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Abstract

Pseudo MV-algebras are a non-commutative extension of MV-algebras introduced recently by Georgescu and Iorgulescu. We introduce states (finitely additive probability measures) on pseudo MV-algebras. We show that extremal states correspond to normal maximal ideals. We give an example that in contrast to classical MV-algebras introduced by Chang, states can fail on pseudo MV-algebras. We prove that representable and normal-valued pseudo MV-algebras admit at least one state.

1 Introduction

Recently, G. Georgescu and A. Iorgulescu [GeIo] introduced pseudo MV-algebras which are a non-commutative generalization of MV-algebras introduced in 1958 by C.C. Chang [Cha] and which are an extension of a two-valued reasoning. A non-commutative generalization of reasoning can be found e.g. in psychological processes. J. Rachůnek [Rac] introduced non-commutative MV-algebras, which in fact are equivalent with pseudo MV-algebras.

[1]Keywords: pseudo MV-algebra, MV-algebra, unital ℓ-group, state, state-morphism, ideal, representable pseudo MV-algebra, normal-valued pseudo MV-algebra.

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States on MV-algebras were introduced by F. Chovanec [Cho] and by D. Mundici [Mun 1] with the intent of capturing the notion of "average degree of truth" of a proposition, see also [RiMu].

States are finitely additive probability measures which are additive on summable elements. By [Dvu 1], there is a one-to-one correspondence between pseudo MV-algebras and unital (not necessary Abelian) ℓ-groups as intervals $[0, u]$, where u is a strong unit, and summable elements correspond to elements with the group addition restricted to the interval.

The state system of a pseudo MV-algebra, if it is non-void, is a convex set with extremal points which are in a one-to-one correspondence with normal maximal ideals. It is well-known that every classical MV-algebra admit at least one state. Unfortunately, for pseudo MV-algebras it is not a case. We show that each linearly ordered pseudo MV-algebra, representable pseudo MV-algebra, and normal-valued pseudo MV-algebras possess at least one state. We present an example of a stateless pseudo MV-algebra. In addition we show that the families of representable MV-algebras and of normal-valued pseudo MV-algebras form varieties, and we present their equational base. We notify that for representable pseudo MV-algebras we give a positive solution to the problem formulated in [GeIo] concerning a functional base for representable pseudo MV-algebras.

The paper is accomplished by examples elucidating some notions.

2 Pseudo MV-algebras

According to [GeIo], we introduce pseudo MV-algebras and present their representation by intervals in unital ℓ-groups.

Definition 2.1 *A* **pseudo MV-algebra** *is an algebra* $(M; \oplus, ^-, ^\sim, 0, 1)$ *of type* $(2, 1, 1, 0, 0)$ *such that the following axioms hold for all* $x, y, z \in M$ *with an additional binary operation* $\odot$ *defined via*

$$y \odot x = (x^- \oplus y^-)^\sim$$

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;

(A2) $x \oplus 0 = 0 \oplus x = x$;

(A3) $x \oplus 1 = 1 \oplus x = 1$;

(A4) $1^\sim = 0$; $1^- = 0$;

(A5) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$;

(A6) $x \oplus x^\sim \odot y = y \oplus y^\sim \odot x = x \odot y^- \oplus y = y \odot x^- \oplus x$;

(A7) $x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y$;

(A8) $(x^-)^\sim = x$.

We shall assume that $0 \neq 1$. If we define $x \leq y$ iff $x^- \oplus y = 1$, then $\leq$ is a partial order such that M is a distributive lattice with $x \vee y = x \oplus x^\sim \odot y$. For basic properties of pseudo MV-algebras see [GeIo].

For example, if u is a strong unit of a (not necessary Abelian) ℓ-group G,

$$\Gamma(G, u) := [0, u]$$

and

$$\begin{aligned} x \oplus y &:= (x + y) \wedge u, \\ x^- &:= u - x, \\ x^\sim &:= -x + u, \\ x \odot y &:= (x - u + y) \vee 0, \end{aligned}$$

then $(\Gamma(G, u); \oplus, ^-, ^\sim, 0, u)$ is a pseudo MV-algebra [GeIo].

Example 2.2 *Let $G = (\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}; +, (0, 0, 0), \leq)$ be the Scrimger 2-group, i.e.,*

$$(k_1, m_1, n_1) + (k_2, m_2, n_2) := \begin{cases} (m_1 + k_2, m_2 + k_1, n_1 + n_2), & \textit{if } n_2 \textit{ is odd} \\ (k_1 + k_2, m_1 + m_2, n_1 + n_2), & \textit{if } n_2 \textit{ is even.} \end{cases}$$

Then $0 = (0, 0, 0)$ is the neutral element, and

$$-(k, m, n) = \begin{cases} (-m, -k, -n), & \textit{if } n \textit{ is odd} \\ (-k, -m, -n), & \textit{if } n \textit{ is even,} \end{cases}$$

and G is a non-Abelian ℓ-group with the positive cone

$$G^+ = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}^+_{>0} \cup \mathbb{Z}^+ \times \mathbb{Z}^+ \times \{0\},$$

or equivalently, $(k_1, m_1, n_1) \leq (k_2, m_2, n_2)$ iff (i) $n_1 < n_2$, or (ii) $n_1 = n_2$, $k_1 \leq k_2$, $m_1 \leq m_2$.

Then

$$(k_1, m_1, n_1) \vee (k_2, m_2, n_2) := \begin{cases} (k_1, k_2, n_1), & \text{if } n_1 > n_2 \\ (k_1 \vee k_2, m_1 \vee m_2, n_1 \vee n_2), & \text{if } n_1 = n_2 \\ (k_2, m_2, n_2), & \text{if } n_1 < n_2, \end{cases}$$

and $u = (1, 1, 1)$ is a strong unit for G. Consequently, the corresponding pseudo MV-algebra has the form

$$\Gamma(G, u) = \mathbb{Z}^+ \times \mathbb{Z}^+ \times \{0\} \cup \mathbb{Z}_{\leq 1} \times Z_{\leq 1} \times \{1\},$$

with

$$\begin{aligned} (k, m, 0)^- &= (1 - k, 1 - m, 1), \\ (k, m, 0)^\sim &= (1 - m, 1 - k, 1), \\ (k, m, 1)^- &= (1 - m, 1 - k, 0), \\ (k, m, 1)^\sim &= (1 - k, 1 - m, 0), \end{aligned}$$

and

$$\begin{aligned} (k_1, m_1, 0) \oplus (k_2, m_2, 0) &= (k_1 + k_2, m_1 + m_2, 0), \\ (k_1, m_1, 0) \oplus (k_2, m_2, 1) &= ((m_1 + k_2) \wedge 1, (m_2 + k_1) \wedge 1, 1), \\ (k_1, m_1, 1) \oplus (k_2, m_2, 0) &= ((k_1 + k_2) \wedge 1, (m_1 + m_2) \wedge 1, 1), \\ (k_1, m_1, 1) \oplus (k_2, m_2, 1) &= (1, 1, 1). \end{aligned}$$

We recall that according to [Dvu 2], every finite pseudo MV-algebra is an MV-algebra.

Let $(M; \oplus, ^-, ^\sim, 0, 1)$ be a pseudo MV-algebra. Define a partial binary operation $+$ on M via: $x + y$ is defined iff $x \leq y^-$, and in this case

$$x + y := x \oplus y.$$

It is clear that $x + y$ is defined iff $x \leq y^-$ iff $y \leq x^\sim$. In addition $a \leq b$ iff there exists $c \in M$ such that $a + c = b$ iff there exists $d \in M$ such that $d + a = b$.

The basic properties of $+$ are studied in [Dvu] or in [DvPu]. We recall that the derived $+$ in $\Gamma(G, u)$ coincides with the usual group $+$ if it exists in the interval $\Gamma(G, u)$.

We recall that the pseudo MV-algebra $(\Gamma(G,u);\oplus,^-,^\sim,0,u)$ is in some sense the archetype of pseudo MV-algebras owing to the following basic representation theorem for pseudo MV-algebras proved in [Dvu 1, Thm 3.9, Thm 6.4] generalizing a famous result of Mundici [Mun] for MV-algebras:

Theorem 2.3 *For any pseudo MV-algebras M, there exists a unique (up to isomorphism) unital ℓ-group G with a strong unit u such that $M \cong \Gamma(G,u)$. The functor Γ defines a categorical equivalence of the category of pseudo MV-algebras with the category of unital ℓ-groups. In addition, if h is a given isomorphism of M with $\Gamma(G,u)$, (G,h) is the universal group for M: that is, for any partially ordered group $(K;+,\leq,0)$ and any order- and $+$ preserving mapping $g: M \to K$ there is a homomorphism of partially ordered groups $g' : G \to K$ such that $g = g' \circ h$.* [2]

3 Ideals of Pseudo MV-algebras

According to [GeIo], we say that an *ideal* of a pseudo MV-algebra M is a subset I of M such that (i) $0 \in I$, (ii) if $x, y \in I$, then $x \oplus y \in I$ and $y \oplus x \in$I, and (iii) if $x \in I$, $y \in M$, and $y \leq x$, then $y \in I$.

An ideal I of M is said to be (i) *normal* if $x \oplus I = I \oplus x$ for any $x \in M$,[3] (ii) *prime* if $x \wedge y \in I$ entails $x \in I$ or $y \in I$, (iii) *maximal* if I is a proper ideal of M and it is not included into any proper ideal of M different of I.

