Mathematical Institute
Slovak Academy of Sciences, Bratislava

Preprint 16/1999

Michal Fečkan

Criteria on the nonexistence of invariant submanifolds for dynamical systems

August 26, 1999

PREPRINT SERIES

CRITERIA ON THE NONEXISTENCE OF INVARIANT SUBMANIFOLDS FOR DYNAMICAL SYSTEMS

MICHAL FEČKAN

Department of Mathematical Analysis, Comenius University, Mlynská dolina, 842 15 Bratislava, Slovakia, and Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, 814 73 Bratislava, Slovakia

ABSTRACT. Conditions are given for smooth finite dimensional mappings which are precluding the existence of invariant smooth orientable compact submanifolds for those mappings. Flows, their discretizations and averagings are studied as well. Mappings and flows with constraints are also investigated.

1. INTRODUCTION

Let us consider a mapping $F \in C^2(\mathbb{R}^n, \mathbb{R}^n)$. We investigate conditions for the nonexistence of k-dimensional compact orientable submanifolds of $\mathbb{R}^n$ $(1 \leq k \leq n)$ invariant for F. The problem on the nonexistence of periodic solutions for autonomous ordinary differential equations is already studied by many authors started with I. Bendixson [1] and H. Dulac [5] for the planar case. Extensions to higher dimensional cases are given by R.A. Smith [12,13], S.N. Busenberg and P. van den Driessche [2], G. Butler, R. Schmid and P. Waltman [3], J.S. Muldowney [10], and Y. Li and J.S. Muldowney [8]. Conditions precluding the existence of nontrivial periodic orbits for mappings in $\mathbb{R}^n$ are given by C.C. McCluskey and J.S. Muldowney [9].

We are motivated in this paper by the results of [8] and [10]. In Section 2, we study mappings and derive the main results of this paper. In Section 3, the results of Section 2 are applied to autonomous ordinary differential equations. There are also recovered certain results of [8] and [10], but we study more general cases. Section 4 deals with discretizations of autonomous ordinary differential equations. We roughly show under certain conditions, that the flow has the property of the nonexistence of an invariant k-dimensional compact orientable submanifold of $\mathbb{R}^n$ if and only if its discretization has the same property with any sufficiently small step size. This result is in the spirit of a recently developed theory of numerical analysis for dynamical systems (see [6] and references in that paper). Periodic ordinary differential equations with a small nonzero parameter are treaded in Section 5.

E-mail address: Michal.Feckan@ fmph.uniba.sk

1991 *Mathematics Subject Classification.* 34A50, 58F08, 58F25, 58F35.

Key words and phrases. smooth mappings, invariant submanifolds, Stoke's theorem, flows, discretizations, averagings.

This work supported by Grant GA-SAV 2/5133/98

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

First, we study the relationship between the nonexistence of invariant k-dimensional compact orientable submanifolds of $\mathbb{R}^n$ partly for the period mapping of the ordinary differential equation with any small nonzero parameter and partly for the averaged ordinary differential equation. Then the same problem is investigated for more general mappings associated with the ordinary differential equation for the small nonzero parameter. The final Section 6 deals with mappings possessing constraints. This means that mapping F given above possesses an invariant submanifold $V \subset \mathbb{R}^n$. Criteria are derived precluding the existence of k-dimensional compact orientable submanifolds of V invariant for F. Applications are given to autonomous ordinary differential equations with invariant submanifolds.

2. The main results

We denote by $\mathcal{T}_k^m$ the lexicographically ordered set of all k-tuples of integers $(i) = (i_1, i_2, \cdots, i_k)$ such that $1 \le i_1 < i_2 < \cdots < i_k \le m$.

Let A be any $m_1 \times m_2$ matrix, and let $a_{(i)}^{(j)}$ denote the minor of A determined by the rows $(i) = (i_1, i_2, \cdots, i_k) \in \mathcal{T}_k^{m_1}$ and the columns $(j) = (j_1, j_2, \cdots, j_k) \in \mathcal{T}_k^{m_2}$. The k-th multiplicative compound $A^{(k)}$ of A is the $\begin{pmatrix} m_1 \\ k \end{pmatrix} \times \begin{pmatrix} m_2 \\ k \end{pmatrix}$ matrix whose entries, written in lexicographic order, are $a_{(i)}^{(j)}$. In particular, when A is $m \times k$ with columns $a_1, a_2, \cdots, a_k$ then $A^{(k)}$ is the exterior product $a_1 \wedge a_2 \wedge \cdots \wedge a_k$ represented as a column vector [10].

Binet-Cauchy's formula states that

$$(AB)^{(k)} = A^{(k)} B^{(k)} .$$

Let $M \subset \mathbb{R}^n$ be an m-dimensional compact smooth orientable submanifold with a nonempty border ∂M [14]. Hence ∂M is an $m-1$-dimensional compact smooth orientable submanifold. Of course, we always assume that $m \ge 2$.

Let $F \in C^2(\mathbb{R}^n, \mathbb{R}^n)$ satisfy the following assumption

H1 $\quad F(\partial M) = \partial M$ and $F \colon \partial M \to \partial M$ is a diffeomorphism.

We note that H1 is equivalent to the statement that $F(\partial M) \subset \partial M$ and $F \colon \partial M \to \partial M$ is an injective local diffeomorphism.

We consider the set

$$\mathcal{M} = \Big\{ \phi \mid \text{it exists an open neighbourhood} \quad U_\phi \subset \mathbb{R}^n \quad \text{of} \quad M \quad \text{such that}$$
$$\phi \in C^2(U_\phi, \mathbb{R}^n), \quad \phi(\partial M) = \partial M \quad \text{and} \quad \phi \colon \partial M \to \partial M \quad \text{is a diffeomorphism} \Big\} .$$

Clearly $\mathcal{M} \neq \emptyset$, since the identity mapping $I \in \mathcal{M}$.

Let ϕ_i be the i-th component of ϕ, i.e. $\phi = (\phi_1, \phi_2, \cdots, \phi_n)$. Each ϕ_i generates the element $d\phi_i$ of TM^* - the cotangent bundle of M - by

$$d\phi_i(u)v = D\phi_i(u)v ,$$

where $u \in M, v \in T_u M$ and $D\phi_i \colon TM \to \mathbb{R}$. Then $d\phi_{i_1} \wedge d\phi_{i_2} \wedge \cdots \wedge d\phi_{i_m} \in \wedge^m TM^*$ for any $(i) = (i_1, i_2, \cdots, i_m) \in \mathcal{T}_m^n$. We note that $\dim \wedge^m T_u M^* = 1 \; \forall \, u \in M$. Since

M is orientable, there is a $\tau \in \wedge^m TM^*$ such that $\tau(u) > 0 \,\forall\, u \in M$ and $\int\limits_M \tau = 1$. Consequently there is a function $a_{(i)} \in C(M, \mathbb{R})$ such that

$$d\phi_{i_1} \wedge d\phi_{i_2} \wedge \cdots \wedge d\phi_{i_m} = a_{(i)}\tau \,.$$

Furthermore, we define the mapping $G_\phi \colon M \to \mathbb{R}^{\mathcal{T}^n_m} \simeq \mathbb{R}^N$, $N = \binom{n}{m}$ given by

$$G_\phi(u) = \left\{ a_{(i)}(u) \mid (i) \in \mathcal{T}^n_m \right\} .$$

We always identify in our considerations the spaces $\mathbb{R}^{\mathcal{T}^n_m}$ and $\mathbb{R}^N$. Let $|\cdot|$ be any norm on $\mathbb{R}^N$. We put

$$S(\phi) = \int\limits_M |G_\phi(u)|\tau \,.$$

We need the following result.

