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II. Group Representations

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Abstract

This paper is the continuation of [4]. We show that any pseudo-effect algebra fulfilling a certain property of Riesz type is representable by a unit interval of some – not necessarily abelian – partially ordered group. The relation of pseudo-effect to pseudo-MV algebras is made clear, and the ℓ-group representation theorem for the latter structure is reproved.

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With this paper, we continue the work [4], where we introduced a new algebraic structure called pseudo-effect algebra. Sections and theorems are numbered in continuation of [4].

5 Representation of pseudo-effect algebras by unit-intervals of groups

Our aim is to develop a structure theory for pseudo-effect algebras. As intervals in *po*-groups served as prototypes, we ask, first of all, about group representations.

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Now, even when assuming commutativity, it is, in spite of its importance for the foundations of quantum mechanics, an open problem how to characterize exactly those pseudo-effect algebras that are intervals of partially ordered groups. On the other hand, a certain Riesz property, introduced in Section 3, is a sufficient condition; the aim of this section is to show that any pseudo-effect algebra that fulfils the Commutational Riesz Decomposition Property, see Definition 3.1(e), is an interval pseudo-effect algebra.

We will use the so-called word technique, which was introduced by Baer [1] and Wyler [9]. It has also been successfully applied to effect-algebras fulfilling the Riesz Interpolation Property [8] and to commutative BCK-algebras with the relative cancellation property (see [3, Chapter 5.2.5]).

In a first step, we embed a given pseudo-effect algebra into a semigroup. The semigroup will then, in a second step, be extended to a *po*-group.

Definition 5.1 Let $(E; +, 0, 1)$ be a pseudo-effect algebra.

(i) A sequence $A = (a_1, \ldots, a_n)$ of finite, but non-zero, length with entries from E is called a *word* in E. We denote by $\mathcal{W}(E)$ the set of all words; that is

$$\mathcal{W}(E) \stackrel{\text{def}}{=} \{(a_1, \ldots, a_n)\colon\ a_1, ..., a_n \in E, n \geq 1\}.$$

We define an *addition* in $\mathcal{W}(E)$ as the concatenation; that is

$$\begin{aligned} +\colon\quad & \mathcal{W}(E) \times \mathcal{W}(E) \to \mathcal{W}(E), \\ & ((a_1, ..., a_m), (b_1, ..., b_n)) \mapsto (a_1, ..., a_m, b_1, ..., b_n). \end{aligned}$$

(ii) We call two words A and B of E *directly similar*, in symbols $A \sim B$, if one of it has the form $(a_1, ..., a_n)$, $n \geq 2$, and the other one has the form $(a_1, ..., a_p + a_{p+1}, ..., a_n)$, $1 \leq p < n$.

We call two words A and B *similar*, in symbols $A \simeq B$, if there are words $A_0, \ldots, A_k$, $k \geq 0$, such that $A = A_0 \sim A_1 \sim \cdots \sim A_k = B$. In such a case we say that A and B are *connected by a chain of length k*.

We set for $a_1, ..., a_n \in E$, $n \geq 1$,

$$[a_1, ..., a_n] \stackrel{\text{def}}{=} \{A \in \mathcal{W}(E)\colon\ A \simeq (a_1, ..., a_n)\},$$

and we put

$$\mathcal{C}(E) \stackrel{\text{def}}{=} \{[a_1, ..., a_n]\colon\ a_1, ..., a_n \in E, n \geq 1\}.$$

Lemma 5.2 Let $(E;+,0,1)$ be a pseudo-effect algebra.

(i) Similarity in $\mathcal{W}(E)$ is an equivalence relation compatible with $+$.

$+$ being the induced relation, $(\mathcal{C}(E);+)$ is a semigroup with the neutral element $[0]$.

(ii) For $a_1,...,a_n, b \in E$, $n \geq 1$, $(a_1,\ldots,a_n) \simeq (b)$ if and only if $a_1+\cdots+a_n$ exists and equals b.

Proof. (i) $\simeq$ is by construction an equivalence relation.

From $A_1 \simeq A$ and $B_1 \simeq B$ it follows $A_1 + B_1 \simeq A + B$, so $+$ is definable in $\mathcal{C}(E)$.

As $\mathcal{W}(E)$ is associative, so is $\mathcal{C}(E)$, i.e. $\mathcal{C}(E)$ is a semigroup. It has $[0]$ as a neutral element, because e.g. $[a_1,\ldots,a_n]+[0] = [a_1,\ldots,a_n,0] = [a_1,\ldots,a_n]$.

(ii) If for a word $(x_1,\ldots,x_m)$ the sum of its elements $x_1+\cdots+x_m$ exists, the same is true for any word directly similar to $(x_1,\ldots,x_m)$. So the "only if" part follows by induction on the minimal length of a chain by which (b) and $(a_1,\ldots,a_n)$ are connected.

The "if" part is obvious. $\square$

We note that (ii) of this Lemma has been proved by Baer [1, Theorem 1] in a much more general context.

We will now prove the crucial Lemma needed for the representation theorem. For the special notation used herein, see the paragraph preceding Lemma 3.9.

Lemma 5.3 Let $(E;+,0,1)$ be a pseudo-effect algebra fulfilling (RDP$_1$). Let

$$(a_1,\ldots,a_m) \simeq (b_1,\ldots,b_n),$$

where $m,n \geq 1$. Then there are elements $d_{11},\ldots,d_{mn} \in E$ such that

$$\begin{array}{cccc} d_{11} & \ldots & d_{1n} & \rightarrow & a_1 \\ \vdots & & \vdots & & \vdots \\ d_{m1} & \ldots & d_{mn} & \rightarrow & a_m \\ \downarrow & & \downarrow & & \\ b_1 & \ldots & b_n & & \end{array}$$

and such that, for $1 \leq i < m$, $1 \leq j < n$ we have

$$d_{i+1,j}+\cdots+d_{mj} \;\; \mathbf{com} \;\; d_{i,j+1}+\cdots+d_{in}.$$

Proof. The proof is by induction on the minimal length k of a chain that connects $(a_1, \dots, a_m)$ and $(b_1, \dots, b_n)$.

If $k = 0$, we have $(a_1, \dots, a_m) = (b_1, \dots, b_n)$, and the elements in the scheme

$$\begin{array}{cccccc} a_1 & 0 & \dots & 0 & \to & a_1 \\ 0 & a_2 & \dots & 0 & \to & a_2 \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ 0 & 0 & \dots & a_m & \to & a_m \\ \downarrow & \downarrow & & \downarrow & & \\ a_1 & a_2 & \dots & a_m & & \end{array}$$

obviously fulfill the statements.

Suppose the statement is true for $k-1$, $k \geq 1$. Let $A = (a_1, \dots, a_m)$ and $B = (b_1, \dots, b_n)$ be connected by a chain of length k, say $A = A_0 \sim \cdots \sim A_k = B$.

There are now two possibilities how $A_k = B = (b_1, \dots, b_n)$ is constructed from A_{k-1}.

1. Let $A_{k-1} = (b_1, \dots, b_p^1, b_p^2, \dots, b_n)$, $1 \leq p \leq n$ and $b_p = b_p^1 + b_p^2$. Then there are by hypothesis elements in E according to the scheme

$$\begin{array}{cccccccc} d_{11} & \dots & d_{1p}^1 & d_{1p}^2 & \dots & d_{1n} & \to & a_1 \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ d_{m1} & \dots & d_{mp}^1 & d_{mp}^2 & \dots & d_{mn} & \to & a_m \\ \downarrow & & \downarrow & \downarrow & & \downarrow & & \\ b_1 & \dots & b_p^1 & b_p^2 & \dots & b_n, & & \end{array} \tag{1}$$

such that the sums of elements placed below and right from any element fulfil the **com**-relation. Consider now the scheme

$$\begin{array}{ccccc} d_{11} & \dots & d_{1p}^1 + d_{1p}^2 & \dots & d_{1n} \\ \vdots & & \vdots & & \vdots \\ d_{m1} & \dots & d_{mp}^1 + d_{mp}^2 & \dots & d_{mn}. \end{array}$$

Here the lines still sum up to $a_1, \dots, a_m$, and the unchanged columns clearly sum up as before to $b_1, \dots, b_{p-1}, b_{p+1}, \dots, b_n$. Due to the **com**-relations, we have that any element in (1) commutes with any other one that is placed further up and right. So also the p-th column adds up to what it should:

$$\begin{aligned} b_p = b_p^1 + b_p^2 &= d_{1p}^1 + \cdots + d_{mp}^1 + d_{1p}^2 + \cdots + d_{mp}^2 \\ &= d_{1p}^1 + d_{1p}^2 + d_{2p}^1 + \cdots + d_{mp}^1 + d_{2p}^2 + \cdots + d_{mp}^2 \\ &= \ldots \\ &= d_{1p}^1 + d_{1p}^2 + d_{2p}^1 + d_{2p}^2 + \cdots + d_{mp}^1 + d_{mp}^2. \end{aligned}$$

Now the only new conditions concerning the **com**-relation are the following, where $1 \leq j < m$:

$$d_{j+1,p}^1 + d_{j+1,p}^2 + \cdots + d_{mp}^1 + d_{mp}^2 \ \mathbf{com}\ d_{j,p+1} + \cdots + d_{jn}. \tag{2}$$

By the same reasoning as before, the first term in (2) is equal to $(d_{j+1,p}^1 + \cdots + d_{mp}^1) + (d_{j+1,p}^2 + \cdots + d_{mp}^2)$. Here, we know by hypothesis that both the first term inside brackets and the second one fulfils the **com**-condition together with the second term in (2). Now by Lemma 3.2(i), (2) follows.