As it was proved in [GeIo, Thm 3.8], normal ideals and congruences are in a one-to-one correspondence; the quotient pseudo MV-algebra over a normal ideal I, M/I, is defined as the set of all elements of the form $x/I := \{y \in M : x \odot y^- \oplus y \odot x^- \in I\}$, or equivalently, $x/I := \{y \in M : x^\sim \odot y \oplus y^\sim \odot x \in I\}$.

We note that $n \odot a := a_1 \oplus \cdots \oplus a_n$, if $a = a_1 = \cdots = a_n$.

Proposition 3.1 *Let J be a normal ideal of a pseudo MV-algebra M, $a \in M$, and let $I(J,a)$ be the ideal of M generated by J and a. Define*

$$\begin{aligned} I(J,a) &= \{y \in M : y \leq n \odot a \oplus h,\ n \geq 1,\ h \in J\}. \\ I_1(J,a) &= \{y \in M : y \leq n \odot a + h,\ n \geq 1,\ h \in J\}. \end{aligned}$$

[2]According to [Bae, Thm 1], we can assume that K is even a group not necessarily partially ordered.

[3]We define $x \oplus I := \{x \oplus y : y \in I\}$; similarly we define $I \oplus x$.

$$I_2(J,a) = \{y \in M : (n \odot a)^\sim \odot y \in J, \; n \geq 1\}.$$
$$I_3(J,a) = \{y \in M : y \odot (n \odot a)^- \in J, \; n \geq 1\}.$$

Then $I(J,a) = I_1(J,a) = I_2(J,a) = I_3(J,a)$.

Proof. By [GeIo, Lem 3.4], we have the first identity. It is clear that $I_1(J,a) \subseteq I(J,a)$. So let $y \in I(J,a)$. Then $y \leq n \odot a \oplus h_1$ for some $h_1 \in J$. Hence, $n \odot a \oplus h_1 = (n \odot a) + (n \odot a)^\sim \odot (n \odot a \oplus h_1) = (n \odot a) + (n \odot a)^\sim \wedge h_1 = n \odot a + h$, where $h = (n \odot a)^\sim \wedge h_1 \in J$.

Let now $y \in I_1(J,a)$. Then $y \leq n \odot a + h$, so that, $(n \odot a)^\sim \odot y \leq h \in J$, and $y \in I_2(J,a)$. Conversely, let $y \in I_2(J,a)$. Then $y = (n \odot a) \wedge y + (n \odot a)^\sim \odot y \leq n \odot a + h$, where $h = (n \odot a)^\sim \odot y \in J$, i.e., $y \in I_2(J,a)$.

Suppose now $y \in I_3(J,a)$. Then $y = y \odot (n \odot a)^- + y \wedge (n \odot a) \leq h_1 \oplus n \odot a = n \odot a \oplus h_2$, where $h_1 = y \odot (n \odot a)^- \in J$, and $h_2 \in J$, i.e., $y \in I_1(J,a)$.

Conversely, let $y \in I(J,a)$. Then there exist $h_1, h_2 \in J$ such that $y \leq (n \odot a) \oplus h_1 = h_2 \oplus (n \odot a) = h + (n \odot a)$, i.e., $y \odot (n \odot a)^- \leq h$ and $y \in I_2(J,a)$. $\square$

Proposition 3.2 *Every maximal ideal J of a pseudo MV-algebra M is a prime ideal.*

Proof. This is a direct consequence of [GeIo, Prop 2.21] saying that each proper ideal is an intersection of prime ideals. $\square$

A pseudo MV-algebra M is *simple* if there is no non-trivial proper ideal of M.

Proposition 3.3 *A normal ideal I of a pseudo MV-algebra M is maximal if and only if M/I is a simple MV-algebra.*

Proof. Let I be a normal ideal of M, and let J be a proper ideal of M/I. Define $J_0 := \{x \in M : x/M \in M\}$. Then J_0 is an ideal of M containing I. The maximality of I entails $J = \{0/M\}$.

Conversely, let M/I be simple and let I_1 be a proper ideal of M containing I. The set $\{a/I : a \in I_1\}$ is an ideal of M/I. Therefore, this set coincides with the zero-ideal $\{0/I\}$ of M/I, i.e., $I_1 = I$, which proves that I is maximal. $\square$

Proposition 3.4 *Let I be a normal and maximal ideal of a pseudo MV-algebra M. Then M/I is Archimedean.*

Proof. Suppose that $n(a/I)$ is defined in M/I for each integer $n \geq 1$. The ideal J_a of M/I generated by a/I coincides by Proposition 3.1 with the set $\{x/I : \ x/I \leq n \odot (a/I),\ n \geq 1\}$. By Proposition 3.3, M/I is simple, and $1/I \notin J_a$, so that $a/M = 0/M$. $\square$

Corollary 3.5 *If I is a normal and maximal ideal of a pseudo MV-algebra M, then M/I is commutative.*

Proof. By Proposition 3.4, M/I is Archimedean and by [Dvu 1, Thm 4.2], any Archimedean pseudo MV-algebra is commutative, i.e., an MV-algebra. $\square$

4 States on Pseudo MV-algebras

In the present Section, we introduce states and state-morphisms on pseudo MV-algebras, and we show their close connections with normal and normal maximal ideals on pseudo MV-algebras,

A *state* on a pseudo MV-algebra M is a mapping $m : \ M \to [0,1]$ such that (i) $m(1) = 1$, and (ii) $m(a+b) = m(a) + m(b)$ whenever $a+b$ is defined in M.

If M is a commutative pseudo MV-algebra (i.e. an MV-algebra), then M possesses at least one state. Take now $M = \Gamma(G,u)$ from Example 2.2, and define $m((k,m,0)) = 0$ and $m((k,m,1)) = 1$. Then m is a unique state on this pseudo MV-algebra M which is not an MV-algebra.

Proposition 4.1 *Let m be a state on a pseudo MV-algebra M. Then, for all $a, b \in M$, we have*

(i) $m(0) = 0$.

(ii) $m(a) \leq m(b)$ *if* $a \leq b$, *and then*

$$m(b \odot a^-) = m(b) - m(a) = m(a^\sim \odot b).$$

(iii) $m(a^-) = 1 - m(a) = m(a^\sim)$.

(iv) $m(a^=) = m(a) = m(a^\approx)$.

(v) $m(a \vee b) + m(a \wedge b) = m(a) + m(b)$.

(vi) $m(a \oplus b) + m(b \odot a) = m(a) + m(b)$.

(vii) *The set*

$$Ker(m) := \{a \in M : \; m(a) = 0\}$$

is a normal ideal of M.

(viii) $a/Ker(m) = b/Ker(m)$ *if and only if* $m(a) = m(a \wedge b) = m(b)$.

(ix) *There is a unique state* $\tilde{m}$ *on* $M/Ker(m)$ *such that* $\tilde{m}([a]) = m(a)$, $[a] \in M/Ker(m)$, *where* $[a]$ *is the coset in* $M/Ker(m)$ *determined by* $a \in M$.

(x) $m(a \oplus b) = m(b \oplus a)$.

Proof. The properties (i)–(iv) are evident. (v) follows from (b) of Proposition 3.3. (vi) follows from the property $a = (a \oplus b) \odot b^- + b \odot a$ (see [GeIo, Prop 1.25]).

(i)–(vi) entail (vii).

(viii) It is evident.

(ix) Let $[a] \le [b]^-$. We define $a_0 = a \wedge b^-$. Then $a_0 \le b^-$ and $[a_0] = [a \wedge b]^-$, so that

$$\begin{aligned}\tilde{m}([a] + [b]) &= \tilde{m}([a \oplus b]) = \tilde{m}([a_0 + b]) = m(a_0 + b) = m(a_0) + m(b) \\ &= \tilde{m}([a_0]) + \tilde{m}([b]) = \tilde{m}([a]) + \tilde{m}([b]).\end{aligned}$$

(x) Due to (vii) and (ix), $\tilde{m}([a]) = 0$ iff $[a] = [0]$. We claim that $M/Ker(m)$ is Archimedean. Indeed, let $n[a]$ be defined in $M/Ker(m)$ for any integer $n \ge 1$. Then $\tilde{m}(n[a]) = n\tilde{m}([a]) = n\, m(a) \le 1$ for any n. Therefore, $\tilde{m}([a]) = m(a) = 0$. The Archimedeanicity of $M/Ker(m)$ entails the commutativity of $M/Ker(m)$ (see [Dvu 1, Thm 4.2]). Therefore, $m(a \oplus b) = \tilde{m}([a] \oplus [b]) = \tilde{m}([b] \oplus [a]) = m(b \oplus a)$. $\square$

Let $([0,1]; \oplus_{\mathbb{R}}, ^*, 0, 1)$ be the standard MV-algebra of the real unit interval $[0,1]$, where $s \oplus t := \min\{s + t, 1\}$, $s^* = 1 - s$ for all $s, t \in [0,1]$.

A mapping m from a pseudo MV-algebra M into the standard MV-algebra $[0,1]$ such that, for all $a, b \in M$, $m(a \oplus b) = m(a) \oplus_{\mathbb{R}} m(b)$, (ii) $m(a^-) = m(a^\sim) = 1 - m(a)$, and (iii) $m(1) = 1$ is said to be a *state-morphism*. In other words, a mapping m is a state-morphism iff m is a homomorphism from M into the standard MV-algebra $[0,1]$. It is evident that any state-morphism is a state. The converse does not hold, in general.