Theorem 2.1. *There is a constant $\delta > 0$ such that*

$$\inf_{\phi \in \mathcal{M}} S(\phi) \geq \delta \,.$$

Proof. Let $\alpha \in C^\infty\big(\mathbb{R}^n, \wedge^{m-1}(\mathbb{R}^n)^*\big)$ be given by

$$\alpha(x) = \sum \alpha_{i_1 i_2 \cdots i_{m-1}}(x)\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_{m-1}} \,.$$

We suppose that the support of α is compact. Stoke's theorem states [14]

$$\int\limits_{\partial M} \alpha \circ \phi = \int\limits_M d(\alpha \circ \phi) = \int\limits_M d\alpha \circ \phi \,.$$

Let us put

$$d\alpha(x) = \sum \tilde{\alpha}_{i_1 i_2 \cdots i_m}(x)\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_m} \,.$$

Then we have

$$\begin{aligned} \int\limits_M d\alpha \circ \phi &= \int\limits_M \sum \tilde{\alpha}_{i_1 i_2 \cdots i_m}(\phi(u))\, d\phi_{i_1} \wedge d\phi_{i_2} \wedge \cdots \wedge d\phi_{i_m} \\ &= \int\limits_M \sum \tilde{\alpha}_{i_1 i_2 \cdots i_m}(\phi(u)) a_{(i)}(u)\tau \\ &\leq \int\limits_M \sum |\tilde{\alpha}_{i_1 i_2 \cdots i_m}(\phi(u))| \cdot |a_{(i)}(u)|\tau \\ &\leq K\, S(\phi) \,, \end{aligned}$$

for a constant $K > 0$ independent of $\phi \in \mathcal{M}$. On the other hand, since $\phi \colon \partial M \to \partial M$ is a diffeomorphism, we have $\int\limits_{\partial M} \alpha \circ \phi = \sum\limits_{i \in \mathcal{C}} \pm \int\limits_{\partial M_i} \alpha$, where ∂M_i, $i \in \mathcal{C}$ are all connected components of ∂M. We note that the sign $\pm$ for ∂M_i in $\pm \int\limits_{\partial M_i} \alpha$ depends on ϕ and does not on α. Finally, there is clearly a nonzero $\bar{\alpha} \in C^\infty(\mathbb{R}^n, \wedge^{m-1}(\mathbb{R}^n)^*)$ with the compact support $\operatorname{supp} \bar{\alpha} \subset \cup_{i \in \mathcal{C}} U_i$, where U_i are open bounded sets satisfying $U_i \cap U_j = \emptyset, i \neq j, \partial M_i \subset U_i$, and such that $\int\limits_{\partial M} \bar{\alpha} > 0$. By changing the sign of $\bar{\alpha}$ on U_i, which does not influence the constant K, we get an α such that $\int\limits_{\partial M} \alpha \circ \phi = \int\limits_{\partial M} \bar{\alpha}$. The proof is finished. $\square$

Condition H1 implies that

$$F \circ \phi \in \mathcal{M} \quad \forall\, \phi \in \mathcal{M} .$$

Hence $S(F \circ \phi)$ is well-defined. Let $\tilde{\phi}_i$ be the i-th component of $F \circ \phi$, i.e. $F \circ \phi = (\tilde{\phi}_1, \tilde{\phi}_2, \cdots, \tilde{\phi}_n)$ or $F_i \circ \phi = \tilde{\phi}_i$. Like above, we take the 1-differential form $d\tilde{\phi}_i$. We have

$$d\tilde{\phi}_i(u)v = \sum_{j=1}^n \frac{\partial F_i}{\partial u_j}(\phi(u)) d\phi_j(u)v .$$

Let $\partial F(u)_{(i)}^{(j)}$ denote the minor of the matrix $DF(u)$ determined by the rows $(i) = (i_1, i_2, \cdots, i_m) \in \mathcal{T}_m^n$ and the columns $(j) = (j_1, j_2, \cdots, j_m) \in \mathcal{T}_m^n$. Hence we have that $DF(u)^{(m)} = \left(\partial F(u)_{(i)}^{(j)}\right)$.

Putting for $(i) = (i_1, i_2, \cdots, i_m) \in \mathcal{T}_m^n$ like above

$$d\tilde{\phi}_{i_1} \wedge \tilde{d}\phi_{i_2} \wedge \cdots \wedge d\tilde{\phi}_{i_m} = \tilde{a}_{(i)} \tau ,$$

Binet-Cauchy's formula implies

$$\tilde{a}_{(i)}(u) = \sum_{(j) \in \mathcal{T}_m^n} \partial F(\phi(u))_{(i)}^{(j)} a_{(j)}(u) ,$$

i.e.

$$G_{F \circ \phi}(u) = DF(\phi(u))^{(m)} G_\phi(u) .$$

Summarizing, we arrive at the following result

$$\begin{aligned} S(F \circ \phi) = \int\limits_M \left|G_{F\circ\phi}(u)\right| \tau &= \int\limits_M \left|DF(\phi(u))^{(m)} G_\phi(u)\right| \tau \\ &\leq \int\limits_M ||DF(\phi(u))^{(m)}|| \cdot \left|G_\phi(u)\right| \tau , \end{aligned}$$

where $||\cdot||$ is the norm on $\mathcal{L}(\mathbb{R}^N)$ generated by $|\cdot|$.

Theorem 2.2. *Let assumption H1 be satisfied. If there is a constant $\theta, 0 < \theta < 1$ such that*

$$||DF(x)^{(m)}|| \leq \theta \quad \forall\, x \in \mathbb{R}^n , \tag{2.1}$$

then

$$S(F(\phi)) \leq \theta S(\phi) \quad \forall\, \phi \in \mathcal{M} .$$

Theorems 2.1 and 2.2 imply the following results.

Theorem 2.3. *If $F \in C^2(\mathbb{R}^n, \mathbb{R}^n)$ satisfies (2.1) then there is no an m-dimensional compact smooth orientable submanifold M of $\mathbb{R}^n$ such that $F(\partial M) \subset \partial M$ and $F \colon \partial M \to \partial M$ is an injective local diffeomorphism.*

Of course, a diffeomorphism is an injective local diffeomorphism.

Corollary 2.4. *A diffeomorphism $F \in C^2(\mathbb{R}^n, \mathbb{R}^n)$ satisfying (2.1) has no an invariant $m-1$-dimensional submanifold ∂M, which is a border of an m-dimensional compact smooth orientable submanifold M of $\mathbb{R}^n$.*

We recall that the norms of a matrix $A = (a_{ij}) \in \mathcal{L}(\mathbb{R}^N)$ with respect to the norms $\max\limits_i |x_i|$, $\sum\limits_i |x_i|$ and $\sqrt{\sum\limits_i x_i^2}$ on $\mathbb{R}^N$ have the values $\max\limits_i \sum\limits_j |a_{ij}|$, $\max\limits_j \sum\limits_i |a_{ij}|$ and the square root of the largest eigenvalue of the symmetric matrix AA^*, respectively.

Proposition 2.5. *Assumption (2.1) is satisfied if one of the following conditions*

$$\sup_{u\in\mathbb{R}^n,(i)\in\mathcal{T}_m^n} \sum_{(j)\in\mathcal{T}_m^n} \left|\partial F(u)_{(i)}^{(j)}\right| < 1\,, \tag{C1}$$

$$\sup_{u\in\mathbb{R}^n,(j)\in\mathcal{T}_m^n} \sum_{(i)\in\mathcal{T}_m^n} \left|\partial F(u)_{(i)}^{(j)}\right| < 1\,, \tag{C2}$$

$$\begin{aligned}&\sup_{u\in\mathbb{R}^n} \Big\{ \textit{the square root of the largest eigenvalue} \\ &\qquad \textit{of the symmetric matrix} \quad DF(u)^{(m)}(DF(u)^{(m)})^* \Big\} < 1\end{aligned} \tag{C3}$$

is satisfied.

When F is a diffeomorphism then Binet-Cauchy's formula implies

$$DF^{-1}(u)^{(m)} = \left(DF(F^{-1}(u))^{(m)}\right)^{-1}.$$

We can replace F with F^{-1} in the above considerations. Hence condition (2.1) can be replaced by

$$|DF(x)^{(m)}v| \geq \theta|v| \quad \forall\, x \in \mathbb{R}^n,\, \forall\, v \in \mathbb{R}^N\,, \tag{2.2}$$

for a constant $\theta > 1$. We can similarly modify conditions C1, C2 and C3 for this case.

Proposition 2.6. *Assumption (2.2) is satisfied if one of the following conditions*

$$\inf_{u\in\mathbb{R}^n,(i)\in\mathcal{T}_m^n} \left(\left|\partial F(u)_{(i)}^{(i)}\right| - \sum_{(i)\neq(j)\in\mathcal{T}_m^n} \left|\partial F(u)_{(i)}^{(j)}\right|\right) > 1\,, \tag{C4}$$

$$\inf_{u\in\mathbb{R}^n,(i)\in\mathcal{T}_m^n} \left(\left|\partial F(u)_{(i)}^{(i)}\right| - \sum_{(i)\neq(j)\in\mathcal{T}_m^n} \left|\partial F(u)_{(j)}^{(i)}\right|\right) > 1\,, \tag{C5}$$

$$\begin{aligned}&\inf_{u\in\mathbb{R}^n} \Big\{ \textit{the square root of the smallest eigenvalue} \\ &\qquad \textit{of the symmetric matrix} \quad DF(u)^{(m)}(DF(u)^{(m)})^* \Big\} > 1\end{aligned} \tag{C6}$$

is satisfied.

