Mathematical Institute
Slovak Academy of Sciences, Bratislava

Preprint 6/2000

Anatolij Dvurečenskij
Thomas Vetterlein

Pseudo-effect algebras.
I. basic properties

May 26, 2000

PREPRINT SERIES

Pseudo-effect algebras.
I. Basic Properties

Anatolij Dvurečenskij and Thomas Vetterlein*

Mathematical Institute, Slovak Academy of Sciences,
Štefánikova 49, SK-814 73 Bratislava, Slovakia
E-mail: `dvurecen@mat.savba.sk` and `vetterl@mat.savba.sk`

Abstract

As a non-commutative generalization of effect algebras, we introduce pseudo-effect algebras and list some of their basic properties. For the purpose of a structure theory, we further define several kinds of Riesz-like properties for pseudo-effect algebras and show how they are interrelated.

AMS classification: 03G12, 81P10, 06F05, 06F15.

Key words and phrases: pseudo-effect algebra, effect algebra, *po*-groups, interval effect algebra.

0 Introduction

One way that possibly leads to a better understanding of the quantum-mechanical formalism is the examination of physically meaningful first-order structures derived from Hilbert space. The structure that probably has been most intensively studied is the orthomodular lattice of closed subspaces of the standard Hilbert space. But in spite of optimal results, concerning the foundational problem not much was gained.

*The paper has been supported by the grant 2/7193/20 SAV, Bratislava, Slovakia.

So during the last about ten years, the attention moved to another part of Hilbert space; instead of the closed subspaces, which correspond to the projection operators, all the positive operators lying below the identity, called (quantum) effects, are taken into consideration. Several different first-order axiom systems were introduced, modelled by structures the ground set of which may be chosen the set of effects; among these are the weak orthoalgebras [7], effect algebras [4], and D-posets [9]. These three theories are equivalent in the sense that each of them possesses definitional extensions to include the other ones. The first two use a partial sum, the last a partial difference operation.

In this paper, we shall generalize the second mentioned structure; we examine properties of pseudo-effect algebras that basically arise from effect algebras by dropping commutativity.

We are working towards a structure theory of pseudo-effect algebras. Unfortunately, not much is known in this respect even for effect algebras. But the situation becomes much better when postulating a certain property that may be compared to the Riesz Decomposition Property as it is known for *po*-groups. It is then in fact possible to show that our algebra is representable by an interval of a – not necessarily abelian – *po*-group. A similar work was done by Ravindran [10] for effect algebras.

The paper is divided into two parts. We give in the first one the axioms and basic properties of the new structure, and we introduce five non-equivalent properties of Riesz type. If a certain one among these is fulfilled, a pseudo-effect algebra is an interval of a *po*-group, and this is proved in the second part. Finally, we give necessary and sufficient conditions for a pseudo-effect being a pseudo-MV algebra; from this it is possible to reprove that pseudo-MV algebras are intervals in ℓ-groups.

1 Pseudo-Effect Algebras

To model algebraically the set of quantum effects, i.e. the set of self-adjoint operators of a Hilbert space between zero and identity, the notion of an effect algebra was introduced in [4]. We recall that this algebra is a structure $(E; +, 0, 1)$, where $+$ is a partial binary operation and 0 and 1 are constants, such that, for all $a, b, c \in E$, (EA1) $a+b$ is defined iff $b+a$ is defined, in which case these elements are equal, (EA2) $(a+b)+c$ is defined iff $a+(b+c)$ is defined, in which case these elements are equal, (EA3) for exactly one $d \in E$ we have $a+d=1$, (EA4) if $1+a$ is defined, then $a=0$.

We shall generalize this type of structure by dropping commutativity, i.e. giving up (EA1).

Definition 1.1 A structure $(E;+,0,1)$, where $+$ is a partial binary operation and 0 and 1 are constants, is called a *pseudo-effect algebra* if, for all $a,b,c \in E$, the following holds.

(E1) $a+b$ and $(a+b)+c$ exist if and only if $b+c$ and $a+(b+c)$ exist, and in this case $(a+b)+c = a+(b+c)$.

(E2) There is exactly one $d \in E$ and exactly one $e \in E$ such that $a+d = e+a = 1$.

(E3) If $a+b$ exists, there are elements $d,e \in E$ such that $a+b = d+a = b+e$.

(E4) If $1+a$ or $a+1$ exists, then $a=0$.

In view of (E2), we may define the two unary operations $^{\sim}$ and $^{-}$ by requiring for any $a \in E$

$$a + a^{\sim} = a^{-} + a = 1. \tag{EC}$$

Remark. We may also, and we will occasionally in the sequel, consider a pseudo-effect algebra as a structure $(E;+,{}^{\sim},{}^{-},0,1)$, where $+$ is a partial binary operation, $^{\sim}$, $^{-}$ are unary operations and 0, 1 are constants, such that (E1), (E2), (E3), (E4), and (EC) hold.

It is obvious how effect algebras are characterized among pseudo-effect algebras.

Definition 1.2 Let $(E;+,0,1)$ be a pseudo-effect algebra. We say that two elements a and b of E *commute* if $a+b$ and $b+a$ both exist and are equal.

We say that E is *commutative* if, for $a,b \in E$, $a+b$ is defined if and only if $b+a$ is defined, in which case $a+b = b+a$.

Proposition 1.3 Let $(E;+,0,1)$ be a pseudo-effect algebra. Then E is an effect algebra if and only if E is commutative.

In the sequel, by any equation to hold we mean: All sums that occur in it exist, and it holds.

Due to the law of associativity, (E1), we may denote finite sums of elements of a pseudo-effect algebra without brackets.

Lemma 1.4 Let $(E; +, 0, 1)$ be a pseudo-effect algebra. For all $a, b, c \in E$ we have:

(i) $a + 0 = 0 + a = a$ (i.e., 0 is a neutral element).

(ii) $a + b = 0$ implies $a = 0$ and $b = 0$ (positivity).

(iii) $0^\sim = 0^- = 1$, $1^\sim = 1^- = 0$.

(iv) $a^{\sim -} = a^{-\sim} = a$.

(v) $a + b = a + c$ implies $b = c$,
$b + a = c + a$ implies $b = c$ (cancellation laws).

(vi) $a + b = c$ iff $a = (b + c^\sim)^-$ iff $b = (c^- + a)^\sim$.

Proof. We will prove first (v), then (iv), (iii), (i), (ii), and finally (vi).

(v) Suppose $a + b = a + c$. Then by (E2), for some d we have $d + (a + b) = d + (a + c) = 1$, and by (E1), $(d + a) + b = (d + a) + c = 1$. By (E2), it follows $b = (d + a)^\sim = c$.

Similarly, from $b + a = c + a$ it follows $(b + a) + d = (c + a) + d = 1$ for some d, so $b + (a + d) = c + (a + d) = 1$ and $b = (a + d)^- = c$.

(iv) By (E2), we have $a + a^\sim = 1 = a^{\sim -} + a^\sim$, and so by (v), $a = a^{\sim -}$.

Similarly, $a^- + a = 1 = a^- + a^{-\sim}$, and so $a = a^{-\sim}$.