Proposition 4.2 *A state m is a state-morphism if and only if*

$$m(a \wedge b) = \min\{m(a), m(b)\}, \quad a, b \in M. \tag{4.1}$$

Proof. Suppose that m is a measure-morphism. Then $m(a \wedge b) = m(a \odot (a^- \oplus b)) = m(a) \odot_{\mathbb{R}} (m(a^-) \oplus_{\mathbb{R}} m(b)) = \min\{m(a), m(b)\}$.

Conversely, let (4.1) hold. Then $a \oplus b = a + a^\sim \odot (a \oplus b) = a + a^\sim \wedge b$ so that $m(a \oplus b) = m(a) + m(a^\sim \wedge b) = m(a) + \min\{1 - m(a), m(b)\} = \min\{m(a) + m(b), 1\}$. $\square$

Proposition 4.3 *Let m be a state on a pseudo MV-algebra M. Then m is a state-morphism if and only if $Ker(m)$ is a maximal ideal of M.*

Proof. Let m be a state-morphism, and let $x \in M$ be such an element that $m(x) \neq 0$. By (vii) of Proposition 4.1, $Ker(m)$ is a normal ideal of M.

Denote by $Ker(m)_x$ the ideal of M generated by $Ker(m)$ and x. By [GeIo, Lem 3.4] or Proposition 4.3, $Ker(m)_x = \{y \in M : \ y \leq n \odot x \oplus h$ for some $n \geq 1$ and some $h \in Ker(m)\}$. Let z be an arbitrary element of M. There exists an integer $n \geq 1$ such that $(n-1)m(x) \leq m(z) < nm(x)$. Then $m((n \odot x)^\sim z) = 0$. Since $z = (n \odot x) \wedge z \oplus (n \odot)^\sim \odot z \leq n \odot x \oplus (n \odot x)^\sim \odot z$, which proves that $z \in Ker(m)_x$, consequently, $M = Ker(m)_x$ which proves that $Ker(m)$ is a maximal ideal of M.

Suppose now $Ker(m)$ is a maximal ideal of M. Since by [GeIo, (i) Prop 1.25], $x \odot y^- \wedge y \odot x^- = 0$ for all $x, y \in M$, we conclude that either $m(x \odot y^-) = 0$ or $m(y \odot x^-)$. In the first case we have $0 = m(x \odot y^-) = m(x \odot (x \wedge y)^-) = m(x) - m(x \wedge y)$. Similarly, $0 = m(y \odot x^-)$ entails $m(y) - m(x \wedge y) = 0$, i.e., $m(x \wedge y) = \min\{m(x), m(y)\}$, which by Proposition 4.2 means that m is a state-morphism. $\square$

Lemma 4.4 (i) *Let G_1 and G_2 be two Abelian lattice ordered subgroups of $(\mathbb{R}; +)$ each containing a common non-zero element g_0. If there is an injective group-homomorphism ϕ of G_1 into G_2 preserving the order such that $\phi(g_0) = g_0$, then $G_1 \subseteq G_2$ and ϕ is the identity on G_1. If, in addition, ϕ is surjective, then $G_1 = G_2$.*

(ii) *Let M_1 and M_2 be two MV-subalgebras of the standard MV-algebra $[0, 1]$. If there is an MV-isomorphism ψ from M_1 onto M_2, then $M_1 = M_2$, and ψ is the identity.*

Proof. (i) Suppose the converse. Then there is a non-zero element $g \in G_1$ such that $\phi(g) = h \neq g$. In this case, there are two integers p and q, $q > 1$, such that either $h < \frac{p}{q} g_0 < g$ or $g < \frac{p}{q} g_0 < h$. In the first case we obtain $qh < pg_0 < qg$, so that $\phi(pg_0) < \phi(qg)$ which gives $p\phi(g_0) = pg_0 < q\phi(g) = qh$, and this is a contradiction. The second case is similar.

(ii) According to the famous Mundici theorem, [Mun], there are two Abelian ℓ-subgroups G_1 and G_1 of $(\mathbb{R};+)$ such that $M_i = \Gamma(G_i, 1)$. The categorical equivalence of the category of MV-algebras and of the category of Abelian unital ℓ-groups entails that $(G_1, 1) \cong (G_2, 1)$, that is G_1 and G_2 are isomorphic unital ℓ-groups. Hence, by (i), $G_1 = G_2$ which yields $M_1 = M_2$. $\square$

Proposition 4.5 *Let m_1 and m_2 be two measure-morphisms on a pseudo MV-algebra M such that $Ker(m_1) = Ker(m_2)$. Then $m_1 = m_2$.*

Proof. According to Proposition 4.3, $Ker(m_1)$ is a normal and maximal ideal of M, so that by Proposition 3.4 and Corollary 3.5, $M/Ker(m_1)$ is a linear, Archimedean, and commutative pseudo MV-algebra. That is, $M/Ker(m_1)$ is an MV-algebra which is an MV-subalgebra of the standard MV-algebra $[0,1]$.

Define $\hat{m}_1$ and $\hat{m}_2$ according to (ix) of Proposition 4.1. Then they are measure-morphisms on the MV-algebra $M/Ker(m_1)$. Define two MV-algebras $M_i := \hat{m}_i(M/Ker(m_1))$ for $i = 1, 2$. Then M_1 and M_2 are MV-subalgebras of the standard MV-algebra $[0,1]$. Define a mapping $\psi : M_1 \to M_2$ by $\psi(\hat{m}_1([x])) = \hat{m}_2([x])$, $x \in M$. Then ψ is an MV-homomorphism which is injective and surjective, so by (ii) of Lemma 4.4, $M_1 = M_2$ which proves $m_1 = m_2$. $\square$

Proposition 4.6 *Let I be a normal and maximal ideal of a pseudo MV-algebra M. Then there is a unique state-morphism m on M such that $Ker(m) = I$.*

Proof. According to Proposition 3.4 and Corollary 3.4., M/I is a linear, Archimedean commutative pseudo MV-algebra. Due to [Dvu 1], there is an ℓ-group G with a strong unit u such that $M/I = \Gamma(G, u)$. In addition, G is linear and Archimedean, so that G is Abelian, and by Hölder's theorem, [Bir, Thm XIII.12], G is an ℓ-subgroup of the real group $(\mathbb{R};+)$. Consequently, we can assume that $M/I \subseteq [0,1]$, and the mapping $m: \ a \mapsto a/I$, $a \in M$, defines a measure-morphism on M such that $Ker(m) = I$. The uniqueness of m follows from Proposition 4.5. $\square$

Proposition 4.7 *Let m be a state on a pseudo MV-algebra M. Then the following statements are equivalent:*

(i) *m is an extremal state on M.*

(ii) *m is a state-morphism on M.*

(iii) $m(x \wedge y) = \min\{m(x), m(y)\}, \ x, y \in M.$

Proof. (i) $\Rightarrow$ (ii). Let m be an extremal state on M. Define $\hat{m}$ by (ix) of Proposition 4.1 on $M/Ker(m)$. We assert that $\hat{m}$ is an extremal state on the commutative pseudo MV-algebra $M/Ker(m)$. Indeed, let $\hat{m} = \lambda\mu_1 + (1-\lambda)\mu_2$, where $0 < \lambda < 1$ and μ_1 and μ_2 are states on $M/Ker(m)$. There exist two states m_1 and m_2 on M such that $m_i(a) := \mu_i([a])$, $a \in M$ for $i = 1, 2$. Then $m = \lambda m_1 + (1-\lambda)m_2$ which gives $m_1 = m_2 = m$, so that $\mu_1 = m_2 = \hat{m}$.

Since $M/Ker(m)$ is an MV-algebra, we conclude from [DvPu, Thm 6.1.30] that $\hat{m}$ is a state-morphism on $M/Ker(m)$. Consequently, so is m on M.

(ii) $\Rightarrow$ (i). Let m be a state-morphism and let $m = \lambda m_1 + (1-\lambda)m_2$, where $0 < \lambda < 1$ and m_1 and m_2 are states on M.

Then $Ker(m) = Ker(m_1) \cap Ker(m_2)$. Since $Ker(m_1)$ and $Ker(m_2)$ are normal proper ideals of M, and $Ker(m)$ is a normal and maximal ideal of M, we have $Ker(m) = Ker(m_1) = Ker(m_2)$. By Propositions 4.3 and 4.5, and 4.6, we conclude that $m = m_1 = m_2$, so that m is an extremal state.

The equivalence of (ii) and (iii) was established in Proposition 4.2. $\square$

Denote by $\mathcal{S}(M)$ the set of all states of a pseudo MV-algebra M. Then $\mathcal{S}(M)$ is a convex set, and let $Ext(\mathcal{S}(M))$ be the set of all its extremal states. A net $\{m_\alpha\}$ of states on M *converges weakly* to a state m iff $m_\alpha(a) \to m(a)$ for every $a \in M$.

Theorem 4.8 *Let M be a pseudo MV-algebra. If $\mathcal{S}(M) \neq \emptyset$, then $\mathcal{S}(M)$ is a compact convex Hausdorff space with respect to the weak topology, and the space of all state-morphisms on M is a non-void compact Hausdorff space. Any state is a weak limit of convex linear combinations of the set of extremal points of $\mathcal{S}(M)$.*

Proof. It is clear that if $\mathcal{S}(M) \neq \emptyset$, then $\mathcal{S}(M)$ is a compact convex Hausdorff space with respect to the weak topology.