Proof. For any $(a_{ij}) = A \in \mathcal{L}(\mathbb{R}^N)$ we have

$$|Av| \geq \min_i \left(|a_{ii}| - \sum_{j\neq i} |a_{ij}|\right)|v|\,,$$

$$|Av| \geq \min_i \left(|a_{ii}| - \sum_{j\neq i} |a_{ji}|\right)|v|\,,$$

$$|Av| \geq \lambda_N^{1/2}|v|\,,$$

for the norms $|v| = \max_i |x_i|$, $|v| = \sum_i |x_i|$, $|v| = \sqrt{\sum_i x_i^2}$, $v = (x_1, x_2, \cdots, x_N)$ on $\mathbb{R}^N$, respectively. Here λ_N is the smallest eigenvalue of AA^*. Clearly proof is finished by taking $A = DF(u)^{(m)}$. □

When $m = 2$, assumption H1 can be weakened to

H2 $F(\partial M) = \partial M$ for $m = 2$, and the winding number $w(F, \partial M_i)$ of F around each ∂M_i, $i \in \mathcal{C}$ is nonzero,

since then $\int_{\partial M} \alpha \circ \phi = \sum_{i\in\mathcal{C}} w(\phi, \partial M_i) \int_{\partial M_i} \alpha$ in the proof of Theorem 2.1 for any $\phi \in C^2(M, \mathbb{R}^n)$ such that $\phi(\partial M) = \partial M$ and $w(\phi, \partial M_i) \neq 0\ \forall\, i \in \mathcal{C}$. So Theorem 2.1 is valid also with assumption H2. Then also the above rest results remain true. Moreover, now ∂M can be piecemeal smooth. We note that H2 provides a permutation Π_F on $\mathcal{C}$ given by $F(\partial M_i) = \partial M_{\Pi_F(i)}\ \forall\, i \in \mathcal{C}$.

Remark 2.7. When there is a bounded open subset D of $\mathbb{R}^n$ such that $F(D) \subset D$ then by restricting conditions C1-C6 on D, we get nonexistence results on D like above.

We can consider instead of the functional S also the following one

$$S(\phi) = \int_M |A(\phi(u))G_\phi(u)|\tau\,,$$

where $A \in C(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^N))$, $A(\cdot)$ are invertible and $||A(x)^{-1}||$ is uniformly bounded on $\mathbb{R}^n$. The corresponding condition like (2.1) has now the form

$$||A(F(x))DF(x)^{(m)}A(x)^{-1}|| \leq \theta \quad \forall\, x \in \mathbb{R}^n\,, \tag{2.3}$$

for a constant $\theta, 0 < \theta < 1$.

3. Criteria for flows

Let us consider the ordinary differential equation

$$\dot{x} = f(x)\,, \tag{3.1}$$

where $f \in C^2(\mathbb{R}^n, \mathbb{R}^n)$. We suppose the assumption

A1 For any $x \in \mathbb{R}^n$, (3.1) has the solution $t \to \psi(t, x)$, $\psi(0, x) = x$ on the interval $[-2, 2]$.

We put $F(x) = \psi(T,x)$ for a fixed $T, 0 < T < 2$. Clearly $F \in C^2(\mathbb{R}^n, \mathbb{R}^n)$ is a diffeomorphism. Moreover, it holds

$$D\dot{\psi}(t,x) = Df\big(\psi(t,x)\big)D\psi(t,x)\,. \tag{3.2}$$

Let A be any $m_1 \times m_1$ matrix. The k-th additive compound $A^{[k]}$ of A is the $\begin{pmatrix} m_1 \\ k \end{pmatrix} \times \begin{pmatrix} m_1 \\ k \end{pmatrix}$ matrix defined by [10]

$$A^{[k]} = D(I + hA)^{(k)}\ |_{h=0}\ . \tag{3.3}$$

It is shown in [10] that (3.2) implies

$$\begin{gathered} \frac{d}{dt}\Big(D\psi(t,x)^{(m)}\Big) = Df\big(\psi(t,x)\big)^{[m]} D\psi(t,x)^{(m)}\,, \\ D\psi(0,x)^{(m)} = I\,. \end{gathered} \tag{3.4}$$

The Lozinskii logarithmic norm of an $N \times N$ matrix A is defined by [10]

$$\mu(A) = \lim_{h \to 0_+} \frac{||I + hA|| - 1}{h}\,. \tag{3.5}$$

It is know that $\mu(A)$ exists and it has the following property that

$$|x(t)| \le \mathrm{e}^{\int_0^t \mu(A(s))\,ds} |x(0)|\,, \tag{3.6}$$

where $x(t)$ is the solution of $\dot{x} = A(t)x$ for a continuous $A(t) \in \mathcal{L}(\mathbb{R}^N)$. The value of $\mu(A)$ depends on $|\cdot|$ [10]. In the cases $\max_i |x_i|$, $\sum_i |x_i|$, $\sqrt{\sum_i x_i^2}$, the Lozinskii norm is given by

$$\mu(A) = \max_i \Big(a_{ii} + \sum_{j \ne i} |a_{ij}|\Big), \quad \mu(A) = \max_j \Big(a_{jj} + \sum_{i \ne j} |a_{ij}|\Big), \quad \mu(A) = \lambda_1\,,$$

respectively, where $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_N$ are the eigenvalues of $(A^* + A)/2$. More generally, we have the following expressions, for $m = 1, 2, \cdots, N$,

$$\mu(A^{[m]}) = \begin{cases} \max\limits_{(i) \in \mathcal{T}_m^N} \Big[a_{i_1 i_1} + \cdots + a_{i_m i_m} + \sum\limits_{j \notin (i)} \big(|a_{i_1 j}| + \cdots + |a_{i_m j}|\big)\Big] \\ \max\limits_{(j) \in \mathcal{T}_m^N} \Big[a_{j_1 j_1} + \cdots + a_{j_m j_m} + \sum\limits_{i \notin (j)} \big(|a_{i j_1}| + \cdots + |a_{i j_m}|\big)\Big] \\ \lambda_1 + \cdots + \lambda_m\,, \end{cases}$$

respectively. We have the following result.

Theorem 3.1. *If the ordinary differential equation (3.1) satisfies condition A1 and also one of the following conditions*

A2 $\qquad \sup\limits_{u \in \mathbb{R}^n} \mu\Big(Df(u)^{[m]}\Big) < 0\,,$

A3 $\qquad \sup\limits_{u \in \mathbb{R}^n} \mu\Big(-Df(u)^{[m]}\Big) < 0\,,$

then there is no an $m-1$-dimensional submanifold ∂M, which is a border of an m-dimensional compact smooth orientable submanifold M of $\mathbb{R}^n$, such that $\psi(T,\partial M)\subset\partial M$ for some $T, 0<|T|<2$.

Proof. Let $2>T>0$. Condition A2 together with (3.4) and (3.6) imply the existence of a positive constant η such that

$$||D\psi(T,x)^{(m)}||\leq \mathrm{e}^{-T\eta}\quad \forall\, x\in\mathbb{R}^n\,.$$

Hence assumption (2.1) is satisfied for $F(x)=\psi(T,x)$ with $\theta=\mathrm{e}^{-T\eta}$. Corollary 2.4 finishes the proof for condition A2. Changing $t\to -t$ in (3.1) we get the result for A3. Since $\psi(T,x)^{-1}=\psi(-T,x)$, the proof is finished also for $-2<T<0$. □

Taking the norms $\sum_i |x_i|$, $\max_i |x_i|$ $\sqrt{\sum_i x_i^2}$ on $\mathbb{R}^n$ and using the above formulas for the Lozinskii norms, Theorem 3.1 implies the following result.

