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ON THE PROBABILITY THEORY ON MV ALGEBRAS

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Abstract. The paper contains a review of some results concerning probability theory on MV algebras (laws of large numbers, central limit theorem, martingale convergence theorem). Also some algebraic and methodical aspects are discussed.

1. Introduction

Starting by von Neumann model some other possibilities have been developed, e.g., orthomodular posets ([55], [16], [36], [53]). The model we shall speak about is based on the fuzzy sets theory. Of course, the starting point was rather structural, not physical. Anyway, such an approach is quite usual in mathematics and sometimes it appears to be useful in applications, too.

Instead of a probability measure considered is the Kolmogorov model a state is considered in the von Neumann model. A state is a mapping $m\colon \mathcal{L} \to \langle 0,1\rangle$ defined on the family $\mathcal{L}$ of all closed subspaces of a given Hilbert space H satisfying the following two conditions:

(i) $m(H) = 1$.

(ii) If $(A_n) \subset \mathcal{L}$ and A_n are pairwise orthogonal, then

$$m\Big(\bigvee_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} m(A_n)\,.$$

(Here $\bigvee_{n=1}^{\infty} A_n$ is the closed subspace of H generated by $\bigcup_{n=1}^{\infty} A_n$).

The Piasecki concept of P-measure ([34]) was one of the first concept in the fuzzy probability theory. Consider a family $\mathcal{F}$ of fuzzy subsets of a given set X, i.e., a family $\mathcal{F}$ of all functions $f\colon C \to \langle 0,1\rangle$. Assume that $\mathcal{F}$ is closed under the operation $f \mapsto 1-f$ and under the countable supremums. A mapping $m\colon \mathcal{F} \to \langle 0,1\rangle$ is called a P-measure, if the following two conditions are satisfied:

(i) $m\big(f \vee (1-f)\big) = 1$ for any $f \in \mathcal{F}$.

(ii) If $(f_n) \subset \mathcal{F}$ and f_n are pairwise orthogonal (i.e., $f_n \le f_m^{\perp} = 1 - f_m$ for $n \ne m$), then $m\left(\bigvee_{n=1}^{\infty} f_n\right) = \sum_{n=1}^{\infty} m(f_n)$.

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It seems to be natural to use the fuzzy measure concept for the construction of a mathematical model of quantum mechanical systems. It was suggested in [39] (so-called F-quantum space) and it was realised in many papers, mainly of Slovak school (e.g., [5], [9], [19], [27], [28], [40], [41]) but also of some foreign authors (Le Ba, Navara, Piasecki, Pták, Pykacz, see [25], [31], [32], [35], [37]). A more important is the fact that the model can be reduced by a quotient process to the Boolean algebra case ([8], [28], [35]).

Of course, there are many possibilities how to define fuzzy connectives and fuzzy operations, e.g.

$$\chi_{A\cap B} = \chi_A \wedge \chi_B\,,\ \chi_{A\cup B} = \chi_A \vee \chi_B$$
$$\chi_{A\cap B} = \chi_A \cdot \chi_B\,,\ \chi_{A\cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B$$
$$\chi_{A\cap B} = \big(\chi_A + \chi_B = 1\big) \vee 0\,,\ \chi_{A\cup B} = \big(\chi_A + \chi_B\big) \wedge 1$$

and continuum other possibilities. In the model mentioned above we used the first pair – the Zadeh connectives (minimum, maximum). In the present paper we use the third pair – the Lukasiewicz connectives

$$f \odot g = (f + g - 1) \vee 0\,,$$
$$f \oplus g = (f + g) \wedge 1\,.$$

What the additivity of a state $m\colon \mathcal{F} \to \langle 0, 1\rangle$ says. If f and g are disjoint events, then the value m of the union of f and g equals the sum of $m(f)$ and $m(g)$. Using Lukasiewicz connectives, we obtain

$$m(f \oplus g) = m(f) + m(g)$$

provided f, g disjoint. The disjointness means that

$$f \odot g = (f + g - 1) \vee 0 = 0\,,$$

hence

$$f + g - 1 \le 0$$

or

$$f \le 1 - g = g^*\,.$$

Of course, under this assumption

$$f \oplus g = (f + g) \wedge 1 = f + g\,.$$

Therefore the additivity of m means that the following implication hold:

$$f + g \le 1 \Rightarrow m(f + g) = m(f) + m(g)\,.$$

Another formulation: If $f, g, h \in \mathcal{F}$ and $h = f + g$, then $m(h) = m(f) + m(g)$.

The Lukasiewicz approach to quantum mechanical systems was suggested implicitly in [40] and explicitly in [4]. In the paper we give a review of results concerning the probability theory based on families of fuzzy sets with Lukasiewicz connectives and its natural algebraic generalizations, mainly MV algebras. Particularly we mention recent results and ideas appeared after publishing the book [53] which presents a basic reference.

2. Probability domains

Although our main attention will be concentrate in MV algebras, we shall consider also some more general structures. This Section contains a review of such structures. Of course, our starting example there will be an MV algebra of fuzzy sets.

The MV algebra of fuzzy sets is a prototype of a general MV algebra. Consider a family $\mathcal{F}$ of fuzzy sets $f\colon X \to \langle 0,1\rangle$. We say that $\mathcal{F}$ is an MV algebra, if the following implications are satisfied:

(i) If $f, g \in \mathcal{F}$, then $f \oplus g = \min(f+g, 1) \in \mathcal{F}$.

(ii) If $f, g \in \mathcal{F}$, then $f \odot g = \max(f+g-1, 0) \in \mathcal{F}$.

(iii) If $f \in \mathcal{F}$, then $f^* = 1 - f \in \mathcal{F}$.

A motivating example for the operation $\oplus$ is the composition $f \oplus g$ of two grew pictures $f\colon X \to \langle 0,1\rangle$, $g\colon X \to \langle 0,1\rangle$. This composition cannot be more grew than the black colour 1, therefore $f \oplus g = \min(f+g, 1)$. On the other hand, the operation $\odot$ can be obtained by the de Morgan rule:

$$\begin{aligned} f \odot g &= (f^* \oplus g^*)^* = 1 - \big((1-f) \oplus (1-g)\big) \\ &= 1 - \min(1-f+1-g\,, 1) = \max(f+g-1, 0)\,. \end{aligned}$$

Evidently $f \oplus f^* = \min(f+1-f, 1) = 1$, hence $1_X \in \mathcal{F}$, and similarly $0_X = 1_X^* \in \mathcal{F}$. A family $\mathcal{F}$ satisfying the above conditions is called (in a modern terminology [22]) a T_L-tribe (L on the honour of Lukasiewicz). One of the most interesting results concerning T_L-tribes is the following theorem by Butnariu and Klement ([2], see also [53]). If $\mathcal{F}$ contains all constant functions c_X, $c \in \langle 0,1\rangle$, then $\mathcal{F}$ is exactly the set of all functions $f\colon X \to \langle 0,1\rangle$ measurable with respect to a σ-algebra of subsets of X.

