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TRANSVERSAL BOUNDED SOLUTIONS FOR DIFFERENCE EQUATIONS

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Abstract. Higher dimensional Melnikov mappings are introduced for detecting the existence of transversal bounded solutions of autonomous difference equations with nonautonomous perturbations.

1. Introduction

In this note, we consider difference equations of the form

$$x_{k+1} = f(x_k) + h(x_k, \mu, k) \tag{1.1}$$

with $x_k \in \mathbb{R}^n$, $\mu \in \mathbb{R}^m$. We make the following assumptions about (1.1):

(i) f, h are C^3-smooth in all non-discrete arguments.

(ii) $f(0) = 0$ and $h(\cdot, 0, \cdot) = 0$.

(iii) The eigenvalues of $Df(0)$ are non-zero and all lie off the unit circle.

(iv) The unperturbed equation $x_{k+1} = f(x_k)$ has a homoclinic solution. That is, there exists a nonzero sequence $\gamma = \{\gamma_k\}_{k\in\mathbb{Z}}$ such that $\lim_{k\to\pm\infty} \gamma_k = 0$ and $\gamma_{k+1} = f(\gamma_k)$. Moreover, $Df(\gamma_k)$, $k \in \mathbb{Z}$ are nonsingular.

The purpose of this paper is to find a set of parameters μ for which (1.1) has a transversal bounded solution $\{\bar{x}_k\}_{k\in\mathbb{Z}}$ near $\{\gamma_k\}_{k\in\mathbb{Z}}$, i.e. the linearization of (1.1) along $\{\bar{x}_k\}_{k\in\mathbb{Z}}$ given by

$$v_{k+1} = \big(Df(\bar{x}_k) + D_x h(\bar{x}_k, \mu, k)\big) v_k, \quad k \in \mathbb{Z}$$

has the only bounded solution $v_k = 0, \forall\, k \in \mathbb{Z}$. When h is independent of k, i.e. (1.1) is a mapping, then the existence of such a bounded solution means the existence of a transversal homoclinic orbit and so chaos [10]. In general, (1.1) can be associated to quasiperiodically perturbed systems [9], [11], [12]. To derive these

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sets, higher dimensional Melnikov mappings are introduced. Simple zero points of those mappings give wedge-shaped regions in $\mathbb{R}^m$ for μ representing the desired sets.

Results of this paper are based on [2-6], where ordinary differential equations are investigated. We establish a complete analogy between the Melnikov theories for difference equations and ordinary differential equations. Discrete cases of two-dimensional mappings are considered in [7] and [11]. Mappings in arbitrary finite dimensions are considered in [1] and [4] but the dimension is 1 in [4], which is released in this paper, for the intersection of tangent spaces to stable and unstable manifolds along a homoclinic solution to a hyperbolic fixed point of the unperturbed mapping, and while the transversality is not proved in [1]. In this paper, no restriction is given on the dimension of the phase space or on the dimension of intersection of stable and unstable manifolds. At the end of the paper, an application is given to a 4-dimensional Hamiltonian system with small impulsive perturbations.

2. The Variational Equation

In this section, we modify Section 2 of [5] to (1.1). The norm and scalar product on $\mathbb{R}^n$ are denoted by $|\cdot|$, $\langle\cdot,\cdot\rangle$, respectively.

Let us consider the unperturbed equation

$$x_{k+1} = f(x_k)\,. \tag{2.1}$$

For (2.1) we adopt the standard notation W^s, W^u for the local stable and local unstable manifolds, respectively, of the origin and $d_s = \dim(W^s)$, $d_u = \dim(W^u)$. Since $x = 0$ is a hyperbolic equilibrium, $\{\gamma_k\}_{k\in\mathbb{Z}}$ must approach the origin along W^s as $k \to +\infty$ and along W^u as $k \to -\infty$.

By the variational equation of (2.1) along $\{\gamma_k\}_{k\in\mathbb{Z}}$ we mean the linear difference equation

$$u_{k+1} = Df(\gamma_k)u_k\,. \tag{2.2}$$

We note that as $k \to \pm\infty$, $Df(\gamma_k) \to Df(0)$, a hyperbolic matrix. Thus, the following result yields two solutions for (2.2) – one solution for $k \in \mathbb{Z}_+$ and one for $k \in \mathbb{Z}_-$, where $\mathbb{Z}_+ = \mathbb{N}\cup\{0\}$ and $\mathbb{Z}_- = -\mathbb{Z}_+$.

Lemma 2.1. *Let $k \to A(k)$ be a matrix valued function on $\mathbb{Z}_+$ and suppose there exists a constant nonsingular matrix, A_0, and a scalar $a > 0$ such that $\sup_{k\in\mathbb{Z}_+} |A(k) - A_0|\mathrm{e}^{4ak} < \infty$. Then there exists a fundamental solution, $X(k)$ for k large, to the difference equation $x_{k+1} = A(k)x_k$ such that $\lim\limits_{k\to\infty} X(k)A_0^{-k} = I$.*

Proof. The proof is very similar to [5, 1. Lemma], but we give it for the reader convenience. Let P be a matrix such that $P^{-1}A_0P = J$, where J is the Jordan form with the block-diagonal form $J = \operatorname{diag}(J_1, J_2, \dots, J_r)$. Let k_i be the order of J_i and λ_i is the eigenvalue corresponding to J_i. We arrange the Jordan blocks such that $|\lambda_i| \le |\lambda_{i+1}|$. By putting $y = P^{-1}x$ and $B(k) = P^{-1}A(k)P$, the equation $x_{k+1} = A(k)x_k$ has the form

$$y_{k+1} = B(k)y_k = Jy_k + (B(k) - J)y_k\,. \tag{2.3}$$

We fix one block J_i and define $p_i = k_1 + k_2 + \cdots + k_{i-1}$. Similarly we define q_i satisfying $|\lambda_{q_{i-1}}| < |\lambda_i|$ and $|\lambda_{q_i}| = |\lambda_i|$. We split the matrix J^k into $U_1(k), U_2(k)$, where

$$\begin{aligned} U_1(k) &= (J_1^k, J_2^k, \dots, J_{q_{i-1}}^k, 0, \dots, 0),\\ U_2(k) &= (0, 0, \dots, 0, J_{q_i}^k, \dots, J_r^k)\,. \end{aligned}$$

Since the spectrum $\sigma(U_1(1))$ is contained inside the circle with the radius $|\lambda_{q_{i-1}}|$, we can assume by [8, 3.126 Lemma]

$$|U_1(1)| \le |\lambda_{q_{i-1}}| + b \le |\lambda_i| - b$$

for $b > 0$ sufficiently small. Consequently, we obtain for $k \ge 0$ that $|U_1(k)| \le |U_1(1)|^k \le (|\lambda_i| - b)^k$. Since $\sigma(U_2(-1)) = \big(\sigma(U_2(1))\big)^{-1}$, we similarly have

$$|U_2(k)| \le (|\lambda_i| - b)^k, \quad \forall\, k \in \mathbb{Z}_-$$

again for $b > 0$ sufficiently small. Let e_k be the k-th column of the $n \times n$ identity matrix.