Theorem 3.2. *If the ordinary differential equation (3.1) satisfies condition A1 and also one of the following conditions*

$$\text{(i)}\qquad \sup_{\mathbb{R}^n,(i)\in\mathcal{T}_m^n}\left\{\frac{\partial f_{i_1}}{\partial x_{i_1}}+\cdots+\frac{\partial f_{i_m}}{\partial x_{i_m}}+\sum_{j\notin(i)}\left(\left|\frac{\partial f_j}{\partial x_{i_1}}\right|+\cdots+\left|\frac{\partial f_j}{\partial x_{i_m}}\right|\right)\right\}<0\,,$$

$$\text{(ii)}\qquad \sup_{\mathbb{R}^n,(i)\in\mathcal{T}_m^n}\left\{\frac{\partial f_{i_1}}{\partial x_{i_1}}+\cdots+\frac{\partial f_{i_m}}{\partial x_{i_m}}+\sum_{j\notin(i)}\left(\left|\frac{\partial f_{i_1}}{\partial x_j}\right|+\cdots+\left|\frac{\partial f_{i_m}}{\partial x_j}\right|\right)\right\}<0\,,$$

$$\text{(iii)}\qquad \sup_{\mathbb{R}^n}\Big\{\lambda_1+\cdots+\lambda_m\mid \lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n$$

$$\text{are the eigenvalues of}\quad \big(Df(u)+Df(u)^*\big)/2\Big\}<0\,,$$

$$\text{(iv)}\qquad \inf_{\mathbb{R}^n,(i)\in\mathcal{T}_m^n}\left\{\frac{\partial f_{i_1}}{\partial x_{i_1}}+\cdots+\frac{\partial f_{i_m}}{\partial x_{i_m}}-\sum_{j\notin(i)}\left(\left|\frac{\partial f_j}{\partial x_{i_1}}\right|+\cdots+\left|\frac{\partial f_j}{\partial x_{i_m}}\right|\right)\right\}>0\,,$$

$$\text{(v)}\qquad \inf_{\mathbb{R}^n,(i)\in\mathcal{T}_m^n}\left\{\frac{\partial f_{i_1}}{\partial x_{i_1}}+\cdots+\frac{\partial f_{i_m}}{\partial x_{i_m}}-\sum_{j\notin(i)}\left(\left|\frac{\partial f_{i_1}}{\partial x_j}\right|+\cdots+\left|\frac{\partial f_{i_m}}{\partial x_j}\right|\right)\right\}>0\,,$$

$$\text{(vi)}\qquad \inf_{\mathbb{R}^n}\Big\{\lambda_{n-m+1}+\cdots+\lambda_n\mid \lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n$$

$$\text{are the eigenvalues of}\quad \big(Df(u)+Df(u)^*\big)/2\Big\}>0\,,$$

then the statement of Theorem 3.1 is valid. Here $f=(f_1,f_2,\cdots,f_n)$.

We note that Propositions 2.5 and 2.6 are discrete analogies of Theorem 3.2. The case $m=2$ is studied in [8], [10] and we get the same criteria on the nonexistence of either periodic or homoclinic solutions of (3.1) (see Theorem 3.3 of [8] and Theorem 4.1 of [10]).

Remark 3.3. Conditions A2, A3 and Theorem 3.2 can be restricted on a bounded open subset D of $\mathbb{R}^n$ which is either positive or negative invariant for (3.1). Then clearly $T\neq 0$ can be arbitrary and assumption A1 is irrelevant.

Remark 3.4. We get the same results for a nonautonomous T-periodic ordinary differential equation $\dot{x} = f(x,t)$. Of course, we take $F(x) = \psi(T,x)$, where ψ is defined like above for $\dot{x} = f(x,t)$.

Finally, we can study more general mappings generated by the flow $\psi(t,x)$ of (3.1). Let $G \in C^2(\mathbb{R}^n, [1, 5/4])$. We again put $F(x) = \psi(G(x), x)$. Clearly $F \in C^2(\mathbb{R}^n, \mathbb{R}^n)$ and F is not a diffeomorphism in general. But it is a diffeomorphism when $DG(u)f(u) + 1 \neq 0 \ \forall\, u \in \mathbb{R}^n$. We have

$$DF(x) = f(\psi(G(x),x))DG(x) + D\psi(G(x),x)\,.$$

We put $Y(t) = f(\psi(t,x))DG(x) + D\psi(t,x)$. Then it holds

$$\dot{Y}(t) = Df(\psi(t,x))Y(t), \quad Y(0) = f(x)DG(x) + I\,. \tag{3.7}$$

So we can repeat for (3.7) the above arguments of (3.2) to get the following result.

Theorem 3.5. *If the ordinary differential equation (3.1) satisfies condition A1 and also the following one*

$$\text{A4} \qquad \sup_{u\in\mathbb{R}^n} \mathrm{e}^{G(u)\eta} ||(f(u)DG(u) + I)^{(m)}|| < 1\,,$$

where $\eta = \sup_{u\in\mathbb{R}^n} \mu\Big(Df(u)^{[m]}\Big)$, *then there is no an* $m-1$*-dimensional submanifold* ∂M, *which is a border of an* m*-dimensional compact smooth orientable submanifold* M *of* $\mathbb{R}^n$, *such that* $\psi(G(\partial M), \partial M) \subset \partial M$ *and* $\psi(G(x), x)\colon \partial M \to \partial M$ *is an injective local diffeomorphism.*

4. Criteria for discretizations of flows

Let us consider for simplicity the Euler discretization of (3.1) given by

$$x_{j+1} = x_j + hf(x_j)\,, \tag{4.1}$$

where $h > 0$ is the step size of discretization. (4.1) generates the mapping $F_h(x) = x + hf(x)$. Hence $DF_h(x) = I + hDf(x)$, and $DF_h(x)^{(m)} = \big(I + hDf(x)\big)^{(m)}$. So by (3.3) we obtain

$$DF_h(x)^{(m)} = I + hDf(x)^{[m]} + O(h^2)\,. \tag{4.1}$$

We suppose the following condition

$$\text{B1} \qquad \max_{1\leq i,j\leq n} \sup_{x\in\mathbb{R}^n} \left|\tfrac{\partial f_i(x)}{\partial x_j}\right| < \infty\,.$$

Here $f = (f_1, f_2, \cdots, f_n)$. Condition B1 implies the uniform boundedness of the term $O(h^2)/h^2$ in (4.1), and also B1 gives that $F_h(x)$ is a diffeomorphism on $\mathbb{R}^n$ for any $h > 0$ sufficiently small. Clearly B1 implies A1.

Furthermore, according to (3.3) and (3.5) we have

$$\begin{aligned} ||DF_h(x)^{(m)}|| &= ||I + hDf(x)^{[m]} + O(h^2)|| \\ &= ||I + hDf(x)^{[m]}|| + O(h^2) = 1 + h\mu\big(Df(x)^{[m]}\big) + O(h^2)\,. \end{aligned} \tag{4.2}$$

We suppose the following assumption

B2 $||I + hA|| = 1 + h\mu(A) + O(h^2)$ and the term $O(h^2)/h^2$ is uniformly bounded with respect to A from any bounded subset of $\mathcal{L}(\mathbb{R}^N)$ and any $h > 0$ sufficiently small.

Proposition 4.1. *Assumption B2 holds for the norms on $\mathcal{L}(\mathbb{R}^N)$ generated by the norms $\max_i |x_i|$, $\sum_i |x_i|$ and $\sqrt{\sum_i x_i^2}$ on $\mathbb{R}^N$.*

Proof. Since the proof is a straightforward verification, we omit it. □

Conditions B1 and B2 imply the uniform boundedness of the term $O(h^2)/h^2$ in (4.2). Since

$$DF_h(x)^{-1} = I + h\big(-Df(x)\big) + O(h^2)\,,$$

we get a similar estimation like (4.2) for $\big(DF_h(x)^{-1}\big)^{(m)}$. The following result is clear.

Proposition 4.2. *Let condition B1 hold for (3.1) and let the norm $||\cdot||$ on $\mathcal{L}\big(\mathbb{R}^N\big)$ satisfy B2. Then condition (2.1) (or (2.2)) holds for F_h with any $h > 0$ sufficiently small if and only if condition A2 (or A3) is satisfied, respectively.*

Proposition 4.2 roughly states that the flow has the property of the nonexistence of an $m-1$-dimensional invariant compact orientable submanifold if and only if its discretization has the same property with any sufficiently small step size. This result is in the spirit of a recently developed theory of numerical analysis for dynamical systems (see [6] and references in that paper).

Let D be an open bounded subset of $\mathbb{R}^n$ with a smooth border ∂D, and the vector $f(x)$ from (3.1) is transversally oriented at any $x \in \partial D$ inside to D. Then using the solutions of (3.1) starting from ∂D, we can construct a tubular neighbourhood of ∂D. In this way, we can derive a function $V \in C^1(\mathbb{R}^n, \mathbb{R})$ such that

B3 $$\langle V'(x), f(x)\rangle < 0 \quad \forall\, x \in \partial D\,,$$
$$D = \big\{x \in \mathbb{R}^n \mid V(x) < 1\big\}\,.$$

Here V' is the gradient of V with respect to a scalar product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^n$. Clearly property B3 implies the positive invariance of D with respect to the ordinary differential equation (3.1).