2. Let $A_{k+1} = (b_1, \ldots, b_p + b_{p+1}, \ldots, b_n)$, $1 \leq p < n$. Then there are by hypothesis elements in E according to the scheme

$$\begin{array}{ccccccc} d_{11} & \ldots & e_1 & \ldots & d_{1n} & \rightarrow & a_1 \\ \vdots & & \vdots & & \vdots & & \vdots \\ d_{m1} & \ldots & e_m & \ldots & d_{mn} & \rightarrow & a_m \\ \downarrow & & \downarrow & & \downarrow & & \\ b_1 & \ldots & b_p + b_{p+1} & \ldots & b_n, & & \end{array}$$

such that the sums of elements places below and right from any element fulfil the **com**-relation. Lemma 3.9 applied to

$$b_p + b_{p+1} = e_1 + \cdots + e_m$$

gives elements $d_{1p}, d_{1,p+1}, \ldots, d_{mp}, d_{m,p+1} \in E$ such that

$$\begin{array}{cccc} d_{1p} & d_{1,p+1} & \rightarrow & e_1 \\ \vdots & \vdots & & \vdots \\ d_{mp} & d_{m,p+1} & \rightarrow & e_m \\ \downarrow & \downarrow & & \\ b_p & b_{p+1} & & \end{array}$$

and for $1 \leq j < m$

$$d_{j+1,p} + \cdots + d_{mp} \ \mathbf{com} \ d_{j,p+1}.$$

Then we have

$$\begin{array}{ccccccccc} d_{11} & \dots & d_{1p} & d_{1,p+1} & \dots & d_{1n} & \to & a_1 \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ d_{m1} & \dots & d_{mp} & d_{m,p+1} & \dots & d_{mn} & \to & a_m \\ \downarrow & & \downarrow & \downarrow & & \downarrow & & \\ b_1 & \dots & b_p & b_{p+1} & \dots & b_n. & & \end{array}$$

As a sum when deleting any summand from it gets smaller, and as the **com**-relation is additive by Lemma 3.2(i), we see that the conditions concerning the **com**-relation are again preserved herein. □

With the help of Lemma 5.3 we will now check for the semigroup $\mathcal{C}(E)$ the properties that are necessary and sufficient for extending it to a *po*-group whose positive cone it is. The extension itself is described in the subsequent Definition 5.5 and Lemma 5.6.

Lemma 5.4 Let $(E; +, 0, 1)$ be a pseudo-effect algebra fulfilling (RDP$_1$). Then $\mathcal{C}(E)$ is a semigroup such that the following holds.

(i) $[0]$ is a neutral element.

(ii) Let $\mathfrak{a}, \mathfrak{b} \in \mathcal{C}(E)$. From $\mathfrak{a} + \mathfrak{b} = [0]$ it follows $\mathfrak{a} = \mathfrak{b} = [0]$.

(iii) Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathcal{C}(E)$. From $\mathfrak{a} + \mathfrak{b} = \mathfrak{a} + \mathfrak{c}$ it follows $\mathfrak{b} = \mathfrak{c}$; from $\mathfrak{b} + \mathfrak{a} = \mathfrak{c} + \mathfrak{a}$ it follows $\mathfrak{b} = \mathfrak{c}$.

(iv) For any pair $\mathfrak{a}, \mathfrak{b} \in \mathcal{C}(E)$ there is a $\mathfrak{c} \in \mathcal{C}(E)$ such that $\mathfrak{a} + \mathfrak{b} = \mathfrak{c} + \mathfrak{a}$, and there is a $\mathfrak{d} \in \mathcal{C}(E)$ such that $\mathfrak{a} + \mathfrak{b} = \mathfrak{b} + \mathfrak{d}$.

Proof. That $\mathcal{C}(E)$ is a semigroup fulfilling (i) has been proved in Lemma 5.2(i). (ii) follows from Lemma 5.2(ii) and Lemma 1.4(ii).

(iii) We may suppose $\mathfrak{a} = [a]$, and let $\mathfrak{b} = [b_1, \ldots, b_m]$, $\mathfrak{c} = [c_1, \ldots, c_n]$. So we will show that $(a, b_1, \ldots, b_m) \simeq (a, c_1, \ldots, c_n)$ implies $(b_1, \ldots, b_m) \simeq (c_1, \ldots, c_n)$; then the first part will follow, and the second one is proved analogously.

By our assumption there are, by Lemma 5.3, elements in E according to the following scheme:

$$\begin{array}{ccccccc} d & d_1 & \dots & d_n & \to & a \\ e_1 & e_{11} & \dots & e_{1n} & \to & b_1 \\ \vdots & \vdots & & \vdots & & \vdots \\ e_m & e_{m1} & \dots & e_{mn} & \to & b_m \\ \downarrow & \downarrow & & \downarrow & & \\ a & c_1 & & c_n, & & \end{array}$$

where for $1 \leq j < i \leq m, \ \ 1 \leq k \leq n$ we have $e_i + e_{jk} = e_{jk} + e_i$, and for $1 \leq k < i \leq m, \ \ 1 \leq j < l \leq n$ we have $e_{ij} + e_{kl} = e_{kl} + e_{ij}$, and for $1 \leq i \leq m, \ 1 \leq j < k \leq n$ we have $e_{ij} + d_k = d_k + e_{ij}$.

$a = d + d_1 + \dots + d_n = d + e_1 + \dots + e_m$ implies $d_1 + \dots + d_n = e_1 + \dots + e_m$. Using this and the commutativity relations we get

$$\begin{aligned} (b_1, \dots, b_m) &\simeq (e_1, e_{11}, \dots, e_{1n}, \ . \ . \ ., e_m, e_{m1}, \dots, e_{mn}) \\ &\simeq (e_1, \dots, e_m, e_{11}, \dots, e_{m1}, \ . \ . \ ., e_{1n}, \dots, e_{mn}) \\ &\simeq (d_1, \dots, d_n, e_{11}, \dots, e_{m1}, \ . \ . \ ., e_{1n}, \dots, e_{mn}) \\ &\simeq (d_1, e_{11}, \dots, e_{m1}, \ . \ . \ ., d_n, e_{1n}, \dots, e_{mn}) \\ &\simeq (c_1, \dots, c_n). \end{aligned}$$

(iv) We will prove the first part only; the second is shown analogously. Moreover, we suppose that $\mathfrak{a} = [a]$ and $\mathfrak{b} = [b]$; the claim then follows by an easy induction argument. So what we will show is that there is a word $\mathfrak{c}$ such that $(a, b) = \mathfrak{c} + (a)$.

By (RDP$_1$) we have for some $d_1, \dots, d_4 \in E$

$$\begin{array}{cccc} d_1 & d_2 & \to & a \\ d_3 & d_4 & \to & a^\sim \\ \downarrow & \downarrow & & \\ b^- & b. & & \end{array}$$

So there are elements $d_2', d_4' \in E$ such that $(a, b) = (d_1 + d_2, d_2 + d_4) \sim (d_2' + d_1, d_2, d_4) \simeq (d_2', a + d_4) = (d_2', d_4' + a) \sim (d_2', d_4') + (a)$, where we used the fact that due to $d_4 \leq a^\sim$ and Lemma 1.6(iv), $a + d_4$ exists. $\square$

Definition 5.5 Let $(E;+,0,1)$ be a pseudo-effect algebra fulfilling (RDP$_1$).

In view of Lemma 5.4 we may define, for any $\mathfrak{a},\mathfrak{b}\in\mathcal{C}(E)$, $\mathfrak{a}_{\mathfrak{b}}$ to be the unique element such that $\mathfrak{a}+\mathfrak{b}=\mathfrak{b}+\mathfrak{a}_{\mathfrak{b}}$.

Define for $\mathcal{C}(E)\times\mathcal{C}(E)$ the binary operation $+$ by

$$(\mathfrak{a},\mathfrak{b})+(\mathfrak{c},\mathfrak{d}) \stackrel{\text{def}}{=} (\mathfrak{a}+\mathfrak{c}_{\mathfrak{b}},\mathfrak{d}+\mathfrak{b})$$

for $\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}\in\mathcal{C}(E)$.

Define for $\mathcal{C}(E)\times\mathcal{C}(E)$

$$\begin{aligned}
(\mathfrak{a},\mathfrak{b})\approx(\mathfrak{c},\mathfrak{d}) &\stackrel{\text{def}}{\leftrightarrow} \mathfrak{a}+\mathfrak{d}_{\mathfrak{b}}=\mathfrak{c}+\mathfrak{b},\\
\langle\mathfrak{a},\mathfrak{b}\rangle &\stackrel{\text{def}}{=} \{(\mathfrak{c},\mathfrak{d})\colon\ (\mathfrak{c},\mathfrak{d})\approx(\mathfrak{a},\mathfrak{b})\},\\
\mathcal{G}(E) &\stackrel{\text{def}}{=} \{\langle\mathfrak{a},\mathfrak{b}\rangle\colon\ \mathfrak{a},\mathfrak{b}\in\mathcal{C}(E)\},\\
\mathcal{G}(E)^+ &\stackrel{\text{def}}{=} \{\langle\mathfrak{a},[0]\rangle\colon\ \mathfrak{a}\in\mathcal{C}(E)\},\\
\iota_E\colon\ E\to\mathcal{G}(E),&\ \ a\mapsto\langle[a],[0]\rangle.
\end{aligned}$$

Lemma 5.6 Let $(E;+,0,1)$ be a pseudo-effect algebra fulfilling (RDP$_1$).