Generally an MV algebra ([3], [18], [30]) is an algebraic system $(M, \oplus, \odot, *, 1, 0)$, where M is a non-empty set, $\oplus, \odot$ are binary operations, $*$ is a unary operation, 1 and 0 are fixed elements satisfying the following properties:

(i) $a \oplus b = b \oplus a$

(ii) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$

(iii) $a \oplus 0 = 1$

(iv) $a \oplus 1 = 1$

(v) $(a^*)^* = a$

(vi) $0^* = 1$

(vii) $a \oplus a^* = 1$

(viii) $(a^* \oplus b)^* \oplus b = (a \oplus b^*)^* \oplus a$

(ix) $a \odot b = (a^* \oplus b^*)^*$.

If we define $a \vee b = (a^* \oplus b)^* \oplus b$ and $a \leq b$ if and only if $a \vee b = b$, then $(M, \leq)$ is a partially ordered set being a distributive lattice and $a \vee b$ is the least upper bound of a, b.

Of course, by the Mundici theorem [30] any MV algebra can be represented by a commutative l-group. Recall that a commutative l-group is an algebraic system

$(G, +, \leq)$, where $(G, +)$ is a commutative group, $(G, \leq)$ is a partially ordered set being a lattice and $a \leq b$ implies $a + c \leq b + c$. Let $(G, +, \leq)$ be a commutative l-group, 0 be the neutral element of $(G, +)$ and $u \in G$, $u > 0$. On the interval $\langle 0, u\rangle$ define the following operations:

$$\begin{aligned} a \oplus b &= (a + b) \wedge u\,, \\ a \odot b &= (a + b - u) \vee 0\,, \\ a^* &= u - a\,. \end{aligned}$$

Then $\mathscr{M}G\big(\langle 0, u\rangle, \oplus, \odot, *, u, 0\big)$ becomes an MV algebra. The Mundici theorem states that to any MV algebra there exists a commutative l-group G with a strong unit u (i.e., to any $a \in G$ there exists $n \in N$ such that $a \leq nu$) such that

$$M \simeq \mathscr{M}G\,.$$

Some methods which has been originally used in MV algebras of fuzzy sets can be surprisely applied also in a very general case. The most general is probably the case of the structure $(M, \leq, +, 0, 1)$, where

(i) $(M, \leq)$ is a partially ordered set with the least element 0 and the greatest element 1,

(ii) $+$ is a partial, commutative binary operation on M.
We shall call such a structure (only for purposes of this paper) a Q-support.

Let us mention some examples of Q-supports.

Evidently any MV algebra $\mathcal{F}$ of fuzzy sets is a Q-support. Here $\leq$ is the usual ordering of functions, 0_X is the least 1_X is the greatest element of $\mathcal{F}$, $+$ is the usual sum of functions. Of course, $f + g$ is defined as an element of $\mathcal{F}$, if and only if $f + g \leq 1_X$, or equivalently, if $f \leq g^* = 1 - g$. Therefore $+$ is in $\mathcal{F}$ only a partial binary operation.

Also any MV algebra $(M, \oplus, \odot, *, 0, 1)$ is an example of Q-support. It is a partially ordered set, where 0 is the least element and 1 is the greatest element. Further define $a + b$ if and only if $a \leq b^*$ by the formula $a + b = a \oplus b$. If we consider the Mundici representation of M by a commutative l-group G, where $a \oplus b = (a+b) \wedge u$, then $+$ is the corresponding group operation. Of course, $a + b \in M$ if and only if $a + b \leq u$, i.e., $a \leq b^*$ and in this case $a \oplus b = (a + b) \wedge u = a + b$.

The third example of Q-support, we mention here explicitly, is so-called D-poset. It was introduced by Kôpka and Chovanec ([24], see also [6]) and independently by Giuntini and Greuling ([15]) and it is very close to orthoalgebras defined by Foulis, Greechie and Rüttiman ([13]). A D-poset is an algebraic structure $(D \leq, \setminus, 0, 1)$, where

(i) $(D, \leq, 0, 1)$ is a partially ordered set with the least element 0 and the greatest element 1.

(i) $\setminus$ is a partial binary operation on D, where $a \setminus b$ is define if and only if $b \leq a$.

(iii) $\setminus$ satisfies the following two implications:

$$\begin{aligned} &a \leq b \Rightarrow b \setminus a \leq b \quad \text{and} \quad b \setminus (b \setminus a) = a\,, \\ &a \leq b \leq c \Rightarrow c \setminus b \leq c \setminus a \quad \text{and} \quad (c \setminus a) \setminus (c \setminus b) = b \setminus a\,. \end{aligned}$$

In any D poset a partial binary operation $+$ can be introduced defining $a+b$ if and only if $a \leq 1 \setminus b$ (what is the same as $b \leq 1 \setminus a$) by the formula

$$\begin{aligned} a+b &= 1 \setminus \big[(1 \setminus a) - b\big] \\ &= 1 \setminus \big[(1 \setminus b) \setminus a\big] = b + a\,. \end{aligned}$$

Also any orthomodular poset $(P, \leq, \perp, 0, 1)$ is a Q-support, if we define $a + b = a \vee b$, whenever $a \leq b^{\perp}$.

Finally recall that any F-quantum space is a Q-support. Again $f+g$ is defined if and only if $f \leq g^* = 1 - g$ and in this case $f + g = f \vee g$.

3. States and observables

As we have mentioned yet, the notion of a state is an analogy of the notion of a probability measure in the Kolmogorov model. The state can be defined also in the Q-support case.

Definition 3.1. If $M = (M, \leq, +, 0, 1)$ is a Q-support, then a state is a mapping $m \colon M \to \langle 0, 1\rangle$ satisfying the following three conditions:

(i) $m(1) = 1$.