If m is a state on M, then in view of (ix)–(x) of Proposition 4.1, $M/Ker(m)$ is an MV-algebra. Hence, there is a state-morphism μ on $M/Ker(m)$, consequently m_0, defined via $m_0(a) = \mu([a])$, $a \in M$, is a state-morphism on M.

Using the Krein–Mil'man theorem, [ReSi], we obtain the fact that any state on M is a weak limit point of the convex hull of the extremal points of $\mathcal{S}(M)$. In addition, by Proposition 4.7 it is now easy to verify that state-morphisms form a non-void compact Hausdorff space. $\square$

We recall that a *state* on a unital ℓ-group (G, u) with strong unit is a mapping $s : G \to \mathbb{R}$ such that (i) $s(g_1 + g_2) = s(g_1) + s(g_2)$ for all $g_1, g_2 \in G$, (ii) $s(g) \geq 0$ if $g \in G^+$, and (iii) $s(u) = 1$. We recall that any Abelian unital ℓ-group (G, u) possesses at least one state, [Goo, Section 4].

If $M = \Gamma(G, u)$, then in view of Theorem 2.3, there is a one-to-one correspondence between the states on M and (G, u). Indeed, this correspondence is given by the fact that if s is a state on (G, u), then $m := s|\Gamma(G, u)$ is a state on M, and conversely, if m is a state on $\Gamma(G, u)$, then it can be extended uniquely to a state s on (G, u) such that $m = s|\Gamma(G, u)$.

On the other hand, there is a one-to-one correspondence between the set of all state-morphisms on M and the set of all normal and maximal ideals on M; this correspondence is given by $m \leftrightarrow Ker(m)$, Propositions 4.5–4.6.

5 Existence of States

In the present Section, we exhibit the existence of states on pseudo MV-algebras. We show that any linearly ordered, and more generally, any representable pseudo MV-algebra admits at least one state.

We start with two examples of non-commutative pseudo MV-algebras with a linear order having states.

Example 5.1 *Let G be the group of all matrices of the form*

$$A = \begin{pmatrix} \xi & \alpha \\ 0 & 1 \end{pmatrix},$$

where ξ and α are rational (real) numbers such that $\xi > 0$; the group-operation is the usual multiplication of matrices. We denote $A = (\xi, \alpha)$.

Then $A^{-1} = (1/\xi, -\alpha/\xi)$, and $(1,0)$ is the neutral element. We define $G^+ := \{(\xi,\alpha) :$ where (i) $\xi > 1$, or (ii) $\xi = 1$ and $\alpha \geq 0\}$. Then G with the positive cone G^+ is a linearly ordered ℓ-group with a strong unit $U = (2,0)$. Define $M = \Gamma(G,U)$. Then $M = M_1 \cup M_2 \cup M_3$, where $M_1 = \{(\xi,\alpha) :\ 1 < \xi < 2\}$, $M_2 = \{(2,\alpha) :\ \alpha \leq 0\}$, and $M_3 = \{(1,\alpha) :\ \alpha \geq 0\}$. M_3 is a unique normal and maximal ideal of M, and there is a unique state-morphism m, namely $m((\xi,\alpha)) = \log_2(\xi)$, $(\xi,\alpha) \in M$.

Example 5.2 *Let G be the set of all pairs (x,y) of real numbers under the decomposition*

$$(x_1,y_1) + (x_2,y_2) = (x_1 + x_2, e^{x_2}y_1 + y_2),$$

and let G^+ be the set of all (x,y) such that either $x > 0$ or $x = 0$ and $y \geq 0$. Then G is a linearly ordered ℓ-group, with a neutral element $(0,0)$ and with the inverse $-(x,y) = (-x, -e^x y)$ of an element (x,y) and with a strong unit $u = (1,0)$. Then $M = \Gamma(G,u)$ possesses a unique state-morphism m, namely $m((x,y)) = x$, $(x,y) \in M$.

Theorem 5.3 *Let (G,u) be a unital ℓ-group and let $M = \Gamma(G,u)$. Then M is linear if and only if G is linear.*

Proof. If G is a linear, then clearly M is linear.

Suppose now that M is linear and let the lattice-ordered group G be not linear. Then there exist two elements $f_1, f_2 \in G$ such that $f_1 \neq f_1 \wedge f_2 \neq f_2$. Let $f_1 = f_1^+ - f_1^-$, $f_2 = f_2^+ - f_2^-$, where $f_i^+, f_i^- \geq 0$ for $i = 1,2$. Then $f_1^- = \sum_{i=1}^n f_i^1$, $f_2^- = \sum_{i=1}^m f_i^2$, where $f_i^1, f_i^2 \in M$. Without loss of generality we can assume that $n = m$ and let $f_0^0 := \bigvee_{i=1}^n (f_i^1 \vee f_i^2)$. Then $f_1^-, f_2^- \leq n f_0^0 =: f_0$. Therefore, $f_i = f_i^+ - f_i^- \geq f_i^+ - f_0$ and $f_i + f_0 \geq f_i^+$ for $i = 1,2$. Then $f_1 + f_0 \neq (f_1 + f_0) \wedge (f_2 + f_0) \neq f_2 + f_0$.

Since $f_1 + f_0$ and $f_2 + f_0$ both are positive, there exist $x_1, \ldots, x_k \in M$ such that $f_i + f_0 \leq v := \sum_{j=1}^k x_j$ for $i = 1,2$. Then $v_0 := (v - f_0 - f_1) \wedge (v - f_0 - f_1) \geq 0$, and $0 = (v - f_0 - f_1 - v_0) \wedge (v - f_0 - f_1 - v_0)$. Put $g_i = v - f_0 - f_i - v_0$ for $i = 1,2$. Then $g_1 > 0$, $g_2 > 0$ and $g_1 \wedge g_2 = 0$.

Due to the Riesz decomposition property [Dvu], there exist positive elements g_j^i in M with $g_j^i \leq x_j$ for $i = 1, \ldots, k$ and $i = 1,2$ such that

$$g_i = \sum_{j=1}^k g_j^i.$$

Since $g_1 \wedge g_2 = 0$, there are two elements $g^1_{j_1} > 0$ and $g^2_{j_2} > 0$ such that $g^1_{j_1} \wedge g^2_{j_2} = 0$, which contradicts the linearity of M. $\square$

We now show that each linear pseudo MV-algebra possesses a unique state.

Proposition 5.4 *Any linear pseudo MV-algebra possesses a maximal ideal which is normal.*

Proof. The set of all proper ideals of a linear pseudo MV-algebra M is nonvoid. Moreover, if I_1 and I_2 are two ideals of M, then $I_1 \subseteq I_2$ or $I_2 \subseteq I_1$. If not, then there exist two elements x, y such that $x \in I_1 \setminus I_2$ and $y \in I_2 \setminus I_1$. Since $x \leq y$ or $y \leq x$, we obtain a contradiction.

Let now I_0 be the set-theoretical union of all proper ideal of M. Then I_0 is a maximal ideal. We note that, for $x \in M$, $x \in I_0$ iff $n\,x$ is defined in M for any integer $n \geq 1$. Without loss of generality we can assume that $M = \Gamma(G, u)$ for some unital ℓ-group (G, u) with strong unit. By Theorem 5.3, G is a linearly ordered ℓ-group.

We claim that I_0 is a normal ideal. Let $x \in M$ and $h \in I_0$ be given. If $x \in I_0$, then $x \oplus h = (x \oplus h) \odot x^- + x$ which shows that $h_1 = (x \oplus h) \odot x^- \in I_0$.

Let now $x \notin I_0$. Then, for any $h \in I_0$, $nh < x$ for every $n \geq 1$. Without loss of generality we can assume that $x + h$ is defined in M. Let $x + h = h_1 + x$ for some $h_1 \in M$. Using the group-representation, we have $x + h - x = h_1$. We finish the proof showing that nh_1 is defined in M for any $n \geq 1$. We have $nh_1 = x + nh - x = x - (x - nh)$. But $0 < x - nh \leq x \leq\leq u$, so that $nh_1 \in M$, and hence $h_1 \in I_0$.

Similarly we prove that for all $x \in M$ and $h \in I_0$ there exists an $h_2 \in I_0$ such that $h \oplus x = x + h_2$.

The uniqueness of a maximal ideal is evident. $\square$

Theorem 5.5 *Every linearly ordered pseudo MV-algebra M possesses a unique state.*

Proof. According to Proposition 5.4, M possesses a unique normal and maximal ideal I_0 of M. Applying Proposition 4.6, we have that there exists a unique state-morphism m on M such that $Ker(m) = I_0$. $\square$

Theorem 5.6 *Every linear unital ℓ-group (G, u) with strong unit possesses a unique state.*

Proof. Define $M = \Gamma(G, u)$. By Theorem 5.5, M possesses a unique state m. According to Theorem 2.3, this state can be uniquely extended to a state s on (G, u) such that $m = s|M$. $\square$

We extend these results to representable pseudo MV-algebras introduced in [GeIo].