Then the relation $\approx$ on $\mathcal{C}(E)\times\mathcal{C}(E)$ is an equivalence relation that is compatible with $+$. $+$ being the derived operation and $\mathcal{G}(E)^+$ being the positive cone, $(\mathcal{G}(E);+,\le)$ is a *po*-group. We have $\mathcal{G}(E)=\mathcal{G}(E)^+-\mathcal{G}(E)^+=-\mathcal{G}(E)^++\mathcal{G}(E)^+$.

$\varepsilon\colon\ \mathcal{C}(E)\to\mathcal{G}(E),\ \ \mathfrak{a}\mapsto\langle\mathfrak{a},[0]\rangle$ establishes a semigroup isomorphism between $\mathcal{C}(E)$ and $\mathcal{G}(E)^+$.

Proof. This Lemma is a consequence of Lemma 5.4. For the details of the proof see [5, Theorem II.4]. □

We arrive at our main theorem.

Theorem 5.7 Let $(E;+,0,1)$ be a pseudo-effect algebra fulfilling (RDP$_1$).

Then $\iota_E\colon\ E\to\mathcal{G}(E),\ \ a\mapsto\langle[a],[0]\rangle$ determines an isomorphism between $(E;+,0,1)$ and $(\,\Gamma(\mathcal{G}(E),\langle[1],[0]\rangle);\ +,\ \langle[0],[0]\rangle,\ \langle[1],[0]\rangle\,)$, where $\langle[1],[0]\rangle$ is a strong unit of $\mathcal{G}(E)$.

In particular, E is an interval pseudo-effect algebra.

Proof. ι_E is injective. Indeed, for $a, b \in E$, $\langle [a], [0] \rangle = \langle [b], [0] \rangle$ implies $[a] = [b]$ by the injectivity of ε from Lemma 5.6, and this implies $a = b$ by Lemma 5.2(ii).

The image of ι_E is $\Gamma(\mathcal{G}(E), \langle [1], [0] \rangle)$. Indeed, any positive element of $\mathcal{G}(E)$ is of the form $\langle \mathfrak{a}, [0] \rangle$ for some $\mathfrak{a} \in \mathcal{C}(E)$ by the surjectivity of ε from Lemma 5.6. Now $\langle \mathfrak{a}, [0] \rangle \leq \langle [1], [0] \rangle$ means for some other positive element $\langle \mathfrak{b}, [0] \rangle$, where $\mathfrak{b} \in \mathcal{C}(E)$, that $\langle \mathfrak{a}, [0] \rangle + \langle \mathfrak{b}, [0] \rangle = \langle [1], [0] \rangle$. But then $\mathfrak{a} + \mathfrak{b} = [1]$, since, by Lemma 5.6, ε is a semigroup isomorphism. It follows by Lemma 5.2(ii) that $\mathfrak{a} = [a]$, $\mathfrak{b} = [b]$ for some $a, b \in E$, that is $\langle \mathfrak{a}, [0] \rangle = \iota_E(a)$.

ι_E: $E \to \Gamma(\mathcal{G}(E), \langle [1], [0] \rangle)$ is an isomorphism with respect to $+$. Indeed, for $a, b \in E$, $a + b$ exists in E and equals c iff $(a, b) \sim (c)$ iff $[c] = [a] + [b]$ iff $\langle [c], [0] \rangle = \langle [a], [0] \rangle + \langle [b], [0] \rangle$ iff $\iota_E(c) = \iota_E(a) + \iota_E(b)$. Here we used again the fact that ε from Lemma 5.6 is a semigroup isomorphism.

It remains to show that $\langle [1], [0] \rangle$ is a strong unit of $\mathcal{G}(E)$. Since $\mathcal{G}(E) = \mathcal{G}(E)^+ - \mathcal{G}(E)^+$ holds and since $\mathcal{G}(E)^+$ is isomorphic to $\mathcal{C}(E)$, it is sufficient to show that any $\mathfrak{a} = [a_1, \ldots, a_n] \in \mathcal{C}(E)$ lies below a multiple of $[1]$. By Lemma 5.4(iv), we conclude that $[\underbrace{1, \ldots, 1}_{n \text{ times}}] = [a_1, a_1^\sim, \ldots, a_n, a_n^\sim] = [a_1, \ldots, a_n] + \mathfrak{b}$ for some $\mathfrak{b} \in \mathcal{C}(E)$, which means $\mathfrak{a} \leq \underbrace{[1] + \cdots + [1]}_{n \text{ times}}$. □

From now on we will consider $\mathcal{C}(E)$ as a subset of $\mathcal{G}(E)$, thus identifying $\mathfrak{a} \in \mathcal{C}(E)$ with $\langle \mathfrak{a}, [0] \rangle \in \mathcal{G}(E)$; in particular, we write $[0]$ and $[1]$ instead of $\langle [0], [0] \rangle$ and $\langle [1], [0] \rangle$ respectively. Note that we then have $\langle \mathfrak{a}, \mathfrak{b} \rangle = \mathfrak{a} - \mathfrak{b}$ for $\mathfrak{a}, \mathfrak{b} \in \mathcal{C}(E)$.

6 Properties of a pseudo-effect algebra and its representing group

We check properties that are preserved from a *po*-group G with positive element u in the pseudo-effect algebra $\Gamma(G, u)$ – see Definition 2.1 –, and from a pseudo-effect algebra E fulfilling (RDP$_1$) in the representing group $\mathcal{G}(E)$ – see Theorem 5.7.

6.1 Lattice Order and Riesz Properties

Proposition 6.1 Let G be a *po*-group with positive element u.

(i) If then G is an ℓ-group, then $\Gamma(G,u)$ is also lattice-ordered.

(ii) Let G be directed. If G has one of the properties (RIP), (RDP_0), (RDP), (RDP_1), or (RDP_2), then $\Gamma(G,u)$ has the equally denoted property.

In the other direction, we have the following.

Proposition 6.2 Let E be a pseudo-effect algebra fulfilling (RDP_1). The embedding ι_E: $E \to \mathcal{G}(E), \quad a \mapsto \langle [a],[0] \rangle$ preserves infima and suprema.

Proof. Let $a, b \in E$ such that $a \wedge b$ exists. As ι_E is order-preserving, $\iota_E(a \wedge b) \leq \iota_E(a) \wedge \iota_E(b)$, that is, under the above identification of $\mathcal{C}(E)$ and $\mathcal{G}(E)^+$, $[a \wedge b] \leq [a] \wedge [b]$. Suppose an element of $\mathcal{G}(E)$ to be a lower bound of $[a]$ and $[b]$, that is $\langle \mathfrak{x}, \mathfrak{y} \rangle \leq [a],[b]$ for some $\mathfrak{x}, \mathfrak{y} \in \mathcal{C}(E)$.

As (RIP) holds in $\mathcal{G}(E)$, we may conclude from $\mathfrak{x}, \mathfrak{y} \;\leq\; [a] + \mathfrak{y}, [b] + \mathfrak{y}$ that for some $\mathfrak{c} \in \mathcal{G}(E)^+$ we have $\mathfrak{x}, \mathfrak{y} \;\leq\; \mathfrak{c} + \mathfrak{y} \;\leq\; [a] + \mathfrak{y}, [b] + \mathfrak{y}$, which means $\langle \mathfrak{x}, \mathfrak{y} \rangle \leq \mathfrak{c} \leq [a],[b]$.

As $\mathfrak{c} = [c]$ for some $c \in E$, we have from $c \leq a, b$ that $c \leq a \wedge b$, and so $\langle \mathfrak{x}, \mathfrak{y} \rangle \leq [c] \leq [a \wedge b]$. It follows $[a \wedge b] = [a] \wedge [b]$. So we have shown that ι_E preserves infima.

From $[a] \vee [b] = -(([1] - [a]) \wedge ([1] - [b])) + [1]$, we see that ι_E preserves also suprema. $\square$

Proposition 6.3 Let E be a pseudo-effect algebra fulfilling (RDP_1).

(i) $\mathcal{G}(E)$ also fulfils (RDP_1).

(ii) The following statements are equivalent.

(α) E fulfils (RDP_2).

(β) E is lattice-orderded.

(γ) $\mathcal{G}(E)$ fulfils (RDP_2).

(δ) $\mathcal{G}(E)$ is an ℓ-group.

Proof. (i) Let $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{b}_1, \mathfrak{b}_2 \in \mathcal{G}(E)$ be positive elements of $\mathcal{G}(E)$ such that $\mathfrak{a}_1 + \mathfrak{a}_2 = \mathfrak{b}_1 + \mathfrak{b}_2$. Let $\mathfrak{a}_1 = [a_1^1, \ldots, a_k^1]$, $\mathfrak{a}_2 = [a_1^2, \ldots, a_l^2]$, $\mathfrak{b}_1 = [b_1^1, \ldots, b_m^1]$, $\mathfrak{b}_2 = [b_1^2, \ldots, b_n^2]$, where $a_1^1, \ldots, b_n^2 \in E$. By Lemma 5.3 there are $d_{11}^1, \ldots, d_{ln}^4 \in E$ such that

$$\begin{array}{cccccccc}
d_{11}^1 & \ldots & d_{1m}^1 & d_{11}^2 & \ldots & d_{1n}^2 & \rightarrow & a_1^1 \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots \\
d_{k1}^1 & \ldots & d_{km}^1 & d_{k1}^2 & \ldots & d_{kn}^2 & \rightarrow & a_k^1 \\
d_{11}^3 & \ldots & d_{1m}^3 & d_{11}^4 & \ldots & d_{1n}^4 & \rightarrow & a_1^2 \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots \\
d_{l1}^3 & \ldots & d_{lm}^3 & d_{l1}^4 & \ldots & d_{ln}^4 & \rightarrow & a_l^2 \\
\downarrow & & \downarrow & \downarrow & & \downarrow & & \\
b_1^1 & \ldots & b_m^1 & b_1^2 & \ldots & b_n^2 & &
\end{array}$$

and such that every two elements in this diagram fulfil the **com**-condition if one of them is placed further up and further right than the other one.