(ii) If $a, b, c \in M$ and $a = b + c$ (i.e., $b + c$ exists and equals a), then $m(a) = m(b) + m(c)$.

(iii) If $a_n \nearrow a$, then $m(a_n) \nearrow m(a)$.

Let us examine the second property – additivity on some examples. In any MV algebra M it means that

$$b + c \text{ defined } \quad \Rightarrow m(b+c) = m(b) + m(c)\,.$$

But $b+c$ is defined, if $b \leq c^* = u - c$. In this case $b \oplus c = (b+c) \wedge 1 = b + c$. Hence we obtain the implication

$$b \leq c^* \Rightarrow m(b+c) = m(b) + m(c)\,. \tag{2.1}$$

In D-posets the additivity is usually defined by the implication

$$a \leq b \Rightarrow m(b) = m(a) + m(b \setminus a)\,. \tag{2.2}$$

We shall show that the implications (2.1) and (2.2) are equivalent.

Let (2.2) be satisfied and let $b \leq 1 \setminus c$. Then $b + c = 1 \setminus \big[(1 \setminus b) \setminus c\}$, hence

$$\begin{aligned} m(b+c) &= m(1) - m\big((1 \setminus b) \setminus c\big) \\ &= m(1) - m(1 \setminus b) + m(c) \\ &= m(1) - m(1) + m(b) + m(c)\,. \end{aligned}$$

An the other hand, let (2.1) be satisfied and let $a \leq b = b \setminus (b \setminus a) \leq 1 \setminus (b \setminus a) = (b \setminus a)^*$. Then by (2.1)

$$m\big(a + (b \setminus a)\big) = m(a) + m(b \setminus a)\,.$$

But $a + (b \setminus a) = 1 \setminus \big[(1 \setminus a) \setminus (b \setminus a)\big] = 1 \setminus (1 \setminus b) = b$.

In orthomodular posets, the additivity means

$$a \le b^{\perp} \Rightarrow m(a \vee b) = m(a) + m(b)$$

and analogous implication holds in F-quantum spaces.

Let us mention an excellent result by Butnariu and Klement ([2]) stating that any state $m\colon \mathcal{F} \to \langle 0,1\rangle$ defined on the MV algebra (i.e., T_L-tribe) of all measurable functions can by represented by an integral, i.e., $m(f) = \int f\,dP$, $f \in \mathcal{F}$, where P is a probability measure on the considered σ-algebra.

We have seen that the notion of a state on a Q-support is a generalization of the notion of a state on many studied quantum mechanical models. A similar situation is also in the case of an observable.

Recall that the notion of an observable is a generalization of the notion of a random variable. If $(\Omega, \mathcal{S}, P)$ is a probability space, then a random variable is an $\mathcal{S}$-measurable mapping $\xi\colon \Omega \to R$. If follows that $\xi^{-1}(E) \in \mathcal{S}$ for any $E \in \mathcal{B}(R)$. Therefore we obtain a mapping $x\colon \mathcal{B}(R) \to \mathcal{S}$ defined by the formula

$$x(E) = \xi^{-1}(E)\,.$$

Definition 3.2. Let M be a Q-support. A weak observable (on M with respect to a state m on M) is a mapping $x\colon \mathcal{B}(R) \to M$ satisfying the following conditions:

(i) $m\big(x(R)\big) = 1$.

(ii) If $A \cap B = \emptyset$, ten $x(A \cup B) = x(A) + x(B)$.

(iii) If $A_n \nearrow A$, then $x(A_n) \nearrow x(A)$.

A weak observable is observable, if $x(R) = 1$.

Theorem 3.1. *If M is a Q-support, $m\colon M \to \langle 0,1\rangle$ is a state and $x\colon \mathcal{B}(R) \to M$ is a weak observable, then the composite mapping $m_x = m \circ x\colon \mathcal{B}(R) \to \langle 0,1\rangle$ is a probability measure.*

Proof. Evidently $m_x(R) = m\big(x(R)\big) = 1$. If $A \cap B = \emptyset$, then $m_x(A \cup B) = m\big(x(A \cup B)\big) = m\big(x(A) + x(B)\big) = m_x(A) + m_x(B)$. We see that m_x is additive and monotone. Since m_x is also lower continuous, m_x is σ-additive, too.

The probability measure m_x corresponds to the notion of the probability distribution of a random variable in the Kolmogorov theory. Indeed, if $\xi\colon \Omega \to R$ is a random variable, then its probability distribution is the mapping $P_\xi\colon \mathcal{B}(R) \to \langle 0,1\rangle$ defined by the formula

$$P_\xi(A) = P\big(\xi^{-1}(A)\big)\,.$$

By the help of P_ξ all important probability characterizations of ξ can be described, e.g., the mean value is

$$E(\xi) = \int_\Omega \xi\,dP = \int_R t\,dP_\xi(t)\,,$$

the dispersion is

$$\begin{aligned}\sigma^2(\xi) &= \int_R \big(\xi - E(\xi)\big)^2 dP = \int_R \big(t - E(\xi)\big)^2 dP_\xi(t)\\ &= \int_R t^2\,dP_\xi(t) - E(\xi)^2\end{aligned}$$

etc.

Definition 3.3. A weak observable $x\colon \mathcal{B}(R) \to M$ belongs to L^1 (by other words, it is integrable), if there exists $E(x) = \int_R t\,dm_x(t)$. It belongs to L^2, if there exists $\int_R t^2\,dm_x(t)$.

Proposition 3.1. $L^2 \subset L^1$.

Proof. It follows by the Cauchy inequality:

$$\left|\int_R t \cdot 1 dm_x(r)\right| \leq \sqrt{\int_R t^2\,dm_x(t)}\sqrt{\int_R 1\,dm_x(t)}\,.$$

Definition 3.4. If $x\colon \mathcal{B}(R) \to M$ is a weak observable belonging to L^1, then we define

$$E(x) = \int_R t\,dm_x(t)\,.$$

If, moreover, x belongs to L^2, then we define

$$\sigma^2(x) = \int_R \big(t - E(x)\big)^2 dm_x(t)\,.$$

4. Weak independence

Limit theorems of probability theory deals with sequences of observables (random variables). Also in our general case we need to define independence of observables.