(iii) By (E2), we have $1 + 1^\sim = 1^- + 1 = 1$, and by (E4), $1^\sim = 1^- = 0$. By (iv), we get $1 = 1^{\sim -} = 0^-$ and $1 = 1^{-\sim} = 0^\sim$.

(i) By (iii) and (E1), we have $a^- + a = 1 = 1 + 1^\sim = 1 + 0 = (a^- + a) + 0 = a^- + (a + 0)$, and by (v), $a = a + 0$.

Similarly, $a + a^\sim = 1 = 1^- + 1 = 0 + 1 = (0 + a) + a^\sim$ and $a = 0 + a$.

(ii) By (i), $a + b = 0$ implies $b^\sim = a + b + b^\sim = a + 1$, which by (E4) means $a = 0$, so by (iv), (i) and (iii), $b = b^{\sim -} = (0 + 1)^- = 1^- = 0$.

(vi) Suppose $c = a + b$. Then $a + b + c^\sim = 1 = a + a^\sim$, so $b + c^\sim = a^\sim$ and $a = a^{\sim -} = (b + c^\sim)^-$. Similarly, $c^- + a + b = 1 = b^- + b$, so $b = (c^- + a)^\sim$.

Suppose $a = (b + c^{\sim})^{-}$. Then $a^{\sim} = b + c^{\sim}$, so $1 = a + b + c^{\sim}$ and $c = a + b$.

Suppose $b = (c^{-} + a)^{\sim}$. Then $b^{-} = c^{-} + a$, so $1 = c^{-} + a + b$ and $c = a + b$. $\square$

We introduce in the usual manner a partial order for pseudo-effect algebras.

Definition 1.5 Let $(E; +, 0, 1)$ be a pseudo-effect algebra. We define for $a, b \in E$

$$a \leq b \quad \text{iff} \quad a + c = b \text{ for some } c \in E.$$

Remark. From (E3) it is clear that

$$a \leq b \quad \text{iff} \quad d + a = b \text{ for some } d \in E,$$

or, in other words, our order is two-sided. This was in fact the main motivation for chosing the axiom (E3).

Lemma 1.6 Let $(E; +, 0, 1)$ be a pseudo-effect algebra. The following holds in E for all $a, a_1, b, b_1, c \in E$.

(i) $\leq$ is a partial order on E.

(ii) $a \leq b$ iff $b^{-} \leq a^{-}$ iff $b^{\sim} \leq a^{\sim}$.

That is, $^{-}$ and $^{\sim}$ are isomorphisms of the order of E onto the dual order of E.

(iii) If $a + b$ exists, $a_1 \leq a$, and $b_1 \leq b$, then also $a_1 + b_1$ exists.

(iv) $a + b$ exists iff $a \leq b^{-}$ iff $b \leq a^{\sim}$.

(v) Suppose $b + c$ exists. Then $a \leq b$ if and only if $a + c$ exists and $a + c \leq b + c$.

Suppose $c + b$ exists. Then $a \leq b$ if and only if $c + a$ exists and $c + a \leq c + b$.

Proof. (i) $a \leq a$, since $a + 0 = a$.

$a \leq b$ and $b \leq a$ imply $a + a_1 = b$ and $b + b_1 = a$ for some a_1 and b_1, so $a + a_1 + b_1 = a = a + 0$; this means $a_1 + b_1 = 0$ by Lemma 1.4(v) and $a_1 = b_1 = 0$ by Lemma 1.4(ii); so $a = b$.

$a \leq b$ and $b \leq c$ imply $a + a_1 = b$ and $b + b_1 = c$ for some b_1 and c_1, so $a + a_1 + b_1 = c$, which means $a \leq c$.

(ii) $a \leq b$ iff, for some c, $c + a = b$ iff, by Lemma 1.4(vi), for some c, $a^- = b^- + c$ iff $b^- \leq a^-$.

Similarly, $a \leq b$ iff, for some c, $a + c = b$ iff, for some c, $a^\sim = c + b^\sim$ iff $b^\sim \leq a^\sim$.

(iii) If $a + b$ exists and $a_1' + a_1 = a$ and $b_1 + b_1' = b$ for some a_1' and b_1', it follows that $(a_1' + a_1) + (b_1 + b_1')$ exists and so, by (E1), that $a_1 + b_1$ exists.

(iv) $a + b$ exists iff, for some d, $a + b + d = 1$ iff, for some d, $a^\sim = b + d$ iff $b \leq a^\sim$.

Similarly, $a + b$ exists iff, for some d, $d + a + b = 1$ iff, for some d, $b^- = d + a$ iff $a \leq b^-$.

(v) Suppose $b + c$ exists. Then $a \leq b$ iff, for some d, $d + a = b$ iff, for some d, $d + a + c = b + c$ iff $a + c$ exists and $a + c \leq b + c$.

Similarly, suppose that $c + b$ exists. Then $a \leq b$ iff, for some d, $a + d = b$ iff, for some d, $c + a + d = c + b$ iff $c + a$ exists and $c + a \leq c + b$. $\square$

Lemma 1.7 Let $(E; +, 0, 1)$ be a pseudo-effect algebra. Then for all $a, b, c \in E$ the following holds.

(i) Let $c + a$, $c + b$ and $(c + a) \wedge (c + b)$ exist. Then $a \wedge b$ and $c + (a \wedge b)$ exist, and we have $c + (a \wedge b) = (c + a) \wedge (c + b)$.

Let $a + c$, $b + c$ and $(a + c) \wedge (b + c)$ exist. Then $a \wedge b$ and $(a \wedge b) + c$ exist, and we have $(a \wedge b) + c = (a + c) \wedge (b + c)$.

(ii) Let $a \vee b$ and $c + (a \vee b)$ exist. Then $c + a$, $c + b$ and $(c + a) \vee (c + b)$ exist, and we have $c + (a \vee b) = (c + a) \vee (c + b)$.

Let $a \vee b$ and $(a \vee b) + c$ exist. Then $a + c$, $b + c$ and $(a + c) \vee (b + c)$ exist, and we have $(a \vee b) + c = (a + c) \vee (b + c)$.

Proof. (i) As $c \leq (c+a) \wedge (c+b)$, we have for some d that $c+d = (c+a) \wedge (c+b)$. Then $c + d \leq c + a, c + b$ and by Lemma 1.6(v), $d \leq a, b$. Suppose $x \leq a, b$. Then again by Lemma 1.6(v), we have $c + x \leq c + a, c + b$, i.e. $c + x \leq (c + a) \wedge (c + b) = c + d$ and $x \leq d$. It follows $d = a \wedge b$.

The second part is proved similarly.

(ii) As $a, b \leq a \vee b$, we have by Lemma 1.6(v) that $c + a$ and $c + b$ exist and $c + (a \vee b) \geq c + a, c + b$. Let $x \geq c + a, c + b$. Then for some y, z

we have $x = c + a + y = c + b + z = c + v$, where $v = a + y = b + z$. From the latter equation, we have $v \geq a, b$, so $v \geq a \vee b$ and by Lemma 1.6, $x = c + v \geq c + (a \vee b)$. It follows $c + (a \vee b) = (c + a) \vee (b + a)$.