Define now $\mathfrak{d}_1 = [d_{11}^1, \ldots, d_{1m}^1, \ . \ . \ ., d_{k1}^1, \ldots, d_{km}^1]$ and in a similar way also $\mathfrak{d}_2, \mathfrak{d}_3, \mathfrak{d}_4$. Due to the commutativity conditions, we have then

$$\begin{array}{cccc}
\mathfrak{d}_1 & \mathfrak{d}_2 & \rightarrow & \mathfrak{a}_1 \\
\mathfrak{d}_3 & \mathfrak{d}_4 & \rightarrow & \mathfrak{a}_2 \\
\downarrow & \downarrow & & \\
\mathfrak{b}_1 & \mathfrak{b}_2. & &
\end{array}$$

Now let $0 \leq \mathfrak{x} \leq \mathfrak{d}_2$ and $0 \leq \mathfrak{y} \leq \mathfrak{d}_3$. So $\mathfrak{x} + \mathfrak{x}' = \mathfrak{d}_2$ for some $\mathfrak{x}' \in \mathcal{C}(E)$, and we may apply to this equation Lemma 5.3 to conclude that $\mathfrak{x}$ is representable as a word such that every element in it lies below some element in $\mathfrak{d}_2$. The same applies to $\mathfrak{y}$ and $\mathfrak{d}_3$.

Now for every $x, y \in E$ such that x lies below an element occuring in the word $\mathfrak{d}_2$ and y lies below an element occuring in $\mathfrak{d}_3$, we have $x + y = y + x$. We conclude that $\mathfrak{x}$ and $\mathfrak{y}$ commute. So we have proved $\mathfrak{d}_2 \,\mathbf{com}\, \mathfrak{d}_3$, which means that ($\mathrm{RDP}_1$) holds in G.

(ii) As E fulfils (RDP_0), we have by Proposition 3.3(ii) that E fulfils (RDP_2) iff E is lattice-ordered. By Proposition 4.2(ii) we have that $\mathcal{G}(E)$ fulfils (RDP_2) iff $\mathcal{G}(E)$ is lattice-ordered. By Proposition 6.1(i), if $\mathcal{G}(E)$ fulfils (RDP_2), then also E does.

Now let E fulfil (RDP_2). Given $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{b}_1, \mathfrak{b}_2 \in \mathcal{G}(E)$ such that $\mathfrak{a}_1 + \mathfrak{a}_2 = \mathfrak{b}_1 + \mathfrak{b}_2$, we construct in the same manner as in (i) elements $\mathfrak{d}_1, \mathfrak{d}_2, \mathfrak{d}_3, \mathfrak{d}_4$ with the only difference that now the infimum of any element of E occuring in $\mathfrak{d}_2$ and any element occuring in $\mathfrak{d}_3$ is required to be 0. This is easily done by modifying the proofs of the Lemmas 3.2(i), 3.9 and 5.3, replacing the **com**- by the zero-infimum condition.

Since the relation of having a zero infimum is by Proposition 6.2 preserved from E in $\mathcal{G}(E)$, and since it is in additive in ℓ-groups, we conclude that $\mathfrak{d}_2 \wedge \mathfrak{d}_3 = 0$. So ($\mathrm{RDP}_2$) holds in $\mathcal{G}(E)$. □

6.2 Commutativity

Proposition 6.4 Let G be a *po*-group with positive element u. If then G is abelian, then $\Gamma(G, u)$ is also commutative.

Proposition 6.5 Let E be a pseudo-effect algebra fulfilling (RDP_1). If then E is commutative, then $\mathcal{G}(E)$ is also commutative.

Proof. Suppose first that $\mathfrak{a}, \mathfrak{b} \in \mathcal{G}(E)^+$. By Lemma 5.4(iv), there is a unique $\mathfrak{c} \in \mathcal{G}(E)^+$ such that $\mathfrak{a} + \mathfrak{b} = \mathfrak{c} + \mathfrak{a}$. By the proof of that Lemma, we see that if E is commutative, then $\mathfrak{c} = \mathfrak{b}$.

Since by Lemma 5.6, $\mathcal{G}(E)^+$ generates $\mathcal{G}(E)$, we conclude that $\mathcal{G}(E)$ is commutative. □

6.3 Linearity

Proposition 6.6 Let G be a *po*-group with positive element u. If then G is linearly ordered, then $\Gamma(G, u)$ is also linearly ordered.

Proposition 6.7 Let E be a pseudo-effect algebra fulfilling (RDP_1). If E is linearly ordered, then $\mathcal{G}(E)$ is also linearly ordered.

Proof. We prove first that every two elements from $\mathcal{G}(E)^+$ are comparable. If $0 \leq \mathfrak{a}, \mathfrak{b} \leq [1]$, this follows by assumption. Suppose now that every two elements from $\mathcal{G}(E)^+$ each of which is the sum of not more that $n-1$ elements from the unit interval, are comparable, where $n \geq 2$, and let $\mathfrak{a} = [a_1, \ldots, a_n]$,

$\mathfrak{b} = [b_1, \ldots, b_n]$, where $a_1, \ldots, a_n, b_1, \ldots, b_n \in E$. To see that then $\mathfrak{a}$ and $\mathfrak{b}$ are also comparable, we check the case $a_1 \leq b_1$ and $[a_2, \ldots, a_n] \geq [b_2, \ldots, b_n]$. Chose $\mathfrak{r}, \mathfrak{s} \in \mathcal{G}(E)^+$ such that $[b_1] = [a_1] + \mathfrak{r}$ and $[a_2, \ldots, a_n] = \mathfrak{s} + [b_2, \ldots, b_n]$. Then $\mathfrak{r} \leq [1]$, and from $\mathfrak{s} \leq [a_2, \cdots, a_n]$ we see by (RDP$_1$), which holds in $\mathcal{G}(E)$ by Proposition 6.3(i), that $\mathfrak{s}$ is the sum of $n-1$ elements of the unit interval. So $\mathfrak{r}$ and $\mathfrak{s}$ are comparable, which means that also $\mathfrak{a} = [a_1] + \mathfrak{s} + [b_2, \ldots, b_n]$ and $\mathfrak{b} = [a_1] + \mathfrak{r} + [b_2, \ldots, b_n]$ are. By induction the claim follows.

Now suppose two elements from $\mathcal{G}(E)$ to be given. Since by Lemma 5.6, $\mathcal{G}(E) = \mathcal{G}(E)^+ - \mathcal{G}(E)^+ = -\mathcal{G}(E)^+ + \mathcal{G}(E)^+$, we may suppose these elements to have the form $\mathfrak{a} - \mathfrak{b}$ and $-\mathfrak{c} + \mathfrak{d}$, respectively, for some $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d} \in \mathcal{G}(E)^+$. But $\mathfrak{a} - \mathfrak{b}$ and $-\mathfrak{c} + \mathfrak{d}$ are comparable iff $\mathfrak{c} + \mathfrak{a}$ and $\mathfrak{d} + \mathfrak{b}$ are comparable, and this is the case. □

6.4 Archimedeanicity

We introduce archimedeanicity for pseudo-effect algebras.

Definition 6.8 A pseudo-effect algebra $(E; +, 0, 1)$ is said to be *archimedean* if for every $a \in E$ the existence of $n\,a$ for every $n \geq 1$ implies $a = 0$.

We note that the algebra from Example 2.3 is non-archimedian: There, we have that $n\,(0,0,1)$ is defined for every $n \geq 1$.

But we see that the following holds.

Proposition 6.9 Let E be a σ-complete lattice pseudo-effect algebra. Then E is archimedean.

Proof. Suppose $a \in E$ and $n\,a$ to be defined for any $n \geq 1$. Equally as in the proof of Proposition 3.11, we conclude $(\bigvee_m m\,a) + a = \bigvee_m (m\,a + a) = \bigvee_m m\,a$, hence $a = 0$. □

For groups, we shall use the following definition of archimedeanicity.

Definition 6.10[1] A *po*-group $(G; +, \leq)$ is said to be *archimedean* if, for every $a, b \in G$, $n\,a \leq b$ for every $n \geq 1$ implies $a \leq 0$.

[1]The property defined here is by some authors called *completely integrally closed*, e.g. in [5].

Proposition 6.11 Let (G, u) be a unital *po*-group such that G is archimedean. Then $\Gamma(G, u)$ is a archimedean and commutative.

Proof. That under the assumptions on G, $\Gamma(G, u)$ is archimedean is obvious. Moreover, since G possesses a strong unit, it is directed, and in this case we know by [5, Chapter V.1.G and Corollary V.20] that G is abelian; so $\Gamma(G, u)$ is commutative. □

We do not yet know if in general archimedeanicity of a pseudo-effect algebra E fulfilling (RDP_1) implies the same property for the representing group $\mathcal{G}(E)$, and so by Proposition 6.11 also commutativity. But we note the following.

Proposition 6.12 Let E be a linear archimedean pseudo-effect algebra fulfilling (RDP_1). Then E is commutative, and $\mathcal{G}(E)$ is archimedean and abelian.