Let us return to the Kolmogorov model. Consider two random variables ξ, η. They induce a random vector $T = (\xi, \eta)\colon \Omega \to R^2$, the corresponding σ-homomorphism

$$h\colon \mathcal{B}(R^2) \to \mathcal{S}, A \mapsto T^{-1}(A)$$

and the probability distribution $P_T\colon \mathcal{B}(R^2) \to \langle 0, 1\rangle$ of the random vector T defined by

$$P_T(A) = P\big(T^{-1}(A)\big) = P\big(h(A)\big)\,.$$

Random variables ξ, η are independent, if

$$P\big(\xi^{-1}(A) \cap \eta^{-1}(B)\big) = P\big(\xi^{-1}(A)\big)\,P\big(\eta^{-1}(B)\big)$$

for any $A, B \in \mathcal{B}(R)$. Since $\xi^{-1}(A) \cap \eta^{-1}(B) = T^{-1}(A \times B)$, the last equality is equivalent with the equality

$$\begin{aligned} P_T(A \times B) &= P_\xi(A) \cdot P_\eta(B) \\ &= P_\xi \times P_\eta(A \times B)\,, \end{aligned}$$

where $P_\xi \times P_\eta$ is the product of measures P_ξ, P_η. Since P_T, $P_\xi \times P_\eta$ coincide on measurable rectangles, they coincide on $\mathcal{B}(R^2)$, hence $P_T = P_\xi \times P_\eta$. Since $P_T = P \circ h$, we obtain the following characterization of independency of ξ, η. Two random variables $\xi, \eta\colon \Omega \to R$ are independent if and only if there exists a σ-homomorphism $h\colon \mathcal{B}(R^2) \to \mathcal{S}$ such that

$$P \circ h = P_\xi \times P_\eta\,.$$

This formulation can be successfully applied in the very general case of Q-support.

Definition 4.1. Let M be a Q-support, $x_1, \dots, x_n \colon \mathcal{B}(R) \to M$ be weak observables. We say that they are strongly independent, if there exists a mapping $h_n \colon \mathcal{B}(R^n) \to M$ satisfying the following properties:

(i) $m\big(h_n(R^n)\big) = 1$.

(ii) If $A, B \in \mathcal{B}(R^n)$, $A \cap B = \emptyset$, then $h_n(A \cup B) = h_n(A) + b_n(B)$.

(iii) If $A_i \in \mathcal{B}(R^n)$ $(i = 1, 2, \dots)$, $A_i \nearrow A$, then $h_n(A) \nearrow h_n(A)$ $(i \to \infty)$.

(iv) $m \circ h_n = m_{x_1} \times m_{x_2} \times \cdots \times M_{x_n}$.

Of course, the probability distribution gives also a possibility to built a calculus with observables. Namely, if $T = (\xi, \eta)$ is a random vector, then

$$\xi + \eta = g(\xi, \eta) = g \circ T\,,$$

where $g \colon R^2 \to R$, $g(u, v) = u + v$. Then

$$(\xi + \eta)^{-1}(A) = T^{-1}\big(g^{-1}(A)\big)\,.$$

Therefore the probability distribution of $\xi + \eta$ is given by the formula

$$P_{\xi+\eta} = P_T \circ g^{-1},$$

hence, if ξ, η are independent (i.e., $P_T = P_\xi \times P_\eta$), then

$$P_{\xi+\eta} = P \circ (\xi + \eta)^{-1} = P_T \circ g^{-1} = (P_\xi \times P\eta) \circ g^{-1}.$$

This property justifies the following definition.

Definition 4.2. Weak observables $x_1, \dots, x_n \colon \mathcal{B}(R) \to M$ are weakly independent, if there exists a weak observable $y_n \colon \mathcal{B}(R) \to M$ such that

$$m \circ y_n = \big(m_{x_1} \times \cdots \times m_{x_n}\big) \circ g_n^{-1},$$

where $g_n \colon R^n \to R$ is defined by $g_n(u_1, \dots, u_n) = u_1 + \cdots + u_n$.

Evidently, if $x_1, \dots, x_n$ are strongly independent, then they are weakly independent, too. Indeed, put $y_n = h_n \circ g_n^{-1}$. Then

$$\begin{aligned} m \circ y_n &= m \circ h_n \circ g_n^{-1} \\ &= m_{x_1} \times \cdots \times m_{x_n} \circ g_n^{-1}. \end{aligned}$$

It is surprising that under weak independency on a Q-support the weak law of large numbers and the central limit theorem can be proved. Of course, they work with expressions of the form

$$\frac{1}{n} \sum_{i=1}^{n} x_i - a\,.$$

The sum $\sum_{i=1}^n x_i$ can be identify with the observable y_n from the definition of weak independency. Of course, it need not be determine uniquely, but if we have two such observables, they have the same probability distribution and this fact is satisfactory from the point of view of probability theory. Further definitions are clear:

$$\big(\frac{1}{n} y_n\big)(B) = y_n(nB)\,,$$

where $nB = \{nb; b \in B\}$ and

$$(z - a)(B) = z(a + B)\,,$$

where $a + B = \{a + b; b \in B\}$.

Theorem 4.1. *(Law of large numbers). Let (x_n) be a weakly independent sequence of weak observables of L^1 with the same probability distribution $m_{x_n} = \mu$ $(n = 1, 2, \dots)$. Denote $E(x_n) = a$ $(n = 1, 2, \dots)$. Then for every $\varepsilon > 0$ there holds*

$$\lim_{n\to\infty} m\Big(\Big(\frac{1}{n}\sum_{i=1}^{n} x_i - a\Big)(-\varepsilon, \varepsilon)\Big) = 1 .$$

Theorem 4.2. *(Central limit theorem). Let (x_n) be a weakly independent sequence of weak observables of L^2 with the same probability distribution $m_{x_n} = \mu$ $(n = 1, 2, \dots)$, $E(x_n) = a$, $\sigma^2(x_n) = \sigma^2 > 0$ $(n = 1, 2, \dots)$. Then*

$$\lim_{n\to\infty} m\Big(\frac{\sqrt{n}}{\sigma}\Big(\frac{1}{n}\sum_{i=1}^{n} x_i - a\Big)\big((-\infty, t)\big)\Big) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-\frac{u^2}{2}}\, du .$$

We shall not to prove the theorems stated above, because the details have been published in [46]. Of course, we shall mention here the main idea: local representation of observables by random variables. Let (x_n) be a sequence of weakly independent observables. To the concrete sequence we construct a probability space $(\Omega, \mathcal{S}, P)$ by the following way:

$\Omega = R^n$, $\mathcal{S} = \sigma(\mathcal{C})$, where $\mathcal{C}$ is the family of all sets of the form $\{t \in R^N ; h \in A_1, \dots, t_n \in A_n\}$, where $n \in N$, $A_i \in \mathcal{B}(R)$, and $P = \prod_{i=1}^{\infty} m_{x_i}$ is the product measure characterized by the equalities

$$P\big(\{t \in R^N ; t_1 \in A_1, \dots, t_n \in A_n\}\big) = m_{x_1}(A_1) \cdot \ldots \cdot m_{x_n}(A_n) .$$

If we define $\xi_n \colon R^N \to R$ as the coordinate random variable $\xi_n(t) = t_n$, then (ξ_n) is a sequence of independent random variables, $P_{\xi_n} = m_{x_n}$ $(n = 1, 2, \dots)$ and moreover, the probability distribution of the random variable $\left(\frac{1}{n}\sum_{i=1}^{n} \xi_i - a\right)$ and the observable $\left(\frac{1}{n}\sum_{i=1}^{n} x_i - a\right)$ coincide. This fact guarantees the convergence in measure as well as the convergence in distribution.

Something another is the situation with respect to the convergence almost everywhere. For this type of convergence we need the strong independency.

5. Strong independence

If $x_1, \dots, x_n$ are strongly independent, then we have also their so-called joint distribution $h_n \colon \mathcal{B}(R^n) \to M$. Recall that $m \circ h_n = m_{x_1} \times \cdots \times m_{x_n}$. By the help of h_n we are able to construct not only the sum $\sum_{i=1}^{n} x_i$, but also the value of any function $g(x_1, \dots, x_n)$. It can be defined by the help of the equality

$$g(x_1, \dots, x_n)(A) = h_n\big(g^{-1}(A)\big) .$$

Namely, if $T = (\xi_1, \dots, \xi_n)$ is a random vector, then

$$g(\xi_1, \dots, \xi_n) = g \circ T ,$$

hence

$$(g \circ T)^{-1} = T^{-1} \circ g^{-1},$$

and the joint observable h_n in our case corresponds to the random vector T.

So $\frac{1}{n}\sum_{i=1}^{n} x_i - a = g_n(x_1, \dots, x_n)$, where $g_n \colon R^n \to R$ is defined by $g_n(u_1, \dots, u_n) = \frac{1}{n}\sum_{i=1}^{n} u_i - a$.

A more important is the problem of almost everywhere convergence. If (η_n) is a sequence of random variables, then η_n converges P-almost everywhere to 0, if

$$P\Big(\bigcap_{p=1}^{\infty}\bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty}\eta_n^{-1}\Big(\Big(-\frac{1}{p},\frac{1}{p}\Big)\Big)\Big) = 1,$$

or equivalently

$$\lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} P\Big(\bigcap_{i=1}^{k+i}\eta_n^{-1}\Big(\Big(-\frac{1}{p},\frac{1}{p}\Big)\Big)\Big) = 1.$$

Definition 5.1. Let a Q-support $(M, \leq, +)$ be a lattice. A sequence (y_n) of observables converges m-a.e., to 0, if

$$\lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} P\Big(\bigwedge_{n=k}^{k+i} y_n\Big(\Big(-\frac{1}{p},\frac{1}{p}\Big)\Big)\Big) = 1.$$

Theorem 5.1. *Let $(M, \leq, +)$ be a Q-support being a lattice. Let (x_n) be a strongly independent sequence of weak observables from L^2. Let $\sum_{n=1}^{\infty}\frac{\sigma^2(x_n)}{n^2} < \infty$. Then*

$$\Big(\frac{1}{n}\sum_{i=1}^{n}(x_i - E(x_i))\Big)_{n=1}^{\infty}$$

converges m-a.e., to 0.

Again we shall not to discuss the details, they are contained in [52]. We shall mention only the key point. If $\eta_n = g_n(\xi_1, \dots, \xi_n)$ and $y_n = g_n(x_1, \dots, x_n)$, then

$$P\Big(\bigcap_{n=k}^{k+i}\eta_n^{-1}\Big(\Big(-\frac{1}{p},\frac{1}{p}\Big)\Big)\Big) \leq m\Big(\bigwedge_{n=k}^{k+i} y_n\Big(\Big(-\frac{1}{p},\frac{1}{p}\Big)\Big)\Big). \tag{5.1}$$

Therefore, if the left side of the inequality (5.1) converges to 1, then the right side converges to 1, too.

Let us return to the case of MV algebra $\mathcal{F}$ of fuzzy sets. In this case the joint observable exists for any observables x, y ([29]). This result can be easily adapted for the case of weak observables. It was realised in [44], of course, the proof will be repeated in the paper:

Theorem 5.2. *([44], Theorem 2). Let $\mathcal{F}$ be an MV algebra of fuzzy sets. Let $x, y\colon \mathcal{B}(R) \to \mathcal{F}$ be weak observables. Then there exists a mapping $h\colon \mathcal{B}(R^2) \to \mathcal{F}$ satisfying the following conditions:*

(i) $m\big(h(R^2)\big) = 1$.
(ii) *If* $A, B \in \mathcal{B}(R^2)$. $A \cap B = \emptyset$, *then* $h(A \cup B) = h(A) + h(B)$.
(iii) *If* $(A_n) \subset \mathcal{B}(R^2)$, $A_n \nearrow A$, *then* $h(A_n) \nearrow h(A)$.
(iv) $h(C \times D) = x(C) \cdot y(D)$ *for any* $C, D \in \mathcal{B}(R)$.

Proof. Fix $\omega \in \Omega$ and define $\mu_\omega, \nu_\omega\colon \mathcal{B}(R) \to \langle 0, 1\rangle$ by the formulas

$$\mu_\omega(A) = x(A)(\omega)\,, \;\; \nu_\omega(A) = y(A)(\omega)\,.$$

Since μ_ω, ν_ω are finite measures, there exists their product $\mu_\omega \times \nu_\omega\colon \mathcal{B}(R^2) \to \langle 0, 1\rangle$. Define $h\colon \mathcal{B}(R^2) \to \mathcal{F}$ by the formula

$$h(A)(\omega) = \mu_\omega \times \nu_\omega(A)\,.$$

It is not difficult to prove the conditions (ii) – (iv). We prove (i). First $m(f) = 1$ implies $m(f \cdot g) = m(g)$. Indeed, $m(f) = 1$ implies $m(f^*) = 1 - m(f) = 0$, hence also $m(f^* \cdot g) = 0$ and therefore $m(g) = m(g \cdot f) + m(g \cdot f^*) = m(g \cdot f)$. Since $m\big(x(R)\big) = 1$, we obtain

$$m\big(h(R^2)\big) = m\big(x(R) \cdot y(R)\big) = m\big(y(R)\big)\,.$$

In an MV algebra $\mathcal{F}$ of fuzzy sets the independency has been introduced as the p-independency (product independency):

$$m\big(h(C \times D)\big) = m\big(x(C)\big) \cdot m\big(y(D)\big)\,.$$

We see that in this case p-independency in equivalent to the strong independency, so that all results of Sections 4 and 4 holds.