The second part is proved similarly. $\square$

Remark 1.8 The previous Lemma 1.7 may be easily generalized to the case that infinite infima or suprema occur rather than infima or suprema of pairs of elements. So e.g., if for $a_\iota, c \in E$, where ι runs over some index set, the term $\bigwedge_\iota (c + a_\iota)$ exist, we may prove similarly as above that $c + \bigwedge_\iota a_\iota = \bigwedge_\iota (c + a_\iota)$.

In the sequel, by a property of partially ordered sets to hold in a pseudo-effect algebra E, we mean that this property holds in $(E, \leq)$, where $\leq$ is the partial order of E as introduced above. In particular, E is said to be complete, σ-complete, or atomic iff E, considered as a partial ordered set, is complete, σ-complete, or atomic, respectively. Moreover, we call E a lattice pseudo-effect algebra if $(E, \leq)$ is a lattice.

2 Interval Pseudo-Effect Algebras

We are interested in pseudo-effect algebras that arise from intervals in partially ordered groups in the following manner.

Definition 2.1 Let $(G; +, \leq)$ be a *po*-group and u a positive element of G.

(i) We denote by (G, u) the structure $(G; +, \leq, u)$, i.e. we add the element u as a constant. (G, u) is called a *unital po-group* if u is a strong unit of G, i.e. if for all $g \in G$ there is an $n \in \mathbb{N}$ such that $-n\,u \leq g \leq n\,u$.

(ii) We call the set

$$\Gamma(G, u) \stackrel{\text{def}}{=} \{g \in G^+ \colon\ g \leq u\}$$

the *unit interval* of (G, u). We denote by $(\Gamma(G, u); +, 0, u)$ the structure consisting of the unit interval of (G, u), the partial binary operation $+$ that is the restriction of the group addition to those pairs of elements of $\Gamma(G, u)$ whose sum lies again in $\Gamma(G, u)$, the neutral element of G, 0, and the positive element u.

As it is easily checked, $(\Gamma(G,u);+,0,u)$ is a pseudo-effect algebra. For $g \in \Gamma(G,u)$ we have here

$$\begin{aligned} g^{\sim} &= -a+u, \\ g^{-} &= u-a. \end{aligned}$$

Furthermore, it is clear that the order defined for $(\Gamma(G,u);+,0,u)$ coincides with the order of the *po*-group G restricted to $\Gamma(G,u)$.

Definition 2.2 (i) A pseudo-effect algebra $(E;+,0,1)$ is called an *interval pseudo-effect algebra*, if there is a unital *po*-group (G,u) such that $(E;+,0,1)$ and $(\Gamma(G,u);+,0,u)$ are isomorphic.

(ii) By a homomorphism from a pseudo-effect algebra $(E;+,0,1)$ into a unital *po*-group $(G;+,\leq,u)$, we mean a function $\varphi\colon\ E \to G$ such that φ maps into the positive cone of G, the sum, whenever defined, is preserved, $\varphi(0)$ is the neutral element and $\varphi(1)=u$.

An example of a non-commutative *po*-group leading to a non-commutative pseudo-effect algebra is the following [2, Example 4.1].

Example 2.3 Let $G = \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$; define for every two elements of G

$$(a_1,b_1,c_1)+(a_2,b_2,c_2) \stackrel{\text{def}}{=} \begin{cases} (a_1+a_2, b_1+b_2, c_1+c_2) & \text{if } a_2 \text{ is even,} \\ (a_1+a_2, b_2+c_1, b_1+c_2) & \text{if } a_2 \text{ is odd;} \end{cases}$$

and define $(a_1,b_1,c_1) \leq (a_2,b_2,c_2)$ to hold if $a_1 < a_2$ or $a_1 = a_2, b_1 \leq b_2$ and $c_1 \leq c_2$.

Then $(G;+,\leq)$ is an ℓ-group, and

$$\Gamma(G,(1,0,0)) = \{(0,b,c)\colon\ b,c \geq 0\} \cup \{(1,b,c)\colon\ b,c \leq 0\}$$

becomes a pseudo-effect algebra with the sum and constants defined according to Definition 2.1.

Both structures are non-commutative, since e.g. $(0,1,2)+(1,-2,-2) = (1,0,-1)$, but $(1,-2,-2)+(0,1,2) = (1,-1,0)$.

3 Pseudo-Effect Algebras with Riesz Properties

Our aim is to develop a structure theory for pseudo-effect algebras. But this is hardly possible in the general case. We will do here is what was proposed for effect algebras by Ravindran [10]; that is, we will assume a property that is comparable to the Riesz Decomposition Property of *po*-groups.

Now there are several different possibilities how to define a property of Riesz type for pseudo-effect algebras. In the present section we shall in fact introduce not less than five different ones. It seems to be of own interest that no one of these is equivalent to any of the other ones – but they are linearly orderable by strength.

Definition 3.1 Let $(E;+,0,1)$ be a pseudo-effect algebra.

(a) For $a, b \in E$, we write $a \mathbf{com}\, b$ to mean that for all $a_1 \leq a$ and $b_1 \leq b$, a_1 and b_1 commute.

(b) We say that E fulfils the *Riesz Interpolation Property*, (RIP) for short, if for any $a_1, a_2, b_1, b_2 \in E$ such that $a_1, a_2 \leq b_1, b_2$ there is a $c \in E$ such that $a_1, a_2 \leq c \leq b_1, b_2$.

(c) We say that E fulfils the *Weak Riesz Decomposition Property*, (RDP_0) for short, if for any $a, b_1, b_2 \in E$ such that $a \leq b_1+b_2$ there are $d_1, d_2 \in E$ such that $d_1 \leq b_1,\ d_2 \leq b_2$ and $a = d_1 + d_2$.

(d) We say that E fulfils the *Riesz Decomposition Property*, (RDP) for short, if for any $a_1, a_2, b_1, b_2 \in E$ such that $a_1 + a_2 = b_1 + b_2$ there are $d_1, d_2, d_3, d_4 \in E$ such that $d_1 + d_2 = a_1,\ d_3 + d_4 = a_2,\ d_1 + d_3 = b_1,$ $d_2 + d_4 = b_2$.

(e) We say that E fulfils the *Commutational Riesz Decomposition Property*, (RDP_1) for short, if for any $a_1, a_2, b_1, b_2 \in E$ such that $a_1 + a_2 = b_1+b_2$ there are $d_1, d_2, d_3, d_4 \in E$ such that (i) $d_1+d_2 = a_1,\ d_3+d_4 = a_2,$ $d_1 + d_3 = b_1,\ d_2 + d_4 = b_2$, and (ii) $d_2 \mathbf{com}\, d_3$.

(f) We say that E fulfils the *Strong Riesz Decomposition Property*, (RDP_2) for short, if for any $a_1, a_2, b_1, b_2 \in E$ such that $a_1+a_2 = b_1+b_2$ there are $d_1, d_2, d_3, d_4 \in E$ such that (i) $d_1 + d_2 = a_1,\ d_3 + d_4 = a_2,\ d_1 + d_3 = b_1,$ $d_2 + d_4 = b_2$, and (ii) $d_2 \wedge d_3 = 0$.

It is clear that the **com**-relation is symmetric.