Theorem 4.3. *D is invariant for mapping F_h with any $h > 0$ sufficiently small.*

Proof. Let us consider the set

$$D_\epsilon = \big\{x \in \mathbb{R}^n \mid V(x) \le 1 - 2\epsilon\big\}\,,$$

where $\epsilon > 0$ is sufficiently small. Let $x \in D_\epsilon$, then we have

$$V(F_h(x)) = V(x) + h\langle V'(F_{h_1}(x)), f(x)\rangle \le 1 - 2\epsilon + h\langle V'(F_{h_1}(x)), f(x)\rangle$$

for $h_1 \in (0, h)$. For $h > 0$ sufficiently small we get $V(F_h(x)) \le 1-\epsilon$. So $F_h(x) \in D$.

Let $x \in D \setminus D_\epsilon$. Consider the function $\Gamma(h) = V(F_h(x))$. We have

$$\Gamma'(h) = \langle V'(F_h(x)), f(x)\rangle\,.$$

Property B3 gives a constant $\xi > 0$ such that if $\epsilon > 0$ and $h > 0$ are sufficiently small then

$$\langle V'(F_h(x)), f(x)\rangle = \langle V'(x + hf(x)), f(x)\rangle < -\xi$$

for any $x \in D \setminus D_\epsilon$. Consequently we get that $\Gamma'(h) < -\xi$ for $h > 0$ sufficiently small and $\Gamma(0) = V(x) < 1$. So $\Gamma(h) < 1$, i.e. $F_h(x) \in D$. Clearly the proof is finished. □

Applying Proposition 4.2, we have the following result.

Theorem 4.4. *Let the norm $||\cdot||$ on $\mathcal{L}\big(\mathbb{R}^N\big)$ satisfy B2. Let D be an open bounded subset of $\mathbb{R}^n$ with a smooth border ∂D, and the vector $f(x)$ from (3.1) is transversally oriented at any $x \in \partial D$ inside to D. If $\max_{u\in\overline{D}} \mu\Big(Df(u)^{[m]}\Big) < 0$, then neither the ordinary differential equation (3.1) nor mapping F_h, with $h > 0$ sufficiently small, has an $m-1$-dimensional submanifold ∂M, which is a border of an m-dimensional compact smooth orientable submanifold M of D, such that either $\psi(T, \partial M) \subset \partial M$ for some $T \neq 0$ or $F_h(\partial M) \subset \partial M$, respectively.*

Finally, let $G \in C^1(\mathbb{R}^n, (0, \infty))$ and let $\tilde{F}_h \in C^2(\mathbb{R}^n, \mathbb{R}^n)$ be a family of mappings with $h > 0$ sufficiently small such that

$$\psi(hG(x), x) - \tilde{F}_h(x) = O(h^2) \tag{4.3}$$

uniformly on $\overline{D}$ in the C^1-topology. We suppose the above relationship between D and $f(x)$. Hence D is invariant for $G_h(x) = \psi(hG(x), x)$. Clearly G_h and $\tilde{F}_h$ are diffeomorphisms on $\overline{D}$ for any $h > 0$ sufficiently small. Theorem 4.3 holds also for $\tilde{F}_h$. (4.3) implies

$$DG_h(x)^{(m)} - D\tilde{F}_h(x)^{(m)} = O(h^2)$$

uniformly on $\overline{D}$. We have

$$DG_h(x) = I + h\Big(f(x)DG(x) + Df(x)G(x)\Big) + O(h^2)\,.$$

If the norm $||\cdot||$ satisfies B2, then we get

$$||D\tilde{F}_h(x)^{(m)}|| = 1 + h\mu\Big((f(x)DG(x))^{[m]} + G(x)Df(x)^{[m]}\Big) + O(h^2)$$

uniformly on D. Consequently, Theorem 4.4 is valid for $\tilde{F}_h$ if

$$\max_{u\in\overline{D}} \mu\Big(Df(u)^{[m]}\Big) < 0 \quad \text{and} \quad \max_{u\in\overline{D}} \mu\Big((f(u)DG(u))^{[m]} + G(u)Df(u)^{[m]}\Big) < 0\,.$$

We note that μ is subadditive, i.e. $\mu(A+B) \le \mu(A)+\mu(B)\ \forall\, A, B \in \mathcal{L}(\mathbb{R}^N)$, hence Theorem 4.4 is valid for $\tilde{F}_h$ if

$$\max_{u\in\overline{D}} \mu\Big((f(u)DG(u))^{[m]}\Big) < -\max_{u\in\overline{D}} \mu\Big(G(u)Df(u)^{[m]}\Big) > 0\,, \tag{4.4}$$

and these inequalities (4.4) are valid when

$$\max_{u\in\overline{D}} ||(f(u)DG(u))^{[m]}|| < -\max_{u\in\overline{D}} \mu\Big(G(u)Df(u)^{[m]}\Big) > 0\,,$$

since $\mu(A) \leq ||A||$ for any $A \in \mathcal{L}(\mathbb{R}^N)$.

Taking concrete norms for $||\cdot||$, we get similar inequalities like in Theorem 3.2. For instance, when $|v| = \max_i |x_i|$, $v = (x_1, x_1, \cdots, x_N)$ then (4.4) possesses the form

$$\max_{\overline{D},(i)\in\mathcal{T}_m^n} \Big\{ f_{i_1}\frac{\partial G}{\partial x_{i_1}} + \cdots + f_{i_m}\frac{\partial G}{\partial x_{i_m}} + \sum_{j\notin(i)} \Big(\Big| f_{i_1}\frac{\partial G}{\partial x_j}\Big| + \cdots + \Big| f_{i_m}\frac{\partial G}{\partial x_j}\Big| \Big) \Big\} <$$
$$-\max_{\overline{D},(i)\in\mathcal{T}_m^n} \Big\{ G\Big(\frac{\partial f_{i_1}}{\partial x_{i_1}} + \cdots + \frac{\partial f_{i_m}}{\partial x_{i_m}} + \sum_{j\notin(i)} \Big(\Big|\frac{\partial f_{i_1}}{\partial x_j}\Big| + \cdots + \Big|\frac{\partial f_{i_m}}{\partial x_j}\Big| \Big) \Big) \Big\} > 0 \,,$$

where again $f = (f_1, f_2, \cdots, f_n)$. We note that generally Proposition 4.2 does not hold for $\tilde{F}_h$, but it does hold with nonzero constant functions G.

5. Criteria for flows with small parameters

In this section, we study the ordinary differential equation

$$\dot{x} = \varepsilon f(x,t)\,, \tag{5.1}$$

where $\varepsilon > 0$ is a small parameter and $f \in C^2(\mathbb{R}^{n+1}, \mathbb{R}^n)$ is T-periodic in $t \in \mathbb{R}$ for a $T > 0$. Let $\psi_\varepsilon(t,x)$ be the flow of (5.1) defined like above for (3.1). We suppose that

D1 $\sup_{(u,t)\in\mathbb{R}^{n+1}} \Big\{ |f_i(u,t)|, \Big|\frac{\partial}{\partial x_j} f_i(u,t)\Big|, \Big|\frac{\partial^2}{\partial x_j \partial x_k} f_i(u,t)\Big| \Big\} < \infty$ for all $i,j,k \in \{1, 2, \cdots, n\}$.

Remark 5.1. Assumption D1 is technical and it can be removed, for instance, when $\psi_\varepsilon(T,x)$ possesses an invariant open bounded subset $D \subset \mathbb{R}^n$ uniformly for $\varepsilon > 0$ small.

D1 implies that the mapping $F_\varepsilon(x) = \psi_\varepsilon(T,x)$ is well-defined and it is a diffeomorphism. Moreover, D1 also gives

$$F_\varepsilon(x) = x + \varepsilon \int_0^T f(x,t)\,dt + O(\varepsilon^2)\,, \tag{5.2}$$

$$DF_\varepsilon(x) = I + \varepsilon \int_0^T Df(x,t)\,dt + O(\varepsilon^2)\,, \tag{5.3}$$

where the terms $O(\varepsilon^2)/\varepsilon^2$ in (5.2) and (5.3) are uniformly bounded for $x \in \mathbb{R}^n$ and $\varepsilon > 0$ small. When the norm $||\cdot||$ satisfies B2, then (5.3) implies

$$||DF_\varepsilon(x)^{(m)}|| = 1 + \varepsilon\mu\Big(\Big(\int_0^T Df(x,t)\,dt\Big)^{[m]}\Big) + O(\varepsilon^2)\,,$$

where again $O(\varepsilon^2)/\varepsilon^2$ is uniformly bounded like above. Applying Corollary 2.4 to F_ε, we get the following result.