6. Sum of observables

We have been that if we have joint observable (i.e., such a σ-morphism h defined on $\mathcal{B}(R^2)$ that $h(C \times D) = x(C) \cdot y(D)\big)$, then it is easy to construct the sum $x + y$:

$$(x + y)(A) = h\big(g^{-1}(A)\big)\,,$$

where $g(u, v) = u + v$. Of course, we can use this formula also in a very general case of Q-support, but only with a hypothetic assumption that there exists h with

$$m \circ h \circ g^{-1} = m_x \times m_y\,.$$

But only in the case of MV algebra of fuzzy sets ($= T_L$-tribe) we are able to prove that the joint observable h exists. The reason is the following. Any T_L-tribe $\mathcal{F}$ is closed under the product ([22], [53]): $f, g \in \mathcal{F} \Rightarrow f \cdot g \in \mathcal{F}$. Moreover, if $f + g \in \mathcal{F}$, $k \in \mathcal{F}$, then $k \cdot (f + g) = k \cdot f + k \cdot g$.

In a general MV algebra we have not the product of elements, only the Lukasiewicz product $\odot$. Unfortunately, $\oplus$ and $\odot$ do not satisfy the distributive law. We have only the semidistributive law:

$$(a + b) \odot c \geq a \odot c + b \odot c\,.$$

On the other hand, also in a general MV algebra the sum of two observables has been constructed ([53], [45]), The idea of the construction is simple, because it is based on the following equality:

$$\big\{(u, v,) \in R^2;\ u + v < t\big\} = \bigcup_{n=1}^{\infty} \bigcup_{(i,j)\in\alpha_n} A_i^n \times A_j^n\,,$$

where $A_i^n = \left\langle \frac{i-1}{2^n}, \frac{1}{2^n}\right)$, $\alpha_n = \big\{(i, j);\ i + j < 2^n t\big\}$. Of course, the realisation is quite complicated because of the absence of a good distributive law. We shall not mention here details (they have been published in the book [53]). We shall only present here two conditions guaranting the existence of the sum $x + y$:

Definition 6.1. The weak observables x, y are $\odot$-independent, if

$$m\big(x(A) \odot y(B)\big) = m\big(x(A)\big)\, m\big(y(B)\big) \quad \text{for any} \quad A, B \in \mathcal{B}(R)\,.$$

Definition 6.2. The MV algebra M is weakly σ-distributive, if it is σ-complete (any subset of M has the supremum) and for any bounded double sequence $(a_{ij}) \subset M$ such that $a_{ij} \searrow 0$ ($j \to \infty$, $i = 1, 2, \dots$) we have

$$\bigwedge_{\varphi\in N^N} \bigvee_{i=1}^{\infty} a_{i\varphi(i)} = 0\,.$$

Recall that the weak σ-distributivity is a necessary and sufficient condition for any G-valued measure could be extended from an algebra of sets to the generated σ-algebra, of course, in the case that M is a Riesz space.

Anyway, the mentioned sum of observables (we shall denote it by $x \oplus y$) satisfies the following equality:

$$m\big((x \oplus y)(-\infty, t)\big) = m_x \times m_y\big(\{(u, v);\ u + v < t\}\big)\,.$$

By this formula and the results of Section 4 the Weak Law of Large Numbers and the Central Limit Theorem follows (for sequences of $\odot$-independent observables). Moreover, the Strong Law of Large Numbers holds ([45]):

Theorem 6.1. *Let M be a weakly σ-distributive MV algebra. Let (x_n) be $\odot$-independent sequence of weak observables of L^2. Put $E(x_n) = a_n$. Let $\sum_{n=1}^{\infty} \frac{\sigma^2(x_n)}{n^2} < \infty$. Then*

$$\frac{1}{n}\bigoplus_{i=1}^{n} x_i - \frac{1}{n}\sum_{i=1}^{n} a_i \to 0$$

m-almost everywhere.

Recall that in a general MV algebra M we have obtained two versions of the strong law of large numbers. The first holds in an arbitraty MV algebra M (Theorem 5.1), but with a strong independency of considered sequence of observables. In the second case only $\odot$-independency is assumed (not the existence of a convenient morphism $h\colon \mathcal{B}(R^2) \to M$), but we have some more restrictive conditions on M (weak σ-distributivity).

Let us return to the case of an MV algebra of fuzzy sets. In this case we have introduced two definitions of the sum of observables. The first has been defined by the help of the joint observable h:

$$x + y = h \circ g^{-1},$$

where $g(u, v) = u + v$, hence

$$(x + y)(-\infty, t) = h\big(\{(u, v);\ u + v < t\}\big).$$

The second definition hase been defined for $\oplus$-independent observables x, y, but again

$$m\big((x \oplus y)(-\infty, t)\big) = m_x \times m_y\big(\{(u, v,);\ u + v < t\}\big).$$

On the other hand, if x, y are p-independent (i.e., $m\big(x(A)\cdot y(B)\big) = m_x(A)\cdot m_y(B)$), then

$$\begin{aligned} m\big((x + y)(-\infty, t)\big) &= m\big(h(\{u, v);\ u + v < t\})\big) \\ &= m_x \times m_y\big(\{(u, v);\ u + v < t\}\big). \end{aligned}$$

We see that $x + y$ and $x \oplus y$ have the same probability distribution provided x and y are $\odot$-independent as well as $\odot$-independent. Moreover, if x, y are $\odot$-independent, then they are p-independent, too ([44], theorem 5).

7. Conditioning

All results presented above were concerned in sequences of independent observables. Now we would like to say a few words about conditional probability.

