Mathematical Institute
Slovak Academy of Sciences, Bratislava

Preprint 5/2000

Ján Jakubík

On free MV-algebras

February 23, 2000

PREPRINT SERIES

ON FREE MV-ALGEBRAS

J. Jakubík, Košice

1. Introduction

Free MV-algebras have been investigated in detail in Chapter 3 of the monograph [2]. The main tool was the notion of McNaughton function (cf. [6]).

Free abelian lattice ordered groups have been studied in [8] and [9].

Each MV-algebra can be constructed by a standard method from some abelian lattice ordered group with a strong unit (cf. [7]). Thus the natural question arises whether we can construct free MV-algebras by applying free abelian lattice ordered groups.

We proceed as follows. For an abelian lattice ordered group G with the strong unit u we consider the MV-algebra $A = \Gamma(G, u)$ defined as in [7].

Let m be a cardinal with $m \neq 0$ and let X be a set, $\operatorname{card} X = m$. We choose an element u_0 which does not belong to X and we put $X_0 = X \cup \{u_0\}$. There exists a lattice ordered abelian group G_1 such that X_0 is the system of free generators for G_1.

We denote by I the ℓ-ideal of G_1 which is generated by the element $u_0 \wedge 0$ and we put $G_2 = G_1/I$. For $g_1 \in G_1$ we set $\overline{g}_1 = g + I$. Then $\overline{u}_0 > \overline{0}$.

Let us denote by G_3 the convex ℓ-subgroup of G_2 which is generated by the element $\overline{u}_0$. Then $\overline{u}_0$ is a strong unit of G_3 and hence we can construct the MV-algebra

$$A_m^0 = \Gamma(G_3, \overline{u}_0).$$

Further, we put

$$Y = \{(\overline{x} \vee \overline{0}) \wedge \overline{u}_0 : x \in X\}.$$

We denote by A_m the subalgebra of the MV-algebra A_m^0 which is generated by the set Y.

We show that $\operatorname{card} Y = m$ and we prove

(A) A_m is a free MV-algebra with the set Y of free generators.

We remark that in this construction we apply only the definition of free abelian lattice ordered group without using any results on the specific properties of free abelian lattice ordered groups which have been proven in [8] and [9].

2. Preliminaries

We apply the same notation and terminology for lattice ordered groups as in [1] and [3].

For the sake of completeness and for fixing the notation we recall some definitions and results on MV-algebras.

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We define an MV-algebra $\mathcal{A}$ as a nonempty set A with binary operations $\oplus, *$, a unary operation $\neg$ and nulary operations $0, 1$ on A such that the conditions (M1)–(M8) from [4] are satisfied; cf. also [5]. For a formally different but equivalent definition cf. [2].

If no misunderstanding can occur then we write A instead of $\mathcal{A}$.

For the following results $(*)$ and $(**)$ cf. [7].

$(*)$ Let G be an abelian lattice ordered group with a strong unig u. Let A be the interval $[0, u]$ of G. For $a, b \in A$ we put

$$a \oplus b = (a + b) \wedge u, \quad \neg a = u - a,$$

$$1 = u, \quad a * b = \neg(\neg a \oplus \neg b).$$

Then the algebraic system $\mathcal{A} = (A; \oplus, *, \neg, 0, 1)$ is an MV-algebra.

The MV-algebra from $(*)$ will be denoted by $\Gamma(G, u)$ (in [5], a different notation has been applied).

$(**)$ For each MV-algebra $\mathcal{A}$ there exists an abelian lattice ordered group G with a strong unit u such that $\mathcal{A} = \Gamma(G, u)$.

The notion of free algebra is applied in the usual sense; cf., e.g., [1], Chapter VI.

We denote by $\mathcal{C}_s$ the class of all algebraic structures

$$\mathcal{G} = (G; +, \vee, \wedge, u(\mathcal{G}))$$

such that

(i) $(G; +, \vee, \wedge)$ is an abelian lattice ordered group;
(ii) $u(\mathcal{G})$ is a nulary operation on G (i.e., a fixed element of G) such that $u(\mathcal{G})$ is a strong unit of $(G; +, \vee, \wedge)$.

Let $\mathcal{G}$ be as above and let $\mathcal{G}_1 = (G_1, +, \vee, \wedge, u(\mathcal{G}_1))$ be another element of $\mathcal{C}_s$. A mapping $\varphi : G \to G_1$ is said to be a homomorphism (with respect to $\mathcal{C}_s$) if

(i_1) φ is a homomorphism of the lattice ordered group $(G; +, \vee, \wedge)$ into the lattice ordered group $(G_1, +, \vee, \wedge)$, and
(i_2) $\varphi(u(\mathcal{G})) = u(\mathcal{G}_1))$.