We note further that from the Riesz Decomposition Property, (RDP), already one special case of what is required in condition (ii) of (RDP_1) easily follows: With respect to the notations used in (d) and (e) above, we have from (RDP) that d_2 and d_3 commute. Indeed, $d_1 + d_2 + d_3 + d_4 = a_1 + a_2 = b_1 + b_2 = d_1 + d_3 + d_2 + d_4$ and so, by Lemma 1.4(v), $d_2 + d_3 = d_3 + d_2$.

Lemma 3.2 Let $(E; +, 0, 1)$ be a pseudo-effect algebra fulfilling (RDP_0).

(i) Let $a, b, c \in E$ and $a + b$ exist. Then from $a\,\mathbf{com}\,c$ and $b\,\mathbf{com}\,c$ it follows $a + b\ \mathbf{com}\ c$.

(ii) If $a \wedge b = 0$, then $a + b,\ b + a$ and $a \vee b$ exist and are all equal.

Proof. (i) Suppose $a\,\mathbf{com}\,c$ and $b\,\mathbf{com}\,c$ and $d \leq a + b,\ c_1 \leq c$. Due to ($\mathrm{RDP}_0$) there are elements d_1 and d_2 such that $d_1 \leq a,\ \ d_2 \leq b$ and $d = d_1 + d_2$. By assumption each of d_1 and d_2 commutes with c_1; so also $d = d_1 + d_2$ commutes with c_1.

(ii) Let c be given such that $c \geq a, b$; there is of course at least one such element. So for some x, $c = x + b$, and by (RDP_0) there are elements $d_1 \leq x,\ d_2 \leq b$ such that $a = d_1 + d_2$. Since $d_2 \leq a, b$ we have $d_2 = 0$, and so $a \leq x$. It follows $c = x + b \geq a + b$. In particular, $a + b$ exists, and as $a + b$ is an upper bound of a and b, it follows $a + b = a \vee b$. □

Proposition 3.3 Let $(E; +, 0, 1)$ be a pseudo-effect algebra.

(i) We have the implications
$$(\mathrm{RDP}_2) \Rightarrow (\mathrm{RDP}_1) \Rightarrow (\mathrm{RDP}) \Rightarrow (\mathrm{RDP}_0) \Rightarrow (\mathrm{RIP}).$$
The converse of any of these implications does not hold.

(ii) E fulfils (RDP_2) if and only if E is lattice-ordered and fulfils (RDP_0).

(iii) Let E be commutative. Then we have the implications
$$(\mathrm{RDP}_2) \Rightarrow (\mathrm{RDP}_1) \Leftrightarrow (\mathrm{RDP}) \Leftrightarrow (\mathrm{RDP}_0) \Rightarrow (\mathrm{RIP}).$$
Any implication not shown here does not hold.

Proof. (i) (RDP_2) implies (RDP_1) by Lemma 3.2(ii); (RDP_1) implies trivially (RDP); and it is evident that (RDP) implies (RDP_0).

Now suppose (RDP_0) and $a, b \leq c, d$. Then there is an $a_1 \in E$ such that $a + a_1 = c$, and from $b \leq a + a_1$ follows by (RDP_0) that there are $e \leq a$ and $\bar{b} \leq a_1$ such that $b = e + \bar{b}$. Moreover, for some $\bar{a} \in E$ we have $a = e + \bar{a}$, and since $e \leq a, b$, for some $\bar{c}, \bar{d}$ also $c = e + \bar{c}$ and $d = e + \bar{d}$.

By Lemma 1.6(v) we have $\bar{a}, \bar{b} \leq \bar{c}, \bar{d}$, and from $e + \bar{a} + \bar{b} \leq a + a_1 = c = e + \bar{c}$ it follows that $\bar{a} + \bar{b} \leq \bar{c}$.

Now chose $b_1 \in E$ such that $b_1 + \bar{b} = \bar{d}$, and since $\bar{a} \leq b_1 + \bar{b}$, there are $\bar{\bar{a}} \leq b_1$ and $f \leq \bar{b}$ such that $\bar{a} = \bar{\bar{a}} + f$. Chose $\bar{\bar{b}} \in E$ such that $\bar{b} = \bar{\bar{b}} + f$.

Let $\bar{x} = \bar{\bar{a}} + \bar{\bar{b}} + f$. Then $\bar{a}, \bar{b} \leq \bar{x} \leq \bar{a} + \bar{b} \leq \bar{c}$ as well as $\bar{x} \leq b_1 + \bar{b} = \bar{d}$. So $x = e + \bar{x}$ is the interpolant required to show that (RIP) holds.

The examples below show that the converse of the implications does not hold in any case.

(ii) Let E fulfil (RDP_2). By (i), E fulfils (RDP_0).

Let $a, b \in E$. Then from $a + a^\sim = b + b^\sim$ it follows that there are elements d_1, d_2, d_3, d_4 such that $d_1 + d_2 = a,\ d_3 + d_4 = a^\sim,\ d_1 + d_3 = b,\ d_2 + d_4 = b^\sim$ and $d_2 \wedge d_3 = 0$. Then according to Lemma 3.2(ii) we have $d_2 + d_3 = d_2 \vee d_3$, and according to Lemma 1.7(ii) we have $d_1 + d_2 + d_3 = d_1 + (d_2 \vee d_3) = (d_1 + d_2) \vee (d_1 + d_3) = a \vee b$. So all suprema exist.

It is easy to see that $(a^\sim \vee b^\sim)^- = a \wedge b$. So also all infima exist. Hence E is lattice-ordered.

Let E be lattice-ordered and fulfil (RDP_0). Let $a_1 + a_2 = b_1 + b_2$. Set $d_1 = a_1 \wedge b_1$ and d_2, d_3 in such a way that $a_1 = d_1 + d_2,\ b_1 = d_1 + d_3$.

We claim that $d_2 \leq b_2$. Indeed, from $a_1 \leq b_1 + b_2$ we get by (RDP_0) that $a_1 = e_1 + e_2$ for some $e_1 \leq b_1,\ e_2 \leq b_2$, and from $e_1 \leq a_1 \wedge b_1 = d_1$ we have $a_1 \leq d_1 + b_2$, so we conclude $d_2 \leq b_2$. Choose $d_4 \in E$ such that $d_2 + d_4 = b_2$. Similarly, we may choose $d'_4 \in E$ such that $d_3 + d'_4 = a_2$.

Furthermore, $d_2 \wedge d_3 = 0$. For, by Lemma 1.7(i) we have $d_1 = a_1 \wedge b_1 = (d_1 + d_2) \wedge (d_1 + d_3) = d_1 + (d_2 \wedge d_3)$. From this we conclude $d_1 + d_2 + d_3 + d'_4 = a_1 + a_2 = b_1 + b_2 = d_1 + d_3 + d_2 + d_4 = d_1 + d_2 + d_3 + d_4$, so $d'_4 = d_4$. So d_1, d_2, d_3, d_4 fulfil the requirements of Definition 3.1(f) of (RDP_2).

(iii) (RDP_1) trivially follows from (RDP). (RDP) is to be derived from (RDP_0) in the obvious manner [10, Lemma 2.12]. The other implications are proved in (i).

