'+a'')) = (f(a')+f(b''))\wedge(f(b')+f(a'')).$$

Subtracting $\hat{f}(b'')$ and $\hat{f}(a'')$, we obtain

$$\hat{f}(a \wedge b) = \hat{f}(a) \wedge \hat{f}(b).$$

Claim 2. If we define $\hat{h}(a) = h(a^*)^*$, $a \in Rad(M_1)$, then $\hat{h}$ is an MV-homomorphism in question.

Step 1. Let $a,b \in Rad(M_1)$. Then $a \oplus b = a+b$. so that $h(a\oplus b) = h(a)+h(b) = h(a)\oplus h(b)$.

Step 2. Let $a,b \in Rad(M_1)^*$. Then $a\oplus b = 1$, and $h(a),h(b) \in Rad(M_1)^*$, so that $h(a\oplus b) = 1 = h(a)\oplus h(b)$.

Step 3. Let $a \in Rad(M_1)$ and $b \in Rad(M_1)^*$. Without loss of generality, we can assume that $M_1 = \Gamma(\mathbb{Z}\times_{lex} G_1,(0,1))$ and $M_2 = \Gamma(\mathbb{Z}\times_{lex} G_2,(0,1))$. Then $a \cong (0,a)$ and $b\cong (1,b)$, where $a \in G_1^+$ and $b \in G_1^-$, and $a\oplus b \cong (1,0)\wedge(1,a+b) = (1,0\wedge(a+b)) \in Rad(M_1)^*$.

From the above we conclude there exists a unique mapping $f:\ G_1^+ \to G_2^+$ such that $h((0,a)) = (0,f(a))$ and $h((1,b)) = (1,-f(-b))$.

It is clear that f preserves $+$, $\wedge$ and $\vee$ in G_1^+, so that by Claim 1, it can be uniquely extended to an ℓ-group-homomorphism $\hat{f}:\ G_1 \to G_2$. Hence

$$\begin{aligned}\hat{h}(a\oplus b) &= \hat{h}(1,0\wedge(a+b)) = (1,\hat{f}(0\wedge(a+b))\\ &= (1,\hat{f}(0)\wedge(\hat{f}(a)+\hat{f}(b)) = (0,\hat{f}(a))\oplus(1,\hat{f}(b)) = \hat{h}(a)\oplus\hat{h}(b).\end{aligned}$$

□

An MV-algebra M is said to be (i) *Rad-Dedekind σ-complete* (*complete*) if for any sequence (any family) $\{a_n\}$ of elements of $Rad(M)$ such that $a_n \le b$ for any $n\ge 1$, where $b \in Rad(M)$, then $\bigvee_n a_n \in Rad(M)$; (ii) *Rad-Archimedean* if $n\,a := a+\cdots+a \le b$ ($n \ge 1$), for $a,b \in Rad(M)$ entail $a = 0$. It is evident that if $M = \Gamma(\mathbb{Z}\times_{lex} G,(1,0))$, then M is Rad-Dedekind σ-complete (complete) iff G is Dedekind σ-complete (complete). We recall that if M is Rad-Dedekind σ-complete, then it is not necessarily σ-complete, see e.g., Chang's MV-algebra [Cha] $\Gamma(\mathbb{Z}(\mathbb{Z}),(1,0))$.

According to [BeDi], we say that a perfect MV-algebra M is with *principal radical* if there is an element $a \in Rad(M)$ such that the ideal of M generated by a coincides with $Rad(M)$. The category of perfect MV-algebras with principal radical is a category whose objects are pairs (M,a), where M is a perfect MV-algebra with a fixed element $a \in Rad(M)$ generating $Rad(M)$, as an ideal, and morphisms are MV-homomorphisms preserving fixed elements a in radicals. This category is categorically equivalent to the category of unital ℓ-groups.

A *tribe* of fuzzy sets on a set $\Omega \ne \emptyset$ is a non-void system $\mathcal{F} \subseteq [0,1]^\Omega$ such that

(i) $1_\Omega \in \mathcal{F}$,
(ii) if $a \in \mathcal{F}$, then $1-a \in \mathcal{F}$,
(iii) if $\{a_n\}_{n=1}^\infty$ is a sequence of elements of $\mathcal{F}$, then

$$\min\{\sum_{n=1}^\infty a_n, 1\} \in \mathcal{F}.$$

(We recall that all above operations with fuzzy sets are defined pointwisely on Ω.)

A *g-tribe* is a non-void system $\mathcal{T}$ of bounded functions on $\Omega \ne \emptyset$ such that

(i) $0_\Omega, 1_\Omega \in \mathcal{T}$;
(ii) $f\pm g \in \mathcal{T}$ whenever $f,g\in\mathcal{T}$;
(iii) if $\{f_n\}_n$ is a sequence of elements from $\mathcal{T}$ for which there exists $f\in\mathcal{T}$ with $f_n \le f$ (pointwisely), $n\ge 1$, then $\sup_n f_n \in \mathcal{T}$.