By fixing $k_0 \in \mathbb{N}$ sufficiently large, let us define a mapping T_j for $k = k_0, k_0{+}1, \dots$ and for $j \in \{1, 2, \dots, k_i\}$ as follows

$$\begin{aligned} T_j(y)_k = J^k e_{p_i+j} &+ \sum_{j=k_0}^{k-1} U_1(k-1-j)(B(j) - J)y_j \\ &- \sum_{j=k}^{\infty} U_2(k-1-j)(B(j) - J)y_j\,. \end{aligned} \tag{2.4}$$

We consider this mapping on the Banach space

$$Y = \big\{\{y_j\}_{j=k_0}^{\infty} :\ y_j \in \mathbb{R}^n, \quad \sup_{j \ge k_0} |y_j|\,(|\lambda_i| + b)^{-j} < \infty\big\}$$

with the norm $|||y||| = \sup_{k \ge k_0} |y_k|(|\lambda_i| + b)^{-k}$ for $y = \{y_j\}_{j=k_0}^{\infty}$. To show that T_j is well-defined, we compute

$$\sup_k |J^k e_{p_i+j}|(|\lambda_i| + b)^{-k} < \infty\,,$$

since $|J_i^k| < c_1(|\lambda_i| + d)^k$ for a $0 < d < b$ and $c_1 > 0$. By taking $b > 0$ satisfying

$$\frac{|\lambda_i| + b}{|\lambda_i| - b} < \mathrm{e}^{4a}\,,$$

we have for a constant $c > 0$

$$\begin{aligned} &\sup_k \sum_{j=k_0}^{k-1} |U_1(k-1-j)(B(j) - J)y_j|\,(|\lambda_i| + b)^{-k} \\ \le& c(|\lambda_i| - b)^{-1}|||y|||\sup_k \left(\frac{|\lambda_i| - b}{|\lambda_i| + b}\right)^k \sum_{j=k_0}^{k-1} \left(\frac{|\lambda_i| + b}{|\lambda_i| - b}\,\mathrm{e}^{-4a}\right)^j < \infty \end{aligned}$$

and

$$
\begin{aligned}
&\sup_k \sum_{j=k}^{\infty} |U_2(k-1-j)(B(j)-J)y_j|\,(|\lambda_i|+b)^{-k} \le \\
&\le c \sup_k \sum_{j=k}^{\infty} (|\lambda_i|-b)^{k-1-j} |||y|||(|\lambda_i|+b)^j \mathrm{e}^{-4aj}(|\lambda_i|+b)^{-k} \\
&\le c(|\lambda_i|-b)^{-1}|||y|||\sup_k \Big(\frac{|\lambda_i|-b}{|\lambda_i|+b}\Big)^k \sum_{j=k}^{\infty} \Big(\frac{|\lambda_i|+b}{|\lambda_i|-b}\mathrm{e}^{-4a}\Big)^j < \infty\,.
\end{aligned}
$$

Consequently, we arrive at $|||T_j(y)||| < \infty$ and so $T_j\colon Y \to Y$. Furthermore, we have

$$\forall\, \varepsilon > 0\ \exists\, n_0 > k_0 : \Big(\frac{|\lambda_i|-b}{|\lambda_i|+b}\Big)^k < \varepsilon\ \forall\, k > n_0\,.$$

By using this property, the contraction of T_j follows by the same arguments as the well-definition of T_j. Consequently, T_j has a fixed point $y(j)$ satisfying by (2.4)

$$|y(j)_k - J^k e_{p_i+j}| \le K_0(|\lambda_i|-b)^k$$

for a constant $K_0 > 0$. By defining the matrix $Y_i(k)$ of the order $n \times k_i$ with $y(j)_k$ in column j, we obtain

$$|Y_i(k) - F_i(k)|(|\lambda_i|-b)^{-k} \le K_0\,,$$

where $F_i(k)$ is the $n \times k_i$-matrix with J_i^k in rows p_i+1 through p_i+k_i and all other rows zero. Let $\overline{G}_i$ be the identity matrix of order $k_i \times k_i$. Then $\lim_{k\to\infty} Y_i(k)J_i^{-k} = G_i$ and G_i is the matrix of order $n \times k_i$ with $\overline{G}_i$ in rows p_i+1 through p_i+k_i and all other rows zero. This construction is done for the block J_i. To get the result, we take the $n \times n$ matrix $Y(k)$ with $Y_i(k)$ in columns p_i+1 through p_i+k_i for $i = 1, 2, \ldots, r$. So $\lim_{k\to\infty} Y(k)J^{-k} = I$. Finally, by putting $X(k) = PY(k)P^{-1}$ we arrive at $X(k+1) = A(k)X(k)$ satisfying

$$X(k)A_0^{-k} \to I \quad \text{as} \quad k \to \infty\,.$$

The proof is finished. □

Our next result matches at $k = 0$ the two solutions of (2.2) provided by the preceding lemma. The following theorem and its proof is a slight extension of [5, 2. Theorem] so we omit the proof.