3. The construction

We apply the same notation as in Section 1. In the present section we investigate in detail some steps of the construction which has been sketched in Section 1.

3.1. Lemma. $\overline{u}_0 > \overline{0}$.

Proof. We have $\overline{u}_0 \wedge \overline{0} = \overline{u_0 \wedge 0} = \overline{0}$, whence $\overline{u} \geqq \overline{0}$. By way of contradiction, suppose that $\overline{u} = \overline{0}$.

Let Z be the the additive group of all integers with the natural linear order. For each $x \in X$ we put $\varphi_0(x) = 0$; next, we set $\varphi_0(u_0) = 1$. There exists a homomorphism φ_0^1 of G_1 into the linearly ordered group Z such that $\varphi_0^1(t) = \varphi_0(t)$ for each $t \in X_0$.

Since $\varphi_0^1(u_0) = 1$, we must have $\varphi_0^1(u_0 \wedge 0) = 0$ and this yields that $I \subseteq (\varphi_0^1)^{-1}(0)$. Now if $\overline{u} = \overline{0}$, then $u_0 \in I$ and thus $\varphi_0^1(u_0) = 0$, which is a contradiction. □

3.2. Lemma. *Let $x_1, x_2 \in X$, $x_1 \neq x_2$, $y_i = (\overline{x}_i \vee \overline{0}) \wedge \overline{u}$, $(i = 1, 2)$. Then $y_1 \neq y_2$.*

Proof. Consider the lattice ordered group

$$G = Z_1 \times Z_2.$$

We definte the mapping $\varphi_0 : X_0 \to G$ as follows. We put

$$\varphi_0(x_1) = (1, 0), \quad \varphi_0(x_2) = (0, 1), \quad \varphi_0(u_0) = (1, 1),$$

and

$$\varphi_0(x) = (0, 0) \quad \text{for each } x \in X \setminus \{x_1, x_2\}.$$

There exists a homomorphism φ_0^1 of G_1 into G such that $\varphi_0^1(t) = \varphi_0(t)$ for each $t \in X_0$.

Similarly as in the proof of 3.1 we can verify that the relation

$$I \subseteq (\varphi_0^1)^{-1}(0) \tag{1}$$

is valid.

By way of contradiction, assume that $y_1 = y_2$. Put

$$z_i = (x_i \vee 0) \wedge u_0 \quad (i = 1, 2).$$

Hence $\overline{z}_i = y_i$ $(i = 1, 2)$. Therefore the element $z = z_1 - z_2$ belongs to I and hence we obtain $|z| \in I$. Thus in view of (1), $\varphi_0^1(|z|) = (0, 0)$. We have

$$\varphi_0^1(|z|) = |\varphi_0^1(z)|.$$

Further,

$$\varphi_0^1(z) = \varphi_0^1(z_1) - \varphi_0^1(z_2) = \varphi_0(z_1) - \varphi_0(z_2) = (1, -1),$$

whence

$$|\varphi_0^1(z)| = |(1, -1)| = (1, 1).$$

We obtain $(-1, 1) = (0, 0)$, which is a contradiction. $\square$

From 3.2 we immediately obtain

3.3. Corollary. $\operatorname{card} Y = m$.

We put $\overline{X}_0 = \{\overline{t} : t \in X_0\}$.

3.4. Lemma. *Let G be an abelian lattice ordered group with a strong unit u. Let φ_0^* be a mapping of the set $\overline{X}_0$ into G such that $\varphi_0^*(\overline{u}) = u$. Then there exists a homomorphism φ_{01}^* of G_2 into G which is an extension of the mapping φ_0^*.*

Proof. We define a mapping φ_0 of X_0 into G such that

$$\varphi_0(t) = \varphi_0^*(\overline{t}) \quad \text{for each } t \in X.$$

Then φ_0 can be extended to a homomorphism φ_0^1 of G_1 into G and we have

$$\varphi_{01}(u_0) = u.$$

Similarly as in the proofs of 3.1 and 3.2 we can verify that the relation (1) is satisfied.

For each $\overline{t} \in G_2$ we put

$$\varphi_{01}^*(\overline{t}) = \overline{\varphi_{01}^*(t)}.$$

In view of (1) we conclude that φ_{01}^* is a homomorphism of G_2 into G. Moreover, according to the definition of φ_0 we obtain that φ_{01}^* is an extension of the mapping φ_0^*.