It is evident that with respect to the pointwise ordering $f \le g$ iff $f(\omega) \le g(\omega)$ for every $\omega \in \Omega$, $\mathcal{T}$ is a Dedekind σ-complete ℓ-group with strong unit 1_Ω. In addition, $\Gamma(\mathcal{T}, 1_\Omega) := \{f \in \mathcal{T} : 0_\Omega \le f \le 1_\Omega\}$ is a tribe of fuzzy functions.

We now present the Loomis–Sikorski theorem for perfect MV-algebras.

Theorem 2.5. *Let M be a Rad-Dedekind σ-complete perfect MV-algebra with principal radical. Then there exists a g-tribe $\mathcal{T}$ of bounded functions on a compact Hausdorff space X and an MV-homomorphism h from the perfect MV-algebra $M(\mathcal{T}) := \Gamma(\mathbb{Z} \times_{lex} \mathcal{T}, (1, 0_X))$ with principal radical onto M such that h preserves all existing countable joins in $M(\mathcal{T})$. In addition, h maps $Rad(M(\mathcal{T}))$ onto $Rad(M)$, and $h(1_X) = u$, where u is a fixed element of $Rad(M)$ generating it as an ideal.*

Proof. Suppose that $M = \Gamma(\mathbb{Z} \times_{lex} G, (1, 0))$ and let u be a fixed element in $Rad(M)$ generating $Rad(M)$ as an ideal. Hence u is a strong unit for G. Since M is Rad-Dedekind σ-complete, then G is Dedekind σ-complete, and by the Loomis–Sikorski theorem, [Dvu 5, Thm 5.1], for the Dedekind σ-complete unital ℓ-group (G, u), there exists a g-tribe $\mathcal{T}$ of bounded real-valued functions defined on a compact Hausdorff space X equal to the set of all extremal states on (G, u)[1] and there exists an ℓ-group-homomorphism h' from $(\mathcal{T}, 1_X)$ onto (G, u) such that if $\sup_n f_n \in \mathcal{T}$, then $h'(\sup_n f_n) = \bigvee_n h'(f_n)$, and $h(1_X) = u$.

In other words, the mapping $h'' : Rad(M(\mathcal{T})) \cong \mathcal{T}^+$ onto $G^+ \cong Rad(M)$ defined by $h''((0, f)) := h'(f)$ $(f \in \mathcal{T}^+)$ preserves preserves $+$, $\wedge$ and $\vee$. Hence by Proposition 2.4, there exists a unique extension h of h'' from the perfect MV-algebra $M(\mathcal{T})$ into G. In any rate, h is surjective, [Dvu 5, Thm 5.1].

Suppose now that $\bigvee_n f_n$ is defined in $M(\mathcal{T})$. Then either all $f_n \in Rad(M(\mathcal{T}))$ or there exists $f_n \in Rad(M(\mathcal{T}))^*$. In the first case by Proposition 2.2, $\bigvee_n f_n \in Rad(M(\mathcal{T}))$ so that $\bigvee_n f_n = \sup_n f_n$, and $h(\bigvee_n f_n) = \bigvee_n h(f_n)$.

In the second case, $\bigvee_n f_n = \bigvee_{n\in D} f_n$, where $D := \{n : f_n \in Rad(M(\mathcal{T}))^*\}$. For $n \in D$, $f_n^* \in Rad(M(\mathcal{T}))$, so that $\bigwedge_{n\in D} f_n^* = \inf_{n\in D} f_n^*$ and $\bigvee_{n\in D} f_n = (\bigwedge_{n\in D} f_n^*)^*$ which entails

$$h(\bigvee_n f_n) = h(\bigvee_{n\in D} f_n) = h((\bigwedge_{n\in D} f_n^*)^*) = (h(\bigwedge_{n\in D} f_n^*))^* = (\bigwedge_{n\in D} h(f_n^*))^* = \bigvee_{n\in D} h(f_n).$$

$\square$

3. Commutative BCK-algebras with the Relative Cancellation Property

In the present section, we generalize the Loomis–Sikorski theorem for commutative BCK-algebras with the relative cancellation property and with a quasi strong unit.

Commutative BCK-algebras with the relative cancellation property have been recently introduced by Dvurečenskij and Graziano [DvGr], and in [DvGr 1] it was shown that they appear from ℓ-groups. In addition, in [Dvu 1], there was shown that they are categorically equivalent to a special category of ℓ-groups.

[1] Let (G, u) be an ℓ-group with strong unit u. A *state* on G is any mapping $s : G \to \mathbb{R}$ such that $s(g_1 + g_2) = s(g_1) + s(g_2)$, $s(u) = 1$, $s(g) \ge 0$ for $g \in G^+$.