Proof. By Proposition 6.7, $\mathcal{G}(E)$ is linearly ordered.

Let $a, b \in \mathcal{G}(E)$ and $n\,a \leq b$ for all $n \geq 1$. We have to show that $a \leq 0$. But if $a < 0$, we are done; so suppose $a \geq 0$, in which case we have to prove $a = 0$.

If $b \leq [1]$, $a = 0$ follows from the archimedeanicity of E. Let us assume that $a = 0$ follows whenever b is the sum of $k - 1$ elements of the unit interval, where $k \geq 2$. Suppose $n\,a < b = b_1 + \cdots + b_k$ for all n. Then $n\,a - b_k < b_1 + \cdots + b_{k-1}$, from which by assumption follows $n\,a - b_k \leq 0$; this means $n\,a \leq b_k$ for all n, so we have, again by assumption, $a = 0$.

So $\mathcal{G}(E)$ is archimedean. As it is a directed group, we again know by [5, Chapter V.1.G and Corollary V.20] that it is abelian. So E is commutative. □

7 Categorical isomorphism of pseudo-effect algebras and *po*-groups with Riesz property

We shall show that pseudo-effect algebras and unital *po*-groups, both fulfilling (RDP_1), are categorically equivalent.

Definition 7.1 Let $(E;+,0,1)$ be a pseudo-effect algebra. Then a pair $((G,u),\iota_E)$ of a unital *po*-group group (G,u) and a homomorphism $\iota_E :$ $(E;+,0,1) \to (G;+,\leq,u)$ is called a *universal group* for E if for every homomorphism h: $(E;+,0,1) \to (H;+,\leq,v)$ of E into a unital *po*-group (H,v) there is a homomorphism k: $(G;+,\leq,u) \to (H;+,\leq,v)$ such that $h = k \circ \iota_E$.

Theorem 7.2 Let $(E;+,0,1)$ be a pseudo-effect algebra fulfilling (RDP_1).

(i) $((\mathcal{G}(E),[1]),\iota_E)$, where ι_E has been defined in Definition 5.5, is a universal group for E.

(ii) Let φ: $(E;+,0_E,1_E) \to (F;+,0_F,1_F)$ be a homomorphism of E into another pseudo-effect algebra fulfilling (RDP_1). Then there is a unique homomorphism $\psi :$ $(\mathcal{G}(E),[1_E]) \to (\mathcal{G}(F),[1_F])$ of unital *po*-groups such that the following diagram commutes.

$$\begin{array}{ccc} E & \stackrel{\varphi}{\to} & F \\ {\scriptstyle \iota_E}\downarrow & & {\scriptstyle \iota_F}\downarrow \\ \mathcal{G}(E) & \stackrel{\psi}{\to} & \mathcal{G}(F) \end{array}$$

Proof. (i) Let H be a *po*-group, v be a strong unit of H, and h: $(E;+,0,1) \to$ $(H;+,\leq,v)$ a homomorphism; that is, $h(E) \subseteq H^+$, $+$ is preserved, $h(0)$ is the group zero, and $h(1) = v$.

As $\iota_E :$ $E \to \Gamma(\mathcal{G}(E),[1])$, $a \mapsto [a]$ is bijective by Theorem 5.7, we may define

$$k^1\colon\ \Gamma(\mathcal{G}(E),[1]) \to H, \quad [a] \mapsto h(a).$$

Obviously, $k^1 \circ \iota_E = h$, k^1 preserves $+$, and $k^1([1]) = h(1) = v$.

We may extend k^1 to $\mathcal{G}(E)^+$ by requiring

$$k^+\colon\ \mathcal{G}(E)^+ \to H, \quad [a_1,\ldots,a_n] \mapsto h(a_1)+\cdots+h(a_n).$$

This is possible since directly similar words $(x_1,\ldots,x_m)$ and $(x_1,\ldots,x_p+x_{p+1},\ldots,x_m)$, where $x_1,\ldots,x_m \in E$, $m \geq 1$ and $1 \leq p < m$, are both mapped to $h(x_1)+\cdots+h(x_m)$.

By construction, also k^+ preserves $+$.

We may further extend k^+ to $\mathcal{G}(E)$ by requiring

$$k\colon\ \mathcal{G}(E) \to H,\ \ \langle \mathfrak{a}, \mathfrak{b} \rangle \mapsto k^+(\mathfrak{a}) - k^+(\mathfrak{b}).$$

For let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d} \in \mathcal{C}(E)$ and suppose $\langle \mathfrak{a}, \mathfrak{b} \rangle = \langle \mathfrak{c}, \mathfrak{d} \rangle$. That means in view of Definition 5.5, $(\mathfrak{a}, \mathfrak{b}) \approx (\mathfrak{c}, \mathfrak{d})$, i.e. $\mathfrak{a} + \mathfrak{d}_\mathfrak{b} = \mathfrak{c} + \mathfrak{b}$, where $\mathfrak{d}_\mathfrak{b}$ is subject to the condition $\mathfrak{b} + \mathfrak{d}_\mathfrak{b} = \mathfrak{d} + \mathfrak{b}$. So we have $k^+(\mathfrak{a}) + k^+(\mathfrak{d}_\mathfrak{b}) = k^+(\mathfrak{c}) + k^+(\mathfrak{b})$, and $k^+(\mathfrak{b}) + k^+(\mathfrak{d}_\mathfrak{b}) = k^+(\mathfrak{d}) + k^+(\mathfrak{b})$. Both equations combined give $-k^+(\mathfrak{b}) + k^+(\mathfrak{d}) = k^+(\mathfrak{d}_\mathfrak{b}) - k^+(\mathfrak{b}) = -k^+(\mathfrak{a}) + k^+(\mathfrak{c})$, that is $k^+(\mathfrak{a}) - k^+(\mathfrak{b}) = k^+(\mathfrak{c}) - k^+(\mathfrak{d})$.

It remains to show that k is a homomorphism. We first check that k preserves $+$. As k^+ does so, this is the case for positive elements. Now let again $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d} \in \mathcal{C}(E)$. Then we have $k(\langle \mathfrak{a}, \mathfrak{b} \rangle + \langle \mathfrak{c}, \mathfrak{d} \rangle) = k(\langle \mathfrak{a} + \mathfrak{c}_\mathfrak{b}, \mathfrak{d} + \mathfrak{b} \rangle) = k(\mathfrak{a}) + k(\mathfrak{c}_\mathfrak{b}) - k(\mathfrak{b}) - k(\mathfrak{d}) = k(\mathfrak{a}) - k(\mathfrak{b}) + k(\mathfrak{c}) - k(\mathfrak{d}) = k(\langle \mathfrak{a}, \mathfrak{b} \rangle) + k(\langle \mathfrak{c}, \mathfrak{d} \rangle)$.

Secondly k preserves the order; for $h(E) \subseteq H^+$, so $k(\mathcal{G}(E)^+) = k^+(\mathcal{G}(E)^+) \subseteq H^+$. Finally, we have $k([1]) = k^1([1]) = v$.

(ii) As $\iota_F \circ \varphi\colon\ E \to \mathcal{G}(F)$ is a homomorphism of E into $\mathcal{G}(F)$, the existence of a function $\psi\colon\ (\mathcal{G}(E), [1_E]) \to (\mathcal{G}(F), [1_F])$ such that $\iota_F \circ \varphi = \psi \circ \iota_E$ follows from (i). The uniqueness is obvious from the proof of (i). □

Definition 7.3 Let $\mathcal{RPEA}$ denote the category whose objects are the pseudo-effect algebras fulfilling (RDP_1) and whose morphisms are the homomorphisms of these structures.

Let $\mathcal{RPOG}$ denote the category whose objects are the unital *po*-groups fulfilling (RDP_1) and whose morphisms are the homomorphisms of these structures.

Define Γ to be the functor from $\mathcal{RPOG}$ to $\mathcal{RPEA}$ that maps an object (G, u) from $\mathcal{RPOG}$ to $\Gamma(G, u)$ according to Definition 2.1 and maps a morphism $\psi\colon\ (G, u) \to (H, v)$ from $\mathcal{RPEA}$ to its restriction to $\Gamma(G, u)$.

Define Δ to be the functor from $\mathcal{RPEA}$ to $\mathcal{RPOG}$ that maps an object E from $\mathcal{RPEA}$ to $(\mathcal{G}(E), [1])$ and maps a morphism $\varphi\colon\ E \to F$ from $\mathcal{RPEA}$ to the morphism $\psi\colon\ \mathcal{G}(E) \to \mathcal{G}(F)$ subject to the condition $\iota_F \circ \varphi = \psi \circ \iota_E$.

That Γ and Δ are well-defined as functors follows by construction.

Theorem 7.4 The functors $\Delta: \mathcal{RPEA} \to \mathcal{RPOG}$ and $\Gamma: \mathcal{RPOG} \to \mathcal{RPEA}$ form an equivalence of categories, i.e. $\Gamma \circ \Delta$ and $\Delta \circ \Gamma$ are naturally equivalent to the identity functors of $\mathcal{RPEA}$ and $\mathcal{RPOG}$, respectively.

Proof. We have to show that any object E from $\mathcal{RPEA}$ is isomorphic to $\Gamma(\Delta(E))$ and that any G from $\mathcal{RPOG}$ is isomorphic to $\Delta(\Gamma(G))$.

So let E be a pseudo-effect algebra fulfilling (RDP$_1$). By Theorem 5.7, $\Gamma(\Delta(E)) = \Gamma(\mathcal{G}(E), [1]) \cong E$.