Let $\{(M_t; \oplus_t, ^{-_t}, ^{\sim_t}, 0_t, 1_t)\}_{t\in T}$ be a family of pseudo MV-algebras. The Cartesian product $M := \prod_{t\in T} M_t$, where $\oplus, ^-, ^\sim, 0, 1$ are defined in a usual way by coordinates, is said to be a *direct product* of $\{(M_t; \oplus_t, ^{-_t}, ^{\sim_t}, 0_t, 1_t)\}_{t\in T}$. Then M is a pseudo MV-algebra. A pseudo MV-algebra M is a *subdirect product* of a family of $\{(M_t; \oplus_t, ^{-_t}, ^{\sim_t}, 0_t, 1_t)\}_{t\in T}$ of pseudo MV-algebras iff there exists a one-to-one homomorphism $h : M \to \prod_{t\in T} M_t$ of pseudo MV-algebras [4] such that, for each $t \in T$, $\pi_t \circ h$ is a homomorphism from M onto M_t, where π_t is the t-th projection $\prod_{t\in T} M_t$ onto M_t.

According to [GeIo], we say that a pseudo MV-algebra M is *representable* if it can be represented as a subdirect product of linear pseudo MV-algebras. It is well-known that every MV-algebra is representable (see e.g., [CDM]).

Theorem 5.7 *Every representable pseudo MV-algebra M possesses at least one state.*

Proof. There exists a family of linear pseudo MV-algebras $\{M_t\}_{t\in T}$ such that M is a subdirect product of $\prod_{t\in T} M_t$. Every M_t possesses due to Theorem 5.5 a unique state m_t. If h is the embedding of M into $\prod_{t\in T} M_t$ such that $h_t := \pi_t \circ h$ is a homomorphism from M onto M_t, then $m(a) = m_t(h_t(a))$, $a \in M$, is a state-morphism on M. $\square$

It is worth to recalling that the pseudo MV-algebra M from Example 2.2 is not representable because there is no family $\{H_t\}$ of normal prime ideals of M with $\bigcap_t H_t = \{0\}$ which is a necessary and sufficient condition to be M representable (see [GeIo]). On the other hand, this M possesses a unique state (see the beginning of Section 4).

6 States on Normal-Valued Pseudo MV-algebras

We extend the class of pseudo MV-algebras possessing at least one state to the class of normal-valued ones. In addition, we show that the families of

[4]We recall that a mapping $h : M_1 \to M_2$ of two pseudo MV-algebras M_1 and M_2 is said to be a *homomorphism* if h preserves $\oplus$, $^-$, $^\sim$ and 0 and 1.

representable and normal-valued pseudo MV-algebras form varieties, and we give their equational bases.

We denote by $\mathcal{I}(M), \mathcal{P}(M)$, and $\mathcal{N}(M)$ the set of all ideals, prime ideals and normal ideals of a pseudo MV-algebra M. Similarly, let $\mathcal{C}(G)$, $\mathcal{P}(G)$ and $\mathcal{L}(G)$ the set of all convex ℓ-subgroups, prime subgroups (i.e., prime convex ℓ-subgroups) and ℓ-ideals of an ℓ-group G.

Let (G, u) be a unital ℓ-group and let $M = \Gamma(G, u)$. On the basis of the representation of pseudo MV-algebras by unital ℓ-groups, J. Rachůnek [Rac 1] find an equivalence between ideals of M and convex ℓ-subgroups of (G, u), and $\mathcal{P}(M)$ and $\mathcal{P}(G)$, respectively. Here we extend his result.

Theorem 6.1 *Let (G, u) be a unital ℓ-group and let $M = \Gamma(G, u)$. For any ideal I of M, we assign*

$$\phi(I) := \{x \in G : \ |x| \wedge u \in I\}. \tag{6.1}$$

(1) *$\phi(I)$ is a convex ℓ-subgroup of G generated be I. The mapping $\phi : I \mapsto \phi(I)$ from $\mathcal{I}(M)$ to $\mathcal{C}(G)$ defines a one-to-one mapping preserving the set-theoretical inclusion. The inverse mapping ψ is given by*

$$\psi(K) := K \cap [0, u], \ K \in \mathcal{C}(G).$$

(2)

$$\phi(I) = \{x \in G : \ \exists\, x_i, y_j \in I, \quad x = x_1 + \cdots + x_n - y_1 - \cdots - y_m\}. \tag{6.2}$$

(3) *The restriction of ϕ to $\mathcal{P}(M)$ or $\mathcal{N}(M)$ gives a bijection between $\mathcal{P}(M)$ and $\mathcal{C}(G)$, and $\mathcal{N}(M)$ and $\mathcal{L}(G)$, respectively, which preserves the set-theoretical inclusion.*

Proof. (1) Formula (6.1) and the first statement were proved in [Rach 1], as well as the statement that $\phi(I)$ is a prime ideal of G iff I is a prime ideal of M, and if K is a prime ℓ-subgroup of G, then $K \cap [0, u]$ is a prime ideal of M.

(2) It is clear that $\phi(I)$ contains all elements $x \in G$ of the form $x = x_1 + \cdots + x_n - y_1 - \cdots - y_m$, where $x_i, y_j \in I$. Let now $x \in \phi(I)$, $x \geq 0$. Then $x \leq nu$ for some integer $n \geq 1$. By the Riesz decomposition property, there exist $x_1, \ldots, x_n \in M$ such that $x = x_1 + \cdots + x_n$. Then $x_i \leq x \wedge u \in I$, so that $x_i \in I$. If $x = x^+ - x^-$, then $x^+, x^- \in \phi(I)$, and we have (6.2).

(3) We now show that if I is a a normal ideal of M, then $\phi(I)$ is an ℓ-ideal of G. This direction was an open problem in [Rach 1].

According to (A6), we see that I is a normal ideal of M iff $g+I = I+g$, where $g+I := \{g+x \in M : x \in I\}$; similarly $I+g := \{x+g \in M : x \in I\}$.

Case 1. $g \in M$, $x \in I$.

Subcase 1a. $g-x = g-x-g+g = -(g+x-g)+g = -(x'+g-g)+g = -x'+g$, where $x' \in I$.

Subcase 1b. $x+g = g+x'$, where $x' \in I$.

Subcase 1c. $-g-x = -(x+g) = -(g+x') = -x'-g$, where $x' \in I$.

Subcase 1d. $-g+x = -g+x+g-g = -g+g+x'-g = x'-g$, where $x' \in I$.

Subcase 1e. $g+x = x'+g$, where $x' \in I$.

Case 2. By induction we have that Case 1 holds also for $g = \sum_{i=1}^n g_i$, and $x = \sum_{j=1}^m x_j$, where $g_i \in M$ and $x_j \in I$. Hence by (6.2), Case 1 holds for all $g \in G$ and $x \in \phi(I)$. $\square$

Let g be a nonzero element of a pseudo MV-algebra M. According to [GeIo], we say that a *value* of g is an ideal V of M such that $g \notin V$, and V is maximal with respect to this property. As it was shown in [GeIo, Thm 2.15] such an ideal always exists in M and is a prime ideal. We denote by $\Gamma(g)$ the set of all values of g. For any value V of an element g we have that there is a unique least ideal V^* properly containing V; it is called a *cover* of V. It coincides with $\bigcap\{I : I \supset V\}$, and it is equal to the ideal generated by V and the element g.

Moreover, by [GeIo, Prop 2.15], each proper ideal of M is an intersection of values, and if $f \wedge g = 0$ and $V \in \Gamma(f), W \in \Gamma(g)$, then V and W are incomparable.

We recall that, in a similar way for ℓ-groups, by a *value* of $g \in G \setminus \{0\}$ we understand a convex ℓ-subgroup V of G not containing g which is maximal with respect to this property. A *cover* of a value V is a minimal convex ℓ-subgroup V^* properly containing V, and it coincides with the convex ℓ-subgroup of G generated by V and g.

Using the language of ℓ-group theory, we introduce known-notions of ℓ-groups for pseudo MV-algebras (for their ℓ-group analogues see e.g. [Dar, Gla, KoMe]).

An element $g \in M \setminus \{0\}$ is *special* if $|\Gamma(g)| = 1$, and in this case, this value is said to be a *special value*; we denote by V_g the unique value of g.

We say that a value V is (i) *normal in* V^* iff $x+V = V+x$ for any

$x \in V^*$; (ii) *essential* iff it contains all the values of some $g \in M \setminus \{0\}$. For example, if 1 is the greatest element of M having a unique value V_1, then $V \subseteq V_1$ for any $V \in \Gamma(g)$ and any $g > 0$.

M is said to be (i) *finite valued* if every non-zero element $g \in M$ has finitely many values; (ii) *normal-valued* if, for any $g > 0$, any values $V \in \Gamma(g)$ is normal in its cover V^*, and (iii) *special* if $|\Gamma(g)| = 1$ for any $g \in M \setminus \{0\}$.

Proposition 6.2 *Let $M = \Gamma(G, u)$.*

(1) *Let V be a value of $g \in M \setminus \{0\}$. Then $\phi(V)$ is a value of $g \in G$ in G. If V is a value of $g \in G \setminus \{0\}$. Then $V \cap [0, u]$ is a value of $|g| \wedge u \in M \setminus \{0\}$. Moreover, V^* is a cover of V if and only if $\phi(V)^*$ is a cover of $\phi(V)$, and $\phi(V^*) = \phi(V)^*$.*

(2) *V is normal in its cover V^* if and only $\phi(V)$ is normal in $\phi(V^*)$.*

(3) *M is finite-valued if and only if G is finite-valued.*

Proof. (1) By (6.1) it si clear that $g \notin \phi(V)$. If there exists a convex ℓ-subgroup K of G such that $g \notin K \supseteq \phi(V)$, then by Theorem 6.1, $K \cap [0, u] \supseteq V$, which means that $\phi(V)$ is a value of g in G.