We recall a *commutative BCK-algebras* is an algebra $(X;*,0)$ of type $(2,0)$ such that the following axioms are satisfied: for all x, y, $z \in X$,

(BCK-1) $((x*y)*(x*z))*(z*y)=0$;
(BCK-2) $(x*(x*y))*y=0$;
(BCK-3) $x*x=0$;
(BCK-4) $x*y=0$ and $y*x=0$ imply $x=y$;
(BCK-5) $0*x=0$;
(BCK-6) $x*(x*y)=y*(y*x)$;

The partial order $\le$ in X is defined via $x \le y$ iff $x*y=0$, and in this case, $x \wedge y = x*(x*y)$.

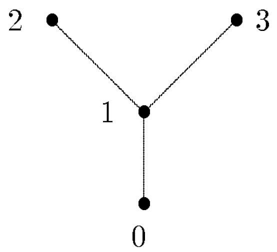
A commutative BCK-algebra $(X;*,0)$ is *bounded* if there is a greatest element 1 in X. Bounded commutative BCK-algebras are categorically equivalent to MV-algebras, [FRT], via (2.3).

According to [DvGr], we say that a commutative BCK-algebra $(X;*,0)$ has the *relative cancellation property* if, for $a,x,y \in X$, $a \le x,y$ with $x*a=y*a$ imply $x=y$. They form a variety.

For example, any upwards directed BCK-algebra (i.e., given $x,y \in X$ there exists $z \in X$ such that $x,y \le z$), in particular, any bounded commutative BCK-algebra, has the relative cancellation property. The following example shows that not any commutative BCK-algebra has the relative cancellation property.

Example 3.1. The example B_{4-3-3} from [MeJu], where $*$ is given by the table (next to it there is its Hasse diagram), gives a commutative BCK-algebra which is not directed upwards, consequently it is not a lattice. It has not the relative cancellation property. Indeed, we have $1 \le 2,3$ and $2*1=1=3*1$ but $2 \neq 3$, consequently, X cannot be embedded into the positive cone of any Abelian ℓ-group.

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	1	0	1
3	3	1	1	0



On the other hand, $([0,\infty);*_{\mathbb{R}},0)$, where

$$s *_{\mathbb{R}} t = \max\{0, s-t\},$$

$s,t \in [0,\infty)$, is an example of a commutative BCK-algebra having the relative cancellation property.

Example 3.2. Suppose that $(G;+,\leq,0)$ is an Abelian ℓ-group with the positive cone $G^+=\{g\in G:\ g\geq 0\}$. Then $(G^+;*_G,0)$ is a commutative BCK-algebra with the relative cancellation property, where $*_G$ is defined via

$$u *_G v := (u-v)\vee 0, \tag{3.1}$$

for $u,v\in G^+$. More generally, if G_0 is a non-void subset of G^+ such that $u,v\in G_0$ imply $u*_G v\in G_0$, then $(G_0;*_G,0)$ is a commutative BCK-subalgebra of $(G^+;*_G,0)$ having the relative cancellation property.

The latter example is in some sense an archetype of commutative BCK-algebras with the relative cancellation property because the following basic representation theorem for commutative BCK-algebras was proved in [DvGr 1, Thm 6.4]:

Theorem 3.3. *Let $(X;*,0)$ be a commutative BCK-algebra with the relative cancellation property. Then there exists an Abelian ℓ-group $(G;+\leq,0)$ with the positive cone G^+ and a non-void subset G_0 of G^+ generating G^+ such that $u,v\in G_0$ imply $u *_G y\in G_0$, and there exists a BCK-isomorphism h from X onto G_0.*

If X has the relative cancellation property, then we can define the partial binary operation $+$ on X with the domain $S=\{(a,b)\in X\times X:$ there exists $c\in X$ with $c\geq b$ and $a=c*b\}$, and we define $c=a+b$ if $c\geq b$, and $a=c*b$ (for more details see [DvGr]), and, for all $a,b,c\in X$, we have [DvGr, Thm 2.5]

(I) $a+0,\ 0+a$ exist for any $a\in X$, and $a+0=a=0+a$.

(II) $a+b$ exists if and only if $b+a$ exists, and then $a+b=b+a$ (the commutativity).

(III) $a+b$ and $(a+b)+c$ exist in X if and only if $b+c$ and $a+(b+c)$ exist in X, and in this case $(a+b)+c=a+(b+c)$ (the associativity).

(IV) If $a+b$ exists in X and if $a_1\leq a,\ b_1\leq b$, then a_1+b_1 exists in X, and $a_1+b_1\leq a+b$.

(V) $a=b$ holds whenever $u+a=u+b$ for some $u\in X$ (the cancellation property).

(VI) If $a+b\leq a+b'$ then $b\leq b'$. Conversely, if $b\leq b'$ and $a+b'$ exists in X, then $a+b\leq a+b'$.