Let (G, u) be a unital *po*-group that fulfils (RDP$_1$). We have to check $(\mathcal{G}(\Gamma(G, u)), [u]) \cong (G, u)$. Now by Theorem 5.7, the unit intervals of $\mathcal{G}(\Gamma(G, u))$ and of G are isomorphic; this isomorphism is given by $\Gamma(\mathcal{G}(\Gamma(G, u)), [u]) \to \Gamma(G, u), \quad \langle [a], [0] \rangle \mapsto a$. By Theorem 7.2(i), we may extend this function to a homomorphism

$$\begin{aligned} h\colon \quad & (\mathcal{G}(\Gamma(G, u)), [u]) \to G, \\ & \langle [a_1, \ldots, a_m], [b_1, \ldots b_n] \rangle \mapsto a_1 + \cdots + a_m - b_n - \cdots - b_1. \end{aligned}$$

We note that words with entrances from $\Gamma(G, u)$ add up to the same element of G iff they are similar. Indeed, for groups in which (RDP$_1$) holds, we may prove a proposition analogous to Proposition 3.9. We conclude from that that two words of $\mathcal{W}(\Gamma(G, u))$ that sum up to the same element are similar. Clearly, similar words add up to the same element of G.

We claim that h is injective. Indeed, suppose we are given $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d} \in \mathcal{C}(\Gamma(G, u))$ such that $h(\langle \mathfrak{a}, \mathfrak{b} \rangle) = h(\langle \mathfrak{c}, \mathfrak{d} \rangle)$. Let $A \in G$ be the sum of the elements of any word representing $\mathfrak{a}$, and define $B, C, D \in G$ be similarly; then the latter condition means $A - B = C - D$. It follows $A + D_B = C + B$, where D_B is chosen such that $B + D_B = D + B$ holds. Now we have, according to Definition 5.5, $\mathfrak{b} + \mathfrak{d}_\mathfrak{b} = \mathfrak{d} + \mathfrak{b}$; hence any word representing $\mathfrak{d}_\mathfrak{b}$ adds up to D_B. So $\mathfrak{a} + \mathfrak{d}_\mathfrak{b} = \mathfrak{c} + \mathfrak{b}$, which means by the same Definition $\langle \mathfrak{a}, \mathfrak{b} \rangle = \langle \mathfrak{c}, \mathfrak{d} \rangle$.

We claim that h is also surjective. Indeed, $\Gamma(G, u)$ generates G. For, since G fulfils (RIP), we have for $0 \le a \le b_1 + \cdots + b_n$ that $a = d_1 + \cdots + d_n$, where $0 \le d_1 \le b_1, \ \ldots, \ 0 \le d_n \le b_n$. We conclude that the unit interval of G generates the positive cone G^+. Moreover, since G possesses a strong unit, it is directed, which means by [5, Proposition II.3(a)] that G^+ generates G.
$\square$

We may apply this result to the case that both the pseudo-effect algebras and the groups are lattice-ordered.

Definition 7.5 Let $\mathcal{LRPEA}$ denote the category whose objects are the lattice pseudo-effect algebras fulfilling (RDP_0) and whose morphisms are the homomorphisms of these structures.

Let $\mathcal{LG}$ denote the category whose objects are the unital ℓ-groups and whose morphisms are the homomorphisms of these structures.

Define Γ to be the functor from $\mathcal{LG}$ to $\mathcal{LRPEA}$ that maps an object (G,u) from $\mathcal{LG}$ to $\Gamma(G,u)$ according to Definition 2.1 and maps a morphism ψ: $(G,u) \to (H,v)$ from $\mathcal{LG}$ to its restriction to $\Gamma(G,u)$.

Define Δ to be the functor from $\mathcal{LRPEA}$ to $\mathcal{LG}$ that maps an object E from $\mathcal{LRPEA}$ to $(\mathcal{G}(E),[1])$ and maps a morphism φ: $E \to F$ from $\mathcal{LRPEA}$ to the morphism ψ: $\mathcal{G}(E) \to \mathcal{G}(F)$ subject to the condition $\iota_F \circ \varphi = \psi \circ \iota_E$.

That Γ and Δ map indeed into $\mathcal{LRPEA}$ and $\mathcal{LG}$, respectively, follows from Propositions 3.3(ii) and 6.3(ii). That Γ and Δ are functors follows again by construction.

Theorem 7.6 The functors Δ: $\mathcal{LRPEA} \to \mathcal{LG}$ and Γ: $\mathcal{LG} \to \mathcal{LRPEA}$ form an equivalence of categories, i.e. $\Gamma \circ \Delta$ and $\Delta \circ \Gamma$ are naturally equivalent to the identity functors of $\mathcal{LRPEA}$ and $\mathcal{LG}$, respectively.

Proof. This follows from Theorem 7.4. □

8 Relations between pseudo-effect and pseudo-MV algebras

In the present section, we show how pseudo-MV algebras, which were introduced in [7], are to be understood as a subfamily of the class of pseudo-effect algebras. With the help of our main Theorem 5.7, we are then able to present a new proof of the fact that pseudo-MV algebras are intervals in ℓ-groups [2].

Pseudo-MV algebras have been introduced in the following manner.

Definition 8.1 A structure $(M; \oplus, {}^-, {}^\sim, 0, 1)$, where $\oplus$ is a binary, ${}^-$ and ${}^\sim$ are unary operations and 0, 1 are constants, is called a *pseudo-MV algebra* if the following axioms hold in it.

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$.

(A2) $x \oplus 0 = 0 \oplus x = x$.

(A3) $x \oplus 1 = 1 \oplus x = 1$.

(A4) $1^\sim = 0$; $1^- = 0$.

(A5) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$.

(A6) $x \oplus (x^\sim \odot y) = y \oplus (y^\sim \odot x) = (x \odot y^-) \oplus y = (y \odot x^-) \oplus x$.

(A7) $x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y$.

(A8) $x^{-\sim} = x$.

Herein, for any $a, b \in M$, we put $a \odot b \stackrel{\text{def}}{=} (b^- \oplus a^-)^\sim$.

Furthermore, we define for $a, b \in M$

$$a \leq_M b \quad \text{iff} \quad a \oplus c = b \ \text{ for some } c \in M.$$

It can be shown that for any pseudo-MV algebra $\leq_M$ is a lattice order [7, Proposition 1.10]; the terms occuring in (A6) and (A7) are in fact the supremum and the infimum of two elements x and y, respectively.

By replacing the total operation $\oplus$ by a partial operation $+$, a pseudo-MV algebra may be considered as a pseudo-effect algebra.

Definition 8.2 Let $(M; \oplus, ^-, ^\sim, 0, 1)$ be a pseudo-MV algebra. Define $+$ to be the partial operation on M that is defined for elements $a, b \in M$ iff $a \leq_M b^-$, and in that case let $a + b \stackrel{\text{def}}{=} a \oplus b$.

Theorem 8.3 Let $(M; \oplus, ^-, ^\sim, 0, 1)$ be a pseudo-MV algebra. Then the following holds.

(i) $(M; +, ^-, ^\sim, 0, 1)$ is a pseudo-effect algebra.

(ii) Let $\leq$ be the order of M considered as a pseudo-effect algebra. Then this order coincides with the original one, i.e. $\leq_M = \leq$.

(iii) For any $a, b \in M$ we have

$$a \oplus b = (a \wedge b^-) + b. \tag{3}$$

Proof. (ii) By [3, Remark 6.4.5], we have for any $a, b \in M$ that $a \leq_M b$ iff, for some $c \in M$, $a + c = b$. So $\leq_M = \leq$, where $\leq$ is defined according to Definition 1.5.

(iii) That equation (3) holds in a pseudo-MV algebra is seen e.g. from [3, Proposition 6.4.8(i) and Exercise 3(1) of 6.4.5].

(i) Let $a, b, c \in M$. To show (E1), suppose $a + b$ and $(a + b) + c$ are defined. Then $b \leq a + b \leq c^-$, which means that $b + c$ is defined. Since we have $x \wedge y = (x \oplus y^\sim) \odot y$ for $x, y \in M$, from the assumptions $a + b \leq c^-$ and $a \leq b^-$ it follows that $a \wedge (b+c)^- = (a+b+c) \odot (b+c)^- = (a+b+c) \odot c^- \odot b^- = [(a + b) \wedge c^-] \odot b^- = (a + b) \odot b^- = a \wedge b^- = a$; so $a + (b + c)$ is defined.

If $b + c$ and $a + (b + c)$ are defined, we conclude similarly that $a + b$ and $(a + b) + c$ are also defined. Now by (A1), (E1) follows.

Since we have $a + a^\sim = a^- + a = 1$ [7, Proposition 1.5] and since cancellation holds for the operation $+$ [3, Proposition 6.4.4], (E2) and (EC) follows.

To show (E3), let $a + b$ be defined. Then $a, b \leq a + b$, and we have $a + b = c + a = b + d$ for exactly one element c and one element d [3, Remark 6.4.5 and Proposition 6.4.4]; so (E3) follows.

If $1 + a$ or $a + 1$ is defined, $a \leq 0$, i.e. $a = 0$. So (E4) follows. $\square$

We are now able to give a new, completely different proof of the main result of [2].

Theorem 8.4 Let $(M; \oplus, ^-, ^\sim, 0, 1)$ be a pseudo-MV algebra. Then there is a ℓ-group $(G; +, \leq)$ with strong unit u such that $(M; \oplus, ^-, ^\sim, 0, 1)$ and $(\Gamma(G, u); \oplus, ^-, ^\sim, 0, u)$ are isomorphic, where for $a, b \in \Gamma(G, u)$ we define

$$a \oplus b = (a + b) \wedge u, \qquad (4)$$
$$a^- = u - a, \qquad (5)$$
$$a^\sim = -a + u. \qquad (6)$$

Proof. We know from Theorem 8.3 that $(M; +, ^-, ^\sim, 0, 1)$ is a pseudo-effect algebra whose order $\leq$ coincides with $\leq_M$. From this fact it follows from [3, Theorem 6.4.12] that M as a pseudo-effect algebra fulfils (RDP_2).