If $(\Omega, \mathcal{S}, P)$ is a probability space, $A \in \mathcal{S}$ and $\mathcal{S}_0 \subset \mathcal{S}$ is a sub-σ-algebra of $\mathcal{S}$, then the conditional probability is any function $P(A|\mathcal{S}_0)\colon \Omega \to R$ that is $\mathcal{S}_0$-measurable and satisfies the equality

$$\int_C P(A|\mathcal{S}_0)dP = P(A \cap C)$$

for any $C \in \mathcal{S}_0$. Of course, if $\mathcal{S}_0$ is generated by a random variable η (we shall write $P(A|\mathcal{S}_0) = P(A|\eta)$ in this case), i.e.,

$$\mathcal{S}_0 = \eta^{-1}(\mathcal{B}(R)) = \{C \in \mathcal{S};\ \exists B \in \mathcal{B}(R),\ C = \eta^{-1}(B)\},$$

there exists also another concept. A conditional probability can be taken as a Borel function $p(A|\eta)\colon R \to R$ satisfying the equaligy

$$\int_B p(A|\eta)dP_\eta = P(A \cap \eta^{-1}(B))$$

for any $B \in \mathcal{B}(R)$. Of course, P-almost everywhere $P(A|\eta) = p(A|\eta) \circ \eta$. We shall use the second concept in our fuzzy case.

Assume that $\mathcal{F}$ is an MV algebra of fuzzy sets. Let $y\colon \mathcal{B}(R) \to \mathcal{F}$ be an observable, $a \in \mathcal{F}$. Then the conditional probability of a with respect to y is a Borel function $p(a|y)\colon R \to R$ such that

$$\int_B p(a|y)dm_y = m(a \cdot y(B)$$

for any $B \in \mathcal{B}(R)$. (Here $\cdot$ is the usual product of functions). Such a function exists by Radon-Nikodym theorem. Namely, if we define $\mu\colon \mathcal{B}(R) \to \langle 0, 1\rangle$ by the formula $\mu(B) = m(a \cdot y(B))$, then μ is a measure and $\mu \ll m_y$ (i.e., $m_y(B) = 0 \Rightarrow \mu(B) = 0$).

It is easy to see that m_y-almost everywhere $p(0|y) = 0$, $p(1|y) = 1$ and $0 \leq p(a|y) \leq 1$. Moreover, if $\sum_i a_i \leq 1$, then $p\left(\sum_i a_i|y\right) = \sum_i (a_i|y)$. A more interesting is a representation process.

Theorem 7.1. *([47], Prop. 3). To every $a \in \mathcal{F}$ and every observable $y\colon \mathcal{B}(R) \to \mathcal{F}$ there exists a probability space $(\Omega, \mathcal{S}, P)$, a random event $A \in \mathcal{S}$ and a random variable $\eta\colon \Omega \to R$ such that $P_\eta = m_y$ and*

$$p(a|y) = p(A|\eta)$$

m_y-almost everywhere.

The construction of A and η is simple. Put $\Omega = R^2$, $\mathcal{S} = \mathcal{B}(R^2)$ and $P = m \circ h$, where h is the joint observable of Q_a and y with

$$Q_a(B) = \begin{cases} 0, & \text{if } 0 \notin B,\ 1 \notin B, \\ a, & \text{if } 0 \notin B,\ 1 \in B, \\ a', & \text{if } 0 \in B,\ 1 \notin B, \\ 1, & \text{if } 0 \in B,\ 1 \in B, \end{cases}$$

Then it suffices to put $A = \{1\} \times R$ and $\eta\colon R^2 \to R$, $(u, v) \mapsto v$.

Let us define now the conditional expectation $E(x|y)$. In the Kolmogorovian model, $E(\xi|\eta)$ can be defined as a Borel function such that

$$\int_B E(\xi|\eta)dP = \int_{\eta^{-1}(B)} \xi dP$$

for any $B \in \mathcal{B}(R)$. Hence we could define $E(x\,|\,y)$ (for given observables x, y) as a Borel measurable function such that

$$\int_B E(x\,|\,y)dm_y = \int_{y(B)} x\,dm\,.$$

Of course, we must first define $\int_a xdm$. Namely we still defined only $\int xdm = \int_1 xdm$ by the formula $\int xdm = E(x) = \int_R t\,dm_x(t)$.

Let us inspire by the classical case, where

$$\int_C \xi\,dP = \int_\Omega \xi\chi_C dP\,.$$

Since

$$(\xi\chi_C)^{-1}(B) = \begin{cases} C \cap \xi^{-1}\,, & \text{if } 0 \notin B\,, \\ \big(C \cap \xi^{-1}(B)\big) \cup C', & \text{if } 0 \in B\,, \end{cases}$$

we can define a mapping $x_c \colon \mathcal{B}(R) \to \mathcal{F}$ by the formulas

$$x_C(B) = \begin{cases} c \cdot x(B)\,, & \text{if } 0 \notin B\,, \\ c \cdot x(B) + c', & \text{if } 0 \in B \end{cases}$$

and then define

$$\int_c xdm = \int x_c dm = \int_R t\,dm_{x_c}(t)\,.$$

Actually, x_c is an observable and it is integrable whenever x is integrable. If we define $\mu \colon \mathcal{B}(R) \to R$ by the formula

$$\mu(B) = \int_{y(B)} xdm\,, \quad \text{then} \quad \mu \text{ is a measure and } \mu \ll m_y\,.$$

Therefore by the Radon-Nikodym theorem there exists a Borel measurable function (denote it by $E(x\,|\,y)$) such that

$$\int_{y(B)} xdm = \int_B VE(x\,|\,y)dm_y$$

for any $B \in \mathcal{B}(R)$. Recall that all details are contained in [47] and [48].

Similarly as in the case of a conditional probability a representation formula can be realised:

$$E(x\,|\,y) = E(\xi\,|\,\eta)\,.$$

This formula is also a key for provind a variant of the martingal convergence theorem. Of course, the sequence (y_n) of conditioning sybalgebras (i.e., observables) is generated by a fixed observable y and a sequence (g_n) of Borel measurable functions $g_n \colon R \to R$.