(VII) If $a+c$ and $b+c$ exist in X, then $(a\wedge b)+c$ exists in X, and

$$(a+c)\wedge(b+c)=(a\wedge b)+c.$$

(VIII) Let $z\geq x,y$. Then $x\vee y$ exists in X, and $x\vee y=x\vee_z y:=z*((z*x)\wedge(z*y))$. Moreover,

$$(x\vee y)*y=x*(x\wedge y)=x*y,$$

and $(x*(x\wedge y))+y$, $(x*y)+y$ exist in X, and

$$x\vee y=(x*(x\wedge y))+y=(x*y)+y.$$

It is worthy to recalling that if M is an MV-algebra, then $+$ defined by (2.4) coincides with that derived from a bounded BCK-algebra M.

Define recursively, for all $x, y \in X$:

$$x *^0 y = x,\ x *^1 y = x * y, \dots ,\ x *^{n+1} y = (x *^n y) * y,\ n \geq 1.$$

An element u of a commutative BCK-algebra $(X; *, 0)$ is said to be a *quasi strong unit* for X if, for each element $x \in X$, there exists an integer $n \geq 1$ such that $x *^n u = 0$.

We say that a BCK-algebra $(X; *, 0)$ is *Dedekind σ-complete* provided that any every non-empty subset of X which has an upper bound in X has a supremum in X.

Example 3.4. *Let $\mathcal{X}$ be a non-void family of real-valued non-negative functions on a set $\Omega \neq \emptyset$ such that* (i) $0_\Omega \in \mathcal{X}$, (ii) *if $f, g \in \mathcal{X}$ then $f * g \in \mathcal{X}$, where*

$$(f * g)(\omega) := \max\{0, f(\omega) - g(\omega)\},\ \omega \in \Omega. \tag{3.2}$$

Then $(\mathcal{X}; *, 0_\Omega)$ is a commutative BCK-algebra with the relative cancellation property, where the BCK-order coincides with the pointwise order of functions.

A *BCK-tribe* is a non-void system $\mathcal{X}$ of bounded non-negative functions defined on a set $\Omega \neq \emptyset$ such that (i) $0_\Omega, 1_\Omega \in \mathcal{X}$; (ii) $f * g \in \mathcal{X}$ whenever $f, g \in \mathcal{X}$, where $*$ is defined via (3.2); (iii) if $\{f_n\}_n$ is a sequence of elements from $\mathcal{X}$ for which there exists $f \in \mathcal{X}$ with $f_n(\omega) \leq f(\omega)$, $\omega \in \Omega$ $(n \geq 1)$, then $\sup_n f_n \in \mathcal{X}$. It is clear that any BCK-tribe is a Dedekind σ-complete commutative BCK-algebra with the relative cancellation property.

The following result is a generalization of the Loomis–Sikorski theorem for BCK-algebras.

Theorem 3.5. *Let $(X; *, 0)$ be a Dedekind σ-complete BCK-algebra with the relative cancellation property and with a quasi strong unit u. Then there exists a BCK-tribe $\mathcal{X}$ of bounded real-valued functions on a compact Haudorff space Ω and there exists a BCK-σ-homomorphism h from $\mathcal{X}$ onto X with $h(1_\Omega) = u$.*

Proof. Let u be a quasi strong unit for X. Then, Theorem 3.3, there exists an ℓ-group G with a subset $G_0 \subseteq G^+$ generating G^+ and $u \in G_0$ such that $X \cong G_0$. Since X is Dedekind σ-complete iff G is Dedekind σ-complete, [DvGr 2, Thm 2.4], we conclude that the unital ℓ-group (G, u) is Dedekind σ-complete. Hence, by [Dvu 5, Thm 5.1], there exists a g-tribe $\mathcal{T}$ of bounded functions on a compact Hausdorff space $\Omega \neq \emptyset$ and an ℓ-group σ-homomorphism h_0 from $\mathcal{T}$ onto G. Define $\mathcal{X} := h_0^{-1}(G_0)$. According to the construction of G (see [Dvu 5, Thm 5.1], $\mathcal{X}$ is a set of positive bounded functions. In addition, $h_0(f * g) = h_0(0_\Omega \vee (f - g)) = h_0(f) *_G h_0(g) \in G_0$, so that $\mathcal{X}$ is a BCK-algebra which si also a BCK-tribe in question. ☐

4. Loomis-Sikorski Theorem for Product BCK-algebras

In the present section, we give the Loomis–Sikorski theorem for product BCK-algebras which were introduced in [Dvu 3]. As a corollary, we obtain the Loomis–Sikorski theorem for product MV-algebras proved by Mundici [Mun 1].