Let V be a value of a nonzero element $g \in G$. Then there exists an integer $n \geq 1$ such that $|g| \leq nu$. Hence $|g| \wedge u > 0$. Put $V_0 = V \cap [0, u]$. Then $|g| \wedge u \notin V_0$, and by Theorem 6.1, V_0 is a value of $|g| \wedge u$ in M. In addition, V is a value of $|g| \wedge u$ in G.

The second part of this statement is evident due to Theorem 6.1.

(2) This follows from (6.2) and from the proof of (3) of Theorem 6.1.

(3) This follows from (1) and Theorem 6.1. $\square$

As a corollary we have that the system $\{\Gamma(g) : g \in M \setminus \{0\}\}$ is a root system; we recall that a *root system* is a poset X such that if $x \leq y, z$ then either $y \leq z$ or $z \leq y$).

According to [Jak], we say that a nonzero element s of M is *basic* if the interval $[0, s]$ is a chain. In particular, an element $a > 0$ is an *atom* if $[0, a] = \{0, a\}$. It is clear that if g and h are two basic elements of M, then either $g \wedge h = 0$ or g and h are mutually comparable. For example, any nonzero element in a linearly ordered pseudo MV-algebra is basic. We recall that if $f \leq g$, then for any $V \in \Gamma(f)$ there is $W \in \Gamma(g)$ such that $V \subseteq W$.

Proposition 6.3 (1) *Let f, g, h be special elements of a pseudo MV-algebra M. If $f \wedge g > 0$, then $V_f \subseteq V_g$ or $V_g \subseteq V_f$.*

(2) *If* g *is a basic element of* M*, the set* $g^\perp := \{x \in M : \ x \wedge g = 0\}$ *is a prime ideal of* M*, and* g *is a special element.*

(3) *If* f *is a special element of* M*, then* V_f *is normal in its cover* V_f^**.*

Proof. (1) Since $f \wedge g > 0$, then either $f \leq g$ or $g \leq f$. In the first case, $g \notin V_f$ so that $V_f \subset V_g$. For the second one, $V_g \subseteq V_f$.

(2) Let g be a basic element of M. Then $g^\perp$ is an ideal of M, [GeIo]. We show that $g^\perp$ is prime: Let $a, b \in M$ and $a \wedge b = 0$. Then $(a \wedge g) \wedge (b \wedge g) = 0$. Since g is basic, $a \wedge g = 0$ or $b \wedge g = 0$, i.e., $a \in g^\perp$ or $b \in g^\perp$.

If now V is a value of g, it must contain $g^\perp$. Since all values form a root system, we have $|\Gamma(g)| = 1$.

(3) Let V_f be a special value of $f \in M$. Then $\phi(V_f)$ is a value of f in G, and in view of (1) of Proposition 6.2, $\phi(V_f)$ is a unique value of f in G. By [Gla, Lem 3.4.2], $\phi(V_f)$ is normal in its cover $\phi(V_f)^*$, so that by (3) of Proposition 6.2, V_f is normal in V_f^*. $\square$

Proposition 6.4 *Let* M *be a pseudo* MV*-algebra.*

(1) *If* $|\Gamma(1)| < \infty$*, then* $|\Gamma(g) < \infty$ *for each* $g \in M \setminus \{0\}$*.*

(2) *If* $|\Gamma(1)| < \infty$*, then each value is a special value, and* M *is normal-valued.*

Proof. (1) Let $|\Gamma(1)| < \infty$ and let $W \in \Gamma(g)$. Since $g \leq 1$, then $1 \notin W$, and there exists $V \in \Gamma(1)$ such that $W \subseteq V$, which yields $|\Gamma(g)| < \infty$.

(2) If $|\Gamma(1)| = 1$, then V_1 is a special value of 1. Suppose now that 1 has the values $V_1, \ldots, V_{n+1}$. They are maximal ideals of M, and every proper ideal of M is contained in at least one of these ideals. Let $f_i \in V_{n+1} \setminus V_i$ and $h_i \in V_i \setminus V_{n+1}$ $(1 \leq i \leq n)$. Let $f = \bigvee_{i=1}^n f_i$ and $h = \bigwedge_{i=1}^n h_i$. Then $f \in V_{n+1} \setminus (\bigcup_{i=1}^n V_i)$ and $h \in (\bigcap_{i=1}^n V_i) \setminus V_{n+1}$. By replacing f by $f \odot (f \wedge g)^-$ and h by $h \odot (f \wedge g)^-$, we may assume that $f \wedge g = 0$. Now $V_1, \ldots, V_n$ are values of f, and V_{n+1} is a value of h. We claim that V_{n+1} is a unique value of h. Indeed, if $V \neq V_{n+1}$ is a value of h, then as $V \subseteq V_i$ for some $i = 1, \ldots, n$, we conclude in view of $f \wedge g = 0$ that $f \in V \subseteq V_i$, which is absurd.

Repeating this process for any permutation $(i_1, \ldots, i_{n+1})$ of $(1, \ldots, n+1)$, we see that each $V_{i_{n+1}}$ is a special value of some element of M.

Let now $g > 1$. By (1), $|\Gamma(g)| \leq |\Gamma(1)| < \infty$. If $|\Gamma(g)| = 1$, then V_g is a unique value of g. If $W_1, \ldots, W_{n'+1}$ are all values of g, we can repeat the above procedure literally also for g.

Since each value is a unique value of some element of M, by (3) of Proposition 6.3, we conclude that M is normal-valued. $\square$

Theorem 6.5 *Each normal-valued pseudo MV-algebra M possesses at least one state. In particular, if $|\Gamma(1)| = n$, then M possesses exactly n state-morphisms.*

Proof. Let $M = \Gamma(G, u)$. Then for $1 = u$ we have that each value $V \in \Gamma(1)$ is normal in $V^* = M$. Hence M/V is a linear pseudo MV-algebra which by Theorem 5.5 possesses a state μ. Consequently, $m(a) := \mu(a/V)$, $a \in M$, is a state on M.

The second statement follows from (2) of Proposition 6.4 and Proposition 4.5. $\square$

We recall that Theorem 6.5 extends the result of Theorem 5.7 in view of the following statement.

Proposition 6.6 *Every representable pseudo MV-algebra is normal-valued.*

Proof. We have, [GeIo, Prop 3.13], that M is representable iff there exists a family $\{H_t : t \in T\}$ of normal prime ideals of M with $\bigcap_{t\in T} H_t = \{0\}$. In view of Theorem 6.1, if $M = \Gamma(G, u)$, then M is representable iff G is representable, i.e., if G is a subdirect product of linearly ordered ℓ-groups G, or equivalently, [Dar, Prop 47.1(c)], iff there exists a system of ℓ-ideals $\{L_t : t \in T\}$ of G such that $\bigcap_{t\in T} L_t = \{0\}$. Applying [Gla, Cor 4.2.5], every representable ℓ-group is normal-valued, so M is normal-valued by (2) of Proposition 6.2. $\square$

We note that the pseudo MV-algebra M from Example 2.2 gives a normal-valued pseudo MV-algebra with $|\Gamma(1)| = 1$ and which is not representable.

In what follows, we show that the family of normal-valued pseudo MV-algebras forms a variety.

For $a \in M$, by $I(a)$ we denote the ideal of M generated by a. We recall that the set of all ideals $\mathcal{I}(M)$ is a complete lattice with respect to the set-theoretical inclusion, [GeIo], where the meet $\wedge = \cap$ and $\vee$ denotes the join in $\mathcal{I}(M)$.

We note that using the Riesz decomposition property in ℓ-groups, we have the following form of the Riesz decomposition property in pseudo MV-algebras: $x \le y_1 \oplus \cdots \oplus y_n$ entails $x = x_1 \oplus \cdots \oplus x_n$, where $x_i \le y_i \in M$, for $i = 1, \ldots, n$.

Theorem 6.7 *Let M be a pseudo MV-algebra. The following statements are equivalent:*

(i) *M is normal-valued.*

(ii) $a \oplus b \leq 2 \odot b \oplus 2 \odot a, \quad a, b \in M.$

(iii) $I(a) \vee I(b) = I(a) \oplus I(b) = I(b) \oplus I(a), \quad a, b \in M.$ [5]

(iv) $I \vee J = I \oplus J = J \oplus I, \quad I, J \in \mathcal{I}(M).$

Proof. Let $M = \Gamma(G, u)$.

(i) $\Rightarrow$ (ii). By (2) of Proposition 6.2, M is normal-valued iff G is normal-valued. It is well-known that an ℓ-group G is normal-valued iff $a+b \leq 2b+2a$ for all $a, b \in G^+$, [Dar, Thm 41.1]. Then if $a, b \in M$, we have $(a+b) \wedge u \leq (2b+2a) \wedge u$, i.e., $a \oplus b \leq 2 \odot b \oplus 2 \odot a$.

(ii) $\Rightarrow$ (iii). We show that, for any integer $n \geq 2$, $n \odot (a \oplus b) \leq k_n \odot a \oplus k_n \odot b \leq 2k_n \odot b \oplus 2k_n \odot a$, where $k_n \geq 2$ is a positive integer.