Proposition 5.2. *Let $||\cdot||$ satisfy B2 and let (5.1) satisfy D1. If*

$$\text{D2} \qquad \sup_{u\in\mathbb{R}^n} \mu\Big(\Big(\int_0^T Df(u,t)\,dt\Big)^{[m]}\Big) < 0\,,$$

then F_ε with any $\varepsilon > 0$ sufficiently small has no an $m-1$-dimensional submanifold ∂M, which is a border of an m-dimensional compact smooth orientable submanifold M of $\mathbb{R}^n$, such that $F_\varepsilon(\partial M) \subset \partial M$.

Since $F_\varepsilon^{-1} = \psi_\varepsilon(-T, x)$, Proposition 5.2 holds when D2 is replaced by

$$\text{D3} \qquad \sup_{u\in\mathbb{R}^n} \mu\Big(-\Big(\int_0^T Df(u,t)\,dt\Big)^{[m]}\Big) < 0\,.$$

We note that

$$\Big(\int_0^T Df(u,t)\,dt\Big)^{[m]} = \int_0^T Df(u,t)^{[m]}\,dt = \Big(D\int_0^T f(u,t)\,dt\Big)^{[m]}\,.$$

Furthermore, mapping F_ε is very similar to F_h of Section 4. Hence Theorems 4.3 and 4.4 remain true when F_h and f are replaced by F_ε and $\frac{1}{T}\int_0^T f(x,t)\,dt$, respectively. Of course, then assumption D1 is irrelevant.

We recall that the averaged equation of (5.1) has the form [11]

$$\dot{x} = \varepsilon \bar{f}(x) = \varepsilon \frac{1}{T}\int_0^T f(x,t)\,dt\,. \tag{5.4}$$

Let $\bar{\psi}_\varepsilon(t,x)$ be the flow of (5.4). Another mapping associated with (5.1) and strongly motivated by the averaging theory, it is given by

$$H_\varepsilon(x) = \psi_\varepsilon\big(G(\varepsilon,x)/\varepsilon, x\big)\,,$$

where $G \in C^2(\mathbb{R}^{n+1}, (0,\infty))$ is such that

$$0 < \inf G \le \sup G < \infty, \quad \sup|D^i_{(\varepsilon,x)}G| < \infty, \quad i = 1,2\,.$$

Theorem 2.6.1 of [11] and D1 imply

$$\psi_\varepsilon(t,x) - \bar{\psi}_\varepsilon(t,x) = O(\varepsilon) \tag{5.5}$$

in the C^1-topology for $x \in \mathbb{R}^n$ uniformly on $[0, L_1/\varepsilon] \times \mathbb{R}^n$, where $L_1 > \sup G$ is a positive constant. By D1 and (5.5) we have

$$DH_\varepsilon(x) = D\bar{\psi}_\varepsilon\big(G(\varepsilon,x)/\varepsilon, x\big) + f\big(\bar{H}_\varepsilon(x), G(\varepsilon,x)/\varepsilon\big)DG(\varepsilon,x) + O(\varepsilon)\,, \tag{5.6}$$

where $\bar{H}_\varepsilon(x) = \bar{\psi}_\varepsilon\big(G(\varepsilon,x)/\varepsilon, x\big)$. We note that $\bar{\psi}_\varepsilon(t,x) = \bar{\psi}_1(\varepsilon t, x)$. Hence (5.6) gives

$$\begin{aligned} DH_\varepsilon(x) &= D\bar{\psi}_1\big(G(0,x), x\big) \\ &+ f\Big(\bar{\psi}_1\big(G(0,x),x\big), \frac{G(0,x)}{\varepsilon} + \frac{\partial}{\partial\varepsilon}G(0,x)\Big)DG(0,x) + O(\varepsilon)\,, \end{aligned} \tag{5.7}$$

uniformly on $\mathbb{R}^n$ when in addition to D1, it also holds

D4 $\sup_{\mathbb{R}^{n+1}} \left|\frac{\partial}{\partial t} f(x,t)\right| < \infty\,.$

Consequently, we get

$$DH_\varepsilon(x)^{(m)} = \Big(D\bar\psi_1\big(G(0,x),x\big) + f\Big(\bar\psi_1\big(G(0,x),x\big), \frac{G(0,x)}{\varepsilon} + \frac{\partial}{\partial\varepsilon}G(0,x)\Big) DG(0,x)\Big)^{(m)} + O(\varepsilon)\,.$$

Since this formula is rather complicated, we suppose that

D5 $G(0,x) = L$ for a positive constant L.

It is very often in the averaging theory that $G(\varepsilon,x) = L$. Then we do not need D4, and (5.5), (5.7) give

$$H_\varepsilon(x) = \bar\psi_1(L,x) + O(\varepsilon), \quad DH_\varepsilon(x) = D\bar\psi_1(L,x) + O(\varepsilon)\,,$$

uniformly on $\mathbb{R}^n$. So $H_\varepsilon(x)$ is a diffeomorphism for any $\varepsilon > 0$ sufficiently small. Since $\dot{\bar\psi}_1 = \bar f(\bar\psi_1)$, Theorem 3.1 implies the following result.

Theorem 5.3. *Let (5.1) satisfy D1 and let D5 hold. If either condition D2 or D3 holds then H_ε with any $\varepsilon > 0$ sufficiently small has no an $m-1$-dimensional submanifold ∂M, which is a border of an m-dimensional compact smooth orientable submanifold M of $\mathbb{R}^n$, such that $H_\varepsilon(\partial M) \subset \partial M$.*

We remark that Theorems 4.3 and 4.4 still remain true when F_h and f are replaced by H_ε and $\bar f$, respectively. Of course, then assumption D1 is again irrelevant, but assumption D5 is considered.

The above results either under conditions D1 and B2, or under conditions D1 and D5, state that if the averaged equation (5.4) of (5.1) satisfies A2 (or A3) then both the T-period mapping F_ε and mapping H_ε, for any $\varepsilon > 0$ sufficiently small, satisfy (2.1) (or (2.2)), respectively. In particular, criteria A2 and A3 precluding the existence of nontrivial periodic solutions of (5.4) also preclude the existence of certain invariant tori for (5.1) with any $\varepsilon > 0$ sufficiently small. This result is an opposite to the well-known result [7] that the existence of a hyperbolic periodic solution of (5.4) gives the existence of a hyperbolic invariant torus of (5.1) for any $\varepsilon > 0$ sufficiently small near the hyperbolic periodic solution.

6. Criteria for mappings with constraints

Let $V \subset \mathbb{R}^n$ be a smooth submanifold with empty border $\partial V = \emptyset$. Let $F \in C^2(\mathbb{R}^n, \mathbb{R}^n)$ be such that $F(V) \subset V$. We are interested in criteria precluding the existence of invariant submanifolds of V for F. So let M be like above, and suppose that $M \subset V$ is also a submanifold of V. Let F satisfy assumption H1. We consider the set

$$\begin{aligned}\mathcal{M}_V = \Big\{\phi \mid & \text{it exists an open neighbourhood } \ U_\phi \subset \mathbb{R}^n \ \text{ of } \ M \\ & \text{such that } \ \phi \in C^2(U_\phi, \mathbb{R}^n), \quad \phi(U_\phi) \subset V, \quad \phi(\partial M) = \partial M \\ & \text{and } \ \phi\colon \partial M \to \partial M \ \text{ is a diffeomorphism}\Big\}.\end{aligned}$$

Clearly $\mathcal{M}_V \neq \emptyset$. A mapping $\phi \in \mathcal{M}_V$ generates a homeomorphism [14]

$$\phi^*_{Vm}(u) \colon \wedge^m T_{\phi(u)} V^* \to \wedge^m T_u M^* \simeq \mathbb{R} \quad \forall\, u \in M\,.$$

Hence $\phi^*_{Vm}(u)$ induces an element $\phi^*_V(u)^{(m)}$ such that

$$\phi^*_V(u)^{(m)} \in \Big(\wedge^m T_{\phi(u)} V^*\Big)^* \to \Big(\wedge^m (\mathbb{R}^n)^*\Big)^* \simeq \mathbb{R}^N,$$

where the embedding

$$i^*_{Vm}(v) \colon \Big(\wedge^m T_v V^*\Big)^* \to \Big(\wedge^m (\mathbb{R}^n)^*\Big)^* \; \forall\, v \in V \tag{6.1}$$

is given by

$$L \in \Big(\wedge^m T_v V^*\Big)^* \to L\Big(\xi / \times_m T_v V\Big) \quad \forall\, \xi \in \wedge^m (\mathbb{R}^n)^*\,.$$

Consequently, we define a continuous mapping $G^V_\phi \colon M \to \mathbb{R}^N$ by

$$G^V_\phi(u) = \phi^*_V(u)^{(m)}\,.$$

We note that this construction of G^V_ϕ coincides with the construction of G_ϕ in Section 2, since it holds that

$$\phi^*_m(u) = \phi^*_{Vm}(u) \circ j^*_{Vm}(\phi(u)) \quad \forall\, u \in M\,,$$

where the homeomorphism $\phi^*_m(u) \colon \wedge^m (\mathbb{R}^n)^* \to \wedge^m T_u M^*$ is generated by mapping $\phi \colon M \to \mathbb{R}^n$, while the homeomorphism $j^*_{Vm}(v) \colon \wedge^m (\mathbb{R}^n)^* \to \wedge^m T_v V^*$ is generated by the inclusion $j_V(v) \colon T_v V \subset \mathbb{R}^n$.