We say that a commutative BCK-algebra $(X;*,0)$ with the relative cancellation property admits a *product* if there is a binary operation $\cdot$ on X satisfying for all $a,b,c \in X$

(i) if $a+b$ is defined in X, then $(a\cdot c)+(b\cdot c)$ and $c\cdot a + c\cdot b$ exist and

$$(a+b)\cdot c = (a\cdot c)+(b\cdot c),$$
$$c\cdot (a+b) = (c\cdot a)+(c\cdot b),$$

and we say that X is a *product BCK-algebra*. Sometimes we write $X=(X;*,0,\cdot)$. An element u of a product BCK-algebra X is said to be a *unity*, if $a\cdot u = u\cdot a = a$ for any $a\in X$.

A product $\cdot$ on X is

(ii) *associative* if $(a\cdot b)\cdot c = a\cdot (b\cdot c)$, $a,b,c\in X$;
(iii) *commutative* if $a\cdot b = b\cdot a$, $a,b\in X$.

It is worth saying that if $\cdot$ is a product on X, then

(iv) $a\cdot 0 = 0 = 0\cdot a$,
(v) if $a\le b$, then for any $c\in X$, $a\cdot c\le b\cdot c$ and $c\cdot a\le c\cdot b$.

The following characterization of product BCK-algebras was proved in [Dvu 3, Thm 3.2].

Theorem 4.1. *Let $(R;+,\cdot,0,\le)$ be an (associative) ℓ-ring with $\emptyset\ne R_0\subseteq R^+$ such that $a,b\in R_0$ entails $a*_R b\in R_0$ and $a\cdot b\in R_0$. Then $(X;*,0,\cdot):=(R_0;*_R,0,\cdot)$ is a product BCK-algebra with an (associative) product $\cdot$, which is the restriction of $\cdot$ to $R_0\times R_0$.*

Conversely, let $(X;*,0,\cdot)$ be a product BCK-algebra with an (associative) product $\cdot$. Then there exists a unique (up to isomorphism) (associative) ℓ-ring $(R;+,\cdot,0,\le)$ with a non-empty set $R_0\subseteq R^+$ generating R^+ as a group-cone such that $a,b\in R_0$ entails $a*_R b\in R_0$, $a\cdot b\in R_0$, and $X\cong R_0$, and $\phi(a\cdot b)=\phi(a)\cdot\phi(b)$, $a,b\in X$, where ϕ is a BCK-isomorphism of X onto R_0.

We have showed in [Dvu 3] that any Dedekind σ-complete product BCK-algebra with unity which is a quasi strong unit has the commutative and associative product.

We recall that the product MV-algebras and their characterizations via ℓ-rings were presented in [DiDv].

Let X be a product BCK-algebra. A non-empty subset I of X is said to be a $\cdot$-*ideal* of X if

(i) $0\in I$;
(ii) $a*b\in I$, $a\in X$, and $b\in I$, entail $a\in I$;
(iii) $a\in I$ and $b\in X$ entail $a\cdot b\in I$ and $b\cdot a\in I$.

An *ideal* of X is a non-void subset I of X such that (i) $0\in I$; (ii) $a*b\in I$, $a\in X$, and $b\in I$ entail $a\in I$. A *maximal ideal* is a maximal proper ideal of X. There exists an example of a commutative BCK-algebra [Dvu, Ex 4.2] having no maximal ideal. If X has a quasi strong unit, then X possesses at least one maximal ideal of X. In addition, [Dvu 4], there exists a one-to-one correspondence between measure-homomorphisms and maximal ideals given by $m\leftrightarrow Ker_m : \{x\in X:\ m(x)=0\}$.

A mapping $m: X \to [0,\infty)$ is said to be a *measure* if $m(x*y) = m(x) - m(y)$ whenever $y \le x$. If $m(x*y) = m(x) *_{\mathbb{R}} m(y)$ for all $x, y \in X$, m is said to be a *measure-homomorphism*. Fix an element $u \in X$ and denote by $\mathcal{S}_u(X)$ the set of all measures m on X such that $m(u) = 1$. In general, $\mathcal{S}_u(X)$ can be an empty set. If u is a quasi strong unit, then $\mathcal{S}_u(X)$ is a non-void compact Hausdorff space under weak limit of measures, [Dvu 4, Thm 5.3], and the set of extremal points of $\mathcal{S}_u(X)$, $Ext(\mathcal{S}_u(X))$, is a non-void compact Hausdorff space, and it coincides with the set of all measure-homomorphisms from $\mathcal{S}_u(X)$.

Let u be a quasi strong unit for X. A measure $m \in \mathcal{S}_u(X)$ is said to be *discrete* if there exists an integer $n \ge 1$ such that $m(X) \subseteq \frac{1}{n}\mathbb{Z}$. If m is a discrete measure-homomorphism, then there exists an integer n such that $m([0,u]) = \{0, 1/n, 2/n, \ldots, n/n\}$.

Now we present the Loomis–Sikorski theorem for product BCK-algebras. We recall that a *product BCK-tribe* is a BCK-tribe $\mathcal{X}$ such that if $f, g \in \mathcal{X}$, then the usual product $fg \in \mathcal{X}$.