Indeed,

$$\begin{aligned}
(n+1) \odot (a \oplus b) &= a \oplus n \odot (b \oplus a) \oplus a \\
&\leq a \oplus k_n \odot b \oplus k_n \odot a \oplus b \\
&\leq a \oplus 2k_n \odot a \oplus 2k_n \odot b \oplus b \\
&= (1+2k_n) \odot a \oplus (1+2k_n) \odot b \\
&= 2(1+2k_n) \odot b \oplus 2(1+2k_n) \odot a.
\end{aligned}$$

We have $I(a) \vee I(b) = I(a \oplus b)$. Hence, if $x \in I(a \oplus b)$, then $x \leq n \odot (a \oplus b) \leq k_n \odot a \oplus k_n \odot b$, and $x \leq x_1 \oplus x_2$, where $x_1 \leq k_n \odot a \in I(a)$, $x_2 \leq k_2 \odot b \in I(b)$. Hence $x \in I(a) \oplus I(b) \subseteq I(a \oplus b)$, which proves $I(a) \vee I(b) = I(a) \oplus I(b)$, and similarly $I(a) \vee I(b) = I(b) \oplus I(a)$.

(iii) $\Rightarrow$ (i). We have $I = \bigvee_{a \in I} I(a)$ and $J = \bigvee_{b \in J} I(b)$. Then $I \vee J = \bigvee_a I(a) \vee \bigvee_b I(b) = \bigvee_{a,b} I(a) \vee I(b) = \bigvee_{a,b} (I(a) \oplus I(b)) \subseteq I \oplus J \subseteq I \vee J$. In a similar way, we have $I \vee J = J \oplus I$.

(iv) $\Rightarrow$ (i). For $A, B \subseteq M$, we denote by $A + B := \{a + b \in M : \ a \in A, b \in B\}$. Then $I \oplus J = I + J$ in view of $x \oplus y = (x \oplus y) \odot y^- + y = x \wedge y^- + y$, and $x \wedge y^- \in I$. Let ϕ be the mapping from Theorem 6.1. Due to (6.2) and the proof of (3) of Theorem 6.1, we have

$$\phi(I) \vee \phi(J) = \phi(I \vee J) = \phi(I \oplus J) = \phi(I + J) \subseteq \phi(I) + \phi(J) \subseteq \phi(I \vee J).$$

[5] For $A, B \subseteq M$, we denote $A \oplus B := \{a \oplus b : \ a \in A, b \in B\}$; similarly by $A + B := \{a + b \in M : \ a \in A, b \in B\}$.

In a similar manner, $\phi(I) \vee \phi(J) = \phi(J) + \phi(I)$. Due to Theorem 6.1, we have $K \vee L = K + L = L + K$ for all $K, L \in \mathcal{C}(G)$, which by [Dar, Thm 41.1] means that G is normal-valued, so that by Proposition 6.2, so is M. $\square$

Theorem 6.8 *The family of normal-valued pseudo MV-algebras forms a variety.*

Proof. The equational base is by Theorem 6.7

$$(a \oplus b) \wedge (2 \odot b \oplus 2 \odot a) = (a \oplus b), \quad a, b \in M. \tag{6.3}$$

$\square$

Finally, we show that representable pseudo MV-algebras form a variety and we present their equational base solving an open problem from [GeIo, Problem 3.15]. For $g \in M$, we denote by $g^\perp := \{x \in M : \ x \wedge g = 0\}$.

Proposition 6.9 *A pseudo MV-algebra M is representable if and only if $g^\perp$ is normal in M for any $g \in M$.*

Proof. Let $M = \Gamma(G, u)$. We have $\phi(g^\perp) = g^{\perp_G}$, where $g^{\perp_G} := \{h \in G : \ |h| \wedge g = 0\}$. Indeed, it is clear that $g^\perp \subseteq g^{\perp_G}$, hence $\phi(g^\perp) \subseteq g^{\perp_G}$. Let $h \in G^+$, and $h \wedge g = 0$. Then $h = h_1 + \cdots + h_n$, where $h_1, \ldots, h_n \in M$. Then each $h_i \in g^\perp$, and by (6.2), $h \in \phi(g^\perp)$.

Moreover, if $g \in G^+$, then $\phi((g \wedge u)^\perp) = (g \wedge u)^{\perp_G} = g^{\perp_G}$.

According to [Gla, Thm 3.C], G is representable iff $g^{\perp_G}$ is normal in G for any $g \in G^+$, so that by (3) of Theorem 6.1, M is representable iff $g^\perp$ is normal in M. $\square$

Let $x, y \in M$; we write $x \ll y$ iff nx exists in M for any $n \geq 1$, and $nx < y$ $(n \geq 1)$. If $x, y \in G^+ \setminus \{0\}$, we write $x \ll y$ if $nx < y$ for any $n \geq 1$.

Proposition 6.10 *A pseudo MV-algebra M is not representable if and only if there are $x, y \in M \setminus \{0\}$ such that $x + y$ and $y + x$ are defined in M, and*

$$(x + y) \wedge (y + x) = y. \tag{6.4}$$

In such a case, $x \ll y$.

Proof. Let $M = \Gamma(G,u)$. Then M is not representable iff so is not G.

(1) We show that G is not representable iff there are $x, y \in M \setminus \{0\}$ such that

$$x \wedge (-y + x + y) = 0. \tag{6.5}$$

(We note that the element $-y + x + y$ is from M, see below.)

If G is not representable, by [Dar, Prop 47.1], there is a polar P of G [6] which is not normal in G. Hence there is $y \in G$ such that $P \neq -y + P + y$. Now if, for any $g > 0$, $P = -g + P + g$, we must have $-y + P + y = y^- - y^+ + P + y^+ - y^- = P$. Since $y \in G^+$ is of the form $y = y_1 + \cdots + y_n$, where $y_i \in M$ for $i = 1, \ldots, n$, without loss of generalization we can assume $y \in M \setminus \{0\}$.

If $P \supset -y + P + y$, assume there exists $t \in P \setminus (-y + P + y)$. Let $x = t - (t \wedge (y + t - y))$; clearly $x > 0$. Then $x \wedge (-y + x + y) = t - (t \wedge (y + t - y)) \wedge ((-y + t + y) - (t \wedge (-y + t + y)) = 0$.

If $P \subset -y + P + y$, we have $y + P - y \supset P$, and repeating the above with $x = t - (t \wedge (y + t - y))$ we have that $x \wedge (-y + x + y) = 0$ for some $x, y \in M \setminus \{0\}$.

If now (6.5) holds for some $x, y \in M \setminus \{0\}$, then $x^{\perp_G}$ is a polar of G which is not normal, so that G is not representable.

From (6.5) we have $(y + x) \wedge (x + y) = y$. Since $x \leq y + x, x + y$, we have $x \leq y$. It is clear that $x < y$, otherwise $y = x$ entails $2x \wedge 2x = x$, a contradiction.

Suppose now $nx < y$. Changing in (6.5) y to $y - nx$, we have $x \wedge [-(y - nx) + x + (y - nx)] = (nx + x - nx) \wedge (nx - y + x + y - nx) = nx + [x \wedge (-y + x + y)] - nx = 0$, which proves $x < y - nx$, i.e., $(n+1)x < y$, and consequently $x \ll y$.

(2) Let $x \in G^+ \setminus \{0\}$. We define $R(x)$ as the set of all $y \in G^+ \setminus \{0\}$ such that (6.5) holds. If $R(x) \neq \emptyset$ and $y \in R(x)$, then (i) $x \ll y$, (ii) $-x + y, y - x \in R(x)$, (iii) $x + y, y + x \in R(x)$, and (iv) if $y \in M$, then $-nx + y - mx \in R(x) \cap M$ for all integers $n, m \geq 1$.

Therefore, if M is not representable, there are $x, y_0 \in M \setminus \{0\}$ satisfying (6.5). Then $y := -x + y_0 - x \in R(x) \cap M$, and x, y satisfy (6.4). Conversely, if $x, y \in M \setminus \{0\}$ satisfy (6.5), then they satisfy (6.5), and G is not representable, so is not M. $\square$

[6] A subset P of G is a polar in G if there exists a subset X of G such that $P = X^\perp := \{g \in G : \ |g| \wedge |x| = 0 \text{ for all } x \in X\}$.

Theorem 6.11 *The family of representable pseudo MV-algebras forms a variety with the equational base*

$$2 \odot (a \wedge b) = (2 \odot a) \wedge (2 \odot b), \quad a, b \in M. \tag{6.6}$$

In addition, it is a proper subvariety of the variety of representable pseudo MV-algebras.

Proof. Since (6.6) holds in any linearly ordered pseudo MV-algebra, so (6.6) holds in any representable pseudo MV-algebra.

Let $M = \Gamma(G, u)$ and suppose that M is not representable. By (1) of the proof of Proposition 6.10, there exist two nonzero elements x and y_0 in M such that (6.5) holds. In view of (iv) of (2) of the proof of Proposition 6.10, $y := -2x + y_0 - 2x \in M \setminus \{0\}$, and x and y satisfy (6.4), moreover, $2x + y \in M$.