Theorem 2.2. *Let $d_s = \dim(W^s)$, $d_u = \dim(W^u)$ for (2.2) and let I_s, I_u denote the identity matrices of order d_s, d_u respectively. There exists a fundamental solution $U(k)$, $k \in \mathbb{Z}$ for (2.2) along with a nonsingular matrix S, constants $M > 1$, $K_0 > 0$ and four projections P_{ss}, P_{su}, P_{us}, P_{uu} such that $P_{ss}+P_{su}+P_{us}+P_{uu} = I$ and such that the following hold:*

(i) $|U(t)(P_{ss}+P_{us})U(s)^{-1}| \le K_0 M^{(s-t)}$ *for* $0 \le s \le t$,

(ii) $|U(t)(P_{su}+P_{uu})U(s)^{-1}| \le K_0 M^{(t-s)}$ *for* $0 \le t \le s$,

(iii) $|U(t)(P_{ss}+P_{su})U(s)^{-1}| \le K_0 M^{(t-s)}$ *for* $t \le s \le 0$,

(iv) $|U(t)(P_{us}+P_{uu})U(s)^{-1}| \le K_0 M^{(s-t)}$ *for* $s \le t \le 0$,

(v) $\sup_{t\ge 0} \left| U(t)(P_{ss}+P_{us})U(t)^{-1} - S\begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix} S^{-1} \right| M^t < \infty$,

(vi) $\sup_{t\ge 0} \left| U(t)(P_{su}+P_{uu})U(t)^{-1} - S\begin{pmatrix} 0 & 0 \\ 0 & I_u \end{pmatrix} S^{-1} \right| M^t < \infty$,

(vii) $\sup_{t\le 0} \left| U(t)(P_{ss}+P_{su})U(t)^{-1} - S\begin{pmatrix} 0 & 0 \\ 0 & I_u \end{pmatrix} S^{-1} \right| M^{-t} < \infty$,

(viii) $\sup_{t\le 0} \left| U(t)(P_{us}+P_{uu})U(t)^{-1} - S\begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix} S^{-1} \right| M^{-t} < \infty$.

Also, $\operatorname{rank} P_{ss} = \operatorname{rank} P_{uu} = d$ *for some positive integer* d.

In the language of dichotomies [10] we see that Theorem 2.2 provides a two-sided exponential dichotomy. For $k \to -\infty$ an exponential dichotomy is given by the fundamental solution $U(k)$ and the projection $P_{us}+P_{uu}$ while for $k \to +\infty$ such is given by $U(k)$ and $P_{ss}+P_{us}$.

Let u_j denote column j of U and assume these are numbered so that

$$P_{uu} = \begin{pmatrix} I_d & 0_d & 0 \\ 0_d & 0_d & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad P_{ss} = \begin{pmatrix} 0_d & 0_d & 0 \\ 0_d & I_d & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here, I_d denotes the $d \times d$ identity matrix and 0_d denotes the $d \times d$ zero matrix.

For each $i = 1, \dots, n$ we define $u_i^\perp(k)$ by $\langle u_i^\perp(k), u_j(k+1) \rangle = \delta_{ij}$. The vectors $u_i^\perp$ can be computed from the formula $U(k)^{\perp *} = U(k+1)^{-1}$ where $U^\perp$ denotes the matrix with $u_j^\perp$ as column j. By using the identity $U(k+1)U(k)^{\perp *} = I$ we obtain that $U(k+1)^\perp = \left(Df(\gamma_{k+1})^*\right)^{-1} U(k)^\perp$. Thus, $U^\perp$ is the adjoint of U (see [4, pp. 103-104]).

We take the Banach space

$$Z = \left\{ \{y_j\}_{j\in\mathbb{Z}} : \; y_j \in \mathbb{R}^n, \quad \sup_{j\in\mathbb{Z}} |y_j| < \infty \right\}$$

with the norm $||y|| = \sup_{k\in\mathbb{Z}} |y_k|$ for $y = \{y_j\}_{j\in\mathbb{Z}}$.

Summation of the inequalities in Theorem 2.2 yields the following result.

Theorem 2.3. *Let* U *be the fundamental solution to* (2.2) *along with the projections* P_{ss}, P_{su}, P_{us}, P_{uu} *as in Theorem* 2.2. *Then there exists a constant* $K > 0$ *such that for any* $z \in Z$ *the following hold:*

(i) $\sum_{k=0}^{j} |U(j)(P_{ss}+P_{us})U(k)^{-1} z_k| \le K\|z\|$ *for* $j \ge 0$,

(ii) $\sum_{k=j}^{\infty} |U(j)(P_{su}+P_{uu})U(k)^{-1} z_k| \le K\|z\|$ *for* $j \ge 0$,

(iii) $\sum_{k=j}^{0} |U(j)(P_{ss}+P_{su})U(k)^{-1} z_k| \le K\|z\|$ *for* $j \le 0$,

(iv) $\sum_{k=-\infty}^{j} |U(j)(P_{us}+P_{uu})U(k)^{-1} z_k| \le K\|z\|$ *for* $j \le 0$.

Let us define a closed linear subspace of Z given by

$$Z_0 = \Big\{ z \in Z : \sum_{k=-\infty}^{\infty} P_{uu} U(k+1)^{-1} z_k = 0 \Big\} .$$

We consider the difference equation

$$z_{k+1} = Df(\gamma_k) z_k + w_k, \quad \{w_k\}_{k\in\mathbb{Z}} \in Z . \tag{2.5}$$

The following result, which is an extension of [4, Theorem 2.3] and [5, p. 9], is a Fredholm-like condition for (2.5).

Theorem 2.4. *Necessary and sufficient condition for the existence of a solution* $\{x_k\}_{k\in\mathbb{Z}} \in Z$ *of* (2.5) *is that* $\{w_k\}_{k\in\mathbb{Z}} \in Z_0$.

Proof. $\Longrightarrow$

Let $z = \{z_k\}_{k\in\mathbb{Z}}$ be a solution of (2.5). Denote $A(k) = Df(\gamma_k)$ and compute

$$P_{uu} U(k+1)^{-1} z_{k+1} = P_{uu} U(k+1)^{-1} A(k) z_k + P_{uu} U(k+1)^{-1} w_k .$$

Since $U(k+1) = A(k)U(k)$ then $U(k+1)^{-1} = U(k)^{-1}A(k)^{-1}$, and hence

$$\sum_{k=-\infty}^{\infty} P_{uu} U(k+1)^{-1} z_{k+1} = \sum_{k=-\infty}^{\infty} P_{uu} U(k)^{-1} z_k + \sum_{k=-\infty}^{\infty} P_{uu} U(k+1)^{-1} w_k$$

$$0 = \sum_{k=-\infty}^{\infty} P_{uu} U(k+1)^{-1} w_k .$$

We note that Theorem 2.3 gives the convergence of these series.