Theorem 4.2. *Let $(X; *, 0, \cdot)$ be a Dedekind σ-complete product BCK-algebra with a quasi strong unit u which is a product unity. Then there exists a product BCK-tribe $\mathcal{X}$ of bounded positive functions defined on a compact Hausdorff space $\Omega \neq \emptyset$ and a BCK-σ-homomorphism h from $\mathcal{X}$ onto X which preserves product in $\mathcal{X}$ and $h(1_\Omega) = u$.*

Proof. Let u be a quasi strong unit for X, and let $\Omega := Ext(\mathcal{S}_u(X))$. Then Ω is a compact Hausdorff space. Denote by $X_u := [0,u] = \{x \in X : 0 \le x \le u\}$. It is clear that X_u is a bounded product commutative BCK-algebra. Any restriction of $\mathcal{S}_u(X)$ onto X_u gives a measure on X_u, and conversely, any measure (measure-homomorphism) on X_u can be uniquely extended to a measure (measure-homomorphism) on X.

Let $(R; +, 0, \cdot, \le)$ be an Archimedean commutative and associative ℓ-ring from Theorem 4.1 with $R_0 \subseteq R^+$ generating R^+ as a group-cone such that $(X; *, 0, \cdot) \cong (R_0, *_R, 0, \cdot)$. Without loss of generality, we can assume that $X = R_0$. Since u is a quasi strong unit for X, so is u for the ℓ-ring R.

Let now m be a measure-homomorphism from $\mathcal{S}_u(X)$. Then Ker_m is a maximal ideal of X. We show that any maximal ideal I of X is a $\cdot$-ideal. Indeed, let $a \in I$ and $b \in X$. Then, [DvGr 1, Lem 6.1], there exists an integer $n \ge 1$ such that $b = b_1 + \cdots + b_n$, where $b_i \in X_u$, $i = 1, \ldots, n$. Hence, $a \cdot b = a \cdot (b_1 + \cdots + b_n) = a \cdot b_1 + \cdots + a \cdot b_n \le a \cdot u + \cdots a \cdot u = n\,a$ which proves that na is in the ℓ-ideal of R generated by I, consequently by [Dvu 2, Thm 2.6], $a \cdot b \in I$. In a similar way we can prove that $b \cdot a \in I$. Whence $m(x \cdot y) = m(x) \cdot m(y)$ for all $x, y \in X$. This can be seen due to property that the quotient BCK-algebra X/Ker_m is a linear Archimedean BCK-algebra[2], so that, [Dvu 2], it is isomorphic to a BCK-subalgebra of $\mathbb{R}$ containing the number 1 as a quasi strong unit which is closed under the natural multiplication. (We can apply Theorem 4.1 to Pickert–Hion's theorem, [Fuc, p. 126].)

Consequently, a measure-homomorphism m is discrete iff $m(X_u) = \{0, 1\}$.

[2] A commutative BCK-algebra with the relative cancellation property is said to be *Archimedean* if $n\,a \le b$ for any $n \ge 1$ entail $a = 0$, where $n\,a =: a_1 + \cdots + a_n$ with $a = a_1 = \cdots = a_n$. If u is a quasi-strong unit, then X is Archimedean iff so is X_u, [DvGr 2].

Hence, the mapping $x \mapsto \hat{x} \in [0,\infty)^\Omega$, $x \in X$, where $\hat{x}(m) := x/Ker_m = m(x)$, $m \in Ext(\mathcal{S}_u(X))$, gives a one-to-one representation of X with $\hat{X} := \{\hat{x} : x \in X\}$ which is a set of continuous functions from $C(\Omega)$, the space of all real-valued continuous functions on Ω. Moreover, $\widehat{x \cdot y} = \hat{x} \cdot \hat{y}$. According to [Goo, Lem 9.12] and [DvGr 2, Thm 2.4], we can show that a continuous function $f : \Omega \to [0,\infty)$ belongs to $\hat{X}$ iff $f(m) \in \mathbb{Z}^+$ for any discrete measure m from $Ext(\mathcal{S}_u(X))$.

Let now $\mathcal{X}$ be a minimal BCK-tribe of bounded non-negative functions on Ω generated by $\hat{X}$, and consider by $\mathcal{X}'$ the class of all functions $f \in \mathcal{X}$ with the property that for some $b \in X$, $N(f-\hat{b}) := \{m \in \Omega : |f(m)-\hat{b}(m)| > 0\}$ is a meager set.

If b_1 and b_2 are two elements of X such that $N(f-\hat{b}_i)$ is a meager set for $i=1,2$. Then $N(\hat{b}_1-\hat{b}_2) \subseteq N(\hat{b}_1 - f) \cup N(f-\hat{b}_2)$ is a meager set. Due to the Baire theorem saying that any non-empty set of a compact Hausdorff space cannot be a meager set, we conclude that $\hat{b}_1 = \hat{b}_2$, i.e., $b_1 = b_2$.