So we put

$$S^V(\phi) = \int_M \big|G^V_\phi(u)\big| \tau\,.$$

Theorem 2.1 still holds for S^V. Since $F \colon V \to V$, we have like above that F generates homeomorphisms

$$F^*_V(v)^{(m)} \colon \Big(\wedge^m T_v V^*\Big)^* \to \Big(\wedge^m T_{F(v)} V^*\Big)^* \quad \forall\, v \in V\,,$$
$$F^*_{Vm}(v) \colon \wedge^m T_{F(v)} V^* \to \wedge^m T_v V^* \quad \forall\, v \in V\,.$$

Hence we have a mapping

$$F^{*(m)}_V \colon \Big(\wedge^m TV^*\Big)^* \to \Big(\wedge^m TV^*\Big)^*. \tag{6.2}$$

The above defined embedding (6.1) given by $i^*_{Vm}(v) \colon \Big(\wedge^m T_v V^*\Big)^* \to \Big(\wedge^m (\mathbb{R}^n)^*\Big)^* \simeq \mathbb{R}^N \;\forall\, v \in V$ induces the norm $|\cdot|_v$ on each $\Big(\wedge^m T_v V^*\Big)^*$, $v \in V$ for a given norm $|\cdot|$ on $\mathbb{R}^N$. Since [14] we know that $\big(F \circ \phi\big)^*_{Vm} = \phi^*_{Vm} \circ F^*_{Vm}$, we get

$$G^V_{F\circ\phi}(u) = F^*_V\big(\phi(u)\big)^{(m)} G^V_\phi(u)\,.$$

Furthermore, F preserves the set $\mathcal{M}_V$. The mapping $DF^{(m)}$ of Section 2 coincides with homeomorphism (6.2) when $V = \mathbb{R}^n$, i.e. $DF^{(m)} = F^{*(m)}_{\mathbb{R}^n}$. It is not hard to see that the following holds

$$i^*_{Vm}(F(v)) \circ F^*_V(v)^{(m)} = DF(v)^{(m)} \circ i^*_{Vm}(v) \quad \forall\, v \in V\,,$$

so $DF^m / \Big(\wedge^m TV^* \Big)^* = F^{*(m)}_V$.

Summarizing, we see that we get a whole analogy with Section 2. So Theorem 2.2 remains true when $DF^{(m)}$ is replaced by $F^{*(m)}_V$ from (6.2). Of course, the computation of $\sup\limits_{v \in V} ||F^*_V(v)^{(m)}||$ is much more involved than like in Section 2 for concrete forms of the norm $|\cdot|$ on $\mathbb{R}^N$ (see Proposition 2.5), so we omit further details and postpone them to another paper.

We remark that autonomous ordinary differential equations possessing several first integrals naturally fit into the framework of this section. We get rough results by restricting criteria of Section 3 on nondegenerate level sets of first integrals for (3.1). To get more precise results, we put

$$\mu_W(A) = \lim_{h \to 0_+} \frac{||(I + hA)/W|| - 1}{h} \tag{6.3}$$

for any $A \in \mathcal{L}(\mathbb{R}^N)$ and any linear subspace $W \subset \mathbb{R}^N$. We note that $\mu_W(A)$ is well-defined (see [4, p. 58]). (6.3) implies that $\mu_W(A) \leq \mu(A)\ \forall\, A \in \mathcal{L}(\mathbb{R}^N)$. Moreover, let $W(t) \subset \mathbb{R}^N$ be a family of linear subspaces and let $A(t) \in \mathcal{L}(\mathbb{R}^N)$ be continuous for $-T \leq t \leq T$ with some $T > 0$. Let $y \in C^1([-T, T], \mathbb{R}^N)$ satisfy

$$\dot{y}(t) = A(t)y(t), \quad y(t) \in W(t) \quad \forall\, t \in (-T, T)\,. \tag{6.4}$$

Then it is not hard to observe (see [4, pp. 58-59]) that it holds

$$|y(t)| \leq \mathrm{e}^{t \times \sup\limits_{t \in [-T,T]} \mu_{W(t)}(A(t))} |y(0)| \quad \forall\, t \in [0, T)\,. \tag{6.5}$$

Let V be invariant for (3.1) satisfying A1, i.e. $\psi(t, V) = V\ \ \forall\, t \in (-2, 2)$. Let $T, 0 < T < 2$ be fixed. Since

$$D\psi(t, v)\colon T_vV \to T_{\psi(t,v)}V \quad \forall\, v \in V\,,$$

we get

$$D\psi(t, v)^{(m)}\colon \Big(\wedge^m T_vV^* \Big)^* \to \Big(\wedge^m T_{\psi(t,v)}V^* \Big)^* \quad \forall\, v \in V\,.$$

Here we consider the embedding i^*_{Vm} and the isomorphism $\Big(\wedge^m (\mathbb{R}^n)^* \Big)^* \simeq \mathbb{R}^N$. So we have

$$D\psi(t, v)^{(m)}\xi \in \Big(\wedge^m T_{\psi(t,v)}V^* \Big)^* \quad \forall\, \xi \in \Big(\wedge^m T_vV^* \Big)^*\,.$$

Moreover, $D\psi(t, v)^{(m)}$ satisfies (3.4). We have $F(x) = \psi(T, x)$ in the notations of Section 3. Hence the above result (6.5) for (6.4) implies that

$$||F^*_V(v)^{(m)}|| \leq \mathrm{e}^{T\eta} \quad \forall\, v \in V$$

for $\eta = \sup_{v\in V} \mu_{\left(\wedge^m T_v V^*\right)^*}\left(Df(v)^{[m]}\right)$. Consequently, condition A2 must be replaced by the assumption

E1 $$\sup_{v\in V} \mu_{\left(\wedge^m T_v V^*\right)^*}\left(Df(v)^{[m]}\right) < 0\,.$$

Similarly for conditions A3 and A4. Now Theorems 3.1 and 3.5 are naturally extended for this section. An extension of Theorem 3.2 depends both on a concrete form of the norm $|\cdot|$ on $\mathbb{R}^N$ and on V. For instance, let us consider the norm $\sqrt{\sum_i x_i^2}$ on $\mathbb{R}^N$. Then $\mu_W(A) = \lambda_1$, where λ_1 is the largest eigenvalue of the symmetric matrix $\frac{1}{2}P_W(A+A^*)/W$ and $P_W\colon \mathbb{R}^N \to W$ is the orthogonal projection. Summarizing, we have the following result.

Theorem 6.1. *Let the ordinary differential equation (3.1) satisfy condition A1 and let a smooth submanifold $V \subset \mathbb{R}^n$ without border be invariant for (3.1). If it holds that*

$$\sup_{v\in V}\Big\{\textit{the largest eigenvalue of the symmetric}$$
$$\textit{matrix}\quad \frac{1}{2}P_v\Big(Df(v)^{[m]} + \Big(Df(v)^{[m]}\Big)^*\Big)\Big/\Big(\wedge^m T_v V^*\Big)^*\Big\} < 0\,,$$

where $P_v\colon \left(\wedge^m (\mathbb{R}^n)^*\right)^* \simeq \mathbb{R}^N \to \left(\wedge^m T_v V^*\right)^*$ is the orthogonal projection, then there is no an $m-1$-dimensional submanifold ∂M, which is a border of an m-dimensional compact smooth orientable submanifold M of V, such that $\psi(T,\partial M) \subset \partial M$ for some $T, 0<|T|<2$.

It would be interesting to derive similar results for another norms on $\mathbb{R}^N$.