Define $a = y + x$, $b = x + y$. Then $a \wedge b = y$. On the other hand, using the group addition, we have in G:

$$\begin{aligned} 2a &= y + x + y + x = 2y + (-y + x + y) + x \\ &> 2y + (-y + x + y), \\ 2b &= x + y + x + y = 2y + (-2y + x + 2y) + (-y + x + y) \\ &= 2y + (-y + x + y) + (-2y + x + 2y) \\ &> 2y + (-y + x + y), \end{aligned}$$

while $(-y+x+y) \wedge (-2y+x+2y) = 0$. We know that $(-y+x+y) \in M \setminus \{0\}$, and $x \ll y$, therefore, $(2 \odot a) \wedge (2 \odot b) = (2a) \wedge u \wedge (2b) \wedge u \geq (2y + (-y + x + y)) \wedge u = (2 \odot y) \oplus (-y + x + y)$.

We claim that the element $(2 \odot y) + (-y + x + y)$ is defined in M. To prove that, we have to show that $(2 \odot y) \leq (-y + x + y)^-$. Calculate

$$\begin{aligned} 2 \odot y &= y \oplus y = (y \oplus y) \odot y^- + y = (y \wedge y^-) + y \\ &= (x \wedge y^-) \wedge y + y = ((x \oplus y) \odot y^-) \wedge y + y \\ &= ((x + y) - y) \wedge y + y = (x \wedge y) + y = x + y. \end{aligned}$$

On the other hand, $(-y + x + y)^- = u - y - x + y$. In the view of the choice of x and y, $2x + y \in M$, i.e., $2x \leq y^-$. Hence $x \leq y^- - x$ which gives $x + y \leq y^- - x + y = (-y + x + y)^-$. This entails

$$(2 \odot a) \wedge (2 \odot b) \geq 2 \odot y + (-y + x + y) > 2 \odot y = 2 \odot (a \wedge b),$$

which proves the assertion in question.

The rest follows from Theorem 6.8, Proposition 6.6, and Example 2.2 gives a normal-valued pseudo MV-algebra which is not representable. $\square$

Remark 6.12 *We note that Theorem* 6.11 *gives a positive answer to the problem posed in* [GeIo, Problem 3.15] *whether does* (6.6) *entail representability of pseudo MV-algebras.*

7 Nonexistence of States

In the present Section, we show that in contrast to MV-algebras, there are pseudo MV-algebras such that no maximal ideal is normal, consequently that there exist pseudo MV-algebras having no state.

We start showing that if M has a nonnormal value, then it has infinitely many nonnormal values.

Proposition 7.1 *Let M be a pseudo MV-algebra. If g has at least one value that is nonnormal, then g has infinitely many nonnormal values.*

Proof. Let $M = \Gamma(G, u)$ and let V be a value of $g \in M \setminus \{0\}$. Due to Proposition 6.2, we see that $\phi(V)$ is a nonnormal value of $g \in G$. Applying [Dar, Prop. 40.7], we see that g has infinitely many values $W_1, W_2, \dots$ of g in G. Consequently, $V_n := W_n \cap [0, u]$ is a nonnormal value of g in M for any $n \geq 1$. $\square$

Proposition 7.2 *Let $g > 0$ be an element of an ℓ-group G, and let $G(g)$ be the convex ℓ-subgroup of G generated by g. If V is a value of g in G, then $V \cap G(g)$ is a value of g in $G(g)$, and V is normal in its cover V^* if and only if $V \cap G(g)$ is normal in $G(g)$.*

Proof. It is clear that $V \cap G(g)$ is a value of g in $G(g)$. The rest of the statement follows from [Dar, Cor 40.3]. $\square$

Suppose now that Ω is any linearly ordered set and let $A(\Omega)$ be the set of all order automorphisms $f : \Omega \to \Omega$. On $A(\Omega)$, we define $g \leq h$ iff $g(\alpha) \leq h(\alpha)$ for all $\alpha \in \Omega$. Then under composition and this ordering, $A(\Omega)$ is an ℓ-group such that $(g \wedge h)(\alpha) = g(\alpha) \wedge h(\alpha)$, $\alpha \in \Omega$. These groups are of the great importance because in view of Holland's theorem [Dar, Thm 29.3], saying every ℓ-group is an ℓ-subgroup of $A(\Omega)$ for some linearly ordered Ω.

For $\alpha \in \Omega$, we define the *stabilizer* of α as the set $S_\alpha := \{g \in A(\Omega) : g(\alpha) = \alpha\}$. By [Dar, Prop 30.2], each stabilizer S_α is a prime subgroup.

Let $g \in A(\Omega)$. The *support* of g, written *supp*(g), is the set $\{\alpha \in \Omega : g(\alpha) \neq \alpha\}$. g is called *bounded* (*above*) (*below*) if its support is bounded (above) (below) in Ω. Let $B(\Omega)$ $(L(\Omega))$ $(R(\Omega))$ be the set of all elements of $A(\Omega)$ whose supports are bounded (bounded above) (bounded below). Then $B(\Omega), L(\Omega)$, and $R(\Omega)$ are ℓ-ideals of $A(\Omega)$ [Dar, Prop 30.1].

If $\Omega = \mathbb{R}$ is the real line, then $A(\mathbb{R})$ consists of all strictly increasing continuous functions f from $\mathbb{R}$ onto $\mathbb{R}$, moreover, $A(\mathbb{R})$ has only three proper ℓ-ideals, namely $B(\mathbb{R}), L(\mathbb{R})$, and $R(\mathbb{R})$ (which are not prime). Moreover, for any $x \in \mathbb{R}$, the stabilizer S_x is a maximal convex ℓ-subgroup and so, is a value, but no S_x is normal in $A(\mathbb{R})$ [Dar, p. 183]. On the other hand, if P is a prime subgroup of $A(\mathbb{R})$, then $\{f \in A(\mathbb{R}) : fP = Pf\} = P$ [Dar, Ex 40.1].

Proposition 7.3 *There exists a pseudo MV-algebra M such that any value of* 1 *is not normal, and* $|\Gamma(1)| = \infty$.

Proof. Take $G = A(\mathbb{R})$ and let $g(x) > x$ for any $x \in \mathbb{R}$. Define $G(g)$ as the convex ℓ-subgroup of G generated by g. Then g is a strong unit for $G(g)$, and put $M = \Gamma(G(g), g)$. Let U be a value of g in M. By Proposition 6.2, $\phi(U)$ is a value of g in $G(g)$. Let V be any value of g in G containing $\phi(U)$. Then $V \cap G(g) \supseteq \phi(U)$. We claim $V \cap G(g) = \phi(U)$. Take $f \in V \cap G(g)$. If $f \notin \phi(U)$, then the minimal convex ℓ-subgroup V^f of $G(g)$ generated by $\phi(U)$ and f has to be equal to $G(g)$, but V^f is also a convex ℓ-subgroup of G, V^f has to be a subset of V while $f \in V$, so that $G(g) \subseteq V$, a contradiction. Hence $f \in \phi(U)$.

Since V is a prime subgroup of G, its normalizer $N(V) := \{f \in A(\mathbb{R}) : fV = Vf\}$ is equal to V [Dar, Ex 40.1]. Therefore, V is a value of g which is not normal in G, consequently by Proposition 7.2, $\phi(U) = V \cap G(g)$ is not normal in $G(g)$, and by Proposition 6.2, V is not a normal value of g in M.

By Proposition 7.1, $|\Gamma(1)| = \infty$. $\square$

Corollary 7.4 *There exists a pseudo MV-algebra which has no states on it.*

Proof. This result follows from the one-to-one correspondence between state-morphisms and normal maximal ideals, Proposition 4.6, and from the existence of a pseudo MV-algebra in Proposition 7.3 admitting no normal value of 1. $\square$

We finish with the note that instead of $A(\mathbb{R})$ we can use $A(\Omega)$ in Proposition 7.1, where Ω is *doubly transitive*, i.e., for all $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2$ there exists $g \in A(\Omega)$ such that $g(\alpha_1) = \beta_1$ and $g(\alpha_2) = \beta_2$. For example, $A(\mathbb{R})$ is doubly transitive. In doubly transitive $A(\Omega)$, any stabilizer is a maximal ℓ-subgroup of G, [Dar, Prop 31.12], and for any prime subgroup P, P is its own normalizer, [Dar, Ex 40.1]. So we can obtain a whole family of pseudo MV-algebras admitting no state.

We now present a pseudo MV-algebra M which is not normal-valued but it possess a unique state.

Example 7.5 *Let $M_1 = [0,1]$ and let M_2 be a pseudo MV-algebra having no state. Define $M = M_1 \times M_2$. Let m_0 be the unique state on the real interval M_1 given by $m_0(t) = t$, $t \in [0,1]$. Then the mapping $m : M \to [0,1]$ defined by $m(a,b) = m_0(a)$, $(a,b) \in M_1 \times M_2$, gives a unique state on M.*

Proposition 7.6 *The class of pseudo MV-algebras is closed with respect to pseudo MV-subalgebras, products, but not with respect to homomorphic images.*

Proof. If M_0 is a pseudo MV-subalgebra of M, and if m is a state on M, then $m|M_0$ is a state on M_0.

Let $M = \prod_t M_t$ and let m_{t_0} be a state on M_{t_0}. Then $m(\{a_t\}_t) := m_{t_0}(a_{t_0})$, $\{a_t\}_t \in \prod_t M_t$, is a state on M.

Take the example $M = M_1 \times M_2$ from Example 7.5. Then π_2 defined by $\pi_2((a,b)) := b$, $(a,b) \in M_1 \times M_2$, is a surjective homomorphism of the pseudo MV-algebra M onto M_2, but M_2 possesses no state. $\square$

Problem 7.7 *Characterize pseudo MV-algebras, or equivalently, unital ℓ-groups admitting at least one state.*

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