$\Longleftarrow$

Let $w = \{w_k\}_{k=-\infty}^{\infty} \in Z_0$. We define the mapping $\mathcal{K}$ as follows

$$\begin{aligned} \mathcal{K}(w)_k = U(k)\Big[\sum_{j=-\infty}^{-1} P_{us} U(j+1)^{-1} w_j + & \sum_{j=0}^{k-1} (P_{ss} + P_{us}) U(j+1)^{-1} w_j \\ & - \sum_{j=k}^{\infty} (P_{su} + P_{uu}) U(j+1)^{-1} w_j \Big] , \end{aligned}$$

for $k \geq 0$,

$$\begin{aligned} \mathcal{K}(w)_k = U(k)\Big[- \sum_{j=0}^{\infty} P_{su} U(j+1)^{-1} w_j + & \sum_{j=-\infty}^{k-1} (P_{us} + P_{uu}) U(j+1)^{-1} w_j \\ & - \sum_{j=k}^{-1} (P_{ss} + P_{su}) U(j+1)^{-1} w_j \Big] , \end{aligned}$$

for $k \leq 0$. Here we define $\sum_{j=0}^{-1} = 0$.

Theorem 2.3 implies the well-definition and continuity of $\mathcal{K} \colon Z_0 \to Z$ and by putting $z_k = \mathcal{K}(w)_k$, $\forall\, k \in \mathbb{Z}$ in (2.5), we easily verify that it is a solution. We note that the general solution of (2.5) has the form

$$z = \sum_{j=1}^{d} \beta_j u_{j+d} + \mathcal{K}(w), \quad \beta_j \in \mathbb{R} .$$

The proof is finished. □

The next result provides an appropriate projection.

Theorem 2.5. *Let U be as in Theorem 2.2 and let Z_0 be as in Theorem 2.4. There exists a bounded projection $\Pi\colon Z \to Z$ such that $Im\,\Pi = Z_0$.*

Proof. We take Π in the form $I - P$, where P is defined by

$$P(w)_k = \frac{U(k+1)}{a_{k+1}} P_{uu} \sum_{j=-\infty}^{\infty} U(j+1)^{-1} w_j \,,$$

and the sequence $\{a_k\}_{k\in\mathbb{Z}}$ satisfies

$$a_k > 0,\, \forall\, k \in \mathbb{Z}, \qquad \sum_{k=-\infty}^{\infty} \frac{1}{a_{k+1}} = 1, \qquad \sup_{k\in\mathbb{Z}} \frac{U(k+1)}{a_{k+1}} < \infty\,.$$

We verify that this P is a projection, i.e. $P^2 = P$.

$$P(P(w))_k = P\Big(\Big\{\frac{U(s+1)}{a_{s+1}} P_{uu} \sum_{j=-\infty}^{\infty} U(j+1)^{-1} w_j\Big\}_{s\in\mathbb{Z}}\Big) =$$
$$= \frac{U(k+1)}{a_{k+1}} P_{uu} \sum_{l=-\infty}^{\infty} U(l+1)^{-1} \Big(\frac{U(l+1)}{a_{l+1}} P_{uu} \sum_{j=-\infty}^{\infty} U(j+1)^{-1} w_j\Big) = P(w)_k\,.$$

Hence P is a projection. Now we verify that $\Pi = I - P$ is such that $\Pi w \in Z_0$.

$$\sum_{k=-\infty}^{\infty} P_{uu} U(k+1)^{-1} \Pi(w)_k = \sum_{k=-\infty}^{\infty} P_{uu} U(k+1)^{-1} (I-P)(w)_k$$
$$= \sum_{k=-\infty}^{\infty} P_{uu} U(k+1)^{-1} \Big(w_k - \frac{U(k+1)}{a_{k+1}} P_{uu} \sum_{j=-\infty}^{\infty} U(j+1)^{-1} w_j\Big)$$
$$= \sum_{k=-\infty}^{\infty} P_{uu} U(k+1)^{-1} w_k - \sum_{k=-\infty}^{\infty} P_{uu} U(k+1)^{-1} \Big(\frac{U(k+1)}{a_{k+1}} P_{uu}$$
$$\sum_{j=-\infty}^{\infty} U(j+1)^{-1} w_j\Big) = 0\,.$$

Consequently, Π has the desired properties. □

3. Perturbation Theory

We study the equation

$$\begin{aligned} F_{\mu,\epsilon,y}(x)_k &= x_{k+1} - f(x_k) - h(x_k,\mu,k) - \epsilon|\mu|\mathcal{L}(x-y-\gamma) = 0 \\ F_{\mu,\epsilon,y}&\colon Z \to Z\,, \end{aligned} \tag{3.1}$$

where $\mathcal{L}\colon Z \to Z$ is a linear continuous mapping such that $||\mathcal{L}|| \le 1$, $y \in Z$, and $\epsilon \in \mathbb{R}$ is small. It is clear that solutions of (3.1) with $\epsilon = 0$ are bounded solutions of (1.1).

We define mappings $L\colon Z \to Z$ and $G\colon Z \times \mathbb{R}^m \times \mathbb{R} \times Z \to Z$ as follows

$$\begin{aligned} L(z)_k &= z_{k+1} - Df(\gamma_k)z_k \\ G(z,\mu,\epsilon,y)_k &= f(z_k+\gamma_k) - f(\gamma_k) - Df(\gamma_k)z_k \\ &\qquad + h(z_k+\gamma_k,\mu,k) + \epsilon|\mu|\mathcal{L}(z-y)\,. \end{aligned}$$

By putting $x = z + \gamma$ in (3.1), this equation has the form

$$L(z) = G(z,\mu,\epsilon,y)\,. \tag{3.2}$$

We decompose (3.2) in the following way

$$\begin{aligned} L(z) &= \Pi G(z,\mu,\epsilon,y) \\ 0 &= (I-\Pi)G(z,\mu,\epsilon,y)\,. \end{aligned}$$

By using Theorem 2.4, the above pair of equations is equivalent to

$$z = \sum_{j=1}^{d} \beta_j u_{j+d} + \mathcal{K}\big(\Pi G(z,\mu,\epsilon,y)\big), \quad \beta_j \in \mathbb{R} \tag{3.3a}$$

$$0 = (I-\Pi)G(z,\mu,\epsilon,y)\,. \tag{3.3b}$$

Moreover by using the Lyapunov-Schmidt procedure like in [5, Theorem 8], the study of the equations (3.3a-b) can be expressed in the following theorem for z, μ, ϵ, $\beta = (\beta_1, \beta_2, \cdots, \beta_d)$, y sufficiently small.