The examples below show that none of the implications not shown holds. □

For every possible combination of the different kinds of Riesz properties to hold or not to hold in a pseudo-effect agebra, we will now give one example.

As an example of a pseudo-effect algebra in which (RDP_2) holds, a unit interval in any ℓ-group may serve; see e.g. [5, Theorem V.1]. In this way, a non-commutative pseudo-effect algebra has already been constructed; see Example 2.3.

Example 3.4 Let E be the set of rational functions from the real unit interval to itself such that no singularities occur. Define $+$ to be the pointwise addition of two such functions whenever this leads to a result within E; let 0 and 1 the constant functions with value 0 and 1 respectively. Then $(E; +, 0, 1)$ is obviously an effect algebra and a fortiori a pseudo-effect algebra. We shall see that (RDP_1) holds in E, but (RDP_2) does not.

If for $f_1, f_2, g_1, g_2 \in E$ the equation $f_1 + f_2 = g_1 + g_2$ holds, then the four, continuously extended whenever necessary, functions $\frac{f_1 g_1}{g_1+g_2}, \frac{f_1 g_2}{g_1+g_2}, \frac{f_2 g_1}{g_1+g_2}, \frac{f_2 g_2}{g_1+g_2}$ obviously fulfil the requirements of the definition of (RDP). So by commutativity, (RDP_1) holds in E.

On the other hand, (RDP_2) would by Proposition 3.3(ii) force an infimum of any two functions of E to exist in E, which is not the case. So (RDP_2) fails to hold.

Example 3.5 (variation of [6, Example 3.10]) Let G be an additive group generated by the countably many elements $g_0, g_1, \ldots$; let $v\colon (G; +) \to (\mathbb{R}, +)$ be the homomorphism determined by the conditions $v(g_i) = (\frac{1}{2})^i$, $i = 0, \ldots$; and let G fulfil the condition that every $a \in G$ such that $v(a) = 0$ commutes with any other $b \in G$. Define a partial order in G by setting $G^+ \stackrel{\text{def}}{=} \{x \in G\colon\ x = 0 \text{ or } v(x) > 0\}$; this means that we have for $a, b \in G$

$$a \leq b \text{ iff } a = b \text{ or } v(a) < v(b). \tag{1}$$

We shall see that $\Gamma(G, g_0)$ fulfils (RDP), but not (RDP_1).

To prove (RDP), let $a_1, a_2, b_1, b_2 \in \Gamma(G, g_0)$ such that $a_1 + a_2 = b_1 + b_2$; we may suppose $a_1, a_2, b_1, b_2 > 0$. We have to show that for some $k \in G$ the scheme

$$
\begin{array}{ccccc}
a_1 - k & k & \to & a_1 \\
k - a_1 + b_1 & -k + b_2 & \to & a_2 \\
\downarrow & \downarrow & & \\
b_1 & b_2 & &
\end{array}
\tag{2}
$$

holds and every element in it is in $\Gamma(G, g_0)$.

If $v(a_1) < v(b_1)$, we put $k = 0$, if $v(a_1) > v(b_1)$, we put $k = -b_1 + a_1$. Let now $v(a_1) = v(b_1)$ and, since the case $a_1 = b_1$ is trivial, let $a_1 \neq b_1$. Since then $-a_1 + b_1$ commutes with k however chosen, we may put $k = g_i$, chosing i large enough to make every element in (2) strictly positive. So (RDP) is proved.

Now consider again an equation $a_1 + a_2 = b_1 + b_2$ where $0 < a_1, a_2, b_1, b_2 \leq g_0$, $v(a_1) = v(b_1)$ and $a_1 \neq b_1$. (2) to hold means that k can be neither 0 nor $-b_1 + a_1$. It follows that for a sufficiently large natural number i, we have $g_i \leq k - a_1 + b_1$ and $g_{i+1} \leq k$; but g_i and g_{i+1} do not commute. So (RDP_1) does not hold.

Example 3.6 Similar to the previous example, let G be the additive group generated freely by countably many elements $g_0, g_1, \ldots$; let $v\colon (G;+) \to (\mathbb{R},+)$ be the homomorphism determined by the conditions $v(g_{2i}) = v(g_{2i+1}) = (\frac{1}{2})^i$, $i = 0, \ldots$. Define a partial order in G by setting $G^+ \stackrel{\text{def}}{=} \{x \in G\colon x = 0 \text{ or } v(x) > 0\}$; then again (1) holds. We shall see that $\Gamma(G, g_0)$ fulfils (RDP_0), but not (RDP).

Indeed, let $a, b_1, b_2 \in \Gamma(G, g_0)$ such that $a \leq b_1 + b_2$. By (1), it is then clear that for some $k \in \Gamma(G, g_0)$ we have $0 \leq a - k \leq b_1$ and $0 \leq k \leq b_2$. So (RDP_0) holds.

On the other hand, consider the equation $g_2 + (-g_2 + 3g_4) = g_3 + (-g_3 + 3g_4)$, and suppose that there are four elements as required by Definition 3.1(d). This means for some $k \in G$ that the following scheme holds:

$$
\begin{array}{ccccc}
g_2 - k & k & \to & g_2 \\
\genfrac{}{}{0pt}{}{k - g_2 + g_3}{\genfrac{}{}{0pt}{}{=}{-g_2 + g_3 + k}} & -k - g_3 + 3g_4 & \to & -g_2 + 3g_4 \\
\downarrow & \downarrow & & \\
g_3 & -g_3 + 3g_4. & &
\end{array}
\tag{3}
$$

We have especially $k+(-g_2+g_3)=(-g_2+g_3)+k$. From this it follows that $k=z(-g_2+g_3)$ for some $z\in\mathbb{Z}$. Now k is required to be in G^+, which is only the case for $z=0$. But then $k-g_2+g_3=-g_2+g_3\notin G^+$. It follows that (RDP) does not hold.

Example 3.7 Let $(E;+,0,1,)$ be the diamond [3, Example 1.9.23]; that is, let $E=\{a,b,0,1\}$, let $+$ be defined iff one argument is 0 or both arguments are a or both are b, in which latter cases the sum be 1. Then E is an effect algebra and a fortiori a pseudo-effect algebra.

As it is easily checked, E fulfils (RIP). But it does not fulfil (RDP_0), as is seen e.g. from the inequality $a\leq b+b$.

Example 3.8 An example of an effect algebra, so also of a pseudo-effect algebra, that does not fulfil (RIP) is the standard effect algebra $(\mathcal{E}(H);+,0,I)$, where $\mathcal{E}(H)$ is the set of positive operators less than identity in an at least two-dimensional complex Hilbert space, $+$ is the sum of two operators defined when it is less than identity, and 0 is the zero, I the identity operator.