So by Theorem 5.7 and Proposition 6.3 there is a unital ℓ-group (G, u) such that $(M; +, ^-, ^\sim, 0, 1)$ is isomorphic to $(\Gamma(G, u); +, ^-, ^\sim, 0, u)$.

Here, according to the remark following Definition 2.1, ${}^{\sim}, {}^{-}:\ \Gamma(G,u) \to \Gamma(G,u)$ are given by (5) and (6), respectively.

It remains to check that under this isomorphism, $\oplus$: $\Gamma(G,u) \to \Gamma(G,u)$ as defined by (4) coincides with the equally denoted operation of the pseudo-MV algebra M. Now, for any $a, b \in \Gamma(G,u)$, we have $a \oplus b = (a+b) \wedge u = (a+b) \wedge (b^{-}+b) = (a \wedge b^{-}) + b$, where $\wedge$ and $+$ is calculated in the group G. Now the last term may be equally calculated in the pseudo-effect algebra $(\Gamma(G,u); +, {}^{-}, {}^{\sim}, 0, u)$. So by Theorem 8.3(iii) the claim follows. □

Now, under certain presumptions a pseudo-effect algebra may also be considered as a pseudo-MV algebra.

Definition 8.5 Let $(E; +, 0, 1)$ be a pseudo-effect algebra. Let $/$ and $\backslash$ be partial binary operations such that, for $a, b \in E$, a / b as well as $a \backslash b$ is defined iff $b \leq a$, in which case the conditions

$$a \;=\; a / b \,+\, b \;=\; b \,+\, a \backslash b$$

are to be fulfilled.

Furthermore, let E be lattice-ordered. Let $\oslash$ and $\obslash$ be the binary operations defined for $a, b \in E$ by

$$\begin{aligned} a \oslash b &\stackrel{\text{def}}{=} a / (a \wedge b), \\ a \obslash b &\stackrel{\text{def}}{=} a \backslash (a \wedge b). \end{aligned}$$

Finally, let for $a, b \in E$

$$a \oplus b \stackrel{\text{def}}{=} (b^{-} \oslash a)^{\sim}.$$

Lemma 8.6 Let $(E; +, 0, 1)$ be a pseudo-effect algebra.

(i) If, for $a, b \in E$, $b \leq a$, we have

$$\begin{aligned} a \backslash b &= b^{\sim} / a^{\sim}, \\ a / b &= b^{-} \backslash a^{-}, \end{aligned} \tag{7}$$

(ii) Let E be lattice-ordered. If $a + b$ is defined, we have $a \oplus b = a + b$.

Proof. (i) Let $b \leq a$. Then $a \setminus b = x$ means $a = b + x$, so by Lemma 1.4(vi), $b^{\sim} = x + a^{\sim}$ and $x = b^{\sim} / a^{\sim}$. So the first equation of (7) follows, from which the second is easily derivable.

(ii) Let $a + b$ exists. So by Lemma 1.6(iv), $a \leq b^{-}$ holds. Then we have $b^{-} \oslash a = b^{-} / (a \wedge b^{-}) = b^{-} / a$. From $b^{-} / a = c^{-}$ it follows $b^{-} = c^{-} + a$, and so $c = a + b$ by Lemma 1.4(vi). This means $a \oplus b = (b^{-} \oslash a)^{\sim} = (b^{-} / a)^{\sim} = c = a + b$. □

The following Lemma contains the conditions under which a pseudo-effect algebra may be considered a pseudo-MV algebra.

Proposition 8.7 Let $(E; +, 0, 1)$ be a lattice pseudo-effect algebra. Then the following statements are equivalent.

(α) E fulfils (RDP_0).

(β) E fulfils (RDP_2).

(γ) For all $a, b \in E$

$$a / (a \wedge b) = (a \vee b) / b. \tag{8}$$

(δ) For all $a, b \in E$

$$a \setminus (a \wedge b) = (a \vee b) \setminus b. \tag{9}$$

If one of these statements is true, we have for $a, b \in E$

$$\begin{array}{rcl} a \obslash b & = & b^{\sim} \oslash a^{\sim}, \\ a \oslash b & = & b^{-} \obslash a^{-}. \end{array} \tag{10}$$

Proof. (α) $\Leftrightarrow$ (β). This has been proved in Proposition 3.3(ii).

(β) $\Rightarrow$ (γ). Let E fulfil (RDP_2). So from $a^{-} + a = b^{-} + b$ we get elements $d_1, \ldots, d_4 \in E$ such that

$$\begin{array}{cccc} d_1 & d_2 & \rightarrow & b^{-} \\ d_3 & d_4 & \rightarrow & b \\ \downarrow & \downarrow & & \\ a^{-} & a & & \end{array}$$

and $d_2 \wedge d_3 = 0$. By Lemma 1.7(i) we have $d_1 = d_1 + (d_2 \wedge d_3) = a^- \wedge b^-$ and similarly $d_4 = a \wedge b$. So by (7) we have $a / (a \wedge b) = a / d_4 = d_2 = b^- \setminus d_1 = b^- \setminus (a^- \wedge b^-) = b^- \setminus (a \vee b)^- = (a \vee b) / b$. So (8) is proved.

$(\gamma) \Rightarrow (\delta)$. Suppose (8). Using (7) it follows for any $a, b \in E$ that $a \setminus (a \wedge b) = (a \wedge b)^\sim / a^\sim = (a^\sim \vee b^\sim) / a^\sim = b^\sim / (a^\sim \wedge b^\sim) = b^\sim / (a \vee b)^\sim = (a \vee b) \setminus b$. So (9) holds.

$(\delta) \Rightarrow (\alpha)$. Suppose (9), and let $a \leq b + c$. Then $a = (a \wedge b) + (a \setminus (a \wedge b))$; we have $a \wedge b \leq b$ and, since $b + ((a \vee b) \setminus b) = a \vee b \leq b + c$, also $a \setminus (a \wedge b) = (a \vee b) \setminus b \leq c$. So (RDP$_0$) is shown.

Now suppose that (δ) holds, and let $a, b \in E$. By (7) we have $a \obslash b = a \setminus (a \wedge b) = (a \vee b) \setminus b = b^\sim / (a^\sim \wedge b^\sim) = b^\sim \oslash a^\sim$. This is the first equation of (10), from which the second is easily derivable. □

Theorem 8.8 Let $(E; +, {}^\sim, {}^-, 0, 1)$ be a lattice pseudo-effect algebra. Then $(E; \oplus, {}^\sim, {}^-, 0, 1)$ is a pseudo-MV algebra if and only if E fulfils one of the equivalent conditions of Proposition 8.7. In that case we have for $a, b \in E$

$$\begin{aligned} a \oplus b = \ & (b^- \oslash a)^\sim = & (a^\sim \obslash b)^-, \qquad (11) \\ a \odot b = \ & a \oslash b^\sim = & b \obslash a^-. \qquad (12) \end{aligned}$$

Moreover, the order $\leq_M$ of E as an MV-algebra then coincides with the order $\leq$ of E as a pseudo-effect algebra, and we have for $a, b \in E$

$$\begin{aligned} a \wedge b = \ & a \oslash (a \obslash b) = & a \obslash (a \oslash b), \qquad (13) \\ a \vee b = \ & (a \oslash b) \oplus b = & a \oplus (b \obslash a). \qquad (14) \end{aligned}$$

Proof. Let $(E; \oplus, {}^\sim, {}^-, 0, 1)$ be a pseudo-MV algebra.

We show first that the order $\leq_M$ of E coincides with the original one, i.e. with $\leq$. Let $a, b \in E$. From $a \leq_M b$ it follows $b = c \oplus a$ for some $c \in E$; so $b^- = a^- \oslash c$, that is $a^- = b^- + (a^- \wedge c)$; but this means $b^- \leq a^-$ and, by Lemma 1.6(iii), $a \leq b$. Conversely, $a \leq b$ means $b = a + d$ for some $d \in E$; so $b = a \oplus d$ and hence $a \leq_M b$.

It follows for $a, b \in E$ that $a + b$ exists iff $a \leq_M b^-$, in which case $a + b = a \oplus b$. This means that the $+$ operation is derivable from the pseudo-MV algebra

M as described in Definition 8.2. So we may conclude from [3, Theorem 6.4.11] that E as a pseudo-effect algebra fulfills (RDP_0), i.e. that condition (α) of Proposition 8.7 is fulfilled.

Let us now assume that conditions (γ), (δ) of Proposition 8.7 are fulfilled, i.e. that (8) and (9) hold.

We see equally as above that the order $\leq$ of E coincides with the relation $\leq_M$ as defined in Definition 8.1 for pseudo-MV algebras.

We will now prove equations (11) to (14) and then every of the axioms (A1) to (A8) of a pseudo-MV algebra.

Let $a, b, x \in E$. We have $a \oplus b = (b^- \oslash a)^\sim = x$ iff $b^- \oslash a = b^- / (a \wedge b^-) = x^-$ iff $b^- = x^- + a \wedge b^-$ iff $x = (a \wedge b^-) + b$ iff $(a \wedge b^-)^\sim = b + x^\sim$ iff $(a^\sim \vee b) \setminus b = x^\sim$ iff $a^\sim \setminus (a^\sim \wedge b) = x^\sim$ iff $(a^\sim \obslash b)^- = x$. That is, (11) holds.