We show that if $f, g \in \mathcal{X}'$, then $f * g \in \mathcal{X}'$, where $*$ is defined by (3.2). In fact, let $\mathcal{I}$ be the set of all $f \in \mathcal{X}$ such that $N(f)$ is a meager set. Then (i) $0_\Omega \in \mathcal{I}$; (ii) if $f \in \mathcal{I}$ and $g \in \mathcal{X}$, then $\min(f,g) \in \mathcal{I}$; (iii) if $f * g \in \mathcal{I}$ and $g \in \mathcal{I}$, then $f * g = f - \min(f,g)$ which proves that $0 < f(m) = (f(m) - \min(f,g)(m)) + (\min(f,g))(m)$, so that $f \in \mathcal{I}$. Since $N(f-\hat{b}) = N(f * \hat{b}) \cup N(\hat{b} * f)$, $N(f-\hat{b})$ is a meager set iff $N(f*\hat{b})$ and $N(\hat{b} * f)$ are meager sets, which entails by [MeJu, Lem II.1.5] $\mathcal{X}'$ is a BCK-algebra.

To show that $\mathcal{X}'$ is a BCK-tribe is necessary to verify that $\mathcal{X}'$ is closed under limits of non-decreasing sequences from $\mathcal{X}'$.

Let $\{f_n\}_n$ be a sequence of non-decreasing elements from $\mathcal{X}'$ with $f_n \le f$ $(n \ge 1)$ for some $f \in \mathcal{X}$. Choose $b_n \in X$ such that $N(f_n - \hat{b}_n)$ is a meager set. Without loss of generality we can assume that $b_n \le b_{n+1}$. Denote $f = \lim_n f_n$, $b = \bigvee_{n=1}^\infty b_n$, $b_0 = \lim_n \hat{b}_n$. Using the proof of theorem of [Dvu 5, Thm 4.10], we can show that $N(f-\hat{b})$ is a meager set, so that $\mathcal{X}' = \mathcal{X}$.

We now show that $\mathcal{X}$ is closed under formation of the natural product of functions. Indeed, let $f, g \in \mathcal{X}$ and let $a, b \in X$ be such elements that $N(f-\hat{a})$ and $N(g-\hat{b})$ are meager sets. Then in view of the inequalities $|f\cdot g - \widehat{a\cdot b}| = |f \cdot g - \hat{a}\cdot\hat{b}| \le |f\cdot g - f\cdot\hat{b}| + |f\cdot\hat{b} - \hat{a}\cdot\hat{b}| = |f|\,|g-\hat{b}| + |f-\hat{a}|\,|\hat{b}|$ we have that $N(f\cdot g - \widehat{a\cdot b})$ is also a meager set.

Consequently, $\mathcal{X}$ is a product BCK-tribe, and the mapping $h : f \mapsto b$ whenever $N(f-\hat{b})$ is a meager set defines a BCK-σ-homomorphism from the product BCK-tribe $\mathcal{X}$ onto X with $h(1_\Omega) = u$. □

Because any MV-algebra can be considered as a bounded commutative BCK-algebra, as a corollary of Theorem 4.2 we obtain the Loomis–Sikorski theorem for product MV-algebras obtained originally by Mundici [Mun 1, Cor 6.3].

Theorem 4.3. *Let M be a σ-complete MV-algebra such that 1 is product unity. Then there exists a tribe $\mathcal{T}$ of fuzzy sets on a compact Hausdorff space $\Omega \neq \emptyset$ which is closed under the natural product of functions and there exists an MV-σ-homomorphism h from $\mathcal{T}$ onto M preserving product in $\mathcal{T}$.*

Proof. Let $\mathcal{T}$ be a tribe generated by $\hat{M} = \{\hat{a} : a \in M\}$, where $\hat{a}(m) := m(a)$,

$m \in \mathcal{S}_1(M)$. Then $\mathcal{T}$ is a BCK-tribe generated by $\hat{M}$, consequently, we can apply Theorem 4.2. $\square$

An *r-tribe* is a g-tribe $\mathcal{T}$ such that $fg \in \mathcal{T}$ whenever $f, g \in \mathcal{T}$. It is evident that r-tribe is a Dedekind σ-complete ℓ-ring with the natural product of functions with unity 1_Ω which is also a strong unit.