References

1. I. BENDIXSON, *Sur les curbes définiés par des équations différentielles*, Acta Math. **24** (1901), 1-88.
2. S.N. BUSENBERG & P. VAN DEN DRIESSCHE, *A method for proving the nonexistence of limit cycles*, J. Math. Anal. Appl. **172** (1993), 463-479.
3. G. BUTLER, R. SCHMID & P. WALTMAN, *Limiting the complexity of limit sets in selfregulating systems*, J. Math. Anal. Appl. **147** (1990), 63-68.
4. W.A. COPPEL, *"Stability and Asymptotic Behavior of Differential Equations"*, Health, Boston, 1965.
5. H. DULAC, *Recherche des cycles limites*, C. R. Acad. Sci. Paris **204** (1937), 1703-1706.
6. B.M. GARAY, *On structual stability of ordinary differential equations with respect to discretization*, Numer. Math. **72** (1996), 449-479.
7. J.K. HALE, *"Ordinary Differential Equations"*, Wiley-Interscience, New-York, 1969.
8. Y. LI & J.S. MULDOWNEY, *On Bendixson's criterion*, J. Differential Equations **106** (1993), 27-39.
9. C. C. McCLUSKEY & J.S. MULDOWNEY, *Bendixson-Dulac criteria for difference equations*, J. Dynamics Differential Equations **10** (1998), 567-575.
10. J.S. MULDOWNEY, *Compound matrices and ordinary differential equations*, Rocky Mountain J. Math. **20** (1990), 857-871.
11. J.A. SANDERS & F. VERHULST, *"Averaging Methods in Nonlinear Dynamical Systems"*, Springer-Verlag, New York, 1985.
12. R.A. SMITH, *An index theorem and Bendixson's negative criterion for certain differential equations of higher dimension*, Proc. Royal Soc. Edinburgh Sect. A **91** (1981), 63-77.

13. R.A. SMITH, *Some applications of Hausdorff dimension inequalities for ordinary differential equations*, Proc. Royal Soc. Edinburgh Sect. A **104** (1986), 235-259.
14. M. SPIVAK, *"Calculus on Manifolds"*, W.A. Benjamin, Inc., New York, 1965.

This preprint series was founded in 1996. Its purpose is to present manuscripts of submitted or unpublished papers and reports of fellows of the Mathematical Institute of the Slovak Academy of Sciences in Bratislava and its Košice branch. The authors are fully responsible for the content of the preprints. Postal addresses:

Mathematical Institute
Slovak Academy of Sciences
Štefánikova 49
SK–814 73 Bratislava
SLOVAKIA

Tel./Fax (+421 7) 5249 7316

Mathematical Institute
Slovak Academy of Sciences
Grešákova 6
SK–040 01 Košice
SLOVAKIA

Tel./Fax (+421 95) 6228291

PostScript files of preprints are available at WWW address:
`http://www.mat.savba.sk/preprints`

Latest preprints of the series:

- **16/1999** *Michal Fečkan*: Criteria on the nonexistence of invariant submanifolds for dynamical systems
- **15/1999** *Anatolij Dvurečenskij*: On Loomis-Sikorski's theorem for MV-algebras and BCK-algebras
- **14/1999** *Otakar Grošek, Ladislav Satko, Karol Nemoga*: ZKDWCNDPDC...
- **13/1999** *Otakar Grošek, Karol Nemoga, Ladislav Satko*: A remark to Luby-Rackoff and Ueli M. Maurer pseudorandom generators
- **12/1999** *Beloslav Riečan*: On the sum of observables in product MV algebras
- **11/1999** *Peter Eliaš*: A hierarchy of thin sets related to the boundedness of trigonometric series
- **10/1999** *Miloslav Duchoň, Peter Maličký*: On Herglotz theorem in partially ordered vector spaces
- **9/1999** *Ján Jakubík*: Torsion classes of specker lattice ordered groups
- **8/1999** *Ján Jakubík*: Graph automorphisms and cells of lattices
- **7/1999** *Miloslav Duchoň, Peter Maličký*: Positive linear maps on trigonometric polynomials
- **6/1999** *Miloslav Duchoň, Anna Kolesárová*: Fourier transforms of vector-valued measures on a certain compact semigroup
- **5/1999** *Miroslav Ploščica*: Separation properties in congruence lattices of lattices
- **4/1999** *Georges Chevalier, Anatolij Dvurečenskij, Karl Svozil*: Piron's and Bell's geometric lemmas and Gleason's theorem
- **3/1999** *Jarmila Hedlíková*: Relatively orthomodular lattices
- **2/1999** *Anatolij Dvurečenskij, Pekka Lahti, Kari Ylinen*: Some remarks on the moment problem
- **1/1999** *Anatolij Dvurečenskij*: Loomis-Sikorski theorem for σ-complete MV-algebras and ℓ-groups
- **27/1998** *Martin Kochol*: Cubic graphs without a Petersen minor have nowhere-zero 5-flows
- **26/1998** *Anatolij Dvurečenskij, Ferdinand Chovanec, Eva Rybáriková*: D-homomorphisms and atomic σ-complete Boolean D-posets
- **25/1998** *Anatolij Dvurečenskij*: Commutative BCK-algebras with product
- **24/1998** *Anatolij Dvurečenskij, Antonio Di Nola*: Product MV-algebras
- **23/1998** *Anatolij Dvurečenskij, Maria Gabriella Graziano*: Dedekind complete commutative BCK-algebras

(continued on inside back cover)

22/1998 *Anatolij Dvurečenskij*: Ideals of commutative BCK-algebras

21/1998 *Anatolij Dvurečenskij, Beloslav Riečan*: Weakly divisible MV-algebras and product

20/1998 *Ivan Korec*: Definability of addition and multiplication from Pascal's triangle modulo n and number-theoretical functions

19/1998 *Roberto Giuntini, Sylvia Pulmannová*: Ideals and congruences in effect algebras and QMV algebras

18/1998 *Ivan Korec*: Definability of exact arithmetical operations from approximate ones

17/1998 *Georges Chevalier, Sylvia Pulmannová*: Some ideal lattices in partial abelian monoids

16/1998 *Anatolij Dvurečenskij*: On categorical equivalences of commutative BCK-algebras

15/1998 *Ivan Korec*: Definability of addition and multiplication from some quadratic forms $ax^2 + cy^2$

14/1998 *Beloslav Riečan*: On the L^p space of observables on product MV algebras

13/1998 *Beloslav Riečan*: On the strong law of large numbers on some ordered structures

12/1998 *Ladislav Mišík, jr., Tibor Žáčik*: A simplified formula for calculation of metric dimension of converging sequences

11/1998 *Beloslav Riečan*: On the joint distribution of observables

10/1998 *Beloslav Riečan*: On the product MV algebras

9/1998 *Roman Frič*: A Stone-type duality and its applications to probability

8/1998 *Roman Frič*: Ring of maps: sequential convergence and completion

7/1998 *Roman Frič, Ján Jakubík*: Sequential convergences on Boolean algebras defined by systems of maximal filters

6/1998 *Roman Frič*: Categorical reflections on fields of events and probability

5/1998 *Beloslav Riečan*: On the probability theory and fuzzy sets

4/1998 *Ivan Korec*: Definability of arithmetical operations from binary quadratic forms

3/1998 *Beloslav Riečan*: On the conditional expectation of observables in MV algebras of fuzzy sets

2/1998 *Beloslav Riečan*: On the conditional probability on MV algebras

1/1998 *Paula B. Cohen, Robin L. Hudson, K. R. Parthasarathy, S. Pulmannová*: Hall's transformation via quantum stochastic calculus

13/1997 *Ivan Korec*: Real-time generation of primes by a one-dimensional cellular automaton with 9 states

12/1997 *Ivan Korec*: Arithmetical operations strongest with respect to the first order definability

11/1997 *Otokar Grošek, Karol Nemoga, Marcel Zanechal*: Why use bijective S-boxes in GOST-algorithm

10/1997 *Beloslav Riečan*: On the L^p space of observables

9/1997 *Ladislav Mišík Jr., Tibor Žáčik*: A formula for calculation of metric dimension of converging sequences

8/1997 *Anatolij Dvurečenskij, Maria Gabriella Graziano*: Commutative BCK-algebras and lattice ordered groups

7/1997 *Anatolij Dvurečenskij, Maria Gabriella Graziano*: On representations of commutative BCK-algebras

6/1997 *Sylvia Pulmannová, Karl Svozil*: Ideals in ortholattices, Bell inequalities and simultaneously definite properties

5/1997 *Camille Debiève, Miloslav Duchoň, Beloslav Riečan*: Moment problem in some ordered vector spaces