Now by (11) and (10) we get $a \odot b = (b^- \oplus a^-)^\sim = b \obslash a^- = a \oslash b^\sim$. That is, (12) holds.

From $(a \wedge b) + (a \obslash b) = a$ we conclude $a \wedge b = a / (a \obslash b) = a \oslash (a \obslash b)$, and similarly we get $a \wedge b = a \setminus (a \oslash b) = a \obslash (a \oslash b)$. So (13) is shown.

Using this result, we conclude by (11) that $a \vee b = (a^- \wedge b^-)^\sim = [b^- \oslash (b^- \obslash a^-)]^\sim = [b^- \oslash (a \oslash b)]^\sim = (a \oslash b) \oplus b$, and similarly we get $a \vee b = (a^\sim \wedge b^\sim)^- = [a^\sim \obslash (a^\sim \oslash b^\sim)]^- = [a^\sim \obslash (b \obslash a)]^- = a \oplus (b \obslash a)$. So (14) is shown.

(A4) and (A8) hold in E by (iii) and (iv) of Lemma 1.4.

For $a \in E$, from $(1 \oslash a) + a = 1$ we have $1 \oslash a = a^-$, so $a \oplus 0 = (0^- \oslash a)^\sim = (1 \oslash a)^\sim = a^{-\sim} = a$, which proves the first part of (A2). The second is shown analogously.

We have $a \oplus 1 = (1^- \oslash a)^\sim = (0 \oslash a)^\sim = (0 / 0)^\sim = 0^\sim = 1$, which proves the first part of (A3). The second is shown analogously.

For $a, b \in E$, we have by (11) and (12) that $(a^\sim \oplus b^\sim)^- = b \oslash a^\sim = a \obslash b^- = (a^- \oplus b^-)^\sim$, which proves (A5).

All the expressions occuring in (A6) equal the supremum of a and b. For by (14) we have $a \oplus (a^\sim \odot b) = a \oplus (b \obslash a) = a \vee b$, hence also $b \oplus (b^\sim \odot a) = a \vee b$. Similarly, $(a \odot b^-) \oplus b = (a \oslash b) \oplus b = a \vee b$, hence also $(b \odot a^-) \oplus a = a \vee b$.

The expressions occuring in (A7) equal the infimum of a and b; for by (13) we have $a \odot (a^- \oplus b) = a \oslash (a \obslash b) = a \wedge b$ and $(a \oplus b^\sim) \odot b = b \obslash (b \oslash a) = a \wedge b$.

To prove assosiativity, i.e. (A1), several intermediate results are needed, denoted by (a0), …, (a5). Let $a, b, c, x, y \in E$.

Claim (a0). From $a \leq x \leq y$ it follows $x \setminus a \leq y \setminus a$.

For: For some x' we have $x + x' = y$ and so $a + (x \setminus a) + x' = a + (y \setminus a)$ and $(x \setminus a) + x' = y \setminus a$.

Claim (a1). The order is monotone from both sides with respect to $\oplus$ and $\odot$.

For: For from $a \leq b$ it follows $b^{\sim} \vee x \leq a^{\sim} \vee x$ and, by (a0), $b^{\sim} \oslash x = (b^{\sim} \vee x) \setminus x \leq (a^{\sim} \vee x) \setminus x = a^{\sim} \oslash x$; so $a \oplus x = (a^{\sim} \oslash x)^{-} \leq (b^{\sim} \oslash x)^{-} = b \oplus x$. Similarly, we conclude $x \oplus a \leq x \oplus b$.

Moreover, from $a \leq b$ it follows $b^{-} \leq a^{-}$, so $b^{-} \oplus x^{-} \leq a^{-} \oplus x^{-}$; so $x \odot a = (a^{-} \oplus x^{-})^{\sim} \leq (b^{-} \oplus x^{-})^{\sim} = x \odot b$. Similarly, we conclude $a \odot x \leq b \odot x$.

Claim (a2). $a \odot b \leq c$ iff $b \leq a^{-} \oplus c$ iff $a \leq c \oplus b^{\sim}$.

For: Suppose $a \odot b \leq c$. Then for some $x \in E$ we have $(a \odot b) + x = b \oslash a^{-} + x = c$. This means $b + x = (a^{-} \wedge b) + c$. So $b \leq (a^{-} \wedge b) + c = (a^{-} \wedge b) \oplus c \leq a^{-} \oplus c$ by (a1).

Suppose $b \leq a^{-} \oplus c$. Then again by (a1) we have $a \odot b \leq a \odot (a^{-} \oplus c) = a \wedge c \leq c$.

Similarly, we prove the equivalence of $a \odot b \leq c$ with $a \leq c \oplus b^{\sim}$.

Claim (a3). $a \oplus (b \wedge c) = (a \oplus b) \wedge (a \oplus c)$; $(b \vee c) \odot a = (b \odot a) \vee (c \odot a)$.

For: $a \oplus (b \wedge c) \leq a \oplus b, a \oplus c$ by (a1). Suppose $x \leq a \oplus b, a \oplus c$. Then by (a2), $a^{\sim} \odot x \leq b, c$, so $a^{\sim} \odot x \leq b \wedge c$, and again by (a2), $x \leq a \oplus (b \wedge c)$. So the first equation follows. The second follows from the first one, applied to a^{-}, b^{-} and c^{-}.

Analogously we may prove

Claim (a3'). $(b \wedge c) \oplus a = (b \oplus a) \wedge (c \oplus a)$; $a \odot (b \vee c) = (a \odot b) \vee (a \odot c)$.

Claim (a4). The order is distributive.

For: By (a3) and (a1) we have $a \wedge (b \vee c) = (b \vee c) \odot [(b \vee c)^{-} \oplus a] = (b \odot [(b \vee c)^{-} \oplus a]) \vee (c \odot [(b \vee c)^{-} \oplus a]) \leq [b \odot (b^{-} \oplus a)] \vee [c \odot (c^{-} \oplus a)] = (a \wedge b) \vee (a \wedge c)$.

Claim (a5). $a \oplus (b \vee c) = (a \oplus b) \vee (a \oplus c)$.

For: By (a1) we have $a \oplus b, a \oplus c \leq a \oplus (b \vee c)$. Suppose $a \oplus b, a \oplus c \leq x$. Then by (a1) $a = a \oplus 0 \leq a \oplus b \leq x$ and $a^{\sim} \wedge b = a^{\sim} \odot (a \oplus b) \leq a^{\sim} \odot x$; similarly $a^{\sim} \wedge c \leq a^{\sim} \odot x$. So by (a3), (a4), (a1) we get $a \oplus (b \vee c) = (a \oplus a^{\sim}) \wedge [a \oplus (b \vee c)] =$

$a \oplus [a^\sim \wedge (b \vee c)] = a \oplus [(a^\sim \wedge b) \vee (a^\sim \wedge c)] \leq a \oplus (a^\sim \odot x) = a \vee x = x.$

Analogously, thereby using (a3') instead of (a3), we may prove

Claim (a5'). $(b \vee c) \oplus a = (b \oplus a) \vee (c \oplus a)$.

We finally prove the associativity of $\odot$; (A1) then easily follows. That is, we claim $(a \odot b) \odot c = a \odot (b \odot c)$.

First, set $d = (a \odot b)^- \vee c$. Then, by (a3'), $(a \odot b) \odot d = (a \odot b) \odot [(a \odot b)^- \vee c] = (a \odot b) \odot c$, and $a \odot (b \odot d) = a \odot [b \odot ((a \odot b)^- \vee c)] = a \odot [(b \odot (b^- \oplus a^-)) \vee (b \odot c)] = a \odot [(a^- \wedge b) \vee (b \odot c)] = [a \odot (a^- \wedge b)] \vee [a \odot (b \odot c)] = a \odot (b \odot c)$.

So we must show $(a \odot b) \odot d = a \odot (b \odot d)$. We have $a^- \leq d$, because $(a \odot b) \wedge c^\sim \leq a \odot b \leq a \odot 1 = a$, i.e. $a^- \leq [(a \odot b) \wedge c^\sim]^- = d$. So $a^- + d^\sim$ exists.

By (a5'), we have $b \vee (a^- + d^\sim) = b \vee d^\sim \vee (a^- + d^\sim) = [(b \vee d^\sim) / d^\sim + d^\sim] \vee (a^- + d^\sim) = [(b \oslash d^\sim) \vee a^-] + d^\sim$ and by (a5) $b \vee (a^- + d^\sim) = a^- \vee b \vee (a^- + d^\sim) = [a^- + ((a^- \vee b) \setminus a^-)] \vee (a^- + d^\sim) = a^- + [(b \obslash a^-) \vee d^\sim]$, so that $[(b \oslash d^\sim) \vee a^-] + d^\sim = a^- + [(b \obslash a^-) \vee d^\sim]$. Thanks to the associativity holding in E, we may rewrite this equation in the form $a^- + [((b \oslash d^\sim) \vee a^-) \setminus a^-] + d^\sim = a^- + [((b \obslash a^-) \vee d^\sim) / d^\sim] + d^\sim$ and conclude $((b \oslash d^\sim) \vee a^-) \setminus a^- = ((b \obslash a^-) \vee d^\sim) / d^\sim$. But this means $(b \oslash d^\sim) \obslash a^- = (b \obslash a^-) \oslash d^\sim$, or $(a \odot b) \odot d = a \odot (b \odot d)$. $\square$

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