Let us first show that (RIP) fails when H is two-dimensional, i.e. $H=\mathbb{C}^2$. The positive operators of H are those self-adjoint operators whose determinant and trace are ≥ 0; so their matrices have the form

$$P=\begin{pmatrix} t-x & y-iz \\ y+iz & t+x \end{pmatrix}$$

for some $x,y,z,t\in\mathbb{R}$ such that $x^2+y^2+z^2\leq t^2$ and $t\geq 0$. Let

$$A_1=\tfrac{1}{18}\begin{pmatrix} 1 & 0 \\ 0 & 7 \end{pmatrix},\ A_2=\tfrac{1}{18}\begin{pmatrix} 7 & 0 \\ 0 & 1 \end{pmatrix},$$
$$B_1=\tfrac{1}{18}\begin{pmatrix} 9 & 4 \\ 4 & 9 \end{pmatrix},\ B_2=\tfrac{1}{18}\begin{pmatrix} 9 & -4 \\ -4 & 9 \end{pmatrix}.$$

Then we have $0\leq A_1,A_2\leq B_1,B_2\leq I$. Suppose for some operator $C=\begin{pmatrix} t-x & y-iz \\ y+iz & t+x \end{pmatrix}$ that $A_1,A_2\leq C\leq B_1,B_2$. The conditions $C-A_1$, $C-A_2$, B_1-C, $B_2-C\geq 0$ lead to

$$\begin{gathered}\tfrac{2}{9} \le t \le \tfrac{1}{2},\\ (|x| + \tfrac{1}{6})^2 + y^2 + z^2 \le (t - \tfrac{2}{9})^2,\\ x^2 + (|y| + \tfrac{2}{9})^2 + z^2 \le (\tfrac{1}{2} - t)^2,\end{gathered}$$

from which we further derive $t \geq \frac{7}{18}$ as well as $t \leq \frac{5}{18}$. So (RIP) does not hold in the two-dimensional case.

Now suppose H to have any dimension greater than or equal to 2, and define the operators A_1, A_2, B_1, B_2, when restricted to some two-dimensional subspace H_0, as above, and let them map to 0 on $H_0^\perp$. Now for another operator C, the condition $C \leq B_1$ means that also the kernel of C includes $H_0^\perp$. So by the same reasoning as before, (RIP) fails to hold.

We note that $\mathcal{E}(H)$ is, like the other examples given above, an interval pseudo-effect algebra; we have $\mathcal{E}(H) = \Gamma(\mathcal{B}(H)_{\mathrm{sa}}, I)$, where $\mathcal{B}(H)_{\mathrm{sa}}$ is the set of all bounded self-adjoint operators of H.

A further property of Riesz type involves any finite number of elements rather than a maximum of four.

To make collections of formulas as they will appear in the sequel easier readable, we use the following abbreviation. Let $d_{ij}, a_i, b_j \in E$, $1 \leq i \leq m$, $1 \leq j \leq n$. By

$$\begin{array}{ccccc} d_{11} & \ldots & d_{1n} & \to & a_1 \\ \vdots & & \vdots & & \vdots \\ d_{m1} & \ldots & d_{mn} & \to & a_m \\ \downarrow & & \downarrow & & \\ b_1 & \ldots & b_n & & \end{array}$$

we mean $d_{i1} + \cdots + d_{in} = a_i$ for $i = 1, ..., m$ and $d_{1j} + \cdots + d_{mj} = b_j$ for $j = 1, ..., n$.

Lemma 3.9 Let $(E; +, 0, 1)$ be a pseudo-effect algebra fulfilling (RDP_1). Let

$$a_1 + \cdots + a_m = b_1 + \cdots + b_n,$$

where $m, n \geq 1$. Then there are elements $d_{11}, \ldots, d_{mn} \in E$ such that

$$\begin{array}{ccccc} d_{11} & \ldots & d_{1n} & \rightarrow & a_1 \\ \vdots & & \vdots & & \vdots \\ d_{m1} & \ldots & d_{mn} & \rightarrow & a_m \\ \downarrow & & \downarrow & & \\ b_1 & \ldots & b_n & & \end{array}$$

and such that for $1 \leq i < m,\ 1 \leq j < n$ we have

$$d_{i+1,j} + \cdots + d_{mj} \ \textbf{com}\ d_{i,j+1} + \cdots + d_{in}.$$

Proof. The Lemma is trivial for $m = 1$ or $n = 1$, and it is true for $m = n = 2$ due to (RDP_1).

Suppose it is true for any pair of integers such that the first is less or equal to m, where $m \geq 2$, and the second is less than n, where $n \geq 3$. From this condition we shall prove that the Lemma is true also for the pair m, n. By complete induction it then follows to be true in general.

So let $a_1 + \cdots + a_m = b_1 + \cdots + b_n$. By the induction hypothesis, there are elements $d_{11}, \ldots, d_{m,n-2}, e_1, \ldots, e_m \in E$ such that

$$\begin{array}{cccccc} d_{11} & \ldots & d_{1,n-2} & e_1 & \rightarrow & a_1 \\ \vdots & & \vdots & \vdots & & \vdots \\ d_{m1} & \ldots & d_{m,n-2} & e_m & \rightarrow & a_m \\ \downarrow & & \downarrow & \downarrow & & \\ b_1 & \ldots & b_{n-2} & b_{n-1} + b_n & & \end{array}$$

and for $1 \leq i < m,\ 1 \leq j \leq n-2$ we have

$$d_{i+1,j} + \cdots + d_{mj} \ \textbf{com}\ d_{i,j+1} + \cdots + d_{i,n-2} + e_i.$$

Moreover, from $e_1 + \cdots + e_m = b_{n-1} + b_n$ we have by the same hypothesis that there are elements $d_{1,n-1}, d_{1n}, \ldots, d_{m,n-1}, d_{mn} \in E$ such that

$$\begin{array}{cccc}
d_{1,n-1} & d_{1n} & \to & e_1 \\
\vdots & \vdots & & \vdots \\
d_{m,n-1} & d_{mn} & \to & e_m \\
\downarrow & \downarrow & & \\
b_{n-1} & b_n & &
\end{array}$$

and for $1 \leq i \leq m-1$

$$d_{i+1,n-1} + \cdots + d_{m,n-1} \ \mathbf{com} \ d_{in}.$$

It follows that the Lemma is true for the pair m, n. □

The following consequences of the Weak Riesz Decomposition Property seem to be notable.

Proposition 3.10 Every finite pseudo-effect algebra $(E;+,0,1)$ fulfilling (RDP_0) is commutative.

Proof. Let $(E;+,0,1)$ be finite. Then every element $a \in E$ is the sum of finitely many atoms of E. Indeed, below every non-zero $a \in E$ lies an atom; otherwise E would not be finite. So choose an atom $e_1 \leq a$; if $a \neq e_1$, choose an atom e_2 below e_1', where $e_1 + e_1' = a$; and so on. Then for some n we have $a = e_1 + \cdots + e_n$; otherwise E would not be finite.

Now suppose that E fulfils (RDP_0). Then any two atoms of E commute. For either they are equal, or their infimum is 0, whence, due to Lemma 3.2(ii), their sum exists and does not depend on the order. So we conclude that any two summable elements of E commute. □

Proposition 3.11 Every complete and atomic lattice pseudo-effect algebra $(E;+,0,1)$ fulfilling (RDP_0) is commutative.

Proof. We claim that for every atom $a \in E$ and every $x \in E$, there is largest $n \in \mathbb{N}$ such that $n\,a \leq x$; we will denote this number by $j_a(x)$. For suppose there is no such n. Then $m\,a$ is defined for any m, so by Lemma 1.6(iv) and the completeness of E, we have $m\,a \leq a^-$ for any m, so $\bigvee_m m\,a \leq a^-$, which means that $(\bigvee_m m\,a) + a$ exists. By Remark 1.8 we conclude $(\bigvee_m m\,a) + a = \bigvee_m (m\,a + a) = \bigvee_m m\,a$, that is, $a = 0$.