Theorem 7.2. *([48], Theorem 2). Let $g_n \colon R \to R$ be Borel measurable functions $g_n^{-1}\big(\mathcal{B}(R)\big) \subset g_{n+1}^{1}\big(\mathcal{B}(R)\big)$ $(n = 1, 2, \dots)$, $\bigcup_{n=1}^{\infty} g_n^{-1}\big(\mathcal{B}(R)\big) = \mathcal{B}(R)$. Let $y \colon \mathcal{B}(R) \to \mathcal{F}$ be an observable, $y_n = y \circ g_n^{-1}$, $m_y \ll m_{y_n}$ $(n = 1, 2, \dots)$. Let $x \colon \mathcal{B}(R) \to \mathcal{F}$ be an integrable observable. Then m_y-a.e., $E(x\,|\,y_n) \to E(x\,|\,y)$.*

8. Structures with product

We have seen some advatages of the existence of the joint observable h of observables x and y. First, by the help of the joint observable, the sum $x+y$ can be defined

$$x + y = h \circ g^{-1},$$

where $g(u,v) = u + v$. (Of course, also some other operations can be defined by this way.) The translation formula (5.1) is the second important application of h. To see the role of the joint observable in (5.1) we shall prove this formula.

For the sake of simplicity we consider $n = 2$:

$$\begin{aligned}
P\Big(\eta_1^{-1}(A) \cap \eta_2^{-1}(A)\big) &= P\big((g_1 \circ \xi_1)^{-1}(A) \cap \big(g_2 \circ (\xi_1, \xi_2)\big)^{-1}(A)\Big) \\
&= P\Big(\xi_1^{-1}\big(g_1^{-1}(A)\big) \cap (\xi_1, \xi_2)^{-1}\big(g_2^{-1}(A)\big)\Big) \\
&= m\Big(h_2\big(\{u, v);\ u \in g_1^{-1}(A),\ (u,v) \in g_2^{-1}(A)\}\big)\Big) \\
&= \Big(h_2\big((g_1^{-1}(A) \times R) \cap g_2^{-1}(A)\big)\Big)\,.
\end{aligned}$$

Here is the crucial point of the proof. The joint observable h_2 is monotone. Therefore

$$\begin{aligned}
&m\Big(h_2\big((g_1^{-1}(A) \times R) \cap g_2^{-1}(A)\big)\Big) \\
&\leq m\Big(h_2\big((g_1^{-1}(A) \times R\big) \wedge h_2\big(g_2^{-1}(A)\big)\Big) \\
&= m\Big(h_1\big(g_1^{-1}(A)\big) \wedge h_2\big(g_2^{-1}(A)\big)\Big)\,.
\end{aligned}$$

Now (5.1) followe, since $y_1 = h_1 \circ g_1^{-1}$, $y_2 = h_2 \circ g_1^{-1}$ by the definition.

In both cases when the joint observable could be applied, we used the fact that the MV algebra $\mathcal{F} \subset \langle 0, 1\rangle^X$ was closed under the product of functions. The third application of the joint observable and the operation of product is in the definition of the conditional probability and its set representation.

How to substitute the methods described above without the product operation? One of possibilities is to introduce a product axiomatically.

The first has been the concept of l-ring of Kacian [21]. Since any MV algebra can be represented by an l-group, this concept seems to be very natural.

Of course, more general are the concepts of the author ([50], [51]) and Dvurečenskij and Di Nola ([10], [12]). Since the terminology has not been fixed yet, we shall use some terminology ad hoc, only for this paper.

Definition 8.1. We shall say that $(M, +, \cdot, \leq, 0, 1)$ is a PQ-support, if the following properties are satisfied:

(i) $(M, \leq, 0, 1)$ is a partially ordered set with the least element 0 and the greatest 1.

(ii) $+$ is a partial binary operation, $\cdot$ is a binary operation, $+$ and $\cdot$ are commutative and associative

(iii) If $a, b, c \in M$ and $b + c$ exists in M, then $a \cdot (b + c) = a \cdot b + a \cdot c$.

(iv) If $a_n \downarrow 0$, $b_n \downarrow 0$, then $a_n \cdot b_n \downarrow 0$.

In [10] the commutativity of $\cdot$ is substituted by two distributive laws:

$$a \cdot (b + c) = a \cdot b + a \cdot c\,, \ \ (b + c) \cdot a = b \cdot a + c \cdot a\,.$$

All considerations in [50] and [51] are correct also under these weaker assumptions.

Definition 8.2. Let M be a PQ-support. A weak state is a mapping $m\colon M \to \langle 0,1\rangle$ satisfying the following conditions:

(i) If $a = b + c$, then $m(a) = m(b) + m()c)$.

(ii) If $a_n \nearrow a$, then $m(a_n) \nearrow m(a)$.

(iii) If $m(a) = 1$, then $m(a \cdot b) = m(b)$ for any $b \in M$.

Theorem 8.1. *([51], Theorem 2.2). Let M be a PQ-support, $x, y\colon \mathcal{B}(R) \to M$ be weak observables. Then there exists a probability measure $\mu\colon \mathcal{B}(R^2) \to \langle 0,1\rangle$ such that*

$$\mu(C \times D) = m\big(x(C) \cdot y(D)\big)$$

for any $C, D \in \mathcal{B}(R)$.

The measure μ is usually called the joint distribution of x and y. As possible applications, the definition of the conditional probability is suggested in [51]. Similarly, a definition of the sum of two observables is suggested, of course, only in the case of weakly σ-distributive MV algebra.

If given PQ-support M is simultaneously an MV algebra, then we say that M is a product MV algebra.

Definition 8.3. A product MV algebra is an algebraic system $(M, \oplus, \odot, \cdot, *, u, 0)$, where $(M, \oplus, \odot, *, u, 0)$ is an MV algebra and $\cdot$ is a binary operation satisfying the following conditions:

(i) $u \cdot u = u$.

(ii) The operation $\cdot$ is commutative and associative.

(iii) If $a + b \le u$, then $c \cdot (a + b) = c \cdot a + c \cdot b$ for any $c \in M$.

(iv) If $a_n \downarrow 0$, $b_n \downarrow 0$, then $a_n \cdot b_n \downarrow 0$.

Theorem 8.2. *([50], Theorem 2). Let M be a weakly σ-distributive product MV algebra (see Definition 6.2). Then to any weak observables x, y there exists their joint observable, i.e. a mapping $h\colon \mathcal{B}(R^2) \to M$ satisfying the following conditions:*

(i) $h(R^2) = 1$.

(ii) *If $A \cap B = \emptyset$, then $h(A \cup B) = h(A) + h(B)$.*

(iii) *If $A_n \nearrow A$, then $h(A_n) \nearrow h(A)$.*

(iv) *$h(C \times D) = x(C) \cdot y(D)$ for any $C, D \in \mathcal{B}(R)$.*

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