Theorem 3.1. *Let U and d be as in Theorem 2.2. Then there exist small neighborhoods $0 \in Q \subset Z$, $0 \in O \subset \mathbb{R}^d$, $0 \in W \subset \mathbb{R}^m$, $0 \in V \subset \mathbb{R}$ and a C^3-function $H\colon Q \times O \times W \times V \to \mathbb{R}^d$ denoted by $(y,\beta,\mu,\epsilon) \to H(y,\beta,\mu,\epsilon)$ with the following properties:*

- (i) *The equation $H(y,\beta,\mu,\epsilon) = 0$ holds if and only if (3.1) has a solution near γ and moreover, each such (y,β,μ,ϵ) determines only one solution of (3.1),*
- (ii) $H(0,0,0,0) = 0$,
- (iii) $\frac{\partial H_i}{\partial \mu_j}(0,0,,0,0) = -\sum_{k\in\mathbb{Z}} \left\langle u_i^{\perp}(k), \frac{\partial h}{\partial \mu_j}(\gamma_k,0,k)\right\rangle$,
- (iv) $\frac{\partial H_i}{\partial \beta_j}(0,0,0,0) = 0$,
- (v) $\frac{\partial^2 H_i}{\partial \beta_k \partial \beta_j}(0,0,0,0) = -\sum_{l\in\mathbb{Z}} \left\langle u_i^{\perp}(l), D^2f(\gamma_l)(u_{d+j}(l), u_{d+k}(l))\right\rangle$.

Proof. Since the proof is the same as of [5, Theorem 8], we omit it. □

We introduce the following notations:

$$\begin{aligned} a_{ij} &= -\sum_{l\in\mathbb{Z}} \left\langle u_i^{\perp}(l), \frac{\partial h}{\partial \mu_j}(\gamma_l,0,l)\right\rangle \\ b_{ijk} &= -\sum_{l\in\mathbb{Z}} \left\langle u_i^{\perp}(l), D^2f(\gamma_l)(u_{d+j}(l), u_{d+k}(l))\right\rangle\,. \end{aligned}$$

Finally, we take the mapping $M_\mu\colon \mathbb{R}^d \to \mathbb{R}^d$ defined by

$$\big(M_\mu(\beta)\big)_i = \sum_{j=1}^{m} a_{ij}\mu_j + \frac{1}{2}\sum_{j,k=1}^{d} b_{ijk}\beta_j\beta_k\,.$$

Now we can state the main result of this note.

Theorem 3.2. *If M_{μ_0} has a simple zero point β_0, i.e. β_0 satisfies $M_{\mu_0}(\beta_0) = 0$ and $D_\beta M_{\mu_0}(\beta_0)$ is a regular matrix, then there is a wedge-shaped region in $\mathbb{R}^m$ for μ of the form*

$$\mathcal{R} = \Big\{ s^2 \tilde{\mu} : \ s, \text{ respectively } \tilde{\mu}, \text{ is from a small open} \\ \text{neighborhood of } 0 \in \mathbb{R}, \text{ respectively of } \mu_0 \in \mathbb{R}^m \Big\}$$

such that for any $\mu \in \mathcal{R} \setminus \{0\}$, the equation (1.1) *possesses a transversal bounded solution.*

Proof. Let us consider the mapping defined by

$$\Phi(y, \tilde{\beta}, \tilde{\mu}, \tilde{\epsilon}, s) = \begin{cases} \frac{1}{s^2} H(y, s\tilde{\beta}, s^2\tilde{\mu}, s^3\tilde{\epsilon}) & \text{for} \quad s \neq 0 \\ M_{\tilde{\mu}}(\tilde{\beta}) & \text{for} \quad s = 0 \,. \end{cases}$$

According to (ii)-(v) of Theorem 3.1, the mapping Φ is C^1-smooth near

$$(y, \tilde{\beta}, \tilde{\mu}, \tilde{\epsilon}, s) = (0, \beta_0, \mu_0, 0, 0)$$

with respect to the variable $\tilde{\beta}$. Since

$$M_{\mu_0}(\beta_0) = 0 \quad \text{and} \quad D_\beta M_{\mu_0}(\beta_0) \quad \text{is a regular matrix}\,,$$

we can apply the implicit function theorem to solve locally and uniquely the equation $\Phi = 0$ in the variable $\tilde{\beta}$. This gives for $\epsilon = 0$, by (i) of Theorem 3.1, the existence of $\mathcal{R}$ on which (1.1) has a bounded solution.

To prove the transversality of these bounded solutions, we fix $\mu \in \mathcal{R} \setminus \{0\}$ and take

$$y = \tilde{\gamma} - \gamma\,,$$

where $\tilde{\gamma}$ is the solution of (3.1) for which the transversality should be proved. Then we vary $\epsilon = s^3\tilde{\epsilon}$ small. Note that $s \neq 0$ is also fixed due to $\mu = s^2\tilde{\mu}$. Since the local uniqueness of solutions of (3.1) near $\tilde{\gamma}$ is satisfied for any $\tilde{\epsilon}$ sufficiently small according to the above application of the implicit function theorem, such equation (3.1) (*with the fixed $\mu \in \mathcal{R} \setminus \{0\}$, $\epsilon = s^3\tilde{\epsilon}$ where $s \neq 0$ is also fixed and the special $y = \tilde{\gamma} - \gamma$*) has the only solution $x = \tilde{\gamma}$ near $\tilde{\gamma}$ for any $\tilde{\epsilon}$ sufficiently small.

Now we apply the following theorem of [2].

Theorem. *Let V, W be Banach spaces and let $||\cdot||_{\mathcal{L}(V,W)}$ be the norm of the space $\mathcal{L}(V, W)$ of all linear, bounded mappings from V to W. Consider a C^2-smooth mapping $F\colon V \to W$ satisfying $F(0) = 0$ and assume that $DF(0)$ is Fredholm with index 0.*

If there exist a neighbourhood $U \subset V$ of 0 and a number $\delta > 0$ such that for any linear bounded mapping $B\colon V \to W$, $||B||_{\mathcal{L}(V,W)} \leq 1$ the perturbation $\epsilon B + F$, $0 \leq \epsilon \leq \delta$ has the only zero point 0 in U, then $DF(0)$ is invertible.