It is now easy to see that for every $x \in E$ we may write $x = \bigvee_{a \in \mathcal{A}(E)} j_a(x)$, where $\mathcal{A}(E)$ is the set of all atoms of E.

As in the proof of the previous Proposition 3.10 we see that multiples of two atoms commute. So by Remark 1.8 we have, for any $x, y \in E$, $x + y = \bigvee_{a \in \mathcal{A}(E)} j_a(x)a + \bigvee_{a \in \mathcal{A}(E)} j_a(y)a = \bigvee_{a \in \mathcal{A}(E)} \bigvee_{b \in \mathcal{A}(E)} (j_a(x)a + j_b(y)b) = y + x$. $\square$

4 *po*-groups with Riesz Properties

We define the different properties of Riesz type for *po*-groups in exact analogy to those of pseudo-effect algebras.

Definition 4.1 Let $(G; +, \leq)$ be a directed *po*-group with neutral element 0.

(a) For $a, b \geq 0$, we write $a \,\mathbf{com}\, b$ to mean that for all a_1, b_1 such that $0 \leq a_1 \leq a$ and $0 \leq b_1 \leq b$, a_1 and b_1 commute.

(b) We say that G fulfils the *Riesz Interpolation Property*, (RIP) for short, if for any $a_1, a_2, b_1, b_2 \geq 0$ such that $a_1, a_2 \leq b_1, b_2$ there is a $c \in G$ such that $a_1, a_2 \leq c \leq b_1, b_2$.

(c) We say that G fulfils the *Weak Riesz Decomposition Property*, (RDP_0) for short, if for any $a, b_1, b_2 \geq 0$ such that $a \leq b_1 + b_2$ there are $d_1, d_2 \in G$ such that $0 \leq d_1 \leq b_1$, $0 \leq d_2 \leq b_2$ and $a = d_1 + d_2$.

(d) We say that G fulfils the *Riesz Decomposition Property*, (RDP) for short, if for any $a_1, a_2, b_1, b_2 \geq 0$ such that $a_1 + a_2 = b_1 + b_2$ there are $d_1, d_2, d_3, d_4 \geq 0$ such that $d_1 + d_2 = a_1$, $d_3 + d_4 = a_2$, $d_1 + d_3 = b_1$, $d_2 + d_4 = b_2$.

(e) We say that G fulfils the *Commutational Riesz Decomposition Property*, (RDP_1) for short, if for any $a_1, a_2, b_1, b_2 \geq 0$ such that $a_1 + a_2 = b_1 + b_2$ there are $d_1, d_2, d_3, d_4 \geq 0$ such that (i) $d_1 + d_2 = a_1$, $d_3 + d_4 = a_2$, $d_1 + d_3 = b_1$, $d_2 + d_4 = b_2$, and (ii) $d_2 \,\mathbf{com}\, d_3$.

(f) We say that G fulfils the *Strong Riesz Decomposition Property*, (RDP_2) for short, if for any $a_1, a_2, b_1, b_2 \geq 0$ such that $a_1 + a_2 = b_1 + b_2$ there are $d_1, d_2, d_3, d_4 \geq 0$ such that (i) $d_1 + d_2 = a_1$, $d_3 + d_4 = a_2$, $d_1 + d_3 = b_1$, $d_2 + d_4 = b_2$, and (ii) $d_2 \wedge d_3 = 0$.

Proposition 4.2 Let $(G; +, \leq)$ be a directed *po*-group.

(i) We have the implications
$$(\mathrm{RDP}_2) \Rightarrow (\mathrm{RDP}_1) \Rightarrow (\mathrm{RDP}) \Rightarrow (\mathrm{RDP}_0) \Leftrightarrow (\mathrm{RIP}).$$
There are no more implications holding between these conditions.

(ii) G fulfils (RDP_2) if and only if G is lattice-ordered.

(iii) Let G be abelian. Then we have the implications
$$(\mathrm{RDP}_2) \Rightarrow (\mathrm{RDP}_1) \Leftrightarrow (\mathrm{RDP}) \Leftrightarrow (\mathrm{RDP}_0) \Leftrightarrow (\mathrm{RIP}).$$
The implication not shown does not hold.

Proof. (ii) Let (RDP_2) hold. Similarly as for Lemma 3.2(ii) we may prove that, for $a, b \in G$, from $a \wedge b = 0$ it follows $a \vee b = a + b$. Furthermore, it is easy to see that, for $a, b, c \in G$, if $a \vee b$ exists we have $c + (a \vee b) = (c + a) \vee (c + b)$. Now we may analogously to Proposition 3.3(ii) prove that, for $a, b \in G^+$, $a \vee b$ always exists.

Using [1, XIII, §3, Eq. (7)], we conclude that, for $a, b \in G^+$, $(a \vee b) - b = (a - b) \vee 0$ holds. Since G is directed, we have $G = G^+ - G^+$ [5, Proposition II.1]. So for all $g \in G$, $g \vee 0$ exists, which by [1, XIII, §3, Lemma 1] means that G is an ℓ-group.

Conversely, in all ℓ-groups, (RDP_2) holds [5, Theorem V.1].

(i) If (RDP_2) holds, G is by (ii) an ℓ-group. Elements a and b in an ℓ-group such that $a \wedge b = 0$ commute [1, XIII, §3, Eq. (13)]; so (RDP_1) holds. From (RDP_1) follows (RDP), and from (RDP) follows (RDP_0) trivially. (RDP_0) and (RIP) are equivalent according to [6, Theorem 2.2].

That the converse implications in three cases do not hold is seen from the examples given above for pseudo-effect algebras. So the group of rational functions from $[0, 1]$ to $\mathbb{R}$ with no singularities – see Example 3.4 – fulfils (RDP_1), but not (RDP_2). The group described in Example 3.5 fulfils (RDP), but not (RDP_1). The group described in Example 3.6 fulfils (RIP), but not (RDP).

(iii) (RDP_1) trivially follows from (RDP). (RDP) is to be derived from (RDP_0) in the obvious manner [8, Proposition 2.1]. The other implications are proved in (i).

Example 3.4 shows that from (RDP_1) (RDP_2) does not follow. □

References

[1] G. Birkhoff, "Lattice Theory," Colloquium Publications 25, American Mathematical Society, Providence, 3rd ed. 1995.

[2] M. R. Darnel, "Theory of Lattice-Ordered Groups," Marcel Dekker Inc., New York, 1995.

[3] A. Dvurečenskij, S. Pulmannová, "New Trends in Quantum Structures," Kluwer Acadamic Publ., Dordrecht, Ister Science, Bratislava, 2000, to appear.

[4] D. J. Foulis, M. K. Bennett, Effect algebras and unsharp quantum logics, *Found. Phys.* **24** (1994), 1325–1346.

[5] L. Fuchs, "Partially Ordered Algebraic Systems," Pergamon Press, Oxford, 1963.