Theorem 4.4. *Let $(R; +, 0, \cdot, \leq)$ be a Dedekind σ-complete ℓ-ring with unity 1 which is a strong unit for R. Then there exists an r-tribe $\mathcal{T}$ of bounded functions on a compact Hausdorff space $\Omega \neq \emptyset$ and an ℓ-ring σ-homomorphism h from $\mathcal{T}$ onto R with $h(1_\Omega) = 1$.*

Proof. According to [Dvu 5, Thm 5.1], there exists a g-tribe $\mathcal{T}$ of bounded functions on the compact Hausdorff space $\Omega \neq \emptyset$ of all measures s on the product BCK-algebra $(R^+; *_R, 0, \cdot)$ with $s(1) = 1$ and an ℓ-group σ-homomorphism h from $\mathcal{T}$ onto R with $h(1_\Omega) = 1$. The construction of h is such one that h maps $\Gamma(\mathcal{T}, 1_\Omega)$ onto the interval $[0, 1]$ in R, in addition, $\Gamma(\mathcal{T}, 1_\Omega)$ is a tribe generated by $\{\hat{x} : x \in [0, 1]\}$, when $[0, 1]$ is understood as a product MV-algebra. Consequently, by the proof of Theorem 4.4, h preserves products in $\Gamma(\mathcal{T}, 1_\Omega)$. Consequently, in view that 1_Ω is a strong unit for $\mathcal{T}$, h preserves product in the whole $\mathcal{T}$, which proves that $\mathcal{T}$ is an r-tribe. $\square$

Finally, we shall discuss a product on perfect MV-algebras. The following result has been proved in [DiDv, Thm 3.7].

Theorem 4.5. *Let M be a perfect MV-algebra and define a binary operation $\cdot$ as follows. Let $a, b \in Rad(M)$. Then*

$$\begin{aligned} a \cdot b &= 0, \\ a \cdot b^* &= a, \\ a^* \cdot b &= b, \\ a^* \cdot b^* &= (a + b)^*. \end{aligned} \tag{4.1}$$

Then $\cdot$ is a commutative product on M such that $a \cdot 1 = a = 1 \cdot a$ for any $a \in M$. If M is Rad-Archimedean, then on M there is a unique product $\cdot$ such that $a \cdot 1 = a = 1 \cdot a$ for any $a \in M$; this product is defined by (4.1).

Consequently, on non-trivial Rad-Archimedean perfect MV-algebras there is no commutative product with unity 1. Hence, a commutative and associative product with unit 1 on perfect algebras can be defined only on their radicals as it was done in [BDG]: A *product* on a perfect MV-algebra is a binary operation $\cdot : Rad(M) \times Rad(M) \to Rad(M)$ such that $a \cdot (b + c) = (a \cdot b) + (a \cdot b)$ and $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$, $a, b, c \in Rad(M)$, and M is said to be a *product perfect MV-algebra*. The following form of the Loomis–Sikorski theorem holds for perfect MV-algebras.

Theorem 4.6. *Let M be a Rad-Dedekind σ-complete product perfect algebra with a fixed element u generating the radical as an ideal and let u be a unity for the product. Then there exists an r-tribe $\mathcal{T}$ of bounded functions on a compact Hausdorff space X and an MV-homomorphism h from the product perfect algebra $M(\mathcal{T}) := \Gamma(\mathbb{Z} \times_{lex} \mathcal{T}, (1, 0_X))$, with principal radical with 1_X generating the radical* *of $M(\mathcal{T})$ as an ideal, onto M such that h preserves all existing countable joins and products in $M(\mathcal{T})$. In addition, h maps $Rad(M(\mathcal{T}))$ onto $Rad(M)$, and $h(1_X) = u$.*

Proof. In view of Theorem 4.5 we can suppose that $M = \Gamma(\mathbb{Z} \times_{lex} R, (1,0))$, where R is an ℓ-group and let u be a fixed element in $Rad(M)$ generating $Rad(M)$ as an ideal. Hence u is a strong unit for R. Since M is Rad-Dedekind σ-complete, R is Dedekind σ-complete, and by the Loomis–Sikorski theorem, [Dvu 5, Thm 5.1], for the Dedekind σ-complete unital ℓ-group (R, u), there exists a g-tribe $\mathcal{T}$ of bounded real-valued functions defined on a compact Hausdorff space X equal to the set of all extremal states on (R, u) and there exists an ℓ-group-homomorphism h' from $(\mathcal{T}, 1_X)$ onto (R, u) such that if $\sup_n f_n \in \mathcal{T}$, then $h'(\sup_n f_n) = \bigvee_n h'(f_n)$, and $h(1_X) = u$.

In other words, the mapping $h'' : \ Rad(M(\mathcal{T})) \cong \mathcal{T}^+$ onto $R^+ \cong Rad(M)$ defined by $h''((0, f)) := h'(f)$ $(f \in \mathcal{T}^+)$ preserves preserves $+$, $\wedge$ and $\vee$. Therefore on R^+ there is a product $\cdot$. Hence by Proposition 2.4 and Theorem 2.5, there exists a unique extension h of h'' from the perfect MV-algebra $M(\mathcal{T})$ onto R which preserves countable supremas in $Rad(M)$. Moreover h preserves the product on $Rad(M) \cong R^+$, consequently R is an ℓ-ring with a unity u, i.e., 1_X is a unity for the natural product of functions in $\mathcal{T}$, i.e., $\mathcal{T}$ is an r-tribe. □

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