Hence the above theorem of [2] gives the invertibility of $DF_{\mu,0,\tilde{\gamma}-\gamma}(\tilde{\gamma})$ and so the only bounded solution on $\mathbb{Z}$ of the equation

$$v_{k+1} = Df(\tilde{\gamma}_k)v_k + D_x h(\tilde{\gamma}_k, \mu, k)v_k$$

is $v_k = 0$, $\forall\, k \in \mathbb{Z}$. The proof is finished. □

4. Bifurcation from a Manifold of Homoclinic Solutions

In many cases, (2.1) has a manifold of homoclinic solutions. Hence we suppose that

(v) There is an open non-empty subset $\mathcal{O} \subset \mathbb{R}^d$ and C^3-smooth mappings $\gamma_k \colon \mathcal{O} \to \mathbb{R}^n$, $\omega \colon \mathcal{O} \to \mathbb{R}^n$, $\forall\, k \in \mathbb{Z}$ satisfying

$$\begin{aligned} &\gamma_{k+1}(\theta) = f(\gamma_k(\theta)) \quad \forall\, k \in \mathbb{Z}, \forall\, \theta \in \mathcal{O} \\ &\omega(\theta) = f(\omega(\theta)) \quad \forall\, \theta \in \mathcal{O} \\ &\lim_{k \to \pm\infty} \gamma_k(\theta) = \omega(\theta) \quad \forall\, \theta \in \mathcal{O}\,. \end{aligned}$$

(vi) The eigenvalues of $Df(\omega(\theta))\,\forall\, \theta \in \mathcal{O}$ are non-zero and all lie off the unit circle. Moreover, $Df(\gamma_k(\theta))$ $\forall\, k \in \mathbb{Z}$, $\forall\, \theta \in \mathcal{O}$ are nonsingular.

(vii) $\frac{\partial \gamma_k}{\partial \theta_i}$ are uniformly bounded on $\mathcal{O}$ with respect to $k \in \mathbb{Z}$ when $\theta = (\theta_1, \theta_2, \cdots, \theta_d)$.

(viii) $\left\{ \frac{\partial \gamma_0}{\partial \theta_i}(\theta) \right\}_{i=1}^d$ are linearly independent for any $\theta \in \mathcal{O}$.

From the identity $\gamma_{k+1}(\theta) = f(\gamma_k(\theta))$, we obtain

$$\frac{\partial \gamma_{k+1}}{\partial \theta_i}(\theta) = Df(\gamma_k(\theta)) \frac{\partial \gamma_k}{\partial \theta_i}(\theta)\,.$$

Consequently, $\left\{ \frac{\partial \gamma_k}{\partial \theta_i}(\theta) \right\}_{i=1,\, k\in\mathbb{Z}}^d$ is a family of bounded solutions of the difference equation

$$v_{k+1} = Df(\gamma_k(\theta)) v_k\,. \tag{4.1}$$

We use the approach of Section 3 by considering θ as a parameter. The difference is only that now $\left\{ \frac{\partial \gamma_k}{\partial \theta_i}(\theta) \right\}_{i=1,\, k\in\mathbb{Z}}^d$ provides a natural family of solutions of (4.1) corresponding to the projections P_{ss}. Hence we suppose that Theorem 2.2 holds parametrically by $\theta \in \mathcal{O}$, i.e. $U = U(\theta, t)$ is smooth in (θ, t) and columns of $U(\theta, t)$ are numbered so that

$$P_{uu} = \begin{pmatrix} I_d & 0_d & 0 \\ 0_d & 0_d & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad P_{ss} = \begin{pmatrix} 0_d & 0_d & 0 \\ 0_d & I_d & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now we take $x = z + \gamma(\theta), \gamma(\theta) = \{\gamma_k(\theta)\}_{k\in\mathbb{Z}}$ in (3.1). The corresponding operators of (3.2) then depend on θ as well:

$$\begin{aligned} L(z,\theta)_k &= z_{k+1} - Df(\gamma_k(\theta)) z_k \\ G(z,\theta,\mu,\epsilon,y)_k &= f(z_k + \gamma_k(\theta)) - f(\gamma_k(\theta)) - Df(\gamma_k(\theta)) z_k \\ &\quad + h(z_k + \gamma_k(\theta), \mu, k) + \epsilon|\mu|\mathcal{L}(z - y)\,. \end{aligned}$$

Consequently, (3.1) has the form

$$L(z,\theta) = G(z,\theta,\mu,\epsilon,y)\,, \tag{4.2}$$

and (3.3a-b) are replaced by

$$z = \mathcal{K}(\theta)\big(\Pi(\theta)G(z,\theta,\mu,\epsilon,y)\big) \tag{4.3a}$$

$$0 = (I - \Pi(\theta))G(z,\theta,\mu,\epsilon,y)\,, \tag{4.3b}$$

where $\mathcal{K}(\theta)$ and $\Pi(\theta)$ are corresponding mappings to $\mathcal{K}$, Π, respectively. We consider in (4.3a-b) the variable θ as a bifurcation parameter.

We take the mapping $N_\mu \colon \mathbb{R}^d \to \mathbb{R}^d$ defined by

$$\big(N_\mu(\theta)\big)_i = \sum_{j=1}^{m} a_{ij}(\theta)\mu_j\,,$$

where

$$a_{ij}(\theta) = -\sum_{l\in\mathbb{Z}} \left\langle u_i^{\perp}(\theta,l), \frac{\partial h}{\partial \mu_j}(\gamma_l(\theta),0,l)\right\rangle .$$

The vectors $u_i^{\perp}(\theta,l)$ are defined by $\langle u_i^{\perp}(\theta,l), u_j(\theta,l+1)\rangle = \delta_{ij}$. By repeating the proof of Theorem 3.2 (see also [5, p. 18]), we can state the main result of this section.