[6] L. Fuchs, Riesz groups, *Annali della Scuola Norm. Sup. Pisa*, III. Ser. **19** (1965), 1-34.

[7] R. Giuntini, H. Greuling, Towards a formal language for unsharp properties, *Found. Phys.* **19** (1989), 931-945.

[8] K. R. Goodearl, "Partially Ordered Abelian Groups with Interpolation," Mathematical Surveys and Monographs 20, American Mathematical Society, Providence, 1986.

[9] F. Kôpka, F. Chovanec, D-posets, *Math. Slovaca* **44** (1994), 21-34.

[10] K. Ravindran, On a structure theory of effect algebras, PhD Thesis, Manhattan, 1996.

This preprint series was founded in 1996. Its purpose is to present manuscripts of submitted or unpublished papers and reports of fellows of the Mathematical Institute of the Slovak Academy of Sciences in Bratislava and its Košice branch. The authors are fully responsible for the content of the preprints. Postal addresses:

Mathematical Institute
Slovak Academy of Sciences
Štefánikova 49
SK–814 73 Bratislava
SLOVAKIA

Tel./Fax (+421 7) 5249 7316

Mathematical Institute
Slovak Academy of Sciences
Grešákova 6
SK–040 01 Košice
SLOVAKIA

Tel./Fax (+421 95) 6228291

PostScript files of preprints are available at WWW address:
`http://www.mat.savba.sk/preprints`

Latest preprints of the series:

6/2000 *Anatolij Dvurečenskij, Thomas Vetterlein*: Pseudo-effect algebras. I. basic properties

5/2000 *Ján Jakubík*: On free MV-algebras

4/2000 *Anatolij Dvurečenskij*: Pseudo MV-algebras are intervals in l-groups

3/2000 *Anna Jenčová*: Geometry of quantum states: dual connections and divergence functions

2/2000 *Anatolij Dvurečenskij*: MV-observables and MV-algebras

1/2000 *Michal Fečkan*: Transversal bounded solutions for difference equations

21/1999 *Roman Frič*: Lukasiewicz tribes are absolutely sequentially closed bold algebras

20/1999 *Miroslav Repický*: Spaces not distinguishing convergences

19/1999 *Beloslav Riečan*: On the probability theory on MV algebras

18/1999 *Beloslav Riečan, Jozefína Petrovičová*: Kolmogorov-Sinai theorem for fuzzy T-dynamical systems

17/1999 *Beloslav Riečan*: On a Banach space valued observables

16/1999 *Michal Fečkan*: Criteria on the nonexistence of invariant submanifolds for dynamical systems

15/1999 *Anatolij Dvurečenskij*: On Loomis-Sikorski's theorem for MV-algebras and BCK-algebras

14/1999 *Otakar Grošek, Ladislav Satko, Karol Nemoga*: ZKDWCNDPDC...

13/1999 *Otakar Grošek, Karol Nemoga, Ladislav Satko*: A remark to Luby-Rackoff and Ueli M. Maurer pseudorandom generators

12/1999 *Beloslav Riečan*: On the sum of observables in product MV algebras

11/1999 *Peter Eliaš*: A hierarchy of thin sets related to the boundedness of trigonometric series

10/1999 *Miloslav Duchoň, Peter Maličký*: On Herglotz theorem in partially ordered vector spaces

9/1999 *Ján Jakubík*: Torsion classes of specker lattice ordered groups

8/1999 *Ján Jakubík*: Graph automorphisms and cells of lattices

7/1999 *Miloslav Duchoň, Peter Maličký*: Positive linear maps on trigonometric polynomials

6/1999 *Miloslav Duchoň, Anna Kolesárová*: Fourier transforms of vector-valued measures on a certain compact semigroup

5/1999 *Miroslav Ploščica*: Separation properties in congruence lattices of lattices

(continued on inside back cover)

4/1999 *Georges Chevalier, Anatolij Dvurečenskij, Karl Svozil*: Piron's and Bell's geometric lemmas and Gleason's theorem

3/1999 *Jarmila Hedlíková*: Relatively orthomodular lattices

2/1999 *Anatolij Dvurečenskij, Pekka Lahti, Kari Ylinen*: Some remarks on the moment problem

1/1999 *Anatolij Dvurečenskij*: Loomis-Sikorski theorem for σ-complete MV-algebras and ℓ-groups

27/1998 *Martin Kochol*: Cubic graphs without a Petersen minor have nowhere-zero 5-flows

26/1998 *Anatolij Dvurečenskij, Ferdinand Chovanec, Eva Rybáriková*: D-homomorphisms and atomic σ-complete Boolean D-posets

25/1998 *Anatolij Dvurečenskij*: Commutative BCK-algebras with product

24/1998 *Anatolij Dvurečenskij, Antonio Di Nola*: Product MV-algebras

23/1998 *Anatolij Dvurečenskij, Maria Gabriella Graziano*: Dedekind complete commutative BCK-algebras

22/1998 *Anatolij Dvurečenskij*: Ideals of commutative BCK-algebras

21/1998 *Anatolij Dvurečenskij, Beloslav Riečan*: Weakly divisible MV-algebras and product

20/1998 *Ivan Korec*: Definability of addition and multiplication from Pascal's triangle modulo n and number-theoretical functions

19/1998 *Roberto Giuntini, Sylvia Pulmannová*: Ideals and congruences in effect algebras and QMV algebras

18/1998 *Ivan Korec*: Definability of exact arithmetical operations from approximate ones

17/1998 *Georges Chevalier, Sylvia Pulmannová*: Some ideal lattices in partial abelian monoids

16/1998 *Anatolij Dvurečenskij*: On categorical equivalences of commutative BCK-algebras

15/1998 *Ivan Korec*: Definability of addition and multiplication from some quadratic forms $ax^2 + cy^2$

14/1998 *Beloslav Riečan*: On the L^p space of observables on product MV algebras

13/1998 *Beloslav Riečan*: On the strong law of large numbers on some ordered structures

12/1998 *Ladislav Mišík, jr., Tibor Žáčik*: A simplified formula for calculation of metric dimension of converging sequences

11/1998 *Beloslav Riečan*: On the joint distribution of observables

10/1998 *Beloslav Riečan*: On the product MV algebras

9/1998 *Roman Frič*: A Stone-type duality and its applications to probability

8/1998 *Roman Frič*: Ring of maps: sequential convergence and completion

7/1998 *Roman Frič, Ján Jakubík*: Sequential convergences on Boolean algebras defined by systems of maximal filters

6/1998 *Roman Frič*: Categorical reflections on fields of events and probability

5/1998 *Beloslav Riečan*: On the probability theory and fuzzy sets

4/1998 *Ivan Korec*: Definability of arithmetical operations from binary quadratic forms

3/1998 *Beloslav Riečan*: On the conditional expectation of observables in MV algebras of fuzzy sets

2/1998 *Beloslav Riečan*: On the conditional probability on MV algebras

1/1998 *Paula B. Cohen, Robin L. Hudson, K. R. Parthasarathy, S. Pulmannová*: Hall's transformation via quantum stochastic calculus

13/1997 *Ivan Korec*: Real-time generation of primes by a one-dimensional cellular automaton with 9 states