□

Proof of (A).

Let A be an MV-algebra. In view of (∗∗) there exists an abelian lattice ordered group G with a strong unit u such that $A = \Gamma(G, u)$.

Let A_m be as in Section 1. Then we have $Y \subseteq A_m$. In view of the definition of A_m, Y generates the MV-algebra A_m.

Assume that ψ_0 is a mapping of the set Y into A such that

$$\psi_0(\overline{u}) = u. \tag{2}$$

We have to verify that the mapping ψ_0 can be extended to a homomorphism of A_m into A.

Let $y \in Y$. In view of 3.1 there exists a uniquely determined element $\overline{x} \in \overline{X}$ such that

$$y = (\overline{x} \vee \overline{0}) \wedge \overline{u}, \tag{3}$$

and for each $\overline{x} \in \overline{X}$ we have $y \in Y$, where y is as in (3). We consider the mapping $\varphi_0^* : \overline{X}_0 \to G$ defined by

$$\varphi_0^*(\overline{x}) = \psi_0(y),$$

where $\overline{x}$ and y are as above.

Then, in particular, in view of (2) we have

$$\varphi_0^*(\overline{u}) = u. \tag{4}$$

Let φ_{01}^* be as in 3.4. The relation (4) yields

$$\varphi_{01}^*([0, \overline{u}]) \subseteq [0, u].$$

Hence we obtain

$$\varphi_{01}^*(A_m) \subseteq A. \tag{5}$$

Consider the partial mapping

$$\varphi_{01}^* | A_m = \chi.$$

From the fact that the operations $\oplus, *$ and $\neg$ can be defined by means of the ℓ-group operation we conclude (by taking into account the relation (5) and 3.4)) that χ is a homomorphism with respect to those operation. This and (5) yield that χ is a homomorphism of A_m into A. Moreover, χ is obviously an extension of the mapping ψ_0. This completess the proof. □

4. The class $\mathcal{C}_s$

Let $\mathcal{C}_s$ be as in Section 2.

In the present section we show that as a by-product of the above investigation we obtain the possibility of constructing the free objects in the category $\mathcal{C}_s$.

Assume that $\mathcal{G}_0 = (G_0; +, \vee, \wedge, u(\mathcal{G}_0))$ is an element of $\mathcal{C}_s$. The notion of subalgebra of $\mathcal{G}$ has the usual meaning. Let $\emptyset \neq X \subseteq G_0$; if each subalgebra of $\mathcal{G}_0$ which contains X as a subset coincides with G_0 then $\mathcal{G}_0$ is said to be generated by the set X.

Suppose that $\mathcal{G}_0$ is generated by the set X and that, whenever $\mathcal{G}' = (G', +, \vee, \wedge, u(\mathcal{G}'))$ and φ_0 is a mapping of X into G', then φ_0 can be extended to a homomorphism of $\mathcal{G}$ into $\mathcal{G}'$. Under these assumption we call X the system of free generators for $\mathcal{G}_0$; we also say that $\mathcal{G}_0$ is a free object in $\mathcal{C}_s$ with m free generators, where $m = \operatorname{card} X$.

Let G_1, G_2, G_3, X and $\overline{X}_0$ be as in the construction above. The algebraic structure

$$\mathcal{G}_3 = (G_3; +, \vee, \wedge, \overline{u}_0)$$

is an element of the class $\mathcal{C}_s$ and $\overline{X}_0 \subseteq \mathcal{G}_3$.

We denote by $\mathcal{G}_3^*$ the subalgebra of $\mathcal{G}_3$ which is generated by the set $\overline{X}_0$. We write

$$\mathcal{G}_3^* = (G_3^*, +, \vee, \wedge, \overline{u}_0).$$

Let $\mathcal{G} = (G; +, \vee, \wedge, u) \in \mathcal{C}_s$. Consider the mappings φ_0^* and φ_{01}^* dealt with in 3.4. Put

$$\varphi_{02}^* = \varphi_{01}^* | G_3^*.$$

In view of 3.4 we conclude that φ_{02}^* is a homomorphism of $\mathcal{G}_3^*$ into $\mathcal{G}$ which extends the mapping φ_0^*.

Hence we have

4.1. Theorem. *$\mathcal{G}_3^*$ is the free object in the category $\mathcal{C}_s$ with the system $\overline{X}_0$ of free generators.*

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Matematický ústav SAV, Grešákova 6, 040 01 Košice, Slovakia

E-mail address: musavke@saske.sk

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