Theorem 4.1. *If N_{μ_0} has a simple zero point θ_0, i.e. θ_0 satisfies $N_{\mu_0}(\theta_0) = 0$ and $D_\theta N_{\mu_0}(\theta_0)$ is a regular matrix, then there is a wedge-shaped region in $\mathbb{R}^m$ for μ of the form*

$$\mathcal{R} = \Big\{ s\tilde{\mu} : \ s, \textit{ respectively } \tilde{\mu}, \textit{ is from a small open neighborhood of } 0 \in \mathbb{R}, \textit{ respectively of } \mu_0 \in \mathbb{R}^m \Big\}$$

such that for any $\mu \in \mathcal{R} \setminus \{0\}$, the equation (1.1) *possesses a transversal bounded solution.*

5. An Example

We consider a 4-dimensional impulsive differential equation given by

$$\begin{aligned} &\dot z = g_1(z), \quad \dot y = g_2(y) \\ &z(i+) = z(i-) + \mu h_1(z(i-), y(i-), \mu) \\ &y(i+) = y(i-) + \mu h_2(z(i-), y(i-), \mu), \quad i \in \mathbb{Z}\,, \end{aligned} \tag{5.1}$$

where

$$g_{1,2} \in C^3(\mathbb{R}^2,\mathbb{R}^2), \quad h_{1,2} \in C^3(\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R},\mathbb{R}^2), \quad \mu \in \mathbb{R}$$

and $\dot z = g_1(z)$, $\dot y = g_2(y)$ are Hamiltonian systems. Let Ψ_1, Ψ_2 be the 1-time Poincaré mappings of $\dot z = g_1(z)$, $\dot y = g_2(y)$, respectively. Here $z(i\pm) = \lim\limits_{s\to i_\pm} z(s)$.

We consider the mapping

$$F(z,y,\mu) = \Big(\Psi_1(z) + \mu h_1\big(\Psi_1(z),\Psi_2(y),\mu\big), \Psi_2(y) + \mu h_2\big(\Psi_1(z),\Psi_2(y),\mu\big)\Big)\,. \tag{5.2}$$

Clearly the dynamics of (5.2) determines the behaviour of (5.1). In the notation of (1.1), we have

$$
\begin{aligned}
(5.3) \qquad & x = (z, y) \in \mathbb{R}^2 \times \mathbb{R}^2, \quad f(x) = \big(\Psi_1(z), \Psi_2(y)\big) \\
& h(x, \mu, k) = \Big(\mu h_1\big(\Psi_1(z), \Psi_2(y), \mu\big), \mu h_2\big(\Psi_1(z), \Psi_2(y), \mu\big)\Big)\,.
\end{aligned}
$$

We suppose

a) $g_{1,2}(0) = 0$ and the eigenvalues of $Dg_{1,2}(0)$ lie off the imaginary axis.

b) There are homoclinic solutions γ_1, γ_2 of $\dot z = g_1(z)$, $\dot y = g_2(y)$, respectively, to 0.

The conditions a) and b) imply that

$$
\begin{aligned}
\gamma_k(\theta) &= \big(\gamma_1(\theta_1 + k), \gamma_2(\theta_2 + k)\big), \quad k \in \mathbb{Z} \\
\omega(\theta) &= (0, 0), \quad \theta = (\theta_1, \theta_2) \in \mathcal{O} = \mathbb{R}^2
\end{aligned}
$$

satisfy (v)-(viii) of Section 4 for (5.3). Now (4.1) has the form

$$
\begin{aligned}
(5.4) \qquad v_{k+1} &= D\Psi_1\big(\gamma_1(\theta_1 + k)\big)v_k \\
w_{k+1} &= D\Psi_2\big(\gamma_2(\theta_2 + k)\big)w_k\,.
\end{aligned}
$$

Hence (4.1) is now decomposed into two difference equations. We note that $\Psi_{1,2}$ are area-preserving. We can take

$$
\begin{aligned}
u_3(\theta, k) &= \big(\dot\gamma_1(\theta_1 + k), 0\big) \\
u_4(\theta, k) &= \big(0, \dot\gamma_2(\theta_2 + k)\big)\,.
\end{aligned}
$$

According to the computations in [4, pp. 104-105], we can take

$$
u_1^\perp(\theta, k) = \big(\bar{\dot\gamma}_1(\theta_1 + k + 1), 0\big), \quad u_2^\perp(\theta, k) = \big(0, \bar{\dot\gamma}_2(\theta_2 + k + 1)\big)\,,
$$

where $\bar z = (z_2, -z_1)$, $\forall\, z = (z_1, z_2) \in \mathbb{R}^2$, and $u_1(\theta, k)$, $u_2(\theta, k)$ are not required to be known.

Consequently, the mapping N_μ of Section 4 has now the form

$$
\begin{aligned}
\big(N_\mu(\theta)\big)_1 &= -\mu \sum_{k\in\mathbb{Z}} h_1\big(\Psi_1(\gamma_1(\theta_1 + k)), \Psi_2(\gamma_2(\theta_2 + k)), 0\big) \wedge \dot\gamma_1(\theta_1 + k + 1) \\
(5.5) \qquad &= \mu \sum_{k\in\mathbb{Z}} \dot\gamma_1(\theta_1 + k) \wedge h_1\big(\gamma_1(\theta_1 + k), \gamma_2(\theta_2 + k), 0\big) \\
\big(N_\mu(\theta)\big)_2 &= -\mu \sum_{k\in\mathbb{Z}} h_2\big(\Psi_1(\gamma_1(\theta_1 + k)), \Psi_2(\gamma_2(\theta_2 + k)), 0\big) \wedge \dot\gamma_2(\theta_2 + k + 1) \\
&= \mu \sum_{k\in\mathbb{Z}} \dot\gamma_2(\theta_2 + k) \wedge h_2\big(\gamma_1(\theta_1 + k), \gamma_2(\theta_2 + k), 0\big)\,,
\end{aligned}
$$

where $\wedge$ is the wedge product defined by $z \wedge y = z_1 y_2 - z_2 y_1$, $z, y \in \mathbb{R}^2$. Theorem 4.1 gives the following result.

Theorem 5.1. *If there is a simple zero point of $N_1(\theta)$ given by (5.5), then (5.1) has a transversal homoclinic solution and so it exhibits chaos for any $\mu \neq 0$ sufficiently small.*

Of course, there are h_1, h_2 satisfying the assumptions of Theorem 5.1. For simplicity, we assume

$$\begin{aligned} g = g_1 = g_2, \quad h_1(z,y,\mu) &= (1+\mu)y + \alpha \\ h_2(z,y,\mu) &= (1+\mu^2)z + \alpha\,, \end{aligned} \tag{5.6}$$

where $\alpha \in \mathbb{R}^2$ is a constant vector. Then we have $\gamma_1 = \gamma_2 = \gamma$ and (5.5) possesses the form

$$\begin{aligned} \big(N_\mu(\theta)\big)_1 &= \mu \sum_{k\in\mathbb{Z}} \dot\gamma(\theta_1+k) \wedge \gamma(\theta_2+k) + \mu\Big(\sum_{k\in\mathbb{Z}} \dot\gamma(\theta_1+k)\Big) \wedge \alpha \\ \big(N_\mu(\theta)\big)_2 &= \mu \sum_{k\in\mathbb{Z}} \dot\gamma(\theta_2+k) \wedge \gamma(\theta_1+k) + \mu\Big(\sum_{k\in\mathbb{Z}} \dot\gamma(\theta_2+k)\Big) \wedge \alpha\,. \end{aligned} \tag{5.7}$$

We put

$$\Omega(\tau) = \sum_{k\in\mathbb{Z}} \dot\gamma(\tau+k) \wedge \gamma(\tau+k) + \Big(\sum_{k\in\mathbb{Z}} \dot\gamma(\tau+k)\Big) \wedge \alpha\,.$$

We note that Ω is 1-periodic. We clearly for $\theta = (\tau,\tau)$ have

$$\begin{aligned} &\big(N_\mu(\theta)\big)_1 = \big(N_\mu(\theta)\big)_2 = \mu\Omega(\tau) \\ &\big(DN_\mu(\theta)\big)_1 = \mu\big(\Omega'(\tau),0\big), \quad \big(DN_\mu(\theta)\big)_2 = \mu\big(0,\Omega'(\tau)\big)\,. \end{aligned}$$

Simple computations give the following result.

Theorem 5.2. *Consider* (5.1) *with* (5.6). *If τ_0 is a simple root of $\Omega(\tau)$ then $\theta_0 = (\tau_0,\tau_0)$ is a simple zero point of $N_1(\theta)$ given by* (5.7)*.*

To be more concrete, we take in (5.6)

$$g(x_1,x_2) = \big(x_2, x_1 - 2x_1^3\big), \quad \alpha = (\beta,\beta)\,.$$

Hence (5.1) has the form

$$\begin{aligned} \ddot z &= x - 2x^3, \quad \ddot y = y - 2y^3 \\ x(i+) &= x(i-) + \mu\big((1+\mu)y(i-) + \beta\big) \\ \dot x(i+) &= \dot x(i-) + \mu\big((1+\mu)\dot y(i-) + \beta\big) \\ y(i+) &= y(i-) + \mu\big((1+\mu^2)x(i-) + \beta\big) \\ \dot y(i+) &= \dot y(i-) + \mu\big((1+\mu^2)\dot x(i-) + \beta\big), \quad i \in \mathbb{Z}\,. \end{aligned} \tag{5.8}$$

(5.8) are two Duffing equations coupled by impulsive effects. We now take $\dot\gamma(t) = \big(\operatorname{sech} t, \operatorname{sech} t\big)$ and Ω has the form

$$\Omega(\tau) = \sum_{k\in\mathbb{Z}} \operatorname{sech}^4(\tau+k) + \beta \sum_{k\in\mathbb{Z}} \frac{3 - \mathrm{e}^{2(\tau+k)}}{2} \operatorname{sech}^3(\tau+k)\,.$$

Consequently, we have

$$\Omega(\tau) = \Omega_1(\tau) - \beta\Omega_2(\tau)\,,$$

where

$$\Omega_1(\tau) = \sum_{k\in\mathbb{Z}} \operatorname{sech}^4(\tau+k), \quad \Omega_2(\tau) = \sum_{k\in\mathbb{Z}} \frac{e^{2(\tau+k)}-3}{2} \operatorname{sech}^3(\tau+k)\,.$$

The functions $\Omega_{1,2}$ are again 1-periodic. Moreover, they are analytic and Ω_1 is positive. Clearly, Ω_2/Ω_1 is non-constant. So the image of $\mathbb{R}$ by Ω_2/Ω_1 is an interval $[a_1,a_2]$, $-\infty < a_1 < a_2 < \infty$ and there is only a finite number of $\beta_1,\cdots,\beta_{j_0} \in [a_1,a_2]$ such that $\Omega = \Omega_1 - \beta\Omega_2$ does have a simple root for any $\beta \neq 0$ satisfying $1/\beta \in [a_1,a_2] \setminus \{\beta_1,\cdots,\beta_{j_0}\}$. Summarizing, we arrive at the following result.

Theorem 5.3. *There are constants $-\infty < a_1 < a_2 < \infty$ and $a_1 = \beta_1 < \cdots < \beta_{j_0} = a_2$ such that for any $\beta \neq 0$ satisfying $1/\beta \in [a_1,a_2] \setminus \{\beta_1,\cdots,\beta_{j_0}\}$, the impulsive system* (5.8) *has a chaotic behaviour for any $\mu \neq 0$ sufficiently small.*

We note that according to computations of [7, Appendix B], the functions $\Omega_{1,2}$ could be expressed in terms of elliptic and elementary functions. We do not follow this line since formulas seem to be very awkward. On the other hand, numerical derivation of the graph of $\Omega_2(\tau)/\Omega_1(\tau)$ shows that it should be

$$a_1 \simeq -0.0199198, \quad a_2 \simeq 0.0190729, \quad j_0 = 2\,.$$

Consequently, Theorem 5.3 should hold for any $\beta \neq 0$ satisfying

$$1/\beta \in (-0.019919, 0.019072)\,.$$

6. Conclusions and Future Directions

Melnikov mappings are derived for detecting transversal bounded solutions of nonautonomous difference equations. No restriction is given on the dimension of the phase space or on the dimension of intersection of stable and unstable manifolds. A complete analogy is established to the know theory for ordinary differential equations (see [3-6]). A connection between our results and similar ones for quasiperiodically perturbed systems is given via Poincaré mappings (see [1]). Finally we note that a coupled two McMillan mappings of [7] can be similarly studied. In general after applying our results, the main difficulty is to find an appropriate form of the Melnikov mapping derived in the above way so that one could be able to detect its simple zero point. The Poisson summation formula like in [7] could help to overcome this